CONVERGENCE IN DISTRIBUTION FOR THE PERIODOGRAM OF CHAOTIC PROCESSES

Lopes, A.O.* and **Lopes, S.R.C.***[1]

* Instituto de Matemática - UFRGS, Porto Alegre, RS, Brazil

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Abstract:

In this work we analyze the convergence in distribution sense for the periodogram function based on a time series of a stationary process obtained from the iterations of a continuous transformation invariant for an ergodic probability μ. We only assume a certain rate of convergence to zero for the autocovariance function of the stochastic process, that is, we assume there exist $C > 0$ and $\beta > 2$ such that $|\gamma_X(h)| \leq C|h|^{-\beta}$, for all $h \in \mathbb{Z}$, where $\gamma_X(\cdot)$ is the autocovariance function of the process.

Keywords: Periodogram Function, Convergence in Distribution, Chaotic Processes.

1. Introduction

Here we consider the stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ obtained from the iterations of a continuous transformation T from the unit interval to itself and μ an ergodic probability invariant under T. This stochastic and stationary process $\{X_t\}_{t\in\mathbb{Z}}$ is given by

$$X_t \equiv (\varphi \circ T^t)(X_0) = \varphi(T^t(X_0)) = (\varphi \circ T)(X_{t-1}), \text{ for } t \in \mathbb{Z}, \tag{1.1}$$

[1]Corresponding author. E-mail: slopes@mat.ufrgs.br

where φ is a continuous map $\varphi : [0, 1) \to \mathbb{R}$ and X_0 is distributed over $[0, 1)$ according to μ.

We shall assume a certain rate of convergence to zero for the autocovariance function of such stochastic process. We will denote by $\gamma_X(\cdot)$ the autocovariance function for the process $\{X_t\}_{t\in\mathbb{Z}}$, that is,

$$\gamma_X(h) \equiv \mathbb{E}_\mu(X_h X_0) - \mathbb{E}_\mu(X_0), \text{ for } h \in \mathbb{Z}.$$

In this work we analyze the convergence in distribution sense, to $f_X(\lambda) = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \gamma_X(h) e^{-ih\lambda}$, for the periodogram function, based on a time series $\{X_t\}_{t=1}^N$ from the stochastic process $\{X_t\}_{t\in\mathbb{Z}}$. This periodogram function is given by

$$I(\lambda_k) = f_N(\lambda_k)\overline{f_N(\lambda_k)},$$

where

$$f_N(\lambda) = \frac{1}{2\pi\sqrt{N}} \sum_{t=1}^{N} \varphi(T^t(x_0)) e^{-i\lambda t}, \quad \lambda \in (0, 2\pi],$$

$\overline{f_N(\cdot)}$ indicates the complex conjugate of $f_N(\cdot)$ and

$$\lambda_k = \frac{2\pi k}{N}, \text{ for } k = 0, 1, \cdots, N,$$

is the k-th Fourier frequency.

The periodogram function allows one to have an idea of the spectral density function $f_X(\cdot)$ (see Brockwell and Davis (1991)). See remark after the claim of Theorem 3.1.

This paper proceeds in the following way: Section 2 presents some definitions related to the chaotic process of the form (1.1) while Section 3 presents the proof of the convergence in distribution sense for the periodogram function $I(\cdot)$, with the *assumption*, given by the expression (2.3), imposed on the autocovariance function $\gamma_X(\cdot)$, after some preliminary lemmas.

2. Chaotic Processes

Let T be a continuous transformation from the circle S to itself, not necessarily invertible, and μ an ergodic probability invariant under T. Suppose that μ is absolutely continuous with respect to the Lebesgue measure.

Considering the identification of the circle $z \in S$ with $x \in [0,1)$ by $z \equiv e^{2\pi x i}$, from now on we can use either one of the two forms $T : S \to S$ or $T : [0,1) \to [0,1)$.

We assume that μ has density function $\phi(\cdot)$ such that $d\mu(x) = \phi(x)dx$. Let φ be a continuous map $\varphi : [0,1) \to \mathbb{R}$. Then, one can define the stochastic process obtained from the iterations of T in the following way

$$X_t \equiv (\varphi \circ T^t)(X_0) = \varphi(T^t(X_0)) = (\varphi \circ T)(X_{t-1}), \text{ for } t \in \mathbb{N}, \tag{2.1}$$

where X_0 is distributed over $[0,1)$ according to μ. From Birkhoff's theorem let $x_0 \in [0,1)$ be a fixed number chosen according to μ such that for all continuous function g (or indicator function of an interval)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} g(T^j(x_0)) = \int g(x)d\mu(x).$$

Assume, without loss of generality, that

$$\mathbb{E}_\mu(X_0) = \int \varphi(x)d\mu(x) = \int \varphi(x)\phi(x)dx = 0.$$

We will assume that T has a natural extension $F : S \times S \to S \times S$ of the form

$$F(x,y) = (T(x), H(x,y)),$$

for some function $H(\cdot,\cdot)$. Then, in this case, the process $\{X_t\}_{t\in\mathbb{N}}$, given by the expression (2.1), can be extended to

$$\tilde{X}_t = (\tilde{\varphi} \circ F^t)(\tilde{X}_0) = \tilde{\varphi}(F^t(\tilde{X}_0)) = (\tilde{\varphi} \circ F)(\tilde{X}_{t-1}), \text{ for } t \in \mathbb{Z}, \tag{2.2}$$

where the function $\tilde{\varphi} : S \times S \to \mathbb{R}$ is given by $\tilde{\varphi}(x,y) = \varphi(x)$ (see Lopes and Lopes (1998)). We refer the reader to Borovkova, Burton and Dehling (2001) for several examples of such general procedure.

The measure $\tilde{\mu}$ has a natural extension of μ on $S \times S$. In this case

$$\int \tilde{\varphi}(x,y)d\tilde{\mu}(x,y) = \int \varphi(x)d\mu(x) = 0.$$

The process $\{\tilde{X}_t\}_{t\in\mathbb{Z}}$ defined by the expression (2.2) is a stationary process with zero mean if the distribution of $\tilde{X}_0$ is $\tilde{\mu}$.

In order to simplify the notation, since $\tilde{\varphi}$ depends only on the first coordinate, that is, $\tilde{\varphi}(x, y) = \varphi(x)$, we will consider, in the sequel, X_t, φ, T and μ instead of $\widetilde{X}_t$, $\tilde{\varphi}$, F and $\tilde{\mu}$.

The process $\{X_t\}_{t\in\mathbb{Z}}$ has *autocovariance function of order* h given by

$$\gamma_X(h) \equiv \mathbb{E}_\mu(X_h X_0), \text{ for } h \in \mathbb{Z},$$

since $\mathbb{E}_\mu(X_t) = \mathbb{E}_\mu[(\varphi \circ T^t)(X_0)] = 0$. We will assume here a certain rate of convergence to zero of $\gamma_X(\cdot)$, that will be specified below.

Assumption: *We will assume that T is such that for any given $\varphi : [0, 1) \to \mathbb{R}$ continuous map there exist $C > 0$ and $\beta > 2$ such that the autocovariance function of the process $\{X_t\}_{t\in\mathbb{Z}}$ has the rate of convergence to zero given by*

$$|\gamma_X(h)| \leq C|h|^{-\beta}, \text{ for all } h \in \mathbb{Z}. \tag{2.3}$$

Examples of such systems appear when T is a transformation of the circle with an indifferent fixed point (see Isola (1999), Fisher and Lopes (2001), Maes et al. (1999) and Young (1999)). In the notation of Fisher and Lopes (2001), $\beta > 2$ is equivalent to $\gamma > 4$.

The above assumption includes the case where $\gamma_X(\cdot)$ exponentially decays to zero, that is, when there exists $0 < \lambda < 1$ such that

$$|\gamma_X(h)| = \mathbb{E}_\mu(X_h X_0) \leq C_1 \lambda^{|h|}, \text{ for } h \in \mathbb{Z}, \tag{2.4}$$

where C_1 is a positive constant and $\mathbb{E}_\mu(X_t) = \mathbb{E}_\mu[(\varphi \circ T^t)(X_0)] = 0$.

We observe that we are not considering *general fractionally integrated* processes (see Reisen and Lopes (1999)).

The *spectral density function* of the process $\{X_t\}_{t\in\mathbb{Z}}$ defined in the expression (2.2) is given by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h\in\mathbb{Z}} \gamma_X(h) e^{-i\lambda h}, \text{ for } \lambda \in (0, 2\pi], \tag{2.5}$$

where $\gamma_X(\cdot)$ is the autocovariance function of the process.

The *periodogram function* is an unbiased estimator for the spectral density function $f_X(\cdot)$, even though it is not consistent (see Brockwell and Davis

(1991)). We will show that the periodogram function gives an approximation of the following truncated function

$$f_{X,r}(\lambda) = \frac{1}{2\pi} \sum_{|h|<r} \gamma_X(h) e^{-i\lambda h}, \text{ for } \lambda \in (0, 2\pi],$$

where $r > 0$ is a large but fixed constant. In practical situations, this is what we really need: an approximation, as close as we want (see Lemma 1 below) of the function $f_X(\cdot)$ by $f_{X,r}(\cdot)$ and an approximation of $f_{X,r}(\cdot)$ by the periodogram function.

The point x_0 will be chosen in a set of measure one. The value r will appear later and it is considered fixed from now on.

The *periodogram function* of a time series $\{X_t\}_{t=1}^{N}$ of size $N \in \mathbb{N}$ for the point x_0 obtained from the stationary process $\{X_t\}_{t\in\mathbb{Z}}$, given by the expression (2.2), is given by

$$I(\lambda_k) = f_N(\lambda_k)\overline{f_N(\lambda_k)}, \tag{2.6}$$

where

$$f_N(\lambda) = \frac{1}{2\pi\sqrt{N}} \sum_{t=1}^{N} \varphi(T^t(x_0)) e^{-i\lambda t}, \quad \lambda \in (0, 2\pi],$$

$\overline{f_N(\cdot)}$ indicates the complex conjugate of $f_N(\cdot)$ and

$$\lambda_k = \frac{2\pi k}{N}, \text{ for } k = 0, 1, \cdots, N,$$

is the k-th Fourier frequency.

In this work we want to show the convergence in distribution of the periodogram function defined by (2.6) under the *Assumption* given by the expression (2.3). This will be done in the next section.

3. Convergence in Distribution for the Periodogram

In this section we will show the convergence in distribution of the periodogram function. First we need Lemma 1.

Lemma 1: *Let $f_X(\cdot)$ be the spectral density function of the process $\{X_t\}_{t\in\mathbb{Z}}$ defined by the expression (2.2). Let $f_{X,r}(\cdot)$ be the truncated r-spectral density* *function of the process* $\{X_t\}_{t\in\mathbb{Z}}$ *given by*

$$f_{X,r}(\lambda) = \frac{1}{2\pi}\sum_{|h|\leq K_0} \gamma_X(h)e^{-i\lambda h}, \text{ for } \lambda \in (0, 2\pi], \tag{3.1}$$

for $r \in \mathbb{N}$. *Then, for all* $\epsilon > 0$, *there exists* $r_0 \in \mathbb{N}$ *such that for all* $K_0 \geq r_0$,

$$||f_X(\lambda) - f_{X,K_0}(\lambda)||_\infty < \epsilon, \text{ for all } \lambda \in (0, 2\pi],$$

where $||g||_\infty$ *means the infinity norm of the function* g.

Proof: Let ϵ be a positive fixed constant. For all $K_0 \in N$, let us consider the spectral density function for all $h \in \mathbb{Z}$ such that $|h| \leq K_0$, that is, the function $f_{X,K_0}(\cdot)$ given by expression (3.1). Then,

$$\begin{aligned} f_X(\lambda) &= \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \mathbb{E}_\mu(X_h X_0)e^{-i\lambda h} \\ &= \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \gamma_X(h)e^{-i\lambda h} \\ &= \frac{1}{2\pi}\sum_{|h|\leq K_0} \gamma_X(h)e^{-i\lambda h} + \frac{1}{2\pi}\sum_{|h|>K_0} \gamma_X(h)e^{-i\lambda h} \\ &= f_{X,K_0}(\lambda) + \frac{1}{2\pi}\sum_{|h|>K_0} \gamma_X(h)e^{-i\lambda h}. \end{aligned} \tag{3.2}$$

Now, since there exist $C > 0$ and $\beta > 2$ such that, for all $h \in \mathbb{Z}$, we have $|\gamma_X(h)| \leq C|h|^{-\beta}$ we also have that $\sum_{h\in\mathbb{Z}} |\gamma_X(h)|$ converges.

The last term in the expression (3.2) goes to zero when r goes to infinity since $\frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \gamma_X(h)e^{-i\lambda h}$ converges. Therefore, given $\epsilon > 0$, there exists such r_0.

Therefore, the Lemma 1 is proved.

□

The Lemma 1 says that $h > K_0$ is not important for considering in the spectral density function.

We will consider, in the sequel, several numbers such as N, q and r that will go to infinity. It will be very important the order we take them, that is, which number goes to infinity first, and then which one will be the next, etc...

However, the value K_0 will be much larger than all of them. It contains the information of the approximation to the function $f_X(\cdot)$, that is, the function we want to approximate.

Let x_0 be a point in a set of μ-probability one. From now on we will denote $T^t(x_0) \equiv x_t$ and $X_t = \varphi(T^t(x_0)) \equiv \varphi(x_t)$.

From the expression (2.6) one has, for N fixed, that

$$I(\lambda_k) = \frac{1}{4\pi^2 N} \sum_{s,t=1}^{N} X_t X_s e^{-i\lambda_k(t-s)} = \frac{1}{4\pi^2 N} \sum_{\substack{1\le s\le N \\ 1\le s+h\le N}} X_s X_{s+h} e^{-i\lambda_k h}, \qquad (3.3)$$

where in the last equality we change variable $t-s$ by h. Therefore, for each s fixed, the range of h is $1-s \le h \le N-s$.

Now, we want to prove the convergence in distribution for the periodogram function $f_X(\cdot)$ based on a time series $\{X_t\}_{t=1}^N$ of the process $\{X_t\}_{t\in\mathbb{Z}}$ given by (2.2) beginning with x_0.

In the sequel, δ_y will denote the Dirac delta function in y.

Note that $I(\lambda_k)$, for $k \in \{0,1,2,\cdots,N\}$, depends on the initial chosen point x_0 and also on N.

Theorem 3.1: *Let $\{X_t\}_{t\in\mathbb{Z}}$ be the stationary zero mean process given by (2.2). Let x_0 be as above and let $\gamma_X(\cdot)$ be the autocovariance function of the process $\{X_t\}_{t\in\mathbb{Z}}$ such that the assumption given by (2.3) holds. Let $f_X(\cdot)$ be the spectral density function of the process $\{X_t\}_{t\in\mathbb{Z}}$ given by (2.5).*

Then, in the distribution sense

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k} = f_X(\lambda), \ \text{for } \lambda \in (0, 2\pi],$$

where $I(\cdot)$ is the periodogram function defined by (2.6),

$$\lambda_k = \frac{2\pi k}{N}, \ \text{for } k \in \{0,1,\cdots,N\},$$

is the k-th Fourier frequency and δ_{λ_k} is the Dirac delta function with concentrated mass at this k-th frequency.

The proof of Theorem 3.1 will be given after some lemmas. In order to have the convergence in distribution sense we will show that for any smooth function $g : S \to \mathbb{R}$

$$\lim_{N\to\infty} \left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle = \langle f_X, g\rangle = \int_0^{2\pi} f_X(\lambda)g(\lambda)d\lambda = $$
$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{h\in\mathbb{Z}} \gamma_X(h)e^{-i\lambda h} g(\lambda)d\lambda,$$

where

$$\left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle \equiv \int_0^{2\pi} g(x)\frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k} =$$
$$= \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)g(\lambda_k) =$$
$$= \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N\\ 1\le s+h\le N}} X_sX_{s+h}e^{-i\lambda_k h},$$

with the last equality obtained from the expression (3.3).

Remark: To find $f_X(x)$ one takes a continuous function g such that it has support in a small neighborhood of x.

Let $r \in \mathbb{N} - \{0\}$ be a fixed value such that Lemma 1 holds. Then, we can write

$$\left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle = \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N\\ 1\le s+h\le N}} X_sX_{s+h}e^{-i\lambda_k h} \quad (3.4)$$
$$= \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} X_sX_{s+h}e^{-i\lambda_k h} \quad (3.5)$$
$$+ \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N\\ 1\le s+h\le N\\ |h|> r}} X_sX_{s+h}e^{-i\lambda_k h}. \quad (3.6)$$

In Lemma 2 below we shall prove that the expression (3.6), goes to zero, when $N \to \infty$.

Lemma 2: *Given $\epsilon_1 > 0$, there exists r such that, for all $K_0 > r$ and for $x_0 \in [0,1)$ μ-almost everywhere, there exists $N_1 \in \mathbb{N} - \{0\}$ such that*

$$\left| \frac{1}{4\pi^2 N^2} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1 \le s \le N \\ 1 \le s+h \le N \\ K_0 > |h| > r}} X_s X_{s+h} e^{-i\lambda_k h} \right| < \epsilon_1, \text{ for all } N > N_1.$$

Proof: Given $\epsilon_1 > 0$, let ϵ be such that

$$\epsilon = \frac{4\pi^2}{2M_1}\epsilon_1.$$

Given r and K_0 fixed, the function

$$v(x) \equiv \sum_{r<|h|<K_0} \left|\varphi(x)\varphi(T^h(x))\right| \left|e^{-i\lambda_k h}\right| = \sum_{r<|h|<K_0} \left|\varphi(x)\varphi(T^h(x))\right|$$

is continuous. If r is large enough one has, for this $\epsilon > 0$ and $K_0 > r$, that

$$\left| \int v(x) d\mu(x) \right| < \frac{\epsilon}{3}. \tag{3.7}$$

Here we use again that $\sum_{h \in \mathbb{Z}} |\gamma_X(h)|$ converges, since the autocovariance function of order h of the process $\{X_t\}_{t\in\mathbb{Z}}$ goes to zero with order of convergence $|h|^{-\beta}$, with $\beta > 2$.

Now we fix r and K_0 (much more larger than r). For such fixed function $v(\cdot)$, we want to estimate the μ-measure of "bad" points x_0 given by

$$P_{N_0}^{\frac{\epsilon}{3}} = \mu\left(\left\{x_0;\ \sup_{N>N_0} \left| \frac{1}{N}\sum_{s=1}^{N} v(T^s(x_0)) - \int v(x)d\mu(x) \right| > \frac{\epsilon}{3} \right\}\right).$$

From Theorem 13, part 1, in Kachurovskii (1996), as $\sigma_f(-\delta,\delta) = o(\delta^{\beta-1})$ as $\delta \to 0$ (since *Assumption* (2.3) holds), then $P_{N_0}^{\frac{\epsilon}{3}} = o(N_0^{-(\beta-1)})$ as $N_0 \to \infty$. Therefore, as $\beta > 2$,

$$\sum_{N_0=1}^{\infty} P_{N_0}^{\frac{\epsilon}{3}} < \infty.$$

Then, from Borel-Cantelli Lemma, one has that

$$\mu\left(\left\{x_0;\ \sup_{N>N_0}\left|\frac{1}{N}\sum_{s=1}^{N}v(T^s(x_0))-\int v(x)d\mu(x)\right|>\frac{\epsilon}{3}\right\}\ \ i.o.\right)=0,$$

that, is, for any x_0 μ-almost everywhere, there exists $N_1 = N_1(x_0) > 0$ such that $x_0 \notin P_{N_0}^{\frac{\epsilon}{3}}$, for all $N_0 > N_1$. Hence,

$$\sup_{N>N_0}\left|\frac{1}{N}\sum_{s=1}^{N}v(T^s(x_0))-\int v(x)d\mu(x)\right|\leq\frac{\epsilon}{3}. \tag{3.8}$$

From the expressions (3.7) and (3.8), for all $N_0 > N_1$ and for all $N > N_0$, we have

$$\left|\frac{1}{N}\sum_{s=1}^{N}v(T^s(x_0))\right|=\left|\frac{1}{N}\sum_{1\leq s\leq N}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|<\frac{2\epsilon}{3}.$$

This is not enough. As r and K_0 are fixed, one has

$$\lim_{N\to\infty}\frac{1}{N}\sum_{s=0}^{1-h}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))=0$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{s=N-h}^{N}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))=0.$$

Therefore, given $\frac{\epsilon}{3} > 0$, there exists $N_2 \in \mathbb{N} - \{0\}$ such that, for all $N > N_2$,

$$\left|\frac{1}{N}\sum_{s=0}^{1-h}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|<\frac{\epsilon}{6}$$

and

$$\left|\frac{1}{N}\sum_{s=N-h}^{N}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|<\frac{\epsilon}{6}.$$

Since

$$\left|\frac{1}{N}\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N}}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|=$$

$$= \left| \frac{1}{N} \sum_{1 \le s \le N} \sum_{r<|h|<K_0} \varphi(T^s(x_0))\varphi(T^{s+h}(x_0)) - \frac{1}{N} \sum_{\substack{1 \le s \le N \\ s<1-h}} \sum_{r<|h|<K_0} \right.$$

$$\left. \varphi(T^s(x_0))\varphi(T^{s+h}(x_0)) - \frac{1}{N} \sum_{\substack{1 \le s \le N \\ s>N-h}} \sum_{r<|h|<K_0} \varphi(T^s(x_0))\varphi(T^{s+h}(x_0)) \right| <$$

$$< \frac{2\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.$$

Since $M_1 = \sup_{\lambda \in [0,2\pi)} |g(\lambda)|$, one has, for large N, that

$$\left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \frac{1}{N} \sum_{\substack{1 \le s \le N \\ 1 \le s+h \le N \\ r<|h|<K_0}} X_s X_{s+h} e^{-i\lambda_k h} \right| \le \frac{1}{4\pi^2 N} \sum_{k=0}^{N} |g(\lambda_k)| \, \epsilon <$$

$$< \frac{N+1}{4\pi^2 N} M_1 \epsilon < \frac{2M_1}{4\pi^2} \epsilon = \epsilon_1.$$

Therefore, Lemma 2 is proved.

□

Important Remark: The constant K_0 is fixed from now on. We choose K_0 and r of Lemma 2 larger than r_0 of Lemma 1. Note that everything depends on the initially chosen x_0. For this x_0 and K_0 fixed (and large) there is a number N_1. All $N > 0$, in the future, will be larger than such N_1. This lemma says that h such that $r < h < K_0$ is not important for the periodogram function.

Now we will return to the proof of Theorem 3.1.

Therefore, from Lemma 2, the equality in expression (3.3), when $N \to \infty$, can be rewritten as

$$\left\langle \frac{1}{N} \sum_{h=0}^{N} I(\lambda_k) \delta_{\lambda_k}, g \right\rangle = \frac{1}{4\pi^2 N^2} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1 \le s \le N \\ 1 \le s+h \le N \\ |h| \le r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1). \quad (3.9)$$

Now, we fix q and let us define $B_1, \cdots, B_q$ a partition of the unit interval. The intervals B_j, $j \in \{1, \cdots, q\}$, are such that $B_i \cap B_j = \phi$, for all $i, j \in \{1, \cdots, q\}$, $i \neq j$, and such that

$$\bigcup_{j=1}^{q} B_j = [0, 1),$$

where

$$B_j = \left[\frac{j}{q}, \frac{j+1}{q}\right), \text{ for } j \in \{1, \cdots, q\}.$$

Let α_j be a fixed interior point of B_j, for $j \in \{1, \cdots, q\}$. Then, the expression (3.9) can be rewritten as

$$\left\langle \frac{1}{N} \sum_{h=0}^{N} I(\lambda_k) \delta_{\lambda_k}, g \right\rangle = \frac{1}{4\pi^2 N^2} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1 \leq s \leq N \\ 1 \leq s+h \leq N \\ |h| \leq r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1) =$$

$$= \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \frac{1}{N} \sum_{\substack{1 \leq s \leq N \\ 1 \leq s+h \leq N \\ |h| \leq r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1) =$$

$$= \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{j=1}^{q} \frac{1}{N} \sum_{\substack{s : x_s \in B_j \\ 1 \leq s \leq N \\ 1 \leq s+h \leq N \\ |h| \leq r}} \left[\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))\right] e^{-i\lambda_k h}$$

$$+ \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1 \leq s \leq N \\ 1 \leq s+h \leq N \\ |h| \leq r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \qquad (3.10)$$

$$+ \; o(1),$$

where $X_s = \varphi(X_{s-1}) = \varphi(T^s(X_0))$ and $x_s = T^s(x_0)$.

Note that the restriction $1 \leq s + h \leq N$, with $|h| < r$ and r fixed, is a mild assumption because the number of s such that $1 - h < s < N - h$ is of the same order as N. By Birkoff's Theorem

$$\lim_{N\to\infty} \frac{1}{N} \# \begin{bmatrix} x_s \in B_j \\ 1 \leq s + h \leq N \end{bmatrix} = \lim_{N\to\infty} \frac{1}{N} \#[x_s \in B_j] = \mu(B_j),$$

given $\epsilon > 0$ take N large enough such that

$$\left| \frac{1}{N} \#[x_s \in B_j] - \mu(B_j) \right| \leq \frac{4\pi^2}{2M_1 r q M_2^2} \epsilon,$$

for any $j \in \{1, \cdots, q\}$, where $M_1 = \sup_{\lambda \in [0,2\pi]} |g(\lambda)|$ and $M_2 = \sup_{x \in S} |\varphi(x)|$. Suppose N goes to infinity faster than q.

Now, we will show the following claim:

Claim: *Given $\epsilon > 0$, the absolute value of the expression (3.10) can be written as*

$$\left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1 \leq s \leq N \\ 1 \leq s+h \leq N \\ |h| \leq r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| <$$
$$< \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h| \leq r} \sum_{j=1}^{q} \mu(B_j) \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| +$$
$$+ \frac{N+1}{N} \epsilon, \tag{3.11}$$

for large N.

Proof: Observe that using Birkoff's Theorem, for large N,

$$\left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1 \leq s \leq N \\ 1 \leq s+h \leq N \\ |h| \leq r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| \leq$$
$$\leq \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h| \leq r} \sum_{j=1}^{q} \mu(B_j) \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| +$$
$$+ \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} |g(\lambda_k)| \sum_{|h| \leq r} \sum_{j=1}^{q} \frac{4\pi^2 \epsilon}{2M_1 r q M_2^2} |\varphi(\alpha_j)| \left|\varphi(T^h(\alpha_j))\right| \leq$$

$$\le \left| \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r}\sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h}\right| +$$

$$+\frac{1}{N}\sum_{k=0}^{N} |g(\lambda_k)| \sum_{|h|\le r}\sum_{j=1}^{q} \frac{\epsilon}{2M_1 rqM_2^2}M_2^2 =$$

$$= \left| \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r}\sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h}\right| +$$

$$+\frac{1}{N}\sum_{k=0}^{N} |g(\lambda_k)| 2rq\frac{\epsilon}{2M_1 rq} <$$

$$< \left| \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r}\sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h}\right| + \frac{N+1}{N}\epsilon.$$

This proves the Claim.

□

As

$$\left\langle \frac{1}{N}\sum_{h=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g\right\rangle = \frac{1}{4\pi^2 N^2}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1) =$$

$$= \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k)\sum_{j=1}^{q}\frac{1}{N}\sum_{\substack{s:x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} \left[\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))\right] e^{-i\lambda_k h}$$

$$+ \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}}\sum_{j=1}^{q}\frac{1}{N}\#[x_s\in B_j]\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} + o(1),$$

from the above Claim, we can write

$$\left\langle \frac{1}{N}\sum_{h=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g\right\rangle =$$

$$= \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k)\sum_{j=1}^{q}\sum_{\substack{x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h}$$

$$+ \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k)\sum_{|h|\le r}\sum_{j=1}^{q}\mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} + o(1). \qquad (3.12)$$

In Lemma 3 we shall prove that the first term in the expression (3.12) can be taken as small as we want if N and q are large enough.

Lemma 3: *Given $\epsilon > 0$,*

$$\frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k)\sum_{j=1}^{q}\frac{1}{N}\sum_{\substack{x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h} < \epsilon,$$

for q sufficiently large but fixed and for all N large enough.

Proof: Since φ is a continuous map on the compact set $[0, 1)$ it is uniformly continuous. The transformation T is also continuous, so $\varphi(\cdot)\varphi(T^h(\cdot))$ is also a uniformly continuous function. Therefore, for fixed $h \in \mathbb{Z}$, for all $\epsilon_1 > 0$, there exists δ_h such that for all x, y with $|x - y| < \delta_h$, then

$$|\varphi(x)\varphi(T^h(x)) - \varphi(y)\varphi(T^h(y))| < \epsilon_1.$$

Since $|h| < r$ is uniformly bounded one can find a uniform δ that works for all $h \in \mathbb{Z}$ such that $|h| < r$.

Take $\epsilon_1 = \frac{4\pi^2}{2M_1}\epsilon > 0$ and q large enough such that $\frac{1}{q} < \delta$. Therefore, if length of $B_j < \delta$, for all $j \in \{1, \cdots, q\}$, we have

$$|\varphi(x)\varphi(T^h(x)) - \varphi(y)\varphi(T^h(y))| < \epsilon_1,$$

for all $x, y \in B_j$ and for any $h \in \mathbb{Z}$ such $|h| < r$.

Therefore,

$$\left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{j=1}^{q} \frac{1}{N} \sum_{\substack{x_s \in B_j \\ 1 \le s \le N \\ 1 \le s+h \le N \\ |h| \le r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))] e^{-i\lambda_k h} \right| \le$$

$$\le \frac{1}{4\pi^2 N} \sum_{k=0}^{N} |g(\lambda_k)| \frac{1}{N} \sum_{j=1}^{q} \sum_{\substack{x_s \in B_j \\ 1 \le s \le N \\ 1 \le s+h \le N \\ |h| \le r}} \left| \varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j)) \right| \le$$

$$\le \frac{1}{4\pi^2 N} \sum_{k=0}^{N} |g(\lambda_k)| \frac{1}{N} \sum_{j=1}^{q} \sum_{\substack{x_s \in B_j \\ 1 \le s \le N \\ 1 \le s+h \le N \\ |h| < r}} \epsilon_1 = \frac{1}{4\pi^2 N} \sum_{k=0}^{N} |g(\lambda_k)| \frac{1}{N} \sum_{\substack{1 \le s \le N \\ 1 \le s+h \le N \\ |h| < r}} \epsilon_1,$$

for x_s and α_j such that $|x_s - \alpha_j| < \delta$, where the double summation, in the above expression, has N terms all of them less than ϵ_1. Therefore,

$$\frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{j=1}^{q} \frac{1}{N} \sum_{\substack{x_s \in B_j \\ 1 \le s \le N \\ 1 \le s+h \le N \\ |h| \le r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))] e^{-i\lambda_k h} <$$

$$< \frac{(N+1)M_1}{4\pi^2 N} \epsilon_1 < \frac{2M_1}{4\pi^2} \epsilon_1 = \epsilon,$$

where

$$M_1 = \sup_{\lambda \in (0, 2\pi]} |g(\lambda)|.$$

Therefore, Lemma 3 is proved.

□

Now, using Lemma 3, the whole expression (3.12) can be rewritten as

$$\left\langle \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k) \delta_{\lambda_k}, g \right\rangle =$$

$$= \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} \tag{3.13}$$
$$+ \quad o(1).$$

The following lemma proves that the expression (3.13) goes to $\langle f_{X,r}, g\rangle$, when $N \to \infty$.

Lemma 4: *For r fixed,*

$$\frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h} \to \langle f_{X,r}, g\rangle ,$$

when $q, N \to \infty$ (N much more faster than q).

Proof: Given $\epsilon_1 > 0$, note that for fixed N, with N much larger than q, where q is also large, one has

$$\left| \sum_{|h|\le r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} - \sum_{|h|\le r} \left(\int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x) \right) e^{-i\lambda_k h} \right|$$
$$\le \sum_{|h|\le r} \frac{\epsilon_1}{2r} = \epsilon_1.$$

This is true since, for fixed h and $|h| < r$,

$$\sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) \to \int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x), \quad \text{when } q \to \infty.$$

Therefore, given $\epsilon_1 > 0$, if q is large then, for all $|h| < r$,

$$\left| \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) - \int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x) \right| < \epsilon_1.$$

Since
$$\int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x) = \gamma_X(h), \quad \text{for fixed } h,$$

and

$$f_{X,r}(\lambda_k) = \frac{1}{2\pi} \sum_{|h| \le r} \gamma_X(h) e^{-i\lambda_k h},$$

one has

$$\left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} - \right.$$

$$\left. - \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r} \left(\int_0^1 \varphi(x)\varphi(T^h(x)) d\mu(x) \right) e^{-i\lambda_k h} \right| \le$$

$$\le \left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) e^{-i\lambda_k h} \right| \epsilon_1,$$

for large q. Therefore, one has

$$\left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\le r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} - \right.$$

$$\left. - \frac{1}{2\pi N} \sum_{k=0}^{N} g(\lambda_k) f_{X,r}(\lambda_k) \right| \le \frac{1}{4\pi^2 N} \sum_{k=0}^{N} |g(\lambda_k)| \epsilon_1, \qquad (3.14)$$

for large q, and N much larger than q. Expression (3.14) suggests to take

$$\epsilon_1 = \frac{2\pi^2}{M_1} \epsilon > 0.$$

Then,

$$\left\langle \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k) \delta_{\lambda_k}, g \right\rangle = \frac{1}{2\pi N} \sum_{k=0}^{N} g(\lambda_k) f_{X,r}(\lambda_k) + \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \epsilon_1 + o(1)$$

$$= \frac{1}{2\pi} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) f_{X,r}(\lambda_k) + \frac{N+1}{N} \epsilon + o(1)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(\lambda) f_{X,r}(\lambda) d\lambda + \frac{N+1}{N} \epsilon + o(1),$$

for large N.
Then, for large N, one has

$$\left| \left\langle \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k) \delta_{\lambda_k}, g \right\rangle - \frac{1}{2\pi} \int_0^{2\pi} f_{X,r}(\lambda) g(\lambda) d\lambda \right| < o(1). \tag{3.15}$$

This proves Lemma 4.

□

Now we shall prove Theorem 3.1. Considering the expression (3.15) and Lemma 1, for given $\epsilon > 0$, one has

$$\left| \left\langle \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k) \delta_{\lambda_k}, g \right\rangle - \frac{1}{2\pi} \int_0^{2\pi} f_X(\lambda) g(\lambda) d\lambda \right| \leq \epsilon,$$

for N large enough.

This proves Theorem 3.1.

□

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