SPECTRAL ANALYSIS OF EXPANDING ONE-DIMENSIONAL CHAOTIC TRANSFORMATIONS

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SUMMARY

The purpose of this paper is to show explicitly the spectral distribution function of some stationary stochastic processes as

$$X_t = F(X_{t-1}), \quad \text{for} \quad t \in \mathbf{Z},$$

where F is a deterministic two-dimensional invertible map. The invertible map F that will be considered in this paper is the natural extension of a map T on a class $\mathcal{F}_2$ (see Section 1 for definition) of one-dimensional piecewise linear expanding monotonic transformations.

Any non-linear expanding piecewise monotonic transformation $g \in \mathcal{F}_1$ (see Section 1 for definition) can be approximated by a map $T \in \mathcal{F}_2$. From the structural stability of the maps we consider here, it will follow that the spectral density function of the natural extension of any non-linear expanding piecewise monotonic transformation $g \in \mathcal{F}_1$ can be approximated by explicit expressions obtained for the spectral density function of the natural extension of maps T in $\mathcal{F}_2$.

Results for the one-dimensional map T can be obtained from results for the two-dimensional map F.

We also show in last section, that the periodogram is a good estimator (in the distribuition sense) in the case of expanding maps.

Keywords: CHAOTIC TIME SERIES; SPECTRAL ANALYSIS; EXPANDING TRANSFORMATION.

1. INTRODUCTION

We shall consider a special class of *non-linear piecewise monotonic expanding C^2 transformations* g in which the image of any interval of monotonicity is all the interval $(0,1)$ (see Lasota and Mackey (1994) or Section 3 for definition). For instance, Figure 1 shows an example of a map g of such class while Figure 2 shows the graph of a map that is not of the above defined class. The number of intervals of monotonicity of g will be assumed to be finite.

We will denote the set of such class of maps g by $\mathcal{F}_1$.

Consider the class of piecewise linear monotonic continuous expanding transformations T of the following form.

Let A_i, $1 \leq i \leq n$, be an open interval and a_i be the length of A_i, $1 \leq i \leq n$, the intervals of monotonicity (we assume that $T(A_i) = (0,1)$, for $1 \leq i \leq n$), and suppose that $a_i = \sum_{j=1}^{m} b_{ij}$, where $m > n$. Denote by B_{ij}, for $1 \leq i \leq n$ and $1 \leq j \leq m$, intervals such that the length of B_{ij} is b_{ij} and $\cup_{j=1}^{m} B_{ij} = A_i$, for all $1 \leq i \leq n$. Suppose there exists C_j, for $1 \leq j \leq m$, such that $g(B_{ij}) = C_j$, independent of i. Finally, assume that each C_j is contained in a unique A_i and each C_j is a union of sets of the form B_{ul}. Denote by c_j the length of C_j, for $1 \leq j \leq m$.

The analytic expression of $T(x)$ is given by

$$T(x) = \sum_{k=0}^{j-1} c_k + \left(x - \sum_{\alpha=1}^{i-1} a_\alpha - \sum_{\beta=1}^{j-1} b_{i\beta} \right) \frac{c_j}{b_{ij}}, \quad \text{for} \quad x \in B_{ij}.$$

We will denote the set of this second class of maps T by $\mathcal{F}_2$.

In Góra and Boyarsky (1989) and Parry and Pollicott (1990) the explicit expression of the invariant density

$$\sum_{j=1}^{m} I_{C_j}(x) p_j, \quad p_j \in (0,1), \ \text{ with } \sum_{j=1}^{m} p_j = 1$$

of $T \in \mathcal{F}_2$ is obtained by finding the eigenvector $(p_1, p_2, \cdots, p_m)$ of a large matrix. In this way the number p_j, for $1 \leq j \leq m$, can be explicitly obtained.

It is easy to see (we refer the reader to Góra and Boyarsky (1989)) that each $g \in \mathcal{F}_1$ can be C^1 approximated by $T \in \mathcal{F}_2$ (up to a finite number of points where T is not differentiable). In Figure 3, we show the graph of a map $T \in \mathcal{F}_2$.

We will show an explicit formula (see Section 4) for the spectral density function of the natural extension of the piecewise linear map $T \in \mathcal{F}_2$ described above.

Góra and Boyarsky (1989), Li (1976), Ding and Li (1991) and also Parry and Pollicott (1990) show that the invariant density $\eta_n(x)$ of a sequence of maps $T_n \in \mathcal{F}_2$ converging to $g \in \mathcal{F}_1$ satisfies the weak convergence

$$\eta_n(x) \to \eta(x),$$

where $\eta(x)$ is the invariant density for the map g.

Therefore, one can obtain an approximation of the spectral density function of $g \in \mathcal{F}_1$ by an explicit formula for the spectral density function of $T_n \in \mathcal{F}_2$.

The spectral density function of such map $g \in \mathcal{F}_1$ will be a meromorphic function (see Ruelle (1987)).

The spectral density function of maps g of the class $\mathcal{F}_1$ are important for the spectral analysis of chaotic time series and also because the zeta function associated with the potential $-\log g'(x)$ has poles on the same values of the poles of the spectral density function (see Ruelle (1978, 1987) and Rugh (1992)).

We show in section 5, a result of independent interest: the consistency of the periodogram in the distribuition sense for the class of maps $\mathcal{F}_1$ (or for the class $\mathcal{F}_2$).

2. STATIONARY STOCHASTIC PROCESSES

The general setting of chaotic time series we shall analyze is the following. Consider K a compact subset of $\mathbf{R}^n$ with a given Borel σ-algebra $\mathcal{F}$, a bijective continuous transformation $F : K \to K$, an invariant probability $\mathcal{P}$ on K (that is, $\mathcal{P}(F^{-1}(A)) = \mathcal{P}(A)$, for any set $A \in \mathcal{F}$) and $\phi : K \to \mathbf{R}$ a continuous function. We will analyze the stationary stochastic process $\{Z_t\}_{t\in\mathbf{Z}}$ given by

$$Z_t = X_t + \xi_t = (\phi \circ F)(X_{t-1}) + \xi_t, \quad \text{for } t \in \mathbf{Z}. \tag{2.1}$$

The natural measure on $K^{\mathbf{Z}}$ is the product measure $\mathcal{P}^{\mathbf{Z}}$ on $K^{\mathbf{Z}}$ and it is invariant for the stationary process $\{X_t\}_{t\in\mathbf{Z}}$ or $\{Z_t\}_{t\in\mathbf{Z}}$. The process $\{\xi_t\}_{t\in\mathbf{Z}}$ is considered to be a Gaussian white noise process (see Brockwell and Davis (1987)) independent of $\{(\phi \circ F)(X_t)\}_{t\in\mathbf{Z}}$, with zero mean and variance σ_ξ^2. One observes that in the model (2.1) the random variables X_t (or Z_t) and X_{t+1} (or Z_{t+1}) are generally not independent.

We shall denote the above system by

$$(K, F, \mathcal{P}, \phi, \mathcal{F}, \sigma_\xi^2). \tag{2.2}$$

Following the terminology in Tong (1990) we may call the system (2.1), when $\sigma_\xi^2 = 0$, the *skeleton* of the system.

Given a certain measurable function $\phi : K \to \mathbf{R}$ *the autocovariance function at lag* $h \in \mathbf{Z}$ (see Brockwell and Davis (1987)) *of the process* $\{X_t\}_{t\in\mathbf{Z}}$ as in (2.1) is given by

$$R_{XX}(h) = E(X_t X_{t+h}) - [E(X_t)]^2 = \int \phi(x)\phi(F^h(x))d\mathcal{P}(x) - \left[\int \phi(x)d\mathcal{P}(x)\right]^2. \tag{2.3}$$

The autocovariance function $R_{XX}(h)$ in (2.3) measures the covariance between two values of the process $\{X_t\}_{t\in\mathbf{Z}}$ separated by lag h. The *autocorrelation function at lag h of the process* $\{X_t\}_{t\in\mathbf{Z}}$ (see Brockwell and Davis (1987)) is given by

$$\rho_X(h) = \frac{R_{XX}(h)}{R_{XX}(0)}, \quad \text{for} \quad h \in \mathbf{Z}, \tag{2.4}$$

where $R_{XX}(0) = E[(X_t - E(X_t))^2] = \mathrm{Var}(X_t)$ is the variance of the process.

From the Herglotz's theorem (see Brockwell and Davis (1987)) a function $\rho_X(h)$ is non-negative definite if and only if

$$\rho_X(h) = \int_{-\pi}^{\pi} e^{i\lambda h} dF_X(\lambda), \quad \text{for any} \quad h \in \mathbf{Z}, \tag{2.5}$$

where $F_X(\cdot)$ is a right-continuous, non-decreasing, bounded function on $[-\pi, \pi]$ with $F_X(-\pi) = 0$. The function $F_X(\cdot)$ is called *the spectral distribution function* of $\{X_t\}_{t\in\mathbf{Z}}$ and if

$$F_X(\lambda) = \int_{-\pi}^{\lambda} f_X(\omega)\, d\omega, \quad \text{for } -\pi < \lambda \leq \pi, \tag{2.6}$$

then $f_X(\cdot)$ is called *the spectral density function* of the process $\{X_t\}_{t\in\mathbf{Z}}$. When

$$\sum_{h=-\infty}^{\infty} |\rho_X(h)| < \infty,$$

then $\rho_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda)\, d\lambda$, for $h \in \mathbf{Z}$, where $f_X(\cdot)$ is given by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \rho_X(h). \tag{2.7}$$

This function has real values if $\rho_X(h) = \rho_X(-h)$, for all $h \in \mathbf{N}$.

The reason to consider F a bijective map and not just a non-invertible map is for defining $R_{XX}(h)$ also for negative values of $h \in \mathbf{Z}$ and, from this, (2.7) will be well defined.

Each particular invertible transformation F will require different technique in order to obtain explicitly the spectral distribution function (see Lopes and Lopes (1995)).

Example: When the compact subset K is equal to $[-\pi, \pi]$, the transformation F is given by $F(x) = \omega_0 + x \pmod{2\pi}$, with $\omega_0 \in (0, \pi)$, and $\phi(x) = \cos(x)$. This is the classical harmonic model $Z_t = \cos(w_0 t + x) + \xi_t$. The spectral measure of the process $\{X_t\}_{t\in\mathbf{Z}} = \{(\phi \circ F)(X_{t-1})\}_{t\in\mathbf{Z}}$ as in (2.1) is not a function but a *distribution function* given by

$$dF_X(\lambda) = \frac{1}{2}(\delta_{\omega_0} + \delta_{-\omega_0}), \tag{2.8}$$

where δ_{ω_0} is the Dirac delta function concentrated at ω_0.

Remark: Expanding maps (see Section 3 for the definition) always have an exponential decay of autocorrelations, for any ϕ Holder continuous function (see Parry and Pollicott (1990)). Therefore, in this case the spectral density function always exists and it is a meromorphic function (see Ruelle (1978, 1987)).

3. THE NATURAL EXTENSION F OF T

It is well known that in general larger the dimension of the set K, more difficult is to analyze the dynamics of the map F.

When K is one-dimensional, that is, when K is a segment, the diffeomorphism $F : K \to K$ has simple dynamics.

In general, the dynamics of an one-dimensional diffeomorphism is very simple.

The simplest example in dimension 2, that is, when K is a square $[0, 1] \times [0, 1]$, is obtained when F is the natural extension of an one-dimensional map T. The map T is not an one-to-one map, but F is.

When the transformation T is an *expanding map*, that is, there exists $\lambda > 1$ such that $|T'(x)| > \lambda$, for all $x \in [0,1]$, then there exists (see Lasota and Yorke (1973) and also Parry and Pollicott (1990)) a density $\eta(x)$ such that $d\mu(x) = \eta(x)\,dx$ is invariant for T (that is, $\mu(T^{-1}(A)) = \mu(A)$, for any Borel set A). The probability μ is *ergodic* (see Cornfeld, et al. (1982) for the definition) for such map T. There exists a natural way to obtain from such T a bijective map F, called *the natural extension of* T. Denote by (x,y) a vector in the domain K and by $(x',y') = F(x,y)$ its image by the map F. Then, (see Bogomolny and Carioli (1995))

$$T(x) = x' \quad \text{and} \quad T(y') = y$$

defines F.

If T is an expanding map the corresponding F is *Axiom A* (see Robinson (1995) for definition).

The invariant probability μ for T on $[0,1]$ has a natural extension to a probability $\nu = \mathcal{P}$ (according to the notation of Section 2) on $K = [0,1] \times [0,1]$ invariant for F.

Consider now the random variable $\phi : K \to \mathbf{R}$ of the form $\phi(x,y) = \phi(x)$. Then, the time series

$$X_t = \phi(F^t(x,y)) = \phi(T^t(x)), \quad \text{for} \quad 1 \le t \le N,$$

and the probability ν define the simplest example of a chaotic time series on dimension 2.

The dynamics comes basically from an one-dimensional map even if the setting is for a two-dimensional bijective map. As we mentioned before the reason to consider bijective maps is to obtain $R_{XX}(h)$, for $h \in \mathbf{Z}$.

The analysis of the dynamics of F is more general and results for T can be derived from the former transformation.

When $\phi(x,y) = x$, for a certain class of maps T in $\mathcal{F}_2$ (see Section 4) we shall be able to show explicitly the spectral density function. This is obtained by solving some linear systems as we shall explain later. We call a stochastic process obtained from the system (F,ϕ) as above *a standard stochastic process* obtained from (T,ϕ).

Any expanding map g in $\mathcal{F}_1$ can be approximated by maps T in $\mathcal{F}_2$ and the corresponding absolutely continuous invariant measure of T will converge to the corresponding one for g (see Góra and Boyarsky (1989)). Therefore, we will be able to approximate the spectral density function of expanding maps by known expressions. This is the main purpose of this paper.

In the sequel, we shall omit the noise process $\{\xi_t\}_{t\in\mathbf{Z}}$ of the system due to the fact that it does not interfere in the dynamics of T and that the spectral density function of the whole system with noise can be easily obtained from the one without noise (see Lopes et al. (1995, 1996)).

4. THE SPECTRAL DENSITY OF PIECEWISE LINEAR EXPANDING TRANSFORMATIONS

In this section we will show the explicit expression of the spectral density function of the system $X_t = T^t(X_0)$, where $T \in \mathcal{F}_2$.

We will denote by μ the T-invariant measure absolutely continuous with respect to the Lebesgue measure. Denote by $p(x)$ the density of such measure μ. It is well known (see Parry and Pollicott (1990)) that $p(x)$ is of the form

$$p(x) = \sum_{j=1}^{m} I_{C_j}(x)\, p_j.$$

Therefore, $\mu(C_j) = \int_{C_j} p(x)dx = p_j$, for $1 \leq j \leq m$. The number p_j, for $1 \leq j \leq m$, can be obtained by finding an eigenvector of a large matrix (see Góra and Boyarsky (1989) and Li (1976)). Denote also by p_{ij} the measure μ of the interval B_{ij}, $1 \leq i \leq n$ and $1 \leq j \leq m$.

It is enough to show the explicit expression of

$$\gamma(z) = \sum_{k=0}^{\infty} \left(\int_0^1 x\, T^k(x) p(x) dx \right) z^k$$

and the corresponding explicit expression for the spectral density function (2.7) will easily follow (see Lopes et al. (1996)).

In the sequel, we shall consider the following notation:

$$A(k,i,j) = \int_{B_{ij}} x\, T^k(x) p(x) dx, \tag{4.1}$$

$$V(k,i,j) = \int_{B_{ij}} T^k(x) dx, \tag{4.2}$$

$$B(k,i,j) = \int_{B_{ij}} x\, T^k(x) dx \tag{4.3}$$

and

$$A(k) = \int_0^1 x\, T^k(x) p(x) dx. \tag{4.4}$$

First of all we shall compute a recursive formula for $V(k,i,j)$. One observes, from the expression (4.2), that

$$V(k+1,r,s) = \int_{B_{rs}} T^{k+1}(x) dx = \int_{B_{rs}} T^k(T(x)) dx =$$

$$= \int_{C_s} T^k(y) dy \frac{b_{rs}}{c_s} = \sum_{B_{uv} \subset C_s} \left(\int_{B_{uv}} T^k(y) dy \right) \frac{b_{rs}}{c_s} = \frac{b_{rs}}{c_s} \sum_{B_{uv} \subset C_s} V(k,u,v). \tag{4.5}$$

Now we shall obtain the recursive formula for $B(k,i,j)$. One observes, from the expression (4.3), that

$$B(k+1,r,s) = \int_{B_{rs}} x\, T^{k+1}(x) dx = \int_{B_{rs}} x\, T^k(T(x)) dx =$$

$$= \int_{B_{rs}} x\,T^k \Big[\sum_{k=0}^{s-1} c_k + \Big(x - \sum_{\alpha=1}^{r-1} a_\alpha - \sum_{\beta=1}^{s-1} b_{r\beta} \Big) \frac{c_s}{b_{rs}} \Big] dx =$$

$$= \int_{C_s} \Big[\Big(y - \sum_{k=0}^{s-1} c_k \Big) \frac{b_{rs}}{c_s} + \sum_{\alpha=1}^{r-1} a_\alpha + \sum_{\beta=1}^{s-1} b_{r\beta} \Big] T^k(y) \frac{b_{rs}}{c_s} dy =$$

$$= \sum_{B_{ls} \subset C_s} \frac{b_{rs}}{c_s} \Big[\int_{B_{ls}} y\,T^k(y) dy \frac{b_{rs}}{c_s} +$$

$$+ \Big(- \sum_{k=0}^{s-1} c_k \frac{b_{rs}}{c_s} + \sum_{\alpha=1}^{r-1} a_\alpha + \sum_{\beta=1}^{s-1} b_{r\beta} \Big) \int_{B_{ls}} T^k(y) dy \Big] =$$

$$= \sum_{B_{ls} \subset C_s} \frac{b_{rs}}{c_s} \Big[B(k,l,s) \frac{b_{rs}}{c_s} + \Big(- \sum_{k=0}^{s-1} c_k \frac{b_{rs}}{c_s} + \sum_{\alpha=1}^{r-1} a_\alpha + \sum_{\beta=1}^{s-1} b_{r\beta} \Big) V(k,l,s) \Big]. \tag{4.6}$$

We denote $\psi_{ij}(z)$ by

$$\psi_{ij}(z) = \sum_{k \geq 0} V(k,i,j)\, z^k = \sum_{k \geq 0} \Big(\int_{B_{ij}} T^k(x) dx \Big) z^k \tag{4.7}$$

and $\varphi_{ij}(z)$ by

$$\varphi_{ij}(z) = \sum_{k \geq 0} B(k,i,j)\, z^k = \sum_{k \geq 0} \Big(\int_{B_{ij}} x\,T^{k+1}(x) dx \Big) z^k. \tag{4.8}$$

Our purpose is to estimate

$$\gamma(z) = \sum_{k \geq 0} A_k z^k = \sum_{k \geq 0} \Big(\int x\,T^k(x) p(x) dx \Big) z^k, \tag{4.9}$$

but first we need to estimate $\psi_{ij}(z)$.

From (4.5), the power series of $V(k,i,j)$ satisfies the following equation

$$\psi_{ij}(z) = \sum_{k \geq 0} V(k,i,j)\, z^k = V(0,i,j) + \sum_{k \geq 0} V(k+1,i,j)\, z^{k+1} =$$

$$= V(0,i,j) + z \sum_{k \geq 0} V(k+1,i,j)\, z^k =$$

$$= V(0,i,j) + z \sum_{k \geq 0} \Big(\frac{b_{ij}}{c_j} \sum_{B_{uv} \subset C_j} V(k,u,v) \Big) z^k =$$

$$= V(0,i,j) + z\,\frac{b_{ij}}{c_j}\sum_{B_{uv}\subset C_j}\left(\sum_{k\geq 0} V(k,u,v)\,z^k\right) =$$

$$= V(0,i,j) + z\,\frac{b_{ij}}{c_j}\sum_{B_{uv}\subset C_j}\psi_{u,v}(z).$$

Therefore, one can estimate $\psi_{i,j}(z)$ by solving the linear system

$$\psi_{ij}(z) = V(0,i,j) + z\,\frac{b_{ij}}{c_j}\sum_{B_{uv}\subset C_j}\psi_{u,v}(z). \tag{4.10}$$

Finally, the power series of $B(k,i,j)$ satisfies the equation

$$\varphi_{i,j}(z) = \sum_{k\geq 0} B(k,i,j)\,z^k = B(0,i,j) + z\sum_{k\geq 0} B(k+1,i,j)\,z^k =$$

$$= B(0,i,j) +$$

$$+z\sum_{k\geq 0} z^k\left(\sum_{B_{lj}\subset C_j}\frac{b_{ij}}{c_j}\left[B(k,l,j)\frac{b_{ij}}{c_j} + \left(-\sum_{k=0}^{j-1} c_k\frac{b_{ij}}{c_j} + \sum_{\alpha=1}^{i-1} a_\alpha + \sum_{\beta=1}^{j-1} b_{i\beta}\right)V(k,l,j)\right]\right) =$$

$$= B(0,i,j) + z\left[\sum_{B_{lj}\subset C_j}\left(\frac{b_{ij}}{c_j}\right)^2\varphi_{lj}(z) +\right.$$

$$\left.+\sum_{B_{lj}\subset C_j}\left(\frac{b_{ij}}{c_j}\right)^2\left(-\sum_{k=0}^{j-1} c_k\frac{b_{ij}}{c_j} + \sum_{\alpha=1}^{i-1} a_\alpha + \sum_{\beta=1}^{j-1} b_{i\beta}\right)\psi_{l,j}(z)\right]. \tag{4.11}$$

As we know the values $\psi_{ij}(z)$ from (4.10), one can obtain $\varphi_{i,j}(z)$ from the linear equation (4.11).

Finally, we obtain $\gamma(z)$ explicitly by

$$\gamma(z) = \sum_{k\geq 0} A_k z^k = \sum_{k\geq 0}\left(\int_0^1 x\,T^k(x)p(x)dx\right)z^k$$

$$= \sum_{i,j}\sum_{k\geq 0}\left(\int_{B_{ij}} x\,T^k(x)p(x)dx\right)z^k =$$

$$= \sum_{i,j} p_{ij}\sum_{k\geq 0}\left(\int_{B_{ij}} xT^k(x)dx\right)z^k = \sum_{i,j} p_{ij}\,\varphi_{i,j}(z). \tag{4.12}$$

The spectral density function of X_t is given by

$$f_X(\lambda) = \frac{1}{2\pi\, Var(X_t)}\left[\gamma(e^{i\lambda}) + \gamma(e^{-i\lambda}) - E(X_t^2)\right], \quad \text{for any} \quad \lambda\in(-\pi,\pi], \tag{4.13}$$

where $Var(X_t) = E(X_t^2) - [E(X_t)]^2$ and $\gamma(z)$ is given by the expression (4.12).

Remark: The power series $\gamma(z)$ is an analytic function on the unit disc $\{z \in \mathbf{C} | \parallel z \parallel < 1\}$ and the expression (4.13) has the meaning of the radial limit

$$\lim_{r \to 1} re^{i\lambda} = e^{i\lambda} = z.$$

In this sense, the series

$$\sum_{n \in \mathbf{Z}} e^{in\,\lambda} = 2\,\mathcal{R}\mathrm{e}\left(\frac{1}{1 - e^{i\lambda}}\right) - 1 = 0, \quad \text{for} \quad \lambda \neq 0,$$

even though the series $\sum_{n \in \mathbf{Z}} e^{in\,\lambda}$ does not converge. We are using this fact in the expression (4.13) above.

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