A CLOSED FORMULA FOR THE DURBIN-LEVINSON'S ALGORITHM IN SEASONAL FRACTIONALLY INTEGRATED PROCESSES

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Abstract - We consider the fractionally integrated ARFIMA Processes with seasonality s, denoted by SARFIMA$(0, D, 0)_s$. This work presents a closed formula for the Durbin-Levinson's algorithm relating the partial autocorrelation and the autocorrelation functions of these processes. In order to obtain the closed formula we show a hypergeometric identity, namely

$$\begin{aligned} &(l-D)\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(l-j+D)}{\Gamma(l-j-D+1)} \\ = \quad & D\Gamma(-D)\Gamma(D-l+1)(l-1)!\,2^{l-1}\cdot\prod_{i=0}^{l-2}\left(D-\frac{i+1}{2}\right), \end{aligned}$$

for any non-negative integer l and for any $D \in (-0.5, 0.5)$.

Any recursive algorithm that requires the use of the left-hand side of the above expression will have smaller error under the use of the right-hand side formula.

The Durbin-Levinson's algorithm is fully calculated for the SARFIMA$(0, D, 0)_s$ processes.

Keywords – Durbin-Levinson's Algorithm, Long Dependence, Partial Autocorrelation Function, Seasonal Fractionally Integrated Models, Hypergeometric Identity.

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1. INTRODUCTION

In practical situations many time series exhibit a periodic pattern. These time series are very common, for instance, in meteorology, economics, hydrology, and astronomy. Sometimes, even in these fields, the seasonality period can depend on time, that is, the autocorrelation structure of the data can vary from season to season. Here, in our analysis, we consider the seasonal period constant over seasons.

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The purpose of this paper is to review the Durbin-Levinson's algorithm for the partial autocorrelation function of the *seasonal fractionally integrated processes*.

The algorithm due to Durbin [1] is a method of efficient estimation for the parameters of a model involving recurrence relations among partial autocorrelation and autocorrelation functions of any stochastic stationary process. Ramsey [2] used the same idea as Durbin-Levinson's algorithm to give a characterization of the partial autocorrelation function of any wide sense stationary process.

We show that for seasonal fractionally integrated processes the Durbin-Levinson's algorithm has a closed formula based on the Gamma function and on the product of monomials. The result is obtained from the Identity of Pfaff-Saalschütz (see [3]) for hypergeometric functions. We point out that the mathematical proof of the main equality envolving the Durbin-Levinson's algorithm becomes simpler using the expression displayed in the abstract of this paper.

The seasonal fractionally integrated processes, given in some detail in Section 2, are treated in [4-7]. For some applications of these models in fields as hydrology, communications, economics, and finances, we refer the reader to [5], [8-10]. The paper [8] uses the full SARFIMA$(p,d,q) \times (P,D,Q)_s$ model considering the maximum likelihood estimation method and also gives a complete analysis for the Nile river monthly flows. Porter-Hudak [9] considers the estimation method proposed by [11] to estimate the parameter D in a SARFIMA$(0,D,0)_s$ model and apply this technique to monetary aggregates data. Ray [10] uses the full SARFIMA$(p,d,q) \times (P,D,Q)_s$ model to forecast the series of monthly IBM product revenues. Bisognin and Lopes [7] analyzes different estimation methods to estimate the parameter D in SARFIMA$(0,D,0)_s$ models. In this work several semi-parametric methods, besides the maximum likelihood one, are proposed to estimate D. For forecasting a future observation from seasonal fractionally integrated models we refer to the papers [7] and [9, 10].

The paper is organized as follows. In the next section we review the seasonal fractionally integrated processes with some definitions and we provide a proof of the main theorem on properties of these processes. Section 3 states the Durbin-Levinson's algorithm for these processes. The main results are in Section 4 and Section 5 concludes.

2. SEASONAL FRACTIONALLY INTEGRATED PROCESSES

We shall consider the *autoregressive fractionally integrated moving average processes with seasonality s*, denoted here by SARFIMA$(p,d,q) \times (P,D,Q)_s$, which are an extension of the ARFIMA(p,d,q) models, proposed by [12-14].

In the following sub-section we give some definitions and some properties for the SARFIMA $(p,d,q) \times (P,D,Q)_s$ processes.

2.1. Some Definitions and Properties

Definition 2.1: For all $D > -1$, the *seasonal difference operator* $\nabla_s^D := (1-\mathcal{B}^s)^D$, where $s \in \mathbb{N}$ is the seasonality, is defined by the binomial expansion

$$\nabla_s^D(\mathcal{B}) := (1-\mathcal{B}^s)^D = \sum_{k\geq 0} \binom{D}{k} (-\mathcal{B}^s)^k = 1 - D\mathcal{B}^s - \frac{D(1-D)}{2!}\mathcal{B}^{2s} - \cdots, \qquad (2.1)$$

where

$$\binom{D}{k} = \frac{\Gamma(1+D)}{\Gamma(1+k)\Gamma(1+D-k)},$$

with $\Gamma(\cdot)$ the Gamma function (see [15]).

Definition 2.2: Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stochastic process with zero mean and autocovariance function $\gamma_X(\cdot)$ such that $\gamma_X(h) \to 0$, as $h \to 0$. The *partial autocorrelation function*, denoted by $\phi_X(k,j)$, $k \in \mathbb{Z}_{\geq}$, $j = 1, \cdots, k$, are the coefficients in the equation

$$\mathcal{P}_{\overline{sp}(X_1,X_2,\cdots,X_k)}(X_{k+1}) = \sum_{j=1}^{k} \phi_X(k,j) X_{k+1-j},$$

where $\mathcal{P}_{\overline{sp}(X_1,X_2,\cdots,X_k)}(X_{k+1})$ is the orthogonal projection of X_{k+1} in the closed span $\overline{sp}(X_1, X_2, \cdots, X_k)$ generated by the previous observations. Then, from the equations

$$\langle X_{k+1} - \mathcal{P}_{\overline{sp}(X_1,X_2,\cdots,X_k)}(X_{k+1}), X_j \rangle = 0, \;\; j = 1, \cdots, k,$$

where $\langle \cdot, \cdot \rangle$ defined the internal product on the Hilbert space $L^2(\Omega, \mathcal{A}, \mathbb{P})$ given by $\langle X, Y \rangle = \mathbb{E}(XY)$, we obtain

$$\begin{bmatrix} 1 & \rho_X(1) & \rho_X(2) & \cdots & \rho_X(k-1) \\ \rho_X(1) & 1 & \rho_X(1) & \cdots & \rho_X(k-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_X(k-1) & \rho_X(k-2) & \rho_X(k-3) & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_X(k,1) \\ \phi_X(k,2) \\ \vdots \\ \phi_X(k,k) \end{bmatrix} = \begin{bmatrix} \rho_X(1) \\ \rho_X(2) \\ \vdots \\ \rho_X(k) \end{bmatrix}, \qquad (2.2)$$

with $\rho_X(\cdot)$ the autocorrelation function of the process $\{X_t\}_{t\in\mathbb{Z}}$. The coefficients $\phi_X(k,j)$, $k \in \mathbb{Z}_{\geqslant}$, $j = 1, \cdots, k$, are uniquely determined by (2.2). For more details, see [15] .

The definition of partial autocorrelation function plays an important role in the Durbin-Levinson's algorithm (see expressions (3.1) and (3.2) in Section 3) and its expression for seasonal fractionally integrated processes is given in Theorem 2.1 (v) below.

Definition 2.3: Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stochastic process given by the expression

$$\phi(\mathcal{B})\Phi(\mathcal{B}^s)\nabla^d\nabla_s^D(X_t - \mu) = \theta(\mathcal{B})\Theta(\mathcal{B}^s)\varepsilon_t, \qquad (2.3)$$

where μ is the *mean* of the process, $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process, s is the seasonal period, $\mathcal{B}$ is the *backward-shift operator*, that is, $\mathcal{B}^k X_t = X_{t-k}$ and $\mathcal{B}^{sk} X_t = X_{t-sk}$, ∇^d and ∇_s^D are, respectively, the difference and the seasonal difference operators, $\phi(\cdot)$, $\theta(\cdot)$, $\Phi(\cdot)$, and $\Theta(\cdot)$ are the polynomials of order p, q, P, and Q, respectively, defined by

$$\begin{aligned} \phi(\mathcal{B}) &= \sum_{i=0}^{p}(-\phi_i)\,\mathcal{B}^i, \quad \theta(\mathcal{B}) = \sum_{j=0}^{q}(-\theta_j)\,\mathcal{B}^j, \\ \Phi(\mathcal{B}) &= \sum_{k=0}^{P}(-\Phi_k)\,\mathcal{B}^k, \quad \Theta(\mathcal{B}) = \sum_{l=0}^{Q}(-\Theta_l)\,\mathcal{B}^l, \end{aligned}$$

where ϕ_i, $1 \leq i \leq p$, θ_j, $1 \leq j \leq q$, Φ_k, $1 \leq k \leq P$, and Θ_l, $1 \leq l \leq Q$ are constants. Then, $\{X_t\}_{t\in\mathbb{Z}}$ is a *seasonal fractionally integrated ARIMA*$(p,d,q) \times (P,D,Q)_s$ process with period s, denoted by SARFIMA$(p,d,q) \times (P,D,Q)_s$, where d and D are, respectively, *the degree of differencing* and *of seasonal differencing* parameters.

Remarks: (1). A particular case of the SARFIMA$(p,d,q)\times(P,D,Q)_s$ process is when $p = q = P = Q = 0$. This process is called *seasonal fractionally integrated ARIMA model with period s*, denoted by SARFIMA$(0,D,0)_s$, which will be the goal of this work and it is given by

$$\nabla_s^D (X_t - \mu) = \varepsilon_t, \quad t \in \mathbb{Z}. \tag{2.4}$$

(2). When $P = Q = 0$, $D = 0$ and $s = 1$ the SARFIMA$(p,d,q) \times (P,D,Q)_s$ process is just the ARFIMA(p,d,q) process (see [16]). In this situation we already know the behaviour of the parameter estimators (see, for instance, [17, 18]).

(3). From the paper [4] one may generate the process $\{X_t\}_{t\in\mathbb{Z}}$, given by the expression (2.4), using its infinite moving average representation given in Theorem 2.1 (ii), whenever $D < 0.5$. However, this technique requires a truncation point r smaller than or equal to the sample size n, given by

$$\nabla_{s;r}^{-D}(\mathcal{B}) = \sum_{k=0}^{r} \binom{-D}{k} (-\mathcal{B}^s)^k = \sum_{k=0}^{r} \psi_k(\mathcal{B}^{sk}), \tag{2.5}$$

where the coefficients ψ_k are given in the expression (2.7). To exemplify the error magnitude of this truncation, one observes than when $\mathcal{B} = 1$, the expression (2.1) gives $\nabla_1^D(1) = 0$. We also point out that using $r = 1,000$ one gets $\nabla_{1;1,000}^{0.75}(1) = 0.00155$ while using $r = 10,000$ one get $\nabla_{1;10,000}^{0.75}(1) = 0.0002758$. These two values show how important is the right choice of the truncation point r. Baillie et al. [20] point out that the use of this technique to generate a stochastic process can give meaningless estimation results. To overcome with this problem we use the generate algorithm, also proposed by [14], based on the partial autocorrelation function of the SARFIMA$(0,D,0)_s$ process. When the goal is mainly to forecast a future value we consider the infinite moving average representation for this process to obtain the forecast error expression (see [7]).

Before giving some properties of the SARFIMA$(0,D,0)_s$ processes it is convenient to introduce the notation $\mathbb{Z}_{\geq} = \{k \in \mathbb{Z} \mid k \geq 0\}$, $\mathbb{Z}_{\leq} = \{k \in \mathbb{Z} \mid k \leq 0\}$ and let A be the set $\{1, \cdots, s-1\} \subset \mathbb{N}$.

For the proof of the following theorem we refer the reader to [4, 5] and [7]. In the following, we use the notation $[\Gamma(0)]^{-1} = 0$.

Theorem 2.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be the SARFIMA $(0,D,0)_s$ process, given by the expression (2.4), with mean zero and $s \in \mathbb{N}$ as the seasonal period. Then,*

(i) *when $D > -0.5$, $\{X_t\}_{t\in\mathbb{Z}}$ is an invertible process with infinite autoregressive representation given by*

$$\Pi(\mathcal{B}^s)X_t = \sum_{k\geq 0} \pi_k X_{t-sk} = \varepsilon_t,$$

where

$$\pi_k = \frac{-D(1-D)\cdots(k-D-1)}{k!} = \frac{(k-D-1)!}{k!(-D-1)!} = \frac{\Gamma(k-D)}{\Gamma(-D)\Gamma(k+1)}\cdot \tag{2.6}$$

When $k \to \infty$, $\pi_k \sim \frac{k^{-D-1}}{\Gamma(-D)}\cdot$

(ii) *when* $D < 0.5$, $\{X_t\}_{t\in\mathbb{Z}}$ *is a stationary process with an infinite moving average representation given by*

$$X_t = \Psi(\mathcal{B}^s)\varepsilon_t = \sum_{k\geq 0}\psi_k \varepsilon_{t-sk},$$

where

$$\psi_k = \frac{D(1+D)\cdots(k+D-1)}{k!} = \frac{(k+D-1)!}{k!(D-1)!} = \frac{\Gamma(k+D)}{\Gamma(D)\Gamma(k+1)}\cdot \tag{2.7}$$

When $k \to \infty$, $\psi_k \sim \frac{k^{D-1}}{\Gamma(D)}\cdot$

In the following, we assume that $D \in (-0.5, 0.5)$.

(iii) *The process* $\{X_t\}_{t\in\mathbb{Z}}$ *has spectral density function given by*

$$f_X(w) = \frac{\sigma_\varepsilon^2}{2\pi}\left[2\sin\left(\frac{sw}{2}\right)\right]^{-2D}, \quad 0 < w \leq \pi. \tag{2.8}$$

At the seasonal frequencies, for $\nu = 0, 1, \cdots, [s/2]$, *where* $[x]$ *means the integer part of* x, *it behaves as*

$$f_X\left(\frac{2\pi\nu}{s} + w\right) \sim f_\varepsilon\left(\frac{2\pi\nu}{s}\right)(sw)^{-2D}, \quad \textit{when} \quad w \to 0\cdot$$

(iv) *The process* $\{X_t\}_{t\in\mathbb{Z}}$ *has autocovariance and autocorrelation functions of order* k, $k \in \mathbb{Z}_{\geq}$, *given respectively by*

$$\gamma_X(sk+\xi) = \begin{cases} \frac{(-1)^k\Gamma(1-2D)}{\Gamma(k-D+1)\Gamma(1-k-D)}\sigma_\varepsilon^2 = \gamma_X(k), & \textit{if } \xi = 0 \\ 0, & \textit{if } \xi \in A, \end{cases} \tag{2.9}$$

and

$$\rho_X(sk+\xi) = \begin{cases} \frac{\Gamma(1-D)\Gamma(k+D)}{\Gamma(D)\Gamma(k-D+1)} = \rho_X(k), & \textit{if } \xi = 0 \\ 0, & \textit{if } \xi \in A. \end{cases} \tag{2.10}$$

When $k \to \infty$, $\rho_X(sk) \sim \frac{\Gamma(1-D)}{\Gamma(D)}k^{2D-1}$.

(v) *The process* $\{X_t\}_{t\in\mathbb{Z}}$ *has partial autocorrelation function given by*

$$\phi_X(sk+\xi, sl+\eta) = \begin{cases} -\binom{k}{l}\frac{\Gamma(l-D)\Gamma(k-l+1-D)}{\Gamma(-D)\Gamma(k-D+1)} = \phi_X(k,l), & \textit{if } \eta = 0 \\ 0, & \textit{if } \eta \in A, \end{cases} \tag{2.11}$$

for any $k, l \in \mathbb{Z}_{\geq}$ *and* $\xi \in A \cup \{0\}$.

From the expression (2.11), when $k = l$, *the partial autocorrelation function of order* k *is given by*

$$\phi_X(sk, sk) = \frac{D}{k-D} = \phi_X(k,k), \quad \textit{for all} \quad k \in \mathbb{Z}_{\geq}. \tag{2.12}$$

Proof: Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SARFIMA$(0, D, 0)_s$ process with zero mean and seasonality s, given by the expression (2.4).

(i) Writing $\Pi(\mathcal{B}^s)X_t = \varepsilon_t$, we have $\Pi(z^s) = (1-z^s)^D$. When $D > -0.5$, the power series expansion of $\Pi(z^s)$ converges for $|z| \leq 1$, and so the process $\{X_t\}_{t\in\mathbb{Z}}$ is stationary. The binomial expansion of $(1-z^s)^D$ gives (2.6). As $k \to \infty$, by Stirling's formula

$$\pi_k \sim \frac{e^{D+1}(k-D-1)^{-D-1}}{\Gamma(-D)} \sim \frac{c_1 k^{-D-1}}{\Gamma(-D)} \sim \frac{k^{-D-1}}{\Gamma(-D)},$$

where $c_1 = e^{D+1}$.

From the above equation it follows that $\sum_{k=0}^{\infty}\pi_k^2<\infty$ holds if and only if $\sum_{k\geqslant N}(1/k)^{2+2D}<\infty$, for N sufficiently large, that is, when $2+2D > 1$. Since $D > -0.5$, than $\sum_{k=0}^{\infty}\pi_k^2 < \infty$. In this case, the series $\sum_{k\geqslant 0}\pi_k X_{t-sk}$ converges in $L^2(\Omega)$. We than say that

$$(1-\mathcal{B}^s)^D X_t = \sum_{k\geqslant 0}\pi_k X_{t-sk} = \varepsilon_t.$$

(ii) The proof is similar to (i) replacing D by $-D$.

(iii) From the definition of a spectral density function for any stationary stochastic process, for a SARFIMA$(0, D, 0)_s$ process, given by expression (2.4), one has

$$f_X(w) = \left[|1-e^{-isw}|^{-D}\right]^2 f_\varepsilon(w) = \frac{\sigma_\varepsilon^2}{2\pi}\left[|1-e^{-isw}|^2\right]^{-D} = \frac{\sigma_\varepsilon^2}{2\pi}\left[2\,\sin\!\left(\frac{sw}{2}\right)\right]^{-2D},$$

for any $w \in (0, \pi]$, where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is the white noise process.

Since $\sin\!\left(\frac{sw}{2}\right) \to \left(\frac{sw}{2}\right)$, when $w \to 0$, one has

$$f_X\left(\frac{2\pi\nu}{s} + w\right) \sim \frac{\sigma_\varepsilon^2}{2\pi}(sw)^{-2D}, \quad \text{when} \quad w \to 0,$$

where $\nu = 0, 1, \cdots, [s/2]$, with $[x]$ meaning the integer part of x.

(iv) From the Herglotz's theorem (see [15, 19]) and expression (2.8), for any $s \in \mathbb{N}$, one has

$$\begin{aligned}
\gamma_X(sk) &= \int_{-\pi}^{\pi} \cos(swk) \frac{\sigma_\varepsilon^2}{2\pi} \left| 2\, \sin\left(\frac{sw}{2}\right) \right|^{-2D} dw \\
&= \frac{\sigma_\varepsilon^2}{2\pi} 2^{-2D} \int_0^{2\pi} \cos(swk) \left| \sin\left(\frac{sw}{2}\right) \right|^{-2D} dw \\
&= \frac{\sigma_\varepsilon^2}{\pi} 2^{-2D} \frac{\pi(-1)^k \Gamma(-2D+1)}{2^{-2D}\Gamma(k-D+1)\Gamma(1-k-D)} \\
&= \frac{(-1)^k \Gamma(-2D+1)}{\Gamma(k-D+1)\Gamma(1-k-D)}\, \sigma_\varepsilon^2, \qquad (2.13)
\end{aligned}$$

where the third equality in expression (2.13) is formula 3.631.8, page 372, in [21]. It is easy to see that

$$\gamma_X(sk+\zeta) = 0, \quad \text{for} \quad \zeta \in A.$$

The other results for the autocorrelation function follow immediately.

(v) The proof of this part is given in Section 3 below.

□

Remarks: (1). The spectral density function of the stationary SARFIMA$(0, D, 0)_s$ process is unbounded when $0 < D < 0.5$ and it has zeroes when D is negative.

(2). In the SARFIMA$(0, D, 0)_s$ processes the spectral density function is unbounded at frequencies $\frac{2\pi\nu}{s}$, for $\nu = 1, \cdots, [s/2]$. While, in ARFIMA(p, d, q) processes, the long memory is characterized by its spectral density function being unbounded at zero frequency only. Between two seasonal frequencies a SARFIMA process has similar behavior as an ARFIMA process (see Figure 2.1: (a) and (d)).

The autocorrelation function of a SARFIMA$(0, D, 0)_s$ process has values different from zero at lags that are multiples of s and zero otherwise. While in an ARFIMA(p, d, q) process the values of the autocorrelation function do not depend on s (see Figure 2.1: (b) and (e)).

The partial autocorrelation function $\phi_X(k, j)$ in SARFIMA$(0, D, 0)_s$ processes has zero values whenever $j \neq sl$, for $l \in \mathbb{Z}_{\geqslant}$ and nonzero otherwise. Although in ARFIMA(p, d, q) processes this function does not depend on s (see Figure 2.1: (c) and (f)).

(3). The SARFIMA$(p, d, q) \times (P, D, Q)_s$ process is stationary when d and D are less than 0.5 and the polynomials $\phi(\mathcal{B}) \cdot \Phi(\mathcal{B}) = 0$ and $\theta(\mathcal{B}) \cdot \Theta(\mathcal{B}) = 0$ have no roots in common and all roots are outside of the unit circle. When $D > 0$, the process is said to have *seasonal long memory*.

(4). If $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary stochastic SARFIMA$(p, d, q) \times (P, D, Q)_s$ process (see expression (2.3)), with $d, D \in (-0.5, 0.5)$ and zero mean, its spectral density function is given by

$$f_X(w) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\theta(e^{-iw})|^2}{|\phi(e^{-iw})|^2} \frac{|\Theta(e^{-isw})|^2}{|\Phi(e^{-isw})|^2} \left[2\sin\left(\frac{w}{2}\right)\right]^{-2d} \left[2\sin\left(\frac{sw}{2}\right)\right]^{-2D},$$

for all $0 < w \leq \pi$, where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process.

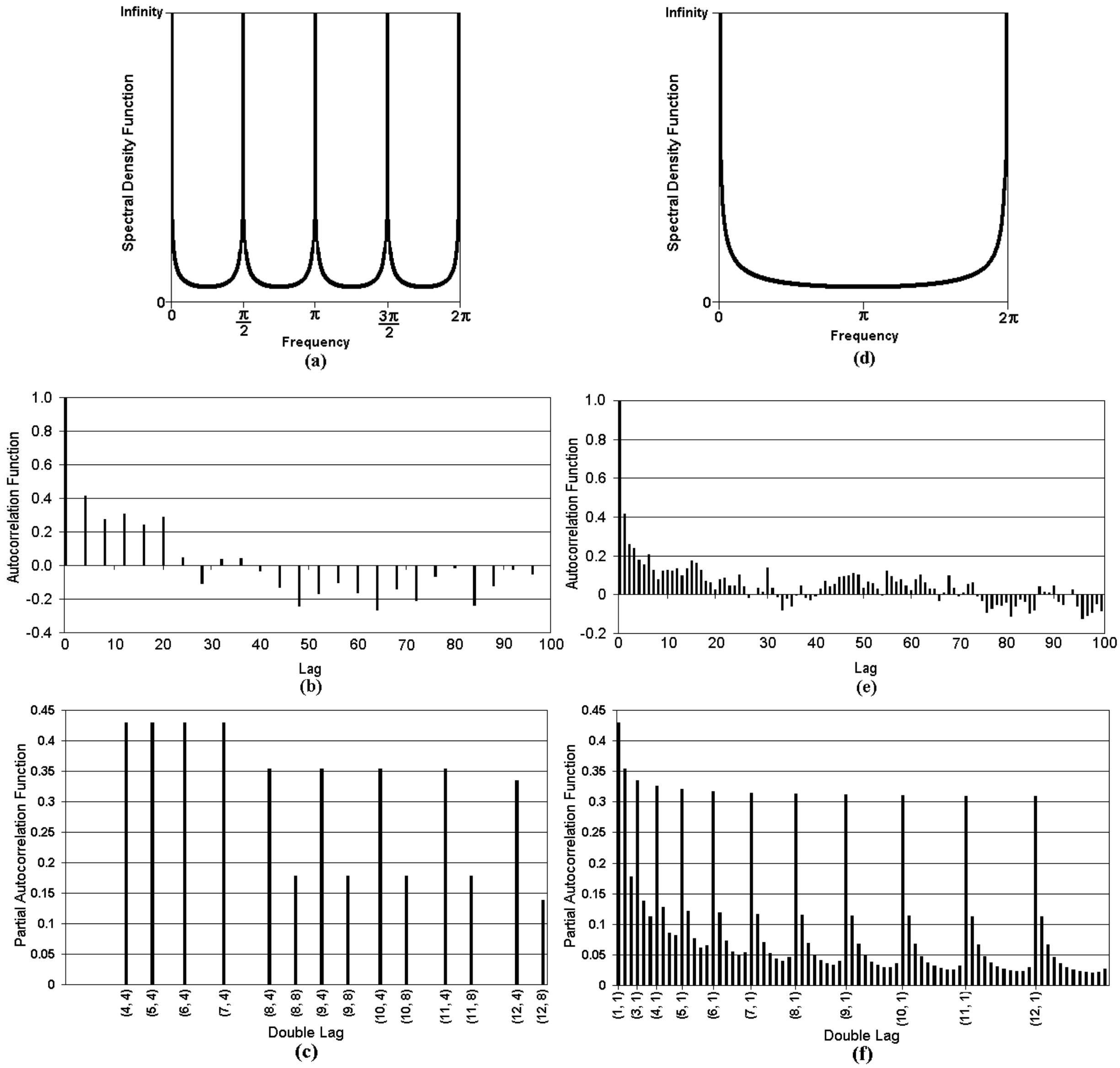

Figure 2.1: The graphs the left-hand side are related to the ARFIMA$(0, 0.3, 0)$ process while in the right-hand side they are related to the SARFIMA$(0, 0.3, 0)_4$: (a) and (d) spectral density functions; (b) and (e) autocorrelation functions; (c) and (f) partial autocorrelation functions.

3. DURBIN-LEVINSON'S ALGORITHM

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SARFIMA$(0, D, 0)_s$ process, given in expression (2.4), with mean μ equal to zero. We want to show that its partial autocorrelation function $\phi_X(\cdot, \cdot)$, given by Theorem 2.1 (v), satisfies the following systems

$$\phi_X(sl,sl) = \frac{\rho_X(sl) - \sum_{j=1}^{sl-1}\phi_X(sl-1,j)\rho_X(sl-j)}{1-\sum_{j=1}^{sl-1}\phi_X(sl-1,j)\rho_X(j)} \tag{3.1}$$

and

$$\phi_X(k+1,sl) = \phi_X(k,sl) - \phi_X(k+1,k+1)\phi_X(k,k+1-sl), \tag{3.2}$$

for any $l \in \mathbb{Z}_{\geq}$ such that $sl < k+1$, where $k+1$ may or may not be a multiple of s.

Recurrence relations (3.1) and (3.2) are known as the Durbin-Levinson's algorithm and they explain how to go from lag k to lag $(k+1)$. We shall prove the recurrence relation (3.1) for any $D \in (-0.5, 0.5)$ with $D \neq 0$.

Considering only the sum in the numerator of expression (3.1) we have

$$\begin{aligned}
&\sum_{j=1}^{sl-1}\phi_X(sl-1,j)\rho_X(sl-j) = \phi_X(sl-1,1)\rho_X(sl-1) \\
&+\phi_X(sl-1,2)\rho_X(sl-2) + \cdots + \phi_X(sl-1,s)\rho_X(sl-s) \\
&+\phi_X(sl-1,s+1)\rho_X(sl-s-1) + \cdots + \phi_X(sl-1,2s)\rho_X(sl-2s) + \cdots \\
&+\phi_X(sl-1,sl-2)\rho_X(sl-sl+2) + \phi_X(sl-1,sl-1)\rho_X(sl-sl+1) \\
&= \phi_X(sl-1,s)\rho_X(s(l-1)) + \phi_X(sl-1,2s)\rho_X(s(l-2)) + \cdots \\
&+\phi_X(sl-1,s(l-1))\rho_X(1) = \sum_{j=1}^{l-1}\phi_X(sl-1,sj)\rho_X(s(l-j)) \qquad (3.3) \\
&= \sum_{j=1}^{l-1}\phi_X(l-1,j)\rho_X(l-j). \qquad (3.4)
\end{aligned}$$

The equality in expression (3.3) is true since $\phi_X(sk+\xi, sl+\eta) = 0$, if $\eta \in A$, for any $k, l \in \mathbb{Z}_{\geq}$ and $\xi \in A \cup \{0\}$. On the other hand, the equality in expression (3.4) is due to Theorem 2.1 (iv) and (v).

Considering only the sum in the denominator of expression (3.1) we have

$$\begin{aligned}
&\sum_{j=1}^{sl-1}\phi_X(sl-1,j)\rho_X(j) = \phi_X(sl-1,1)\rho_X(1) + \phi_X(sl-1,2)\rho_X(2) + \cdots \\
&+\phi_X(sl-1,s)\rho_X(s) + \cdots + \phi_X(sl-1,2s)\rho_X(2s) + \cdots \\
&+\phi_X(sl-1,s(l-1))\rho_X(s(l-1)) + \cdots + \phi_X(sl-1,sl-2)\rho_X(sl-2) \\
&+\phi_X(sl-1,sl-1)\rho_X(sl-1) = \sum_{j=1}^{l-1}\phi_X(sl-1,sj)\rho_X(sj) \qquad (3.5) \\
&= \sum_{j=1}^{l-1}\phi_X(l-1,j)\rho_X(j). \qquad (3.6)
\end{aligned}$$

The equality in expression (3.5) is true since $\phi_X(sk+\xi, sl+\eta) = 0$, if $\eta \in A$, for any $k, l \in \mathbb{Z}_{\geq}$ and $\xi \in A \cup \{0\}$. On the other hand, the equality in expression (3.6) is due to Theorem 2.1 (iv) and (v).

From the expressions (3.4) and (3.6) the system (3.1) is then given by

$$\phi_X(sl, sl) = \frac{\rho_X(l) - \sum_{j=1}^{l-1} \phi_X(l-1, j)\rho_X(l-j)}{1 - \sum_{j=1}^{l-1} \phi_X(l-1, j)\rho_X(j)} = \phi_X(l, l). \tag{3.7}$$

Also, from Theorem 2.1 (v), the system (3.2) is given by

$$\phi_X(k+1, l) = \phi_X(k, l) - \phi_X(k+1, k+1)\phi_X(k, k+1-l), \tag{3.8}$$

for any $l \in \mathbb{Z}_{\geq}$ such that $l < k+1$.

We want to show that the partial autocorrelation functions of the SARFIMA$(0, D, 0)_s$ process, given in Theorem 2.1 (v) satisfy the systems in expressions (3.7) and (3.8), for any $k, l \in \mathbb{Z}_{\geq}$, where $\rho_X(\cdot)$ is given in Theorem 2.1 (iv).

Lemma 3.1: *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a process given by (2.4). For any $k, l \in \mathbb{Z}_{\geq}$ such that $l < k+1$, the system given in (3.8) is true for the partial autocorrelation function of $\{X_t\}_{t\in\mathbb{Z}}$.*

Proof: The right-hand side of the expression (3.8) can be written as

$$\begin{aligned}
&\phi_X(k, l) - \phi_X(k+1, k+1)\phi_X(k, k+1-l) = \\
&-\binom{k}{l}\frac{\Gamma(l-D)\Gamma(k-D-l+1)}{\Gamma(-D)\Gamma(k-D+1)} - \binom{k+1}{k+1}\frac{\Gamma(k+1-D)\Gamma(-D+1)}{\Gamma(-D)\Gamma(k+2-D)} \\
&\cdot\binom{k}{k+1-l}\frac{\Gamma(k+1-l-D)\Gamma(k-D-k-1+l+1)}{\Gamma(-D)\Gamma(k-D+1)} \\
&= -\binom{k}{l}\frac{\Gamma(l-D)\Gamma(k-D-l+1)}{\Gamma(-D)\Gamma(k-D+1)} - \binom{k}{k+1-l}\frac{\Gamma(k+1-D)\Gamma(1-D)}{\Gamma(-D)\Gamma(k+2-D)} \\
&\cdot\frac{\Gamma(k+1-l-D)\Gamma(l-D)}{\Gamma(-D)\Gamma(k+1-D)} \\
&= -\binom{k}{l}\frac{\Gamma(l-D)\Gamma(k+1-l-D)}{\Gamma(-D)\Gamma(k+1-D)} - \binom{k}{k+1-l}\frac{(-D)\Gamma(k+1-l-D)\Gamma(l-D)}{(k+1-D)\Gamma(-D)\Gamma(k+1-D)} \\
&= -\binom{k}{l}\frac{\Gamma(l-D)\Gamma(k+1-l-D)}{\Gamma(-D)\Gamma(k+1-D)} - \binom{k}{k+1-l}\frac{(-D)\Gamma(k+1-l-D)\Gamma(l-D)}{\Gamma(-D)\Gamma(k+2-D)} \\
&= -\frac{\Gamma(l-D)\Gamma(k+2-l-D)}{\Gamma(-D)\Gamma(k+2-D)}\left[\binom{k}{l}\frac{k+1-D}{k+1-l-D} - \binom{k}{k+1-l}\frac{D}{k+1-l-D}\right].
\end{aligned} \tag{3.9}$$

The terms in brackets in (3.9) can be rewritten as

$$\begin{aligned}
&\binom{k}{l}\frac{k+1-D}{k+1-l-D} - \binom{k}{k+1-l}\frac{D}{k+1-l-D} \\
&= \frac{k!}{(l)!(k-l)!}\cdot\frac{k+1-D}{k+1-l-D} - \frac{k!}{(k+1-l)!(l-1)!}\cdot\frac{D}{k+1-l-D} \\
&= \frac{(k+1)!}{l!(k+1-l)!}\left[\frac{(k+1-l)(k+1-D)}{(k+1)(k+1-l-D)} - \frac{lD}{(k+1)(k+1-l-D)}\right] \\
&= \binom{k+1}{l}\left(\frac{(k+1-l)(k+1-D)-lD}{(k+1)(k+1-l-D)}\right) = \binom{k+1}{l}.
\end{aligned} \tag{3.10}$$

Therefore, from the equality (3.10), the expression (3.9) can be rewritten as

$$\begin{aligned}&\phi_X(k,l) - \phi_X(k+1,k+1)\phi_X(k,k+1-l)\\&= -\binom{k+1}{l}\frac{\Gamma(l-D)\Gamma(k+2-l-D)}{\Gamma(-D)\Gamma(k+2-D)} = \phi_X(k+1,l).\end{aligned} \tag{3.11}$$

In view of the expression (3.11) this lemma holds.

□

Lemma 3.2: *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a process given by (2.4), where $D \in (-0.5, 0.5)$ with $D \neq 0$. Then, the quocient in expression (3.7) is given by*

$$\frac{\rho_X(l) - \sum_{j=1}^{l-1}\phi_X(l-1,j)\rho_X(l-j)}{1-\sum_{j=1}^{l-1}\phi_X(l-1,j)\rho_X(j)} = \frac{\displaystyle\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(l-j+D)}{\Gamma(l-j-D+1)}}{\displaystyle\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(j+D)}{\Gamma(j-D+1)}}. \tag{3.12}$$

Proof: Let us consider the left-hand side of expression (3.12). Then, we observe that its numerator is given by

$$\begin{aligned}&\rho_X(l) - \sum_{j=1}^{l-1}\phi_X(l-1,j)\rho_X(l-j) = \frac{\Gamma(1-D)\Gamma(l+D)}{\Gamma(D)\Gamma(l-D+1)}\\&-\sum_{j=1}^{l-1}(-1)\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-1-D-j+1)}{\Gamma(-D)\Gamma(l-1-D+1)}\cdot\frac{\Gamma(1-D)\Gamma(l-j+D)}{\Gamma(D)\Gamma(l-j-D+1)}\\&=\frac{\Gamma(1-D)\Gamma(l+D)}{\Gamma(D)\Gamma(l-D+1)}+\frac{(-D)}{\Gamma(D)\Gamma(l-D)}\\&\cdot\sum_{j=1}^{l-1}\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(l-j+D)}{\Gamma(l-j-D+1)} \qquad (3.13)\\&=\frac{(-D)}{\Gamma(D)\Gamma(l-D)}\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(l-j+D)}{\Gamma(l-j-D+1)}, \qquad (3.14)\end{aligned}$$

since the $j = 0$ term in the sum of expression (3.14) is equal to the first term in the expression (3.13).

Now we observe that the denominator of the left-hand side of expression (3.12) is given by

$$\begin{aligned}&1-\sum_{j=1}^{l-1}\phi_X(l-1,j)\rho_X(j) = 1+\sum_{j=1}^{l-1}\binom{l-1}{j}\frac{\Gamma(j-D)\Gamma(l-j-D)}{\Gamma(-D)\Gamma(l-D)}\\&\cdot\frac{\Gamma(1-D)\Gamma(j+D)}{\Gamma(D)\Gamma(j-D+1)}\end{aligned}$$

$$= 1 + \frac{(-D)}{\Gamma(D)\Gamma(l-D)} \sum_{j=1}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\Gamma(j+D)\Gamma(l-j-D)}{\Gamma(1+j-D)} \tag{3.15}$$

$$= \frac{(-D)}{\Gamma(D)\Gamma(l-D)} \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\Gamma(j+D)\Gamma(l-j-D)}{\Gamma(j-D+1)}, \tag{3.16}$$

since the $j = 0$ term in the sum of expression (3.16) is equal to the first term in expression (3.15).

From expressions (3.14) and (3.16), the equality (3.12) holds. Hence, this lemma is proved. □

We still need to show that (3.12) is equal to $\phi_X(l, l)$. This will follow from Theorem 4.2 in Section 4.

4. MAIN RESULTS

In this section we will show that the numerator of the left-hand side of expression (3.12) times $(l - D)$ (or its denominator times D) is equal to $\phi_X(l, l)$, that is,

$$(l-D) \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(l-j+D)}{\Gamma(l-j-D+1)}$$
$$= D \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(j+D)}{\Gamma(j-D+1)}. \tag{4.1}$$

Moreover, we will also show that

$$(l-D) \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\Gamma(l-j-D)\Gamma(l-j+D)}{\Gamma(l-j-D+1)}$$
$$= \quad D\Gamma(-D)\Gamma(D-l+1)(l-1)!\, 2^{l-1} \cdot \prod_{i=0}^{l-2} \left(D - \frac{i+1}{2}\right). \tag{4.2}$$

The equalities (4.1) and (4.2) will be proved, respectively, in Corollary 4.3 and 4.1. The Durbin-Levinson's algorithm is a consequence of Theorem 2.1 and equality (4.2) above.

We shall first define the *hypergeometric function*.

Definition 4.1: If a_i, b_i and x are complex numbers, with $b_i \notin \mathbb{Z}_{\leq}$, we define the *hypergeometric function* by

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{x^n}{n!},$$

where $(a)_n$ stands for the *Pochhammer symbol*

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

This series is absolutely convergent for all $x \in \mathbb{C}$ such that $|x| < 1$, and also for $|x| = 1$, provided $\Re(b_1 + b_2) > \Re(a_1 + a_2 + a_3)$. Furthermore, it is said to be *balanced* if $b_1 + b_2 = 1 + a_1 + a_2 + a_3$. Note that in case some a_i is a nonpositive integer the above sum is finite and it suffices to let n range from 0 to $-a_i$.

The following identity for a *terminating balanced hypergeometric sum* is of fundamental importance in the sequel. For the identity's proof we refer the reader to [3], Thm. 2.2.6, page 69.

Theorem 4.1 (Identity of Pfaff–Saalschütz). *Let $k \in \mathbb{Z}_{\geq}$, and a, b, and c be complex numbers such that $c, 1 + a + b - c - k \notin \mathbb{Z}_{\leq}$. Then,*

$$_3F_2(-k, a, b; c, 1 + a + b - c - k; 1) = \frac{(c-a)_k (c-b)_k}{(c)_k (c-a-b)_k} . \tag{4.3}$$

Remark. If $(c_n)_{n \geq 0}$ is a sequence of complex numbers satisfying

$$\frac{c_{n+1}}{c_n} = \frac{(a_1 + n)(a_2 + n)(a_3 + n)x}{(n+1)(b_1 + n)(b_2 + n)} \text{ for all } n,$$

straightforward computations show that

$$\sum_{n=0}^{\infty} c_n = c_0 \cdot {}_3F_2(a_1, a_2, a_3; b_1, b_2; x) . \tag{4.4}$$

Theorem 4.2. *Let x and z be complex numbers, with $x \notin \mathbb{Z}$ and $z \notin \mathbb{Z}_{\geq}$. Then*

$$\sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-x)\,\Gamma(l-j+x)}{z-j} = \frac{\Gamma(-x)\,\Gamma(1+x)\,\Gamma(1-z)}{z\,\Gamma(l-z)} \cdot (l-1)! \prod_{i=1}^{l-1} (x - z + i) . \tag{4.5}$$

For $z \in \{l, l+1, \ldots\}$ the right-hand side of expression (4.5) has a removable singularity and by analytic continuation the result is still true.

Proof: Setting

$$c_j := \binom{l-1}{j} \frac{\Gamma(j-x)\,\Gamma(l-j+x)}{z-j} ,$$

we have

$$\frac{c_{j+1}}{c_j} = \frac{j!(l-1-j)!}{(j+1)!(l-2-j)!} \cdot \frac{\Gamma(j+1-x)\,\Gamma(l-1-j+x)}{\Gamma(j-x)\,\Gamma(l-j+x)} \cdot \frac{z-j}{z-j-1}$$

$$= \frac{l-1-j}{j+1} \cdot \frac{j-x}{l-1-j+x} \cdot \frac{z-j}{z-j-1}$$

$$= \frac{j-l+1}{j+1} \cdot \frac{j-x}{j-x-l+1} \cdot \frac{j-z}{j+1-z}\,.$$

Hence, from (4.4) we have

$$\sum_{j=0}^{l-1} c_j = c_0 \cdot {}_3F_2(-l+1, -x, -z; -x-l+1, 1-z; 1)\,.$$

We now apply (4.3) with $a = -x$, $b = -z$, and $c = 1-z$. Note that the hypotheses of Pfaff-Saalschütz's Identity are satisfied, since $1 + a + b - c - l + 1 = -x - l + 1$ and $c = 1-z$ do not belong to $\mathbb{Z}_{\leq}$. It follows that

$$\sum_{j=0}^{l-1} c_j = c_0 \cdot \frac{(1+x-z)_{l-1}(1)_{l-1}}{(1-z)_{l-1}(1+x)_{l-1}}\,.$$

Therefore,

$$\sum_{j=0}^{l-1} c_j = \frac{\Gamma(-x)\,\Gamma(x+l)}{z} \cdot \frac{(x-z+1)(x-z+2)\cdots(x-z+l-1)(l-1)!}{\dfrac{\Gamma(l-z)}{\Gamma(1-z)} \cdot \dfrac{\Gamma(x+l)}{\Gamma(x+1)}}$$

$$= \frac{\Gamma(-x)\,\Gamma(1+x)\,\Gamma(1-z)}{z\,\Gamma(l-z)} \cdot (l-1)! \prod_{i=1}^{l-1}(x-z+i)\,.$$

□

Corollary 4.1. *If $l \in \mathbb{N} - \{1\}$ and D is a noninteger complex number, then*

$$(l-D) \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\,\Gamma(l+D-j)}{l-D-j}$$
$$= D\,\Gamma(-D)\,\Gamma(D-l+1)\,(l-1)!\,2^{l-1} \cdot \prod_{i=0}^{l-2}\left(D - \frac{i+1}{2}\right). \tag{4.6}$$

Proof: Taking $x = D$ and $z = l - D$ in (4.5) and multiplying both sides by $l - D$ yields

$$(l-D) \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\,\Gamma(l+D-j)}{l-D-j}$$
$$= (l-D)\,\frac{\Gamma(-D)\,\Gamma(1+D)\,\Gamma(D-l+1)}{(l-D)\,\Gamma(D)} \cdot (l-1)! \prod_{i=1}^{l-1}(2D-l+i)$$

$$= \frac{\Gamma(-D)\, D\, \Gamma(D)\, \Gamma(D-l+1)}{\Gamma(D)} \cdot (l-1)! \prod_{i=0}^{l-2} (2D-i-1)$$

$$= D\, \Gamma(-D)\, \Gamma(D-l+1)\, (l-1)!\, 2^{l-1} \prod_{i=0}^{l-2} \left(D - \frac{i+1}{2} \right).$$

□

Corollary 4.2. *If $l \in \mathbb{N} - \{1\}$ and D is a noninteger complex number, then*

$$D \sum_{k=0}^{l-1} \binom{l-1}{k} \frac{\Gamma(l-1-k+D)\, \Gamma(k-D+1)}{l-1-k-D}$$
$$= D\, \Gamma(-D)\, \Gamma(D-l+1)\, (l-1)!\, 2^{l-1} \cdot \prod_{i=0}^{l-2} \left(D - \frac{i+1}{2} \right). \tag{4.7}$$

Proof: Use the same idea as in the proof of Corollary 4.1, taking $x = D - 1$ and $z = l - 1 - D$ in (4.5) and multiplying both sides by D.

□

Corollary 4.3. *If $l \in \mathbb{N}$ and D is a noninteger complex number, then*

$$(l-D) \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\Gamma(j-D)\, \Gamma(l+D-j)}{l-D-j}$$
$$= D \sum_{k=0}^{l-1} \binom{l-1}{k} \frac{\Gamma(l-1-k+D)\, \Gamma(k-D+1)}{l-1-k-D}. \tag{4.8}$$

Proof: For $l \geq 2$, combine the expression (4.6) with (4.7). If $l = 1$, expression (4.8) holds trivially.

□

5. CONCLUSIONS

In this paper we gave some results for the seasonal fractionally integrated SARFIMA$(0, D, 0)_s$ processes. The Durbin-Levinson's algorithm recurrent expression was fully calculated for these processes.

Based on some properties of the hypergeometric functions, we derived a simpler and closed formula for the Durbin-Levinson's algorithm to obtain the partial autocorrelation functions of order k for SARFIMA$(0, D, 0)_s$ processes.

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