AMPLITUDE ESTIMATION IN MULTIPLE FREQUENCY SPECTRUM

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ABSTRACT

Here we analyze the mixed spectrum stationary process

$$Z_t = \sum_{j=1}^{p} A_j \cos(\omega_j t + \phi_j) + \varepsilon_t, \quad t \in \mathbf{Z},$$

where p is not necessarily known, $A_1, \cdots, A_p$ are unknown constants, $\omega_1, \cdots, \omega_p$ are unknown frequencies with values in $(-\pi, \pi]$, $\{\varepsilon_t\}_{t\in\mathbf{Z}}$ is white noise with mean 0 and variance σ_ε^2, and $\phi_1, \cdots, \phi_p$ are random variables uniformly distributed in $(-\pi, \pi]$, independent of each other and of the noise process $\{\varepsilon_t\}_{t\in\mathbf{Z}}$. The assumption of white noise is not really needed, but it simplifies the exposition. In fact, any continuous spectrum noise will do just as well. We estimate the amplitudes A_j, $1 \leq j \leq p$, and the variance σ_ε^2 of the noise component based on *HOC (higher order correlations)* sequences $\rho_1(\theta_k)$ from a parametric family of linear filters (α^n-*filters*).
We also present an updating iterative procedure of a certain mapping ρ_1 that gives rise to a periodic orbit of period 2 (*Slutsky filter*).

1. Introduction

In general, when a filter is applied to a time series, it changes the series mode of oscillation. Thus, when a bank of filters is applied to the same series, we obtain a sequence or family of oscillation patterns. The resulting family of zero-crossing counts is referred to as *higher order crossings* or simply *HOC*. The corresponding first order autocorrelations are referred to as *higher order correlations*, or simply *HOC* again. Because the first order autocorrelation and the expected zero-crossing rate of a real valued stationary Gaussian time series are essentially equivalent (see Kedem (1986)), the practice of using the same acronym is quite tolerable. Here we will just consider *higher order correlations*. The process we will analyze is not Gaussian.

In this paper we show how to construct a convergent sequence of *HOC* (*higher order correlations*) to estimate all parameters in the following model

$$Z_t = \sum_{j=1}^{p} A_j \cos(\omega_j t + \phi_j) + \varepsilon_t = X_t + \varepsilon_t, \quad \text{for} \quad t \in \mathbf{Z}, \tag{1}$$

where p is not necessarily known and, for each $j \in \{1, 2, \cdots, p\}$, A_j is an unknown constant, ω_j is an unknown frequency with value in $(-\pi, \pi]$ and the phase ϕ_j is a random variable uniformly distributed in $(-\pi, \pi]$ independent of each other and of the noise component. We assume that the noise process is Gaussian and white for simplicity of the exposition, that is, $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, however for any stationary and ergodic process with continuous spectral density function $f_\varepsilon(\lambda)$ the results follow similarly. Observe that the process $\{Z_t\}_{t\in\mathbf{Z}}$ is not Gaussian.

In Lopes and Kedem (1991) we show how to estimate the frequencies ω_j, $1 \leq j \leq p$, of the model (1). Here we will show how to estimate the amplitudes A_j, $1 \leq j \leq p$, and the variance σ_ε^2 of the noise process.

The gist of the idea is to employ *HOC* sequences in the fine tuning of parametric filters. This is done iteratively as follows. A time series is filtered by a parametric filter, and the resulting first order autocorrelation is immediately used in adjusting the filter parameter. The adjusted filter is then applied again, giving rise to a new first order autocorrelation, and the procedure is repeated. By choosing the filters appropriately, *the scheme gives convergent sequences of higher order correlations, or equivalently, convergent sequences of higher order crossings, depending on what one chooses to observe, correlations or zero-crossing counts.*

The *HOC* method is a faster way to estimate the frequencies than the traditional method of the periodogram analysis based on "Fast Fourier Transform".

The estimation of a finite number of frequencies of signals buried in random noise, as the model (1) with mixed spectrum is an old and very important problem related to different fields as seismology, radar, sonar, radioastronomy and it has received many research work in several areas as communication, signal processing and statistics.

In Section 2 we give an outline of the main properties of Lopes and Kedem (1991) and in Section 3 we present a new result: a filter that with an updating procedure gives rise to periodic orbits of period 2. In Section 4 we present the main result of this paper, namely, a method to estimate the amplitudes A_j, $1 \leq j \leq p$, and the variance σ_ε^2 of the noise component. We remark here that Section 3 is independent of Section 4.

We applied the method to a simulated model with $p = 2$, $A_1 = A_2 = 1.0$, $\sigma_\varepsilon = 0.3$, $\omega_1 = 0.7$ and $\omega_2 = 2.2$. We previously find (see Lopes and Kedem (1991)) the strong consistent estimates $\hat{\omega}_1 = 0.7044$ and $\hat{\omega}_2 = 2.1965$. We simulate a time series with N = 3000 observations. Using these informations we apply the method presented in Section 4 and we find the estimates $\hat{A}_1 = 0.9783$, $\hat{A}_2 = 0.9525$ and $\hat{\sigma}_\varepsilon = 0.3976$.

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2. Complex Filter and Fixed Points

Let $\{Z_t\}$, $t = 0, \pm 1, \pm 2, \cdots$, be a zero-mean stationary process, and let $\{\mathcal{L}_\theta(\cdot)\}_{\theta \in \Theta}$, be a parametric family of time invariant linear filters, where θ is a finite dimensional parameter in the parameter space Θ. Denote by $\{Z_t(\theta)\}_{t \in \mathbf{Z}}$ the filtered process,

$$Z_t(\theta) = \mathcal{L}_\theta(Z)_t.$$

Then $\{\rho_1(\theta)\}_{\theta \in \Theta}$, defined by

$$\rho_1(\theta) = \frac{\mathcal{R}\{E[Z_t(\theta)\overline{Z_{t+1}(\theta)}]\}}{E|Z_t(\theta)|^2}$$

is a *HOC* family defined from a parametrized first order autocorrelation. Here and elsewhere, a bar denotes complex conjugate and $\mathcal{R}\{z\}$ the real part of z.

He e Kedem (1989), using the *alpha filter* (see Definition 4.1 with $n = 1$), present an iterative method for the case $p = 1$ (that is, only one frequency in the model (1)) obtaining the frequency of the discrete spectrum part of the process $\{Z_t\}_{t \in \mathbf{Z}}$. However, with the *alpha filter* one can not obtain the frequencies through the iterative procedure when $p \geq 2$. To present an iterative procedure that works well when $p \geq 2$, it is necessary to use a different filter. We show in Lopes and Kedem (1991) that the *complex filter* (see Definition 2.1) is the filter which enables one to find, through an iterative method, the frequencies ω_j, $1 \leq j \leq p$, when $p \geq 2$. Note that we also can find the value of p.

Now we will present a brief outline of this method in order the reader can understand how the estimates $\hat{\omega}_j$, $1 \leq j \leq p$, are found. These considerations are the main motivation for what will be developed in Sections 3 and 4.

Consider the stochastic process $\{Z_t\}_{t \in \mathbf{Z}}$ as in (1).

Definition 2.1: The *complex filter* applied to the process $\{Z_t\}_{t\in\mathbf{Z}}$ is defined by the transformation

$$Z_t(\alpha, M) = (1 + e^{i\theta(\alpha)}\mathcal{B})^M Z_t, \quad \text{for } t \in \mathbf{Z}, \quad -1 < \alpha < 1 \quad \text{and} \quad -\pi < \theta(\alpha) < \pi,$$

where M is a positive integer and $\mathcal{B}$ is the *shift operator* $\mathcal{B}Z_t = Z_{t-1}$. We think of $\theta(\alpha)$ as the *"center of the filter"* and define it by $\theta(\alpha) = \cos^{-1}(\alpha)$.

For the properties of the *complex filter* see Lopes (1991) and Lopes and Kedem (1991).

Let $\{Z_t\}_{t=1}^{N+M}$ be a time series of length $N + M$ obtained from the process (1) and $Z_t(\alpha, M)$ be the correspondent complex-filtered time series version. We consider M fixed and large (typically, $M = 30$) and consider the first order autocorrelation function given by

$$\rho_1(\alpha) = \rho_1(\alpha, M) = \frac{\mathcal{R}\{E[Z_t(\alpha, M)\overline{Z_{t+1}(\alpha, M)}]\}}{E|Z_t(\alpha, M)|^2} \tag{2}$$

as a function of the variable α. The function $\rho_1(\alpha)$ is a mapping onto $[-1, 1]$.

From an initial value α_0, chosen at random in $(-1, 1)$, define the recursion

$$\alpha_{k+1} = \rho_1(\alpha_k) = \rho_1^k(\alpha_0), \quad \text{para } k \in \mathbf{N}. \tag{3}$$

One calls $\alpha_{k+1} = \rho_1^k(\alpha_0)$ the k^{th} *-iterate of* α_0.

In Figures 1 and 2 we present, for $M = 15$, the graph of $\rho_1(\alpha)$ and $\rho_1^5(\alpha)$ as functions of the variable α. We also plot the diagonal line: the fixed points are located in the intersection of the graph of ρ_1 and the diagonal line.

Observe that the graph of $\rho_1^5(\alpha)$ (Figure 2) shows that, for any α_0, the iterated value $\rho_1^5(\alpha_0)$ is always very close to the fixed points and these for there turn are very close to the values $\cos(\omega_1)$ and $\cos(\omega_2)$ to be estimated.

Definition 2.2: The updating scheme of the form (3) is said to be *approximately globally convergent* if for each fixed $M \in \mathbf{N} - \{0\}$ there exists a set C_M of full measure in [-1,1] such that for any $\alpha_0 \in C_M$ there exists the limit

$$\lim_{k\to\infty} \alpha_k = \alpha_M^*. \tag{4}$$

The values α_M^* are fixed points of the mapping ρ_1, that is, $\rho_1(\alpha_M^*) = \alpha_M^*$.

The iterative updated procedure is considered with respect to the filter with parameter M and the value of α_M^* can depend on α_0. We shall require in this definition that there exist p of these possible values α_M^* and for each one of them there exists $l \in \{1, 2, \cdots, p\}$ such that

$$\lim_{M\to\infty} \alpha_M^* = \cos(\omega_l). \tag{5}$$

The following result guarantees that the estimation of the frequencies of the process (1) is obtained through the fixed points of the mapping $\rho_1(\alpha)$ in (2). When M is large (for instance, $M = 30$), from (4) and (5), the attracting fixed points of $\rho_1(\alpha)$ are very close to the values $\cos(\omega_j)$, $1 \le j \le p$.

Theorem 2.1: *Consider the stochastic process $\{Z_t\}_{t\in\mathbf{Z}}$ as in (1) where the additive noise is white and independent of the process $\{X_t\}_{t\in\mathbf{Z}}$. Let $\{\mathcal{L}_\theta(Z)_t\}_{\theta\in\Theta} = \{Z_t(\alpha, M)\}_{(\alpha,M)\in\Theta}$, where $\theta = (\alpha, M) \in (-1,1)\times\mathbf{N} = \Theta$, be a family of complex filters. Consider the iterative updating scheme (3). Then, the family $\{\mathcal{L}_\theta(Z)_t\}_{\theta\in\Theta}$ is approximately globally convergent.*

The property of strong consistency of the estimators is obtained through the following result (Lopes (1991)).

Theorem 2.2: *Consider the stochastic process $\{Z_t\}_{t\in\mathbf{Z}}$ as in (1) where the additive noise is white and independent of the process $\{X_t\}_{t\in\mathbf{Z}}$. Then, the process $\{Z_t\}_{t\in\mathbf{Z}}$ is stationary and ergodic whenever, for all $j \in \{1, 2, \cdots, p\}$, the values $\frac{\omega_j}{2\pi}$ are irrational and rationally independent.*

The condition of being irrational and rationally independent is general (with probability one) in the set of all possible frequencies.

The above theorem also assures the strong consistency property of the estimates to be used in Section 4.

In Lopes and Kedem (1991) we show that one can estimate the cosine of the frequencies ω_j, $1 \le j \le p$, in the following way: consider a value $\alpha_0 \in (-1, 1)$ at random, apply the iterative updating procedure with $M = 30$ fixed and obtain the values

$$\alpha_0, \quad \alpha_1 = \rho_1(\alpha_0), \quad \alpha_2 = \rho_1(\alpha_1), \quad \cdots, \quad \alpha_{k+1} = \rho_1(\alpha_k).$$

From expression (4) the value α_k with k large (for instance, $k = 20$) is close to an attracting fixed point α_M^*. Since M is large, from expression (5) we know that α_M^* is close to a certain $\cos(\omega_{l_0})$, $1 \le l_0 \le p$. In this way we find an approximated value for one of the frequencies of the discrete spectrum part. After that, with a bandpass filter, one filters out this value α_k, that is approximately $\cos(\omega_{l_0})$, and applies the same above procedure to the resulting time series. Considering another α_0 at random, one estimates, through the updating iterative procedure (3), another value $\cos(\omega_{l_1})$, $1 \le l_1 \le p$ and $l_1 \ne l_0$. In this way and successively, one obtains all frequencies ω_j, $1 \le j \le p$.

After one finds the last frequency ω_p, the mapping ρ_1 will not move anymore the initial value α_0, that is, $\rho_1(\alpha_0) = \alpha_0$, for all $\alpha_0 \in (-1, 1)$. In this way one knows that the procedure is finished and one found the value of p.

A different procedure is the one obtained from decreasing the bandwidth (that is, by increasing M) at each iteration of $\alpha_{k+1} = \rho_1(\alpha_k, M_k)$, until one gets the convergence

$$\lim_{k\to\infty} \alpha_k = \cos(\omega_1),$$

where the initial value $\alpha_0 \in (-1,1)$ is in a small neighborhood of the $\cos(\omega_1)$ with no other $\cos(\omega_j)$, for $j \neq 1$. For this procedure, see Kedem and Lopes (1991).

With any one of these procedures one obtains all frequencies ω_j, $1 \le j \le p$, of the discrete spectrum part of the process $\{Z_t\}_{t\in\mathbf{Z}}$ defined by (1).

In Section 4 we will show how to estimate the amplitudes A_j, $1 \le j \le p$, and the variance σ_ε^2 of the noise process. Before that we will consider the important question: Is it possible to find a family of filters that produces an updating scheme with attractor points that are not fixed ones? The answer is yes, the *Slutsky filter*. With this filter we locate a periodic orbit with period 2 that gives us directly the maximal and the minimal frequencies in a particular case of model (1).

Our purpose in the example below is to show the rich variety of possible kinds of dynamics of the mapping ρ_1 one can observe applying different filters.

3. Slutsky Filter: Periodic Points of Period 2

In this section we will analyze the action of the *Slutsky Filter* (see Definition 3.1) instead of the *complex filter* as we consider in the previous section.

We want to analyze the model as in (1). Assume that the frequencies ω_j, $1 \le j \le p$, can be written in an increasing order

$$0 < \omega_1 < \omega_2 < \cdots < \omega_p.$$

Our interest now is to find, respectively, the lowest and largest frequencies ω_1 and ω_p.

If we apply the *Slutsky filter* to a time series $\{Z_t\}_{t=1}^N$ of length N in a particular example of the process (1) and we apply the updating procedure given by the mapping ρ_1 then its sucessive iterations will allow one to find ω_1 and ω_p, that is, the lowest and the highest frequencies of the model (1). We will explain later how this property is achieved.

We point out that the case considered in this section is an example of what can happen when one applies filters to a time series. It is not a general method for finding the maximal and minimal frequencies.

Now we state some definitions and address some properties of periodic points.

Definition 3.1: The *Slutsky filter* applied to the process $\{Z_t\}_{t\in\mathbf{Z}}$ is defined as the time invariant linear transformation

$$Z_t(\theta,k) = [(1-e^{i\theta}\mathcal{B})^m(1+e^{i\theta}\mathcal{B})^n]^k\, Z_t, \quad \text{for } t \in \mathbf{Z}, \quad -\pi < \theta < \pi, \quad \text{and m, n, k} \in \mathbf{N} \tag{6}$$

where $\mathcal{B}$ is the *shift operator* $\mathcal{B}Z_t = Z_{t-1}$.

The transfer function is given by

$$H_k(\lambda;\theta) = [(1-e^{i(\theta-\lambda)})^m(1+e^{i(\theta-\lambda)})^n]^k, \quad \text{for} \quad -\pi < \lambda \leq \pi,$$

and the corresponding squared gain function is given by

$$|H_k(\lambda;\theta)|^2 = 2^{k(m+n)}[(1-\cos(\theta-\lambda))^m(1+\cos(\theta-\lambda))^n]^k, \quad \text{for} \quad -\pi < \lambda \leq \pi. \tag{7}$$

Define the mapping $\rho_1\colon(-1,1)\to(-1,1)$ such that

$$\rho_1(\alpha,k) = \frac{\mathcal{R}\{E[Z_t(\alpha,k)\overline{Z_{t+1}(\alpha,k)}]\}}{E|Z_t(\alpha,k)|^2}, \quad \text{for k} \in \mathbf{N} \quad \text{fixed}, \tag{8}$$

where $\alpha = \cos(\theta) \in (-1,1)$, with $\theta \in (-\pi,\pi]$.

Definition 3.1 appears in He and Kedem (1989), page 364. Some of the properties of the *Slutsky filter* are derived in that paper. The novelty here is that we consider $m = n = 1$, $k = 10$ and then we analyze the iterative procedure of updating the filter parameter through the transformation ρ_1 as in the *complex filter*. Here we will consider s large, that is, high iterates $\rho_1^s(\alpha,k)$ and we want to derive substantial information from them. We will consider the case $m = n = 1$, that is, the transformation

$$\mathcal{L}_{(\theta,k)}(\cdot) = [(1-e^{i\theta}\mathcal{B})(1+e^{i\theta}\mathcal{B})]^k. \tag{9}$$

Our purpose here is to show that in a particular example if $k = 10$ then the iterative procedure of updating the parameter $\theta = \cos^{-1}(\alpha)$ in the *Slutsky filter* ($m = n = 1$) will locate the lowest and the highest frequencies.

We will consider the stochastic process given by (1) with $p = 4$, $A_1 = 20$, $A_2 = A_3 = 0.1$, $A_4 = 1.0$, $\cos(\omega_1) = 0.2$, $\cos(\omega_2) = 0.4$, $\cos(\omega_3) = 0.6$, $\cos(\omega_4) = 0.8$ and $\sigma_\varepsilon = 0.01$. We will take $m = n = 1$ and $k = 10$ in the definition of the *Slustky filter*.

Now we will introduce some definitions about periodic orbits.

Definition 3.2: A point $\alpha^* \in [-1,1]$ is a *periodic point of period 2* for a function $f\colon[-1,1]\to[-1,1]$ if

$$f(\alpha^*) = \alpha^{**} \quad \text{and} \quad f(\alpha^{**}) = \alpha^*.$$

In other terms, $f^2(\alpha^*) = \alpha^*$, that is α^* is a fixed point for f^2. Note that in this case $\alpha^{**} = f(\alpha^*)$ is also a periodic point of period 2. We say that $\{\alpha^*, \alpha^{**}\}$ constitutes *a periodic orbit of period 2 for f*. More generally, for each value α (not necessarily periodic) the set $\{\alpha, f(\alpha), f^2(\alpha), \cdots, f^n(\alpha), \cdots\}$ is called the *orbit of* α. We say that α is a *periodic point* if its orbit is finite.

We have analogous definitions and properties as in the case of fixed points.

Definition 3.3: The set $\{\alpha^*, \alpha^{**}\}$ is an *attracting periodic orbit of period 2* if

$$|(f^2(\alpha^*))'| < 1.$$

It also follows that

$$|(f^2(\alpha^{**}))'| < 1.$$

If $\{\alpha^*, \alpha^{**}\}$ is an *attracting periodic orbit of period 2*, then there exists a neighborhood V of α^* such that for any $\alpha \in V$ we have that

$$\lim_{s\to\infty} f^{2s}(\alpha) = \alpha^*.$$

For $\alpha \in V$ it also holds that

$$\lim_{s\to\infty} f^{2s+1}(\alpha) = \alpha^{**}.$$

There also exists a neighborhood $\tilde{V}$ of α^{**} such that the analogous property for $\tilde{V}$ is also true, that is, for any $\alpha \in \tilde{V}$

$$\lim_{s\to\infty} f^{2s}(\alpha) = \alpha^{**} \quad \text{and} \quad \lim_{s\to\infty} f^{2s+1}(\alpha) = \alpha^*.$$

If almost every any point in (-1,1) is attracted by iterations of f to α^* or α^{**}, we say that the attracting periodic orbit $\{\alpha^*, \alpha^{**}\}$ is a *global attractor*. In other terms, if we can find an open set U such that $A = [-1,1] - U$ has Lebesgue measure zero on [-1,1] and for all $\alpha \in A$ the sequence $\{f^s(\alpha)\}_{s\in\mathbf{N}}$ has limit points only in the set $\{\alpha^*, \alpha^{**}\}$, then we say that $\{\alpha^*, \alpha^{**}\}$ is a *global attractor*. Let us explain more carefully what we would obtain as the orbit of the point α_0 when it is chosen at random. Suppose, for simplification of the argument, that 0.3 and 0.5 constitutes an *attracting global periodic orbit of period 2 for f*, that is, $f(0.3) = 0.5$, $f(0.5) = 0.3$ and $|(f^2)'(0.3)| < 1$. If we take an initial value α_0, typically we would have, for some large value $s > 0$ that

$$f^s(\alpha_0) = 0.357, \quad f^{s+1}(\alpha_0) = 0.524, \quad f^{s+2}(\alpha_0) = 0.308, \quad f^{s+3}(\alpha_0) = 0.510,$$
$$f^{s+4}(\alpha_0) = 0.3001, \quad f^{s+5}(\alpha_0) = 0.502 \quad \text{and so on.}$$

In our situation the mapping f will be ρ_1 and α^* and α^{**} are points very close, respectively, to $\cos(\omega_1)$ and $\cos(\omega_4)$.

The *Slutsky filter* has an extra parameter k (as the *complex filter* has a parameter M). For each k, $\rho_1(\alpha, k)$ will denote the first-order autocorrelation function of the Slutsky filtered stochastic process $\{Z_t(\theta, k)\}_{t\in T}$, where $\alpha = \cos(\theta)$. The figures obtained in the simulations show that $\rho_1(\alpha, k)$ has a global attracting periodic orbit of period 2 (see Figure 3 for the graph of $\rho_1(\alpha, k)$). For each $k \in \mathbf{N}$, denote $\{\alpha_k^*, \alpha_k^{**}\}$ this orbit of period 2.

How to locate α_k^* and α_k^{**}? Let $\alpha_0 \in (-1, 1)$ be chosen at random and consider the orbit of α_0 given as

$$\rho_1^s(\alpha_0, k), \quad \rho_1^{s+1}(\alpha_0, k), \quad \rho_1^{s+2}(\alpha_0, k), \quad \rho_1^{s+3}(\alpha_0, k), \cdots .$$

Notice that the iterations are in the variable α while $k = 10$ is fixed. If s is large enough then one will notice that

$$\rho_1^s(\alpha_0, k), \quad \rho_1^{s+2}(\alpha_0, k), \quad \rho_1^{s+4}(\alpha_0, k), \ldots$$

will be approximately the same number. It is also true that

$$\rho_1^{s+1}(\alpha_0, k), \quad \rho_1^{s+3}(\alpha_0, k), \quad \rho_1^{s+5}(\alpha_0, k), \ldots$$

will be approximately the same number but different from the previous one. These two numbers that we obtain will be approximately α_k^* and α_k^{**}. This is the iterative procedure for finding α_k^* and α_k^{**}.

The above fact can be easily seen from the figures we obtain by plotting the graph of the mapping $\rho_1(\cdot\,, k)$ and its iterates. Notice that in Figure 3 (with the graph of $\rho_1(\alpha, k)$) a large interval of points in the middle of the interval $(-1, 1)$ are mapped by $\rho_1(\cdot\,, k)$ to a value very close to the cosine of the highest frequency ω_4. Another set of points, in the two external parts of this interval, are mapped by $\rho_1(\cdot\,, k)$ to a value close to the cosine of the lowest frequency ω_1. Figures 3 through 5 are all related to the same model with $p = 4$ frequencies and $k = 10$. In this case, 0.2 and 0.8 are in an attracting orbit of period 2 and any initial value α_0, will be attracted to 0.2 and 0.4 by iterations of ρ_1. Therefore, $\{0.2, 0.4\}$ is a *global attractor*.

Note that there exists a fixed point for the mapping ρ_1 between $\cos(\omega_1)$ and $\cos(\omega_4)$, but it is a repelling (not attracting) fixed point and will not be detected by iterations of ρ_1.

In Figure 3(b) we show the same graph of Figure 3(a), but we add the horizontal dotted lines corresponding to $\cos(\omega_j)$, $1 \le j \le 4$.

Figure 4 shows the graph of $\rho_1^2(\cdot\,, 10)$ from the *Slutsky filter*. The intersection of the diagonal line and the graph determine the fixed points of the mapping ρ_1^2. These two points (very close to $\cos(\omega_1)$ and $\cos(\omega_4)$) constitute an orbit of period 2 for ρ_1.

Finally, in Figure 5 we show the graph of $\rho_1^4(\cdot\,, k)$. Note that any point α_0, after four (4) iterations, will be very close to either $\cos(\omega_1)$ or $\cos(\omega_4)$.

Now we give here the expression of $\rho_1(\alpha, k)$.

Lemma 3.1: *Let $\{Z_t(\theta, k)\}_{t\in T}$ be the Slutsky filtered stochastic process obtained from the application of the transformation (2.4) to the process $\{Z_t\}_{t\in T}$. Then, the first-order autocorrelation function of a stochastic process $\{Z_t(\theta, k)\}_{t\in T}$ is given by*

$$\rho_1(\alpha, k) = \frac{\sum_{j=1}^{p} \frac{A_j^2}{2}[\sin^{2k}(\theta - \omega_j) + \sin^{2k}(\theta + \omega_j)]\cos(\omega_j) + \frac{\sigma_\varepsilon^2}{\pi}\int_{-\pi}^{\pi}\cos(\lambda)\sin^{2k}(\theta - \lambda)\, d\lambda}{\sum_{j=1}^{p} \frac{A_j^2}{2}[\sin^{2k}(\theta - \omega_j) + \sin^{k} 2(\theta + \omega_j)] + \frac{\sigma_\varepsilon^2}{\pi}\int_{-\pi}^{\pi}\sin^{2k}(\theta - \lambda)\, d\lambda}, \tag{10}$$

where $\theta = \cos^{-1}(\alpha)$.

Proof:

We use the Slutsky filter as in expression (9) and we apply the squared gain function given by (7) with $m = n = 1$ to the first order autocorrelation $\rho_1(\alpha, k)$ given by the expression (8). The result follows after few calculations.

■

An heuristic explanation for the existence of the periodic orbit of period 2 is the following. First note that the *Slutsky filter* is, in some sense, the opposite of the *complex filter*. From the expression of $\rho_1(\alpha, M)$ in the complex filter one can see that there exists a tendency of an initial value α_0 to converge to the closest frequency (see expression (4.3) in Theorem 4.1 of Lopes and Kedem (1991)). This is due to the weighted average condition and the terms

$$\frac{A_j^2}{2}\left[\cos^{2k}\left(\frac{\theta - \omega_j}{2}\right) + \cos^{2k}\left(\frac{\theta + \omega_j}{2}\right)\right]\cos(\omega_j).$$

That is, $\cos^{2k}\left(\frac{\theta - \omega_j}{2}\right)$ is relatively larger than $\cos^{2k}\left(\frac{\theta - \omega_l}{2}\right)$, $l \neq j$, when k is large and $\theta \approx \omega_j$. Now we have the opposite situation because in the expression (10) we have terms of the kind

$$\frac{A_j}{2}\left[\sin^{2k}(\theta - \omega_j) + \sin^{2k}(\theta + \omega_j)\right]\cos(\omega_j).$$

In this way, when $\theta \approx \omega_j$ the above term is small and not large. In fact, the above expression is large for θ more distant of ω_j. Therefore, we have a tendency, by applying the mapping ρ_1, to oscilate from the largest ω_4 to the smallest ω_1 and vice-versa.

Remark 3.1: Note that for different values of A_j, $1 \leq j \leq 4$, and k the *Slutsky filter* will not produce the above mentioned phenomenon.

Conclusion: When $p = 4$, $A_1 = 20$, $A_2 = A_3 = 0.1$, $A_4 = 1.0$, $\cos(\omega_1) = 0.2$, $\cos(\omega_2) = 0.4$, $\cos(\omega_3) = 0.6$, $\cos(\omega_4) = 0.8$, $\sigma_\varepsilon = 0.01$ and $k = 10$ the updating procedure generates an *attracting periodic orbit of period 2*.

4. Estimating the Amplitudes and the Noise Variance Using the α^n-Filter

Now we will introduce the following generalization of the *alpha filter* (see He and Kedem (1989).

Definition 4.1: The α^n- *filter* is defined as the transformation

$$Z_t(\alpha, n) = Z_t + \alpha^n Z_{t-1}(\alpha, n), \quad -1 < \alpha < 1 \text{ and } n \in \mathbf{N} - \{0\}. \tag{11}$$

It has transfer function given by

$$H(\lambda; \alpha, n) = \frac{1}{1 - \alpha^n e^{i\lambda}}, \quad -\pi < \lambda \leq \pi$$

and squared gain function given by

$$|H(\lambda; \alpha, n)|^2 = (1 - 2\alpha^n \cos(\lambda) + \alpha^{2n})^{-1}, \quad -\pi < \lambda \leq \pi. \tag{12}$$

Proposition 4.1: *For $n \in \mathbf{N} - \{0\}$ and $\alpha \in (-1, 1)$ the first-order autocorrelation $\rho_{1,n}(\alpha)$ for the α^n- filtered process (1) is given by*

$$\begin{aligned} \rho_{1,n}(\alpha) &= \frac{E[Z_t(\alpha, n) Z_{t+1}(\alpha, n)]}{E[Z_t^2(\alpha, n)]} \\ &= \frac{\sum_{j=1}^{p} \frac{A_j^2}{2} \frac{\cos(\omega_j)}{1 - 2\alpha^n \cos(\omega_j) + \alpha^{2n}} + \sigma_\varepsilon^2 \frac{\alpha^n}{1 - \alpha^{2n}}}{\sum_{j=1}^{p} \frac{A_j^2}{2} \frac{1}{1 - 2\alpha^n \cos(\omega_j) + \alpha^{2n}} + \sigma_\varepsilon^2 \frac{1}{1 - \alpha^{2n}}}. \end{aligned} \tag{13}$$

where the noise process $\{\varepsilon_t\}_{t \in \mathbf{Z}}$ is assumed to be Gaussian and white with $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$.

Proof:

Observe that

$$E[Z_t(\alpha,n)Z_{t+1}(\alpha,n)] = \int_{-\pi}^{\pi} \cos(\lambda)|H(\lambda;\alpha,n)|^2 dF_X(\lambda) + \int_{-\pi}^{\pi} \cos(\lambda)|H(\lambda;\alpha,n)|^2 dF_\varepsilon(\lambda)$$

$$= \sum_{j=1}^{p} \frac{A_j^2}{4}[\cos(\omega_j)|H(\omega_j;\alpha,n)|^2 + \cos(-\omega_j)|H(-\omega_j;\alpha,n)|^2]$$

$$+ \frac{\sigma_\varepsilon^2}{2\pi} \int_{-\pi}^{\pi} \cos(\lambda)|H(\lambda;\alpha,n)|^2 d\lambda$$

$$= \sum_{j=1}^{p} \frac{A_j^2}{2} \cos(\omega_j)|H(\omega_j;\alpha,n)|^2 + \frac{\sigma_\varepsilon^2}{2\pi} \int_{-\pi}^{\pi} \cos(\lambda)|H(\lambda;\alpha,n)|^2 d\lambda$$

and

$$\begin{aligned} E[Z_t^2(\alpha,n)] &= \int_{-\pi}^{\pi} |H(\lambda;\alpha,n)|^2 dF_X(\lambda) + \int_{-\pi}^{\pi} |H(\lambda;\alpha,n)|^2 dF_\varepsilon(\lambda) \\ &= \sum_{j=1}^{p} \frac{A_j^2}{4}[|H(\omega_j;\alpha,n)|^2 + |H(-\omega_j;\alpha,n)|^2] + \frac{\sigma_\varepsilon^2}{2\pi} \int_{-\pi}^{\pi} |H(\lambda;\alpha,n)|^2 d\lambda \\ &= \sum_{j=1}^{p} \frac{A_j^2}{2}|H(\omega_j;\alpha,n)|^2 + \frac{\sigma_\varepsilon^2}{2\pi} \int_{-\pi}^{\pi} |H(\lambda;\alpha,n)|^2 d\lambda \end{aligned}$$

where $|H(\lambda;\alpha,n)|^2$ is given by (12).

From Gradshteyn and Ryzhik (1980), we have

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(\lambda)|H(\lambda;\alpha,n)|^2 d\lambda &= 2\pi \frac{\alpha^n}{1-\alpha^{2n}} \\ &= \alpha^n \frac{2\pi}{1-\alpha^{2n}} = \alpha^n \int_{-\pi}^{\pi} |H(\lambda;\alpha,n)|^2 d\lambda \end{aligned}$$

where $|H(\lambda;\alpha,n)|^2$ is given by (12). Therefore, the expression (13) holds.

■

In the case $p = 1$ we have only one fixed point for the mapping ρ_1. This fixed point is $\cos(\omega_1)$ and, in this case, the problem of finding the only one frequency is solved by using the *alpha filter* (see He and Kedem (1989)). An interesting result related with this situation is the following.

Proposition 4.2: *Suppose* $\alpha^* = \cos(\omega_1)$ *is the fixed point for* ρ_1. *Then,*

$$\rho_1'(\alpha^*) = \frac{Var[\varepsilon_t(\alpha^*)]}{Var[Z_t(\alpha^*)]}. \tag{14}$$

Proof:

Let

$$C(\alpha) = \frac{Var[\varepsilon_t(\alpha)]}{Var[Z_t(\alpha)]}.$$

Then it is known (see expression (2.16) of Kedem and Yakowitz (1990)) that for any $\alpha \in (-1, 1)$, we have

$$\rho_1(\alpha) = \alpha^* + C(\alpha)(\alpha - \alpha^*).$$

Hence, since $\alpha^* = \rho_1(\alpha^*)$

$$\begin{aligned}
\rho_1'(\alpha^*) &= \lim_{\alpha \to \alpha^*} \frac{\rho_1(\alpha) - \rho_1(\alpha^*)}{\alpha - \alpha^*} \\
&= \lim_{\alpha \to \alpha^*} \frac{\rho_1(\alpha) - \alpha^*}{\alpha - \alpha^*} \\
&= \lim_{\alpha \to \alpha^*} \frac{[\alpha^* + C(\alpha)(\alpha - \alpha^*)] - \alpha^*}{\alpha - \alpha^*} \\
&= C(\alpha^*) = \frac{Var[\varepsilon_t(\alpha^*)]}{Var[Z_t(\alpha^*)]}.
\end{aligned}$$

■

From Proposition 4.2, when the *alpha filter* is used we calculate ρ_1' at $\alpha^* = \cos(\omega_1)$ and we obtain

$$\rho_1'(\alpha^*) = \frac{\sigma_\varepsilon^2}{\frac{A_1^2}{2} + \sigma_\varepsilon^2}.$$

Observe that $|\rho_1'(\alpha^*)| < 1$.

Now we return to the general case $p > 1$.

We need to make some considerations before analyzing the linear system associated with the fixed points of $\rho_{1,n}(\alpha)$. Notice that $E[Z_t^2(\alpha, 1)]$ for the value $\alpha = 0$ is given by

$$E[Z_t^2(0,1)] = \sum_{j=1}^{p} \frac{A_j^2}{2} + \sigma_\varepsilon^2 = b$$

(see the denominator of expression (13) when n = 1, that is, the case of the *alpha filter*). We can estimate $E[Z_t^2(\alpha, 1)]$ at $\alpha = 0$ by

$$\hat{b} = \frac{1}{N} \sum_{j=1}^{N} [Z_j(0,1) - \overline{Z(0,1)}]^2. \tag{15}$$

Here we are using no filter, just the original series since $\alpha = 0$. The bar in (15) denotes the mean average value. In the linear system (18) below we will need the condition

$$\sum_{j=1}^{p} \frac{A_j^2}{2} + \sigma_\varepsilon^2 = b. \tag{16}$$

The equation (16) will be the first one of the linear system (18).

In this section we suppose that we already know how to find all the frequencies $\hat{\omega}_j$, $1 \leq j \leq p$, using the approach of Section 2 (see also Kedem and Lopes (1991) and Lopes and Kedem (1991)). The equation of the fixed points of $\rho_{1,n}(\alpha)$ is given by (see expression (13))

$$\begin{aligned}
\rho_{1,n}(\alpha) = \alpha \Longleftrightarrow & \sum_{j=1}^{p} \frac{A_j^2}{2} \frac{\cos(\omega_j)}{1 - 2\alpha^n \cos(\omega_j) + \alpha^{2n}} + \sigma_\varepsilon^2 \frac{\alpha^n}{1 - \alpha^{2n}} \\
& = \alpha[\sum_{j=1}^{p} \frac{A_j^2}{2} \frac{1}{1 - 2\alpha^n \cos(\omega_j) + \alpha^{2n}} + \sigma_\varepsilon^2 \frac{1}{1 - \alpha^{2n}}] \\
& \Longleftrightarrow \sum_{j=1}^{p} \frac{A_j^2}{2} (\frac{\cos(\omega_j) - \alpha}{1 - 2\alpha^n \cos(\omega_j) + \alpha^{2n}}) + \sigma_\varepsilon^2 (\frac{\alpha^n - \alpha}{1 - \alpha^{2n}}) = 0.
\end{aligned} \tag{17}$$

We consider the above expression for $n \in \{2, 3, \cdots, p+1\}$. In order to use for each n a different notation, the fixed points of $\rho_{1,n}(\alpha)$ will be denoted by α_n^*. So, the equation of the fixed points is given by $\rho_{1,n}(\alpha_n^*) = \alpha_n^*$. Therefore, we consider the following linear system of p+1 equations and p+1 unknowns $(A_1^2, \cdots, A_p^2, \sigma_\varepsilon^2)$ given by

$$\begin{aligned}
&\sum_{j=1}^{p} \frac{A_j^2}{2} + \sigma_\varepsilon^2 = b \\
&\sum_{j=1}^{p} \frac{A_j^2}{2} \left(\frac{\cos(\omega_j) - \alpha_2^*}{1 - 2\alpha_2^{*2} \cos(\omega_j) + \alpha_2^{*4}} \right) + \sigma_\varepsilon^2 \left(\frac{\alpha_2^{*2} - \alpha_2^*}{1 - \alpha_2^{*4}} \right) = 0 \\
&\sum_{j=1}^{p} \frac{A_j^2}{2} \left(\frac{\cos(\omega_j) - \alpha_3^*}{1 - 2\alpha_3^{*3} \cos(\omega_j) + \alpha_3^{*6}} \right) + \sigma_\varepsilon^2 \left(\frac{\alpha_3^{*3} - \alpha_3^*}{1 - \alpha_3^{*6}} \right) = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{j=1}^{p} \frac{A_j^2}{2} \left(\frac{\cos(\omega_j) - \alpha_{p+1}^*}{1 - 2\alpha_{p+1}^{*(p+1)} \cos(\omega_j) + \alpha_{p+1}^{*2(p+1)}} \right) + \sigma_\varepsilon^2 \left(\frac{\alpha_{p+1}^{*(p+1)} - \alpha_{p+1}^*}{1 - \alpha_{p+1}^{*2(p+1)}} \right) = 0.
\end{aligned} \tag{18}$$

Notice that the system is linear in $A_1^2, \cdots, A_p^2, \sigma_\varepsilon^2$. Since we have p+1 variables $A_1, \cdots, A_p$, σ_ε that we want to estimate, if we are able to find the constant b and also $\alpha_2^*, \cdots, \alpha_p^*, \alpha_{p+1}^*$ fixed points, respectively, of $\rho_{1,2}(\alpha)$, $\rho_{1,3}(\alpha), \cdots, \rho_{1,p+1}(\alpha)$ then we are able to solve the linear system and, therefore, to obtain the estimated values $\hat{A}_1, \cdots, \hat{A}_p, \hat{\sigma}_\varepsilon$. The value $\hat{b}$ is obtained from (15).

Remark 4.1: The reason to use the condition (16) is to avoid a system with the trivial solution $A_1 = A_2 = \cdots = A_p = \sigma_\varepsilon = 0$. We do not use the equation for the fixed point of the α-filter (that is, n = 1) because the resulting equation of the fixed point does not depend on $A_1, \cdots, A_p, \sigma_\varepsilon$ in the case where $A_1 = A_2 = \cdots = A_p$.

Now we analyze the map $\rho_{1,n}(\alpha)$ for each $n \in \{1, 2, \cdots, p+1\}$. Let us consider, for simplicity, $\rho_{1,2}(\alpha)$. In Figure 6 we plot a graph of $\rho_{1,2}(\alpha)$. When we iterate $\rho_{1,2}$ at an initial value α_0, $\rho_{1,2}^k(\alpha_0)$ converges to some attracting fixed point (a.s.). Therefore, we consider $\rho_{1,2}^8(\alpha_0)$ a good approximation of one of the fixed points α_1^*.

In an analogous way, when $p \in \mathbf{N}$ and $n \in \{2, 3, \cdots, p+1\}$ we have a finite number of fixed points for each map $\rho_{1,n}(\alpha)$. This follows from the fact that the set of fixed points α^* is the solution of a polynomial equation (see (17)). In the same way we take an initial value α_0 at random in (-1,1) and we can consider $\rho_{1,n}^8(\alpha_0)$ as a good approximation for one of the fixed points α_n^*. Now we plug the p+1 solutions $\hat{\alpha}_n^* = \rho_{1,n}^8(\alpha_0)$, $2 \le n \le p+1$ in the system (18) and we get

$$\sum_{j=1}^{p} \frac{A_j^2}{2} + \sigma_\varepsilon^2 = \hat{b}$$

$$\sum_{j=1}^{p} \frac{A_j^2}{2} \left(\frac{\cos(\hat{\omega}_j) - \hat{\alpha}_2^*}{1 - 2\hat{\alpha}_2^{*2} \cos(\hat{\omega}_j) + \hat{\alpha}_2^{*4}}\right) + \sigma_\varepsilon^2 \left(\frac{\hat{\alpha}_2^{*2} - \hat{\alpha}_2^*}{1 - \hat{\alpha}_2^{*4}}\right) = 0$$

$$\sum_{j=1}^{p} \frac{A_j^2}{2} \left(\frac{\cos(\hat{\omega}_j) - \hat{\alpha}_3^*}{1 - 2\hat{\alpha}_3^{*3} \cos(\hat{\omega}_j) + \hat{\alpha}_3^{*6}}\right) + \sigma_\varepsilon^2 \left(\frac{\hat{\alpha}_3^{*3} - \hat{\alpha}_3^*}{1 - \hat{\alpha}_3^{*6}}\right) = 0$$

....

$$\sum_{j=1}^{p} \frac{A_j^2}{2} \left(\frac{\cos(\hat{\omega}_j) - \hat{\alpha}_{p+1}^*}{1 - 2\hat{\alpha}_{p+1}^{*(p+1)} \cos(\hat{\omega}_j) + \hat{\alpha}_{p+1}^{*2(p+1)}}\right) + \sigma_\varepsilon^2 \left(\frac{\hat{\alpha}_{p+1}^{*(p+1)} - \hat{\alpha}_{p+1}^*}{1 - \hat{\alpha}_{p+1}^{*2(p+1)}}\right) = 0 \qquad (19)$$

where $\hat{\omega}_j$, $1 \le j \le p$, was calculated as in Section 2.

Now we apply a numerical method to obtain the solution $\hat{A}_1, \cdots, \hat{A}_p$, $\hat{\sigma}_\varepsilon$ of the linear system. In this way, we obtain all the relevant information of the model (1).

In the simulated model mentioned in the introduction, where $p = 2$, $A_1 = A_2 = 1.0$, $\sigma_\varepsilon = 0.3$, $\omega_1 = 0.7$ and $\omega_2 = 2.2$, with $N = 3000$, we applied the above method and we obtained the strong consistent estimates $\hat{b} = 1.0901$, $\hat{\alpha}_2^* = 0.1005$ and $\hat{\alpha}_3^* = 0.0919$. By solving the linear system (19) and using the values previuosly obtained in Lopes and Kedem (1991) $\hat{\omega}_1 = 0.7044$ and $\hat{\omega}_2 = 2.1965$ we get $\hat{A}_1 = 0.9783$, $\hat{A}_2 = 0.9525$ and $\hat{\sigma}_\varepsilon = 0.3976$.

The strong consistency property of the estimates we used here can be derived from Theorem 2.2 in Section 2.

Conclusion: Using the updating procedure *(HOC)* associated to the *complex filter* and the α^n-*filter* one is able to estimate all the relevant parameters in model (1).

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Figure 1: Fixed points in $\rho_1(\alpha) = \rho_1(\alpha, 15)$ from the complex filter for $p = 2$, $A_1 = A_2 = \sigma_\varepsilon = 1.0$, $\omega_1 = 0.7$ and $\omega_2 = 2.2$ ($\cos(\omega_1) = 0.7684$ and $\cos(\omega_2) = -0.5885$).

Figure 2: Fixed points in $\rho_1^5(\alpha) = \rho_1^5(\alpha, 15)$ from the complex filter for $p = 2$, $A_1 = A_2 = \sigma_\varepsilon = 1.0$, $\omega_1 = 0.7$ and $\omega_2 = 2.2$ ($\cos(\omega_1) = 0.7684$ and $\cos(\omega_2) = -0.5885$).

Figure 3: Graph of $\rho_1(\cdot, 10)$ from Slutsky filter for $p = 4$ and the diagonal line. The dotted lines are the constant functions $\cos(\omega_j)$, $1 \leq j \leq 4$.

Figure 4: Graph of $\rho_1^2(\cdot, 10)$ from Slutsky filter for $p = 4$ and the diagonal line. The two fixed points of ρ_1^2 constitute a periodic orbit of period 2 for ρ_1. The dotted lines are the constant functions $\cos(\omega_j)$, $1 \leq j \leq 4$.

Figure 5: Graph of $\rho_1^4(\cdot, 10)$ from Slutsky filter for $p = 4$ and the diagonal line. The dotted lines are the constant functions $\cos(\omega_j)$, $1 \leq j \leq 4$.

Figure 6: Fixed points in $\rho_{1,2}(\alpha)$ from α^2-filter and the diagonal line.