Diffusion Processes: entropy, Gibbs states and the continuous time Ruelle operator

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Abstract

We consider a Riemmaniann compact manifold M, the associated Laplacian Δ and the corresponding Brownian motion X_t, $t \geq 0$. Given a Lipschitz function $V : M \to \mathbb{R}$ we consider the operator $\frac{1}{2}\Delta + V$, which acts on differentiable functions $f : M \to \mathbb{R}$ via the operator

$$\frac{1}{2}\Delta f(x) + V(x)f(x)\,,$$

for all $x \in M$.

Denote by P_t^V, $t \geq 0$, the semigroup acting on functions $f : M \to \mathbb{R}$ given by

$$P_t^V(f)(x) := \mathbb{E}_x\big[e^{\int_0^t V(X_r)\,dr} f(X_t)\big].$$

We will show that this semigroup is a continuous-time version of the discrete-time Ruelle operator.

Consider the positive differentiable eigenfunction $F : M \to \mathbb{R}$ associated to the main eigenvalue λ for the semigroup P_t^V, $t \geq 0$. From the function F, in a procedure similar to the one used in the case of discrete-time Thermodynamic Formalism, we can associate via a coboundary procedure a certain stationary Markov semigroup. The probability on the Skhorohod space obtained from this new stationary Markov semigroup can be seen as a stationary Gibbs state associated with the potential V. We define entropy, pressure, the continuous-time Ruelle operator and we present a variational principle of pressure for such a setting.

Keywords: Laplacian, Difusions, Entropy, Gibbs states, Pressure, continuous time Ruelle operator, eigenfunction, eigenvalue, Feynman-Kac formula, Thermodynamic Formalism.

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In the present text, we will work with $\{X_t;\, t \geq 0\}$ the Brownian Motion with state-space on a Riemannian compact manifold M. One particular example could be $\mathbb{S}^1$, where $\mathbb{S}^1$ is the interval $[0, 1]$ with $0 \equiv 1$. We call M the state space.

We will denote by $\frac{1}{2}\frac{\partial^2}{\partial x^2}$ the Laplacian on the Riemannian manifold M. The Laplacian operator is selfadjoint (see [22]).

The Brownian Motion has as its infinitesimal generator, the Laplacian, which is the operator $L = \frac{1}{2}\Delta = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ acting on functions $f \in C^2(M)$. The trajectories of this process are $\mathcal{C}$, the space of all continuous functions defined in $[0, T]$ taking values in M. Denote $\mathbb{P}_\mu$ the probability in $\mathcal{C}$ induced by $\{X_t;\, t \geq 0\}$ and the initial probability μ. If the initial measure is δ_x (Delta of Dirac), for some $x \in M$, we will denote $\mathbb{P}_{\delta_x}$ only by $\mathbb{P}_x$. And the expectation (integral) with respect to $\mathbb{P}_\mu$ or $\mathbb{P}_x$ we will denote by $\mathbb{E}_\mu$ or $\mathbb{E}_x$, respectively.

For a fixed $T > 0$ we denote by $\Omega^T \subset \{\, w : [0, T) \to M \,\}$ the associated Skhorohod space (see [4]). This T can be taken as large as we want but fixed.

We denote by $\mathcal{S} \subset \{\, w : [0, \infty) \to M \,\}$ the Skhorohod space where paths $w : [0, \infty) \to M$ are càdlàg

We define for each fixed $s \in \mathbb{R}^+ \cup \{0\}$ the $\mathcal{B}$-measurable transformation $\Theta_s : \mathcal{S} \to \mathcal{S}$ given by $\Theta_s(w_t) = w_{t+s}$. A stationary diffusion Process defines a probability on $\mathcal{S}$, which is invariant for the continuous time flow $\Theta_s : \mathcal{S} \to \mathcal{S}$, $s \geq 0$.

When μ is the volume form on M the associated Markov Process $\{X^\mu_t;\, t \geq 0\}$ is stationary for the flow Θ_s, $s \geq 0$. The $P = P_\mu$ on the Skhorohod space obtained in this way will play the role of our *a priori* probability. Θ_s, $s \geq 0$, is called the continuous time shift.

General results for continuous-time Markov chains that were specially designed to be applicable to our setting appear on [20].

The results we describe here are inspired by the previous results presented on [16] when the state space is the Bernoulli symbolic space $\{1, 2, .., d\}^{\mathbb{N}}$.

Let $V : M \to \mathbb{R}$ a Lipschitz function and consider the operator $L + V$, which acts on functions $f \in C^2(M)$ by the expression

$$(L + V)(f)(x) \,=\, \frac{1}{2}\tfrac{\partial^2}{\partial x^2} f(x) +\, V(x) f(x)\,,$$

for all $x \in M$.

It is a classical result that there exists a positive differentiable eigenfunction $F : M \to \mathbb{R}$ associated to an eigenvalue λ_V (the smallest) for the above operator $L + V$ (see proposition 2.9 chapter 8 in [23], [5] or [13]).

For $t \geq 0$, we consider

$$P^V_t(f)(x) \,:=\, \mathbb{E}_x\big[e^{\int_0^t V(X_r)\,dr} f(X_t)\big]\,, \tag{0.1}$$

for all continuous function $f : M \to \mathbb{R}$ and $x \in M$. By Feynman-Kac, $\{P^V_t,\, t \geq 0\}$ defines a semigroup associated to the infinitesimal operator $L+V$ (see chapter 11 in [23]) which is not Markovian (stochastic).

Given such V we can normalize this semigroup (associated with the infinitesimal operator $L + V$) in order to get a new Markov semigroup. This is done via a kind of *coboundary procedure* using the positive eigenfunction F and the eigenvalue (for the analogous discrete-time procedure see [21]). We call the new associated stationary Markov Process - get in this way - the Gibbs Process associated to the perturbation V. The shift-invariant probability on the Skhorohod space $\mathcal{S}$ obtained from the Gibbs process for V will be called the Gibbs state probability associated to the potential V (see also [3], [16] and [12]).

As the infinitesimal generator we consider is selfadjoint (see [22]), one can show (see [12]) that for any x and $t > 0$

$$\mathbb{E}_x\big[e^{\int_0^t V(X_r)\,dr} f(X_t)\big] = \mathbb{E}_{X_t=x}\big[e^{\int_0^t V(X_r)\,dr} f(X_0)\big]. \tag{0.2}$$

Therefore, the right-hand side of the above expression, as a function of x and t, also satisfies the Feynman-Kac formula (see [8]). In figure 1, we show schematically the difference of looking the integration of paths by $\mathbb{P}_\mu$ via the left-hand side or the right-hand side of (0.2). It is important to notice that the Laplacian operator is selfadjoint (see [22]).

The right-hand side expression (0.2) is the natural generalization to a continuous-time setting of the classical discrete-time Ruelle operator (see also [16])). We elaborate on this claim.

Points in the symbolic space $\{1, 2, .., d\}^{\mathbb{N}}$ are denoted by $x = (x_0, x_1, x_2, .., x_n, ...)$, $x_j \in \{1, 2, ..., d\}$.

Given a Holder continuous potential $A : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$, the discrete time Ruelle operator $\mathcal{L}_A$ acts on continuous functions $\psi : \{1, 2, .., d\}^{\mathbb{N}} \to \mathbb{R}$ via

$$\varphi(x_0, x_1, x_2, x_3, ...) \; = \; \mathcal{L}_A(\psi)(x) =$$

$$\sum_{a=1}^{d} e^{A(a,x_0,x_1,x_2,x_3,...)} \psi(a, x_0, x_1, x_2, x_3, ...) = \sum_{\sigma(y)=x} e^{A(y)} \psi(y),$$

where σ is the discrete time shift acting on $\{1, 2, .., d\}^{\mathbb{N}}$. In this case the *a priori* measure is the counting measure on $\{1, 2, ..., d\}$ (see [14]). Given $n \in \mathbb{N}$

$$\mathcal{L}_A^n(\psi)(x) = \sum_{\sigma^n(y)=x} e^{A(y)+A(\sigma(y)+...+A(\sigma^{n-1}(y))} \psi(y). \tag{0.3}$$

Comparing the two expressions, it is fair to say that (0.2) is a continuous time version of (0.3) in the case the function $A(x) = A(x_0, x_1, x_2, .., x_n, ...)$ depends just on the first coordinate x_0.

For related results and applications to Physics see [10], [11], [5], [19], [7] and [18].

1 On the continuous time Gibbs state for the potential V

General references for basic results on diffusions and semigroups that we use here appear, for example, in [23], [9], [1], [2] and [8].

Let λ_V the biggest eigenvalue of $L+V$ and F_V the differentiable eigenfunction associated to λ_V, then $F_V > 0$ (for the existence theorems see [1], [6] or [23]). To make simply the notation we will denote F_V by F. For $t \geq 0$, if one defines

$$\mathcal{P}_t^V(f)(x) = \mathbb{E}_x\Big[e^{\int_0^t V(X_r)dr}\,\frac{F(X_t)}{e^{\lambda_V t}F(x)}\,f(X_t)\Big] = \frac{P_t^V(Ff)(x)}{e^{\lambda_V t}F(x)}\,, \tag{1.1}$$

where F and λ_V are the eigenfunction and the eigenvalue, respectively. Then $\mathcal{P}_t^V(1)(x) = 1$, $\forall x \in M$. This defines a stochastic semigroup, which is what we were looking for. From this, we will get a new continuous-time Markov process, which will help to define the Gibbs state for V.

Proposition 1.1. *If $V : M \to \mathbb{R}$ is a Lipschitz function and we define the operator $\mathcal{L}^V$ acting on $f \in C^2(M)$ as*

$$\mathcal{L}_V(f)(x) = \frac{1}{F(x)}(L+V)(Ff)(x) - f(x)\lambda_V = \frac{1}{2}\frac{\partial^2}{\partial x^2}f(x) + \frac{\partial}{\partial x}\log(F(x))\frac{\partial}{\partial x}f(x)\,. \tag{1.2}$$

Then, this operator, $\mathcal{L}_V$, is the infinitesimal generator associated to a semigroup $\{\mathcal{P}_t^V, t \geq 0\}$ defined in (1.1).

Notice that a process induced by this kind of infinitesimal generator corresponds to a Brownian Motion with non-homogeneous drift: $\frac{\partial}{\partial x}\log(F(x))$.

Proof. It is easy to see that $\{\mathcal{P}_T^V, T \geq 0\}$ is a semigroup, see Proposition 7 in [16]. To prove that the infinitesimal generator (1.2) is associated to this semigroup, we need to observe that

$$\frac{\mathcal{P}_t^V(f)(x) - f(x)}{t} = \frac{1}{e^{\lambda_V t}F(x)}\left(\frac{P_t^V(Ff)(x) - (Ff)(x)}{t}\right) + f(x)\left(\frac{e^{-\lambda_V t} - 1}{t}\right).$$

Taking the limit as t goes to zero the expression above converges to

$$\frac{1}{F(x)}(L+V)(Ff)(x) - f(x)\lambda_V\,.$$

In order to find the second expression in the statement of the proposition, we can rewrite the expression above as

$$\tfrac{1}{F(x)}\frac{1}{2}\frac{\partial^2}{\partial x^2}(Ff)(x) + (V(x) - \lambda_V)f =$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}f(x) + \frac{\frac{\partial}{\partial x}F(x)}{F(x)}\frac{\partial}{\partial x}f + \left(V(x) - \lambda_V + \frac{1}{2}\frac{\frac{\partial^2}{\partial x^2}F(x)}{F(x)}\right)f(x)\,. \tag{1.3}$$

Using that F is an eigenfunction associated to the eigenvalue λ_V, we have that

$$V(x) - \lambda_V = -\frac{1}{2}\frac{\partial^2}{\partial x^2}F(x)/F(x)\,.$$

Then the last expression in (1.3) becomes the operator $\mathcal{L}_V(f)(x)$, defined in (1.2).

□

From now on, we will elaborate on the properties of initial invariant probability μ_V, for the operator $\mathcal{L}_V$. In other words, μ_V is a probability in M such that, for any $f \in \mathcal{C}^2(M)$, we have for any $T \geq 0$

$$\int \mathcal{P}_t^V(f)\, \mathrm{d}\mu_V \;=\; \int f\, \mathrm{d}\mu_V \quad \text{or equivalently} \quad \int \mathcal{L}_V(f)\, \mathrm{d}\mu_V \;=\; 0\,.$$

Since $\mathcal{L}_V(f)(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x) + \frac{\partial}{\partial x}\log(F(x))\frac{\partial}{\partial x} f(x)$ (see (1.2)), the following lemma will give for us the invariant measure.

Lemma 1.2. *Let $G \in C^1(M)$ and define an operator $A : C^2(M) \to \mathbb{R}$ as*

$$A(f) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f + \frac{\partial}{\partial x} G \frac{\partial}{\partial x} f,$$

for all $f \in C^2(M)$. Then a measure μ such that $\frac{d\mu}{dx} = e^{2G}$ satisfies

$$\int Af\, d\mu = 0,$$

for all $f \in C^2(M)$.

Proof. This proof follows from the Radon-Nikodym theorem and integration by parts. ☐

Thus, taking $G = \log F$, we get that $\tilde{\mu}_V$ satisfies $\frac{d\tilde{\mu}_V}{dx} = F^2$ is the invariant measure for $\mathcal{L}_V$. This measure maybe is not a probability, then we will consider the normalized measure

$$d\mu_V(x) = \frac{F^2(x)}{\gamma_V}\, dx\,, \tag{1.4}$$

where $\gamma_V = \int_M F^2(x)\, dx$.

Remark 1.3. *There is another way to find an invariant measure for $\mathcal{L}_V$. Following the reasoning of [16] one can find an eigenprobability ν_V of $L + V$ associated to eigenvalue λ_V. Then consider*

$$\mu_V = F\nu_V,$$

where F is the eigenfunction associated to eigenvalue λ_V. Since $\mathcal{L}_V(f)(x) = \frac{1}{F(x)}(L+V)(Ff)(x) - f(x)\lambda_V$, we have

$$\int \mathcal{L}_V(f)\; d\mu_V \;=\; \int \Big((L+V)(Ff) - Ff\,\lambda_V\Big)\; d\nu_V \;=\; 0\,. \tag{1.5}$$

Definition 1.4. *Given a Lipschitz function $V : M \to \mathbb{R}$, we define a continuous-time Markov process $\{Y_t^V, t \geq 0\}$ with state-space M whose infinitesimal generator $\mathcal{L}_V$ acts on functions $f \in C^2(M)$ by the expression* (1.2) *and the initial stationary probability μ_V defined in* (1.4). *We call this process $\{Y_t^V, t \geq 0\}$ the continuous time Gibbs state for the potential V. This process induced a probability $\mathbb{P}^V_{\mu_V}$ on the space $\mathcal{C}$, which we call the Gibbs probability for the potencial V.*

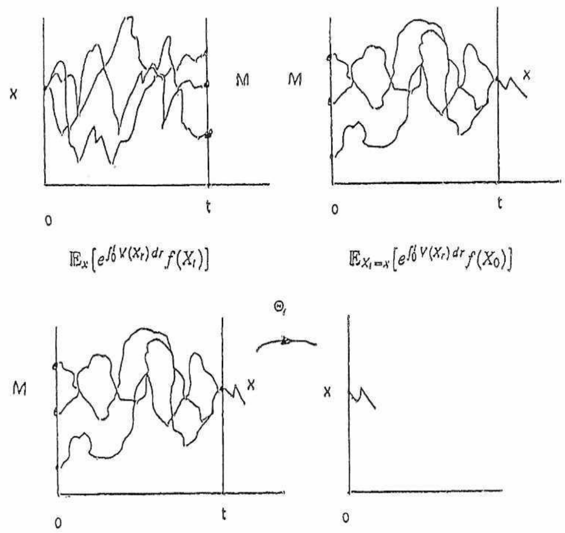

Figure 1: The point $x \in M$ is the value at time $t = 0$ of the path obtained as the image - by the continuous time shift Θ_t - of the set of paths described above.

Remark 1.5. *Suppose V is of class C^∞ and has a finite number of points with derivative zero. Let λ the biggest eigenvalue of $L + V$ and F the eigenfunction associated to λ. One can show an interesting property relating oscillations of V and the oscillations of the main eigenfunction F.*

Suppose that $V : \mathbb{S}^1 \to \mathbb{R}$ has only two points with derivative zero (V has a unique point of maximum and a unique point of minimum). Then, the eigenfunction F has less than four points with derivative zero.

From the hypothesis on given a value c there exist at most two values x such that $V(x) = c$. Suppose F has many values with derivative zero. Then, between each two of these points there exists another one x_1 with $F''(x_1) = 0$. From, $F''(x_1) + V(x_1)F(x_1) = \lambda F(x_1)$ we get that $V(x_1) = \lambda$. But, by hypothesis one can get at most two points with this property.

One can generalize this for V with more oscillations in a similar way. The analogous property for potentials and eigenfunctions in the setting where the state space has no differentiable structure is not so clear how to get it.

2 Relative Entropy, Pressure and the equilibrium state for the potential V

One can ask: "Does the Gibbs state (of the last section) satisfy a variational principle?" We will address this question in the present section.

Given a Lipschitz function $V : M \to \mathbb{R}$, we will consider a *variational problem in the continuous-time setting*, which is analogous to the pressure problem in the discrete-time setting (thermodynamic formalism). This requires a meaning for *entropy*. A continuous-time stationary Markov process, which maximizes our variational problem, will be the *continuous-time equilibrium state for* V. The different probabilities $\tilde{\mathbb{P}}_x$ on $\mathcal{C}$ will describe the possible candidates for being the *stationary equilibrium continuous-time Markov process for* V. These probabilities will be called admissible.

First of all, we will analyze the Radon-Nikodym derivative of $\mathbb{P}_x^V$ with respect to the measure $\mathbb{P}_x$ induced by the Brownian Motion and initial probability δ_x restricted to $\mathcal{F}_t$, where $\{\mathcal{F}_t,\, t \geq 0\}$ is the canonical filtration for the Brownian Motion $\{X_t,\, t \geq 0\}$. The Radon-Nikodym derivative will be denoted as $\frac{d\mathbb{P}_x^V}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t}$, where δ_x is any initial probability in M. In order to find an expression to $\frac{d\mathbb{P}_x^V}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t}$, we remember that it must satisfy

$$\int_{\mathcal{C}} G(w_{T_1}, w_{T_2}, \ldots, , w_{T_k})\, d\mathbb{P}_x^V(w) = \int_{\mathcal{C}} G(w_{T_1}, w_{T_2}, \ldots, , w_{T_k})\, \frac{d\mathbb{P}_x^V}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t} d\mathbb{P}_x(w),$$

for all $k \in \mathbb{N}$, $0 = T_0 < T_1 < \cdots < T_k = t < T$ and $G : (M)^k \to \mathbb{R}$. For this is enough to consider, for any $k \in \mathbb{N}$, functions $f_i : M \to \mathbb{R}$, $i \in \{1, \ldots, k\}$, a time partition as above and study the following integral:

$$\begin{aligned}
&\int_{\mathcal{C}} f_1(w_{T_1}) f_2(w_{T_2}) \ldots f_k(w_{T_k})\, d\mathbb{P}_x^V(w)\\
&= \int_M P_{T_1}(x, x_1) f_1(x_1) \int_M P_{T_2-T_1}(x_1, x_2) f_2(x_2) \cdots \int_M P_{T_k-T_{k-1}}(x_{k-1}, x_k) f_k(x_k)\\
&= \int_M P_{T_1}(x, x_1) f_1(x_1) \cdots \int_M P_{T_{k-1}-T_{k-2}}(x_{k-2}, x_{k-1}) f_{k-1}(x_{k-1}) \mathcal{P}^V_{T_k-T_{k-1}}(f_k)(x_{k-1})\\
&= \cdots\\
&= \mathcal{P}^V_{T_1}(f_1 \ldots (\mathcal{P}^V_{T_{k-1}-T_{k-2}}(f_{k-1} \mathcal{P}^V_{T_k-T_{k-1}}(f_k))) \ldots)(x)
\end{aligned}$$

To fix ideas consider $k = 2$ and analyze:

$$\mathcal{P}^V_{T_1}\big(f_1(\mathcal{P}^V_{T_2-T_1}(f_2)\big)(x),$$

where

$$\mathcal{P}^V_{T_1}\big(f_1(\mathcal{P}^V_{T_2-T_1}(f_2))(x)$$
$$= \mathbb{E}_x\left[e^{\int_0^{T_1} V(X_r)dr}\,\frac{F(X_{T_1})}{e^{\lambda_V T_1}\,F(x_0)}\,f_1(X_{T_1})\,\mathbb{E}_{X_{T_1}}\left[e^{\int_0^{T_2-T_1} V(X_r)dr}\,\frac{F(X_{T_2-T_1})}{e^{\lambda_V(T_2-T_1)}\,F(X_{T_1})}\,f_2(X_{T_2-T_1})\right]\right]$$
$$= \mathbb{E}_x\left[e^{\int_0^{T_1} V(X_r)dr}\,\frac{1}{e^{\lambda_V T_2}\,F(x_0)}\,f_1(X_{T_1})\,\mathbb{E}_{X_{T_1}}\left[e^{\int_0^{T_2-T_1} V(X_r)dr}\,F(X_{T_2-T_1})\,f_2(X_{T_2-T_1})\right]\right]$$
$$= {}^1\mathbb{E}_x\left[e^{\int_0^{T_1} V(X_r)dr}\,\frac{1}{e^{\lambda_V T_1}\,F(x_0)}\,f_1(X_{T_1})\,e^{\int_{T_1}^{T_2} V(X_r)dr}\,F(X_{T_2})\;f_2(X_{T_2})\right]$$
$$= \mathbb{E}_x\left[f_1(X_{T_1})\,f_2(X_{T_2})\,e^{\int_0^{T_2} V(X_r)dr}\,\frac{F(X_{T_2})}{e^{\lambda_V T_2}\,F(x_0)}\right].$$

In this case,

$$\int_{\mathcal{C}} f_1(w_{T_1})f_2(w_{T_2})\,d\mathbb{P}^V_\mu(w)$$
$$= \int_{\mathcal{C}} f_1(w_{T_1})\,f_2(w_{T_2})\,e^{\int_0^{T_2} V(w_r)dr}\,\frac{F(w_{T_2})}{e^{\lambda_V T_2}\,F(w_0)}\,d\mathbb{P}_\mu(w)\,.$$

Remembering that $T_k = t$, we have that, for all $T > t \geq 0$,

$$\frac{d\mathbb{P}^V_x}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t}(w) \;=\; \exp\Big\{\;\log F(w_t)\;-\;\log F(w_0)\;-\int_0^t(\lambda_V - V(w_r))\,dr\Big\},\quad \mathbb{P}_x - a.s. \tag{2.1}$$

or, using another notation,

$$\frac{d\mathbb{P}^V_x}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t} \;=\; \exp\Big\{\;\log F(X_t)\;-\;\log F(X_0)\;-\int_0^t(\lambda_V - V(X_r))\,dr\Big\},$$

where $\{X_s, s \geq 0\}$ is the Brownian Motion.

Definition 2.1. *The probability $\tilde{\mathbb{P}}_\mu$ on $\mathcal{C}$ is called admissible, if, for all $T \geq 0$,*

$$\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x}\Big|_{\mathcal{F}_T} \;=\; \exp\Big\{g(X_T) - g(X_0) - \frac{1}{2}\int_0^T\Big[\tfrac{\partial^2}{\partial x^2}g(X_r) + (\tfrac{\partial}{\partial x}g)^2(X_r)\Big]\,dr\Big\}\,, \tag{2.2}$$

for some function $g \in C^2(M)$.

Notice that according to the above the Gibbs Markov process $\mathbb{P}^V_x$ with the initial probability δ_x is admissible, it is enough take $g = \log F$ and observe that

$$\frac{1}{2}[\tfrac{\partial^2}{\partial x^2}g+(\tfrac{\partial}{\partial x}g)^2] = \frac{1}{2}[\tfrac{\partial}{\partial x}\Big(\frac{\frac{\partial}{\partial x}F}{F}\Big)+\Big(\frac{\frac{\partial}{\partial x}F}{F}\Big)^2] = \frac{\frac{1}{2}\,\frac{\partial^2}{\partial x^2}F}{F} = \frac{LF}{F} = \lambda_V - V\,. \tag{2.3}$$

The last equality is due to $(L+V)F = \lambda_V F$. And, the unperturbed system with the initial measure δ_x is also admissible, just take $g = 0$.

Denote by $\{\tilde{X}_t,\, t \geq 0\}$ which has law in $\mathcal{C}$ the probability $\tilde{\mathbb{P}}_x$. The question is: who is the process $\{\tilde{X}_t,\, t \geq 0\}$? The answer is in [24, Chapter VIII.3] thanks to the Girsanov's Theorem. More specifically, by Proposition 3.4 in [24, Chapter VIII.3] the infinitesimal generator of the process $\{\tilde{X}_t,\, t \geq 0\}$ is $\tilde{L} = L + \Gamma(g, \cdot)$, where Γ is the *opérateur carré du champ* defined in $C^2(M) \times C^2(M)$ as

$$\Gamma(f, g) = L(fg) - fLg - gLf.$$

Since $L = \frac{1}{2}\frac{\partial^2}{\partial x^2}$, the *opérateur carré du champ* is just

$$\Gamma(f, g) = \tfrac{\partial}{\partial x} g \tfrac{\partial}{\partial x} f.$$

Thus the Radom-Nikodym derivative $\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x}$, defined in (2.2), can be rewritten as

$$\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t} \;=\; \exp\Big\{ g(X_t) - g(X_0) - \int_0^t \big[Lg(X_r) + \Gamma(g, g)(X_r)\big]\, dr \Big\}.$$

By [24], if the process has a Radon-Nikodym derivative with respect to $\mathbb{P}_x$ as above, the generator this process is $\tilde{L} = L + \Gamma(g, \cdot)$.

The conclusion is that in our model the generator of $\{\tilde{X}_t,\, t \geq 0\}$ acts on functions $f \in C^2(M)$ as

$$\tilde{L}f = \frac{1}{2}\tfrac{\partial^2}{\partial x^2} f + \tfrac{\partial}{\partial x} g \tfrac{\partial}{\partial x} f. \tag{2.4}$$

Then, the process $\{\tilde{X}_t,\, t \geq 0\}$ is a Brownian Motion with drift $\frac{\partial}{\partial x} g$, i.e., this process is in the same class of the Gibbs Markov process $\mathbb{P}_x^V$ as it should be.

To follow the same way as in [16], we should find the invariant measure for $\tilde{L}$, which we will denote by $\tilde{\mu}$. By Lemma 1.2, the invariant measure $\tilde{\mu}$ for $\tilde{L}$ is such that $d\tilde{\mu}(x) = e^{2g(x)}/\tilde{\gamma}\, dx$, where $\tilde{\gamma} = \int_M e^{2g(x)}\, dx$

Now, we want to give a meaning for the relative entropy of any admissible probability $\tilde{\mathbb{P}}_{\tilde{\mu}}$ with respect to $\mathbb{P}_{\tilde{\mu}}$. The reason why we use the same initial measure for both processes is that we need that the associated probabilities, $\tilde{\mathbb{P}}_{\tilde{\mu}}$ and $\mathbb{P}_{\tilde{\mu}}$, on $\mathcal{C}$ are absolutely continuous with respect to each other. Anyway, the final numerical result for the value of entropy will not depend on the common $\tilde{\mu}$ we chose as the initial probability.

For a fixed $T \geq 0$, we consider the relative entropy of the $\tilde{\mathbb{P}}_{\tilde{\mu}}$, with respect to $\mathbb{P}_{\tilde{\mu}}$ up to time $T \geq 0$ as

$$H_T(\tilde{\mathbb{P}}_{\tilde{\mu}} | \mathbb{P}_{\tilde{\mu}}) \;=\; -\int_M \int_{\mathcal{C}} \log\left(\frac{\mathrm{d}\tilde{\mathbb{P}}_x}{\mathrm{d}\mathbb{P}_x}\Big|_{\mathcal{F}_T} \right)(\omega)\, \mathrm{d}\tilde{\mathbb{P}}_x(\omega)\, \mathrm{d}\tilde{\mu}(x)\,. \tag{2.5}$$

Using the property that the logarithm is a concave function and Jensen's inequality, we obtain that for any g we have $\int \log g\, \mathrm{d}\mu \;\leq\; \log \int g\, \mathrm{d}\mu$. Then $H_T(\tilde{\mathbb{P}}_{\tilde{\mu}} | \mathbb{P}_{\tilde{\mu}}) \leq 0$. Negative entropies appear in a natural way when one analyzes a dynamical system with the property that each point has an uncountable number of preimages (see [14] and [19]).

Using the expression (2.2), we get

$$\begin{aligned} H_T(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}}) &= -\int_{\mathcal{C}} \Big[g(w_T) - g(w_0) - \frac{1}{2} \int_0^T \Big[\tfrac{\partial^2}{\partial x^2} g(w_r) + (\tfrac{\partial}{\partial x} g)^2(w_r) \Big] \, dr \Big] \, d\tilde{\mathbb{P}}_{\tilde{\mu}}(w) \\ &= \int_M \Big\{ \tilde{P}_0 g(x) - \tilde{P}_T g(x) + \frac{1}{2} \int_0^T \tilde{P}_r[\tfrac{\partial^2}{\partial x^2} g + (\tfrac{\partial}{\partial x} g)^2](x) \, dr \Big\} \, dv\tilde{\mu}(x) \, , \end{aligned} \tag{2.6}$$

where $\tilde{P}_t$ is the semigroup associated to $\tilde{L}$.

From the previous expression and ergodicity we get that there exists the limit $\lim_{T\to\infty} \frac{1}{T} H_T(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}})$.

Definition 2.2. *For a fixed initial probability $\tilde{\mu}$ on M, we will denote the limit*

$$\lim_{T\to\infty} \tfrac{1}{T} H_T(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}})$$

as $H(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}})$. Moreover, we will call $H(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}})$ as the relative entropy *of the measure $\tilde{\mathbb{P}}_{\tilde{\mu}}$ with respect to the measure $\mathbb{P}_{\tilde{\mu}}$.*

By the Definition 2.2, the expression (2.6) and the Ergodic Theorem, the relative entropy

$$H(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}}) = \frac{1}{2} \int_M \Big[\tfrac{\partial^2}{\partial x^2} g + (\tfrac{\partial}{\partial x} g)^2 \Big] \, d\tilde{\mu} \, .$$

Definition 2.3. *For a given Lipschitz potential V, we denote the Pressure (or, Free Energy) of V as the value*

$$\boldsymbol{P}(V) := \sup_{\substack{\tilde{\mathbb{P}}_{\tilde{\mu}} \\ admissible}} \Big\{ H(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}}) + \int_M V \, d\tilde{\mu} \Big\} ,$$

where $\tilde{\mu}$ is the initial stationary probability for the infinitesimal generator $\tilde{L}$, defined in (2.4). Moreover, any admissible element which maximizes $\boldsymbol{P}(V)$ is called a continuous-time equilibrium state for V.

Finally, we can state the main result of this section:

Proposition 2.4. *The pressure of the potential V is given by*

$$\boldsymbol{P}(V) \;=\; H(\mathbb{P}^V_{\mu_V}|\mathbb{P}_{\mu_V}) + \int_M V \, d\mu_V = \lambda_V \, .$$

Therefore, the equilibrium state for V is the Gibbs state for V.

Proof. The second equality in the statement of the theorem comes from

$$H(\mathbb{P}^V_{\mu_V}|\mathbb{P}_{\mu_V}) + \int_M V\, \mathrm{d}\mu_V =$$

$$\int_M \frac{1}{2}\Big[\tfrac{\partial^2}{\partial_x^2}\log F + (\tfrac{\partial}{\partial_x}\log F)^2] + V\Big]\, \mathrm{d}\mu_V = \int_M \tfrac{LF}{F} + V\, \mathrm{d}\mu_V = \lambda_V\,,$$

by (2.3) and $(L+V)F = \lambda_V F$.

In order to finish the proof, we need to analyze

$$H(\tilde{\mathbb{P}}_{\tilde{\mu}}|\mathbb{P}_{\tilde{\mu}})\; + \int_M V\, d\tilde{\mu}, \tag{2.7}$$

which is equal to

$$\frac{1}{\tilde{\gamma}}\frac{1}{2}\int_M \Big[\tfrac{\partial^2}{\partial x^2}g + (\tfrac{\partial}{\partial x}g)^2\Big]\, e^{2g}\, dx\; + \frac{1}{\tilde{\gamma}}\int_M V\; e^{2g}\, dx = \frac{1}{\tilde{\gamma}}\int_M \Big[V - \frac{1}{2}(\tfrac{\partial}{\partial x}g)^2\Big]\, e^{2g}\, dx.$$

The last equality follows from integration by parts and the expression of $\tilde{\mu}$. Using that $V = \lambda_V - \frac{LF}{F}$, we can rewrite the last integral above as

$$\lambda_V + \frac{1}{\tilde{\gamma}}\frac{1}{2}\int_M \Big[-\frac{\frac{\partial^2}{\partial x^2}F}{F} - (\tfrac{\partial}{\partial x}g)^2\Big]\, e^{2g}\, dx.$$

Applying integration by parts the integral above becomes

$$\frac{1}{2}[\int_M \tfrac{\partial}{\partial x}F\,\tfrac{\partial}{\partial x}\Big(\frac{e^{2g}}{F}\Big)\, dx\; - \int_M (\tfrac{\partial}{\partial x}g)^2\, e^{2g}\, dx\,].$$

The expression above can be rewritten as

$$-\frac{1}{2}\int_M \Big(\tfrac{\partial}{\partial x}(\log F) - \tfrac{\partial}{\partial x}g\Big)^2\, e^{2g}\, dx.$$

Therefore, the expression in (2.7) is less than or equal to λ_V.

□

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