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A Note on δ-Equilibrium Measures for Rational Maps

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Here we will consider the δ-potential $|z-x|^{-\delta}$, $0<\delta\leqq 2$, $z, x\in\mathbb{C}$, and we will obtain some results relating the dynamics of a rational map and equilibrium potential properties.

A classical problem in potential theory is: what is the minimal state of energy for such a δ-potential when we consider measures with support in a certain compact set?

In the case this measure exists, it is called the equilibrium measure.

For a fixed rational map f we ask the following question: what is the minimal state of energy for such δ-potential when we consider measures that are invariant by f?

When the parameter δ is equal to the Hausdorff dimension of the Julia set, the Gibbs state of a uniformly expanding rational map has infinite δ-energy integral, but it is, analogous to the situation in classical potential theory, an equilibrium measure when we restrict ourselves to the f-invariant measures (see Theorem 2 in § 4).

The results of δ-potential theory that we refer to are the ones that state that arc length (respectively normalised 2-dimensional Lebesgue measure) is the equilibrium measure in a weak sense for the value of the parameter $\delta=1$ (respectively $\delta=2$), for a smooth compact curve (respectively a compact set of positive 2-dimensional Lebesgue measure) in the plane (see Frostman [4], Lithner [8] and Wallin [14]). In this case the arc length (respectively normalised Lebesgue measure) has infinite 1-energy integral (respectively 2-energy integral), which resembles the statement above about Gibbs states.

Recently another result has appeared about Riesz transform of measures with support in the Julia set in "iterated functions systems and the global construction of fractals" (M.F. Barnsley and S. Demko, Proc. R. Soc. London A 399, pp. 243–275, 1985). In this paper the balanced measure is considered instead of the Gibbs state.

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Introduction

Suppose f is an uniformly expanding rational map and the Julia set is bounded (see [3], [11] for these concepts).

Here we will consider the Riesz transform of a general f-invariant probability measure v, that is

$$U_v^\delta(z)=\int |z-x|^{-\delta}\,dv(x) \qquad 0<\delta\leqq 2, \quad z,x\in\mathbb{C}.$$

We will be interested here in the problem of finding the f-invariant probability measure that attains the infimum of $\int\int |z-x|^{-\delta}\,dv(z)\,dv(x)$, among all the f-invariant probability measures v.

This resembles a very important problem in δ-potential theory (see [7] for definitions) where you consider not just the f-invariant probability measures but all probability measures with support in a certain compact set.

It is classical in δ-potential theory to consider the potential $\log|z-x|^{-1}$ (see [7]), which corresponds to the value $\delta=0$.

The problem that we will be interested in here was motivated by a previous result, which states that the equilibrium measure for the logarithmic potential is the maximal entropy measure for a polynomial map [3], [5], [10].

In [9] it was shown that for rational map such that $f(\infty)=\infty$ and the Julia set is bounded, the two measures are different.

The theorems that we will prove here (Theorems 1 and 2 in § 4), relate the topological pressure, Gibbs states and the measure that attains the infimum of $\int\int |z-x|^{-\delta}\,dv(z)\,dv(x)$, for a value δ close to the Hausdorff dimension of the Julia set.

In [1], D. Bessis, J.S. Geronimo and P. Moussa consider the Mellin transform associated with the maximal entropy measure u of a rational map (some authors call u the balanced measure), that is $\int (z-x)^{-\delta}\,du(x)$ for $0<\delta\leqq 2$.

In that paper they obtained some asymptotic values of this transform in some points of the Julia set. This problem is associated with application in quantum and statistical mechanics.

As the maximal measure is, in general, different from the Gibbs state, the problem that we will be interested in here is of a different nature to that of Bessis, Geronimo and Moussa. However there seem to be some analogies between the two problems.

The main results of this paper will be Theorem 1 and 2 that will be proved in § 4. In § 1 we will recall the classical definitions of δ-potential theory. In § 2 we will introduce analogous definitions in the context of dynamics. In § 3 we will give the necessary results of ergodic theory of rational maps.

The theorems that we will prove here, seem to have application to the correlation integral and the dimension defined by Procaccia [2]. You can define the correlation integral and correlation dimension of Gibbs states instead of the balanced measure and investigate the corresponding relation of these concepts.

§ 1. Review of Potential Theory

Let K be a compact set and $M(K)$ the set of Borel measures u such that $u(K)=1$ and $u(\mathbb{C}-K)=0$. Such measures will be called probability measures on K.

Definition 1. For $0<\delta\leqq 2$ the δ-potential of $u\in M(K)$ at the point z is by definition

$$U_u^\delta(z)=\int |z-x|^{-\delta}\,du(x).$$

Definition 2. For $0<\delta\leqq 2$ the δ-energy integral of $u\in M(K)$ is by definition

$$I^\delta(u)=\int\int |z-x|^{-\delta}\,du(x)\,du(z).$$

Remark. The value ∞ is allowed in all these definitions.

Definition 3. For $0<\delta\leqq 2$, let $C(\delta)$ be the δ-capacity of K, the value

$$C(\delta)=\frac{1}{\operatorname*{Inf}_{u\in M(K)}\{I^\delta(u)\}}.$$

Note that, if $C(\delta)=0$, then all measures $u\in M(K)$ have δ-energy integral equal to ∞. This can happen for certain values of δ. In this case $\operatorname*{Inf}_{u\in M(K)}\{I^\delta(u)\}=\infty$.

Definition 4. Suppose $\operatorname*{Inf}_{u\in M(K)}\{I^\delta(u)\}<\infty$, then the measure u_δ such that $I^\delta(u_\delta) = \operatorname*{Inf}_{u\in M(K)}\{I^\delta(u)\}$ is called the δ-equilibrium measure for K.

If $\operatorname*{Inf}_{u\in M(K)}\{I^\delta(u)\}$ is finite, then there always exists such a measure u_δ and it is unique (see [7]).

Definition 5. The capacity dimension of K is the value $\sup\{\delta \mid C(\delta)>0\}$.

Theorem (Frostman [4]). *For K a compact set the capacity dimension of K is equal to the Hausdorff dimension of K.*

For sets with a certain regularity condition the value $C(\delta)$ is zero for δ equal to the Hausdorff dimension of K [14].

In a very few cases the equilibrium measure is known for all values of δ.

Example 1. Let K be the unit circle. Then for any $0<\delta\leqq 1$ the δ-equilibrium measure is the normalized arc length.

Example 2. (Polya-Szegö, [4]). For the segment $K=[-2,2]$ and for any $0<\delta<1$, the δ-equilibrium measure for K has density depending on t of the form

$$\Gamma_\delta\left(1-\frac{t^2}{4}\right)^{-(1-\delta)/2},$$

where Γ_δ is a constant depending on δ.

In Ex. 2, when $\delta\to 1$, then the density goes to the arc length in the segment [4].

This is a particular case of the following theorem.

Theorem (Frostman [4]). *Let K be a compact smooth curve in the plane and u_δ the δ-equilibrium measure for K. Then, u_δ converges to the normalized arc length as $\delta \to 1$.*

The Theorem above asserts that, even if it is not possible to compute the equilibrium measure u_δ, you know that for δ close to one, this measure is close to the arc length.

For δ equal to one it is easy to see that the δ-energy integral of the arc length is infinite.

Denote by λ the 2-dimensional Lebesgue measure on the plane.

Theorem (Lithner [8], Wallin [14]). *Let K be a compact set in the plane such that $\lambda(K)>0$, then when $\delta \to 2$, u_δ converges to the two-dimensional Lebesgue measure restricted to K divided by $\lambda(K)$.*

The 2-energy integral of the limit measure u is infinite.

Let $HD(K)$ denote the Hausdorff dimension of K.

Now we will introduce a definition:

Definition 6. For each $0<\delta<HD(K)$, let u_δ denote the δ-equilibrium measure for K. If u_δ converges to some limit u in the weak topology as $\delta \to HD(K)$ then we will call u the equilibrium measure in the weak sense for K (for the parameter $\delta = HD(K)$).

The two theorems that we just mentioned, give us example of measures satisfying Def. 6.

The δ-energy integral $I^\delta(u)$ of a measure u can be thought of as a norm in the set of measures (see [7]).

The arc length (respectively 2-dimensional Lebesgue measure) of smooth curve (a set of positive 2-dimensional Lebesgue measure) has therefore a peculiar geometric position in the set of all probabilistic measures with support in the smooth curve (respectively with support in the set K). It is the closest one to the origin for δ infinitesimally close to $HD(J)$.

In general it is much more difficult to find equilibrium solutions for the kernel $|z-x|^{-\delta}$ than for the kernel $\log|z-x|^{-1}$. The potential $\int \log|z-x|^{-1}\, du(x)$ is a harmonic function and $\int |z-x|^{-\delta}\, du(x)$ is not. In the first case you can use of conformal maps and harmonic functions in the second case you cannot.

The equilibrium measure for the logarithm potential of a smooth curve can be obtained from the Rieman map of the complement of this curve (see [7]).

§ 2. Potential Theory for Invariant Measures

Let f be a rational map uniformly expanding such that the Julia set of f is bounded in $\mathbb{C}$.

Definition 7. Let $M(f)$ be the set of probabilistic measures such that $u(f^{-1}(A)) = u(A)$ for any Borel set A. These measures will be called f-invariant measures.

Remark. For K equal to the Julia set of f, one can show that $M(f) \subset M(K)$ (except for sinks).

Definition 8. For $0<\delta\leqq 2$, let $C(\delta,f)$ denote the δ-capacity of f, defined as

$$C(\delta,f)=\frac{1}{\operatorname*{Inf}_{u\in M(f)}\{I^\delta(u)\}}$$

In general the $C(\delta)$ capacity of the set K equal to the Julia set can be larger that the δ-capacity of f.

Proposition 1. *Suppose* $\operatorname*{Inf}_{u\in M(f)}\{I^\delta(u)\}<\infty$, *then there exists a unique* u_δ *such that* $I^\delta(u_\delta)=\operatorname*{Inf}_{u\in M(f)}\{I^\delta(u)\}$.

Proof. Let be ξ_δ the set of all invariant signed measures u for f such that $\iint|z-x|^{-\delta}du(z)\,du(x)$ is finite. Then the inner product

$$\langle u,v\rangle=\iint|z-x|^{-\delta}du(z)\,dv(x)\qquad\text{for } u,v\in\xi_\delta,$$

gives a pre-Hilbert structure on ξ_δ (see [7]). The convex cone ξ_δ^+ of f-invariant measures, is a complete metric space with the induced metric structure.

The set $M(f)$ is a convex set and $I^\delta(u)$ is a convex function of u. Now the proof of existence and uniqueness of u_δ is exactly the same as in the classical potential theory case (see [7] p. 132).

Definition 9. If $\operatorname*{Inf}_{u\in M(f)}\{I^\delta(u)\}<\infty$, then the measure u_δ such that $I^\delta(u_\delta)=\operatorname*{Inf}_{u\in M(f)} I^\delta(u)$ will be called the δ-equilibrium measure for f.

We can suppose by means of a linear change of coordinates that the diameter of the Julia set is smaller than one.

In this case, for a fixed a rational map f, the function $C(\delta,f)$ is a non-increasing function of δ.

Definition 10. The capacity dimension of f, is by definition

$$\sup\{\delta\,|\,\delta\text{-capacity of } f \text{ is positive}\}.$$

Let $J(f)$ denote the Julia set of f.

In § 4 we will show that the capacity dimension of f is the Hausdorff dimension of the Julia set.

It is natural therefore to introduce the following definition:

Definition 11. Let be u_δ for each $0<\delta<HD(J(f))$ the δ-equilibrium measure for f. If there exist the limit $u=\lim_{\delta\to HD(J(f))} u_\delta$ in the weak sense, then we will call this measure the equilibrium measure in the weak sense for f (for the value of the parameter $\delta=HD(J(f))$.

All these definitions are analogs of the classical potential theory case.

The following example was shown to me by *S.* Vaienti, and it will appear in a forthcoming paper about correlation integrals and Mellin transform [2].

Example 3. The polynomial map $f(z)=z^2-2$ has the Julia set equal to the real interval $[-2,2]$.

The measure with density $\frac{1}{\pi\sqrt{4-t^2}}$, $-2\leqq t\leqq 2$ on the interval $[-2,2]$ is invariant by f. For any δ, the δ-energy integral is equal to $2\pi\left[\frac{\Gamma(1-\delta)}{\Gamma^2(1-\delta/2)}\right]^2$, where Γ is the gamma function.

§ 3. Ergodic Theory of Rational Maps

Definition 12. For each $0\leqq\delta\leqq 2$, the δ-topological pressure is by definition

$$P(\delta)=\sup_{u\in M(f)}\{h(u)-\delta\int\log|f'(z)|\,du(z)\},$$

where $h(u)$ is the entropy of the measure u.

If f is uniformily expanding rational map, there exists a unique $v_\delta\in M(f)$ that attains the supremum and $P(\delta)$ is a convex function of δ [13]. For any $\delta<HD(J(f))$ we have $P(\delta)>0$ and for any $\delta>HD(J(f))$ we have $P(\delta)<0$ [13].

Definition 13. The measure v_δ such that

$$h(v_\delta)-\delta\int\log|f'(z)|\,dv_\delta(z)=P(\delta)$$

is called the Gibbs state for the parameter δ.

For the parameter $\delta=0$, the Gibbs state is the measure of maximal entropy. We call this measure the maximal measure.

For polynomial maps the maximal measure is the equilibrium measure for the logarithmic potential [3], [5], [10].

Therefore for polynomials, the Gibbs state for the value $\delta=0$ and the $\log|z-x|^{-1}$ potential are closely related (remember now that $|z-x|^{-\delta}$ for $\delta=0$ corresponds to the $\log|z-x|^{-1}$ potential).

We can ask, therefore, if there exists a relation between the Gibbs state for a parameter $0<\delta<H(J(f))$ and the δ-equilibrium measure for f.

In general, if we consider K equal to the Julia set of f, the δ-equilibrium measure for K is not invariant. For example, the measures given in Ex. 2 are not invariant for the polynomial $f(z)=z^2-2$.

For this reason we had to formulate the problem under the context of f-invariant measures.

Anyway, the δ-Gibbs state is not always the δ-equilibrium measure for f, as we will show now.

In [5], it was shown that for some Blaschke products the Julia set is the unit circle, the arc length is invariant and it is not the maximal measure. As we have seen in Ex. 1, for each value of $0<\delta<1$, the δ-equilibrium measure for the circle is arc length. Therefore the δ-equilibrium measure for f is the arc length. If the arc length was the Gibbs state for δ close to zero, then by upper semi-continuity arguments [10], [12] the arc length would be the maximal entropy measure. As we claimed before this is not true. Therefore the two measures, Gibbs state and δ-equilibrium measure for f are different for δ close to zero.

It can be shown that the arc length is the Gibbs state for $\delta=1$. This is a particular case of Theorem 2 that we will show in § 4, that claims that the Gibbs state for the value δ equal to the Hausdorff dimension of the Julia set, is the equilibrium measure in the weak sense for f.

The Julia set of a rational map, in general, has a fractal dimension. For example, the Julia set of polynomials of the form $f(z)=z^2+\varepsilon$ for ε small but nonzero, is a nowhere-differentiable Jordan curve.

Therefore, it will follow from Theorem 2, that the Gibbs state for the value of the parameter $\delta=HD(J(f))$, plays the role of the arc length in the corresponding theorem of Frostman. This is again a geometric property of the Gibbs state for the parameter $\delta=HD(J(f))$, in the set of f-invariant probabilistic measures.

§ 4. Gibbs State and the Equilibrium Measure for f

Theorem 1. *Let f be a uniformly expanding rational map, then the capacity dimension of f is the Hausdorff dimension of the Julia set.*

Proof. Let be $HD(J)$ the Hausdorff dimension of the Julia set and δ_0 the capacity dimension of f.

First let's show:

I) $HD(J)\leqq\delta_0$.

Suppose $HD(J)>0$ (the other case is trivial). The proof will be carried out by contradiction. Suppose there exists $\delta_0<B<HD(J)$. Then from [11] the Gibbs state u for $\delta=HD(J(f))$ satisfies $U(B(x,r))\approx r^{HD(J)}$. Let be $\xi>0$, then

$$\begin{aligned}\int_{B(x,\xi)} |z-x|^{-B}\,du(x) &= \frac{u(B(x,\xi))}{B\xi^B}-\int_0^\xi r^{-B-1}u(B(x,r))\,dr\\ &\leqq \frac{u(B(x,\xi)}{B\xi^B}-\int_0^\xi K\frac{r^{HD-B}}{r}\,dr<\infty\end{aligned}$$

where K is positive constant uniform in the Julia set.

Therefore $I^B(u)<\infty$ and this is a contradiction with $C(B,f)=0$. Hence $HD(J)\leqq\delta_0$. Now let's show that

II) $HD(J)\geqq\delta_0$.

Suppose not, then there exists δ such that $HD(J)<\delta<\delta_0$. Therefore there exists $v\in M(f)$ such that $U_v^\delta(z)<\infty$, v-almost everywhere. Therefore for $\eta>0$

$$C\geqq\int_{B(z,\eta)} |z-x|^{-\delta}\,dv(x)\geqq v(B(z,\eta))\,\eta^{-\delta} \tag{$*$}$$

Let $B(z,n,\varepsilon)$ be $\{y\in J \,\|\, f^j(y)-f^j(z)|<\varepsilon, \forall j\in\{0,1,\ldots,n-1\}\}$. From [11] we have

$$h(v)-\delta\int \log|f'(z)|\,dv(z) \approx \lim_{n\to\infty} -\frac{1}{n}\log v(B(z,n,\varepsilon))+\frac{1}{n}\log|(f^n)'(z)|^{-\delta}$$

for ε small and v-almost every z.

Following the argument of [11], we have

$$h(v)-\delta\int \log|f'(z)|\,dv(z) \approx \lim_{n\to\infty}\frac{1}{n}\log\frac{(\text{diameter of } B(z,\eta,\varepsilon))^\delta}{v(B(z,\eta,\varepsilon))}$$

Now from (*) we conclude that $h(v)-\delta\int \log f'(z)|\,dv(z)\geqq 0$. In [13] it was shown that for $\delta<HD(J)$ we have $P(\delta)>0$ and for $\delta>HD(J)$ we have $P(\delta)<0$.

Therefore $h(v)-\delta\int\log|f'(z)|\,dv(z)\geqq 0$ is a contradiction and therefore $HD(J)\geqq\delta_0$.

Theorem 2. *For the value δ equal to the Hausdorff dimension of the Julia set, the Gibbs state is the equilibrium measure for f in the weak sense.*

Proof. The Gibbs state for $\delta=HD(J)$ of an uniformily expanding rational map is equivalent to the Hausdorff measure of the Julia set [13] and therefore its δ-energy is infinite. Therefore as we have seen before in the end of the proof of the last theorem, there is no measure of finite δ-energy, and $C(HD(J),f)=0$.

For each $\delta<HD(J)$, let be u_δ the δ-equilibrium measure for f. Let be A such that $u_\delta(A)=1$, then

$$\int_A\int_A \frac{1}{|z-x|^\delta}\,du_\delta(z)\,du_\delta(x)=\int\int\frac{1}{|z-x|^\delta}\,du_\delta(z)\,du_\delta(x)<\infty$$

and therefore $HD(A)\geqq\delta$. Therefore

$$HD(u_\delta)=\inf\{HD(A)\,|\,u_\delta(A)=1\}\geqq\delta.$$

From [11] we have that

$$\frac{h(u_\delta)}{\int\log|f'(z)|\,du_\delta(z)}=HD(u_\delta),$$

and finally

$$h(u_\delta)-\delta\int\log|f'(z)|\,du_\delta(z)\geqq 0.$$

If we consider now $\delta\to HD(J)$, then any subsequence of the u_δ that is convergent, has to converge, by upper semi-continuity arguments [10], [12], to a measure u such that

$$h(u)-HD(J)\int\log|f'(z)|\,du(z)\geqq 0.$$

As $P(HD(J))=0$ and the Gibbs state for the parameter $\delta=HD(J)$ is the unique measure that attains the supremum, the result follows.

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Note added in proof. The capacity dimension considered in the paper is not the box-counting dimension.