Cohomology and Subcohomology for expansive geodesic flows

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Abstract. We show that there are examples of expansive, non-Anosov geodesic flows of compact surfaces with non-positive curvature, where the Livsic Theorem holds in its classical (continuous, Hölder) version. We also show that such flows have continuous subaction functions associated to Hölder continuous observables.

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Introduction

The existence of solutions of cohomology and sub-cohomology problems is one of the remarkable subjects connecting analysis, dynamics and rigidity. Given a smooth flow $\psi_t : N \longrightarrow N$, where $t \in \mathbb{R}$, acting on a complete Riemannian manifold N, and a continuous function $f : N \longrightarrow \mathbb{R}$, a cohomology equation associated to ψ_t is an equation of the type

$$F(\psi_t(p)) = F(p) + \int_0^t f(\psi_s(p))ds,$$

where $F : N \longrightarrow \mathbb{R}$ is a continuous function, and the equation is verified for every $t \geq 0$ and $p \in N$. The function F is often called an action function or an action potential associated to f, and in case of existence of F, the function f is said to be cohomologous to zero. There is a discrete version of the above definition for diffeomorphisms, taking $t \in \mathbb{N}$ and sums instead of integrals in the cohomology equation.

The existence of action functions is a typical property of two completely different systems: hyperbolic systems and torus diffeomorphisms and flows without singularities with diophantine rotation numbers. The existence theories of action functions are radically different in both cases: the exponential contraction of the invariant sets of hyperbolic systems allows to construct explicitly the action function under certain hypothesis, whilst in the case of diophantine systems the action function is a solution of a problem of resonances with small divisors. The main basic result of the hyperbolic theory of action functions is the Livsic Theorem [14]: given an Anosov system acting on a compact manifold N and a Hölder continuous function $f : N \longrightarrow \mathbb{R}$ that is homologous to zero along closed orbits there exists a (up to additive constants) unique action potential F associated to f. The Livsic Theorem has a wide range of applications to the study of hyperbolic dynamical systems, mainly related to analysis (see for instance [12], [15]), and rigidity via invariants of smooth conjugacies [15], [8], [23]. The theory of action functions in the case of torus dynamics is based in the works of Moser (see [19], [20] just to give some references), who developed a deep analytic theory to solve small divisors problems in functional equations.

A subaction function F is defined as a solution of an inequality involving the same terms occurring in a cohomology equation: given a smooth flow $\psi_t : N \longrightarrow N$ and a Hölder continuous function $f : N \longrightarrow \mathbb{R}$, a continuous function $F : N \longrightarrow \mathbb{R}$ is a subaction function associated to f if

$$F(\psi_t(p)) \geq F(p) + \int_0^t (f(\psi_s(p)) - m(f))ds,$$

where $m(f)$ is the supremum over all ψ_t-invariant probability measures of the action $\int f d\mu$. Subaction functions have been considered in [23], [16], [17], [3] and [6] and used to study Mather sets of geodesic flows of constant negative curvature [18]. The theory of existence of subaction functions relies strongly in hyperbolic dynamics, as well as the theory of action functions for hyperbolic systems.

The main motivation of this paper is to study the existence of action and subaction functions of systems which are neither hyperbolic nor systems in the torus with diophantine rotation numbers. There are very few examples of such systems in the literature. There is a measurable version of the Livsic Theorem for non-uniformly hyperbolic diffeomorphims of compact surfaces due to Katok: if a smooth difeomorphism $\phi : N \longrightarrow N$ acting on a compact surface N has a hyperbolic measure, then given a continuous function $f : N \longrightarrow \mathbb{R}$ that is homologous to zero along closed orbits in the support of the hyperbolic measure there exists a measurable action potential F associated to f. There are some results involving subaction functions of non-uniformly hyperbolic systems in the interval and in compact surfaces. Such a system in the interval is essentially an expanding map of the interval $[0, 1]$ with one indifferent fixed point [29]. An example is constructed in [29] in a compact surface by making the product of the previous example in the interval and an exponentially contracting map. The former system cannot be conservative by obvious reasons. The contribution of this paper to the theory is the proof of existence of **continuous** action and subaction potentials associated to continuous functions and non-uniformly hyperbolic systems which are conservative.

The examples mentioned in Theorems 1 and 2 bellow arise from a family of surfaces which are studied in two papers [7] and [28]: these are special compact surfaces having negative curvature in all points but along a single closed geodesic γ_0 where the curvature is identically zero. The geodesic flows of such surfaces are expansive, and the rate of contraction of the distance between the closed geodesic γ_0 and an asymptotic geodesic γ satisfies

$$d(\gamma_0(t), \gamma(t)) \leq C d(\gamma_0(0), \gamma(0)) \frac{1}{(1+t)^{\beta}},$$

where $\beta > 1$, and $t \geq 0$. Let us say that a compact surface (M, g) of non-positive curvature is of **type** $\beta > 0$ if there exists a constant $C > 0$ with the following property:

Given any two geodesics $\gamma(t)$ and $\sigma(t)$ in the universal covering $(\tilde{M}, \tilde{g})$ endowed with the pullback $\tilde{g}$ of g satisfying $d(\gamma(t), \sigma(t)) \to 0$ as $t \to +\infty$, we have

$$d(\gamma(t), \sigma(t)) \leq C d(\gamma(0), \sigma(0)) \frac{1}{(1+t)^{\beta}},$$

for every $t \geq 0$.

Our main results are the following:

Theorem 1: There exist compact surfaces of type $\beta > 1$ with non-positive curvature whose geodesic flows are not Anosov, which satisfy the Livsic Theorem in the class of Hölder functions with exponents $\alpha > \frac{1}{\beta}$. Namely, given a Hölder continuous function f with Hölder exponent $\alpha > \frac{1}{\beta}$ in the unit tangent bundle of the surface that is cohomologous to zero along closed orbits, there exists a Hölder, action function F associated to f. The function F is unique up to an additive constant and its Hölder exponent is α.

Theorem 2: There exists compact surfaces of type $\beta > 1$ with non-positive curvature whose geodesic flows are not Anosov, such that any Hölder continuous, real valued function f defined in the unit tangent bundle of the surface with Hölder exponent $\alpha > \frac{1}{\beta}$ has an associated continuous subaction function.

Surfaces of type $\beta > 1$ have the particular feature of being not too smooth: they are at most of class C^3 (see Section 2 for more details). These surfaces provide examples of two-dimensional manifolds with non-Anosov geodesic flows all of whose ideal triangles have finite area (like surfaces of negative curvature) [7]. However, in [28] it is proved that assuming C^4 smoothness, the finiteness of the area of ideal triangles implies that the geodesic flow is Anosov.

We show Theorems 1 and 2 for surfaces of type $\beta > 1$. Roughly speaking, the above theorems proceed from the fact that the geodesic flows of surfaces of type β can be considered in some sense as fake Anosov flows. Such geodesic flows have stable and unstable sets with local product structure, and enough contraction to allow us to generalize the theory of comology and subcomology problems of the Anosov case. As long as surfaces of type β are not too smooth, the above examples might be considered as "pathologies" in the theory of cohomology problems: we might guess that Theorems 1 and 2 hold for a sufficiently smooth expansive geodesic flow in a compact surface if and only if the flow is in fact Anosov. As far as we know, the former statement is an (interesting) open question.

The paper is divided in 5 sections: the first 4 sections contain preliminaries of the theory of expansive geodesic flows in compact manifolds without conjugate points (main references here are [25], [27]) and the proofs of Theorems 1 and 2; and the last section is an appendix concerning expansive geodesic flows in manifolds without conjugate points and some rigidity problems introduced in [13]. In the appendix we employ some general results of expansive geodesic flows in compact manifolds without conjugate points (introduced in Section 1) to give a complete proof of an assertion given in [10] related to the Dirac measures of the geodesic flow. This last assertion provides sufficient conditions to show some rigidity properties of the manifold in terms of the marked length spectrum.

1 Preliminaries about non-positive curvature geometry and expansive geodesic flows

Let us establish some notations. The pair (M, g) is a compact C^3 Riemannian manifold, TM will denote its tangent bundle and T_1M will denote its unit tangent bundle. The canonical coordinates in T_1M are $\theta = (p, v)$, where $p \in M$ and $v \in T_pM$ is a unit vector, and $\pi : T_1M \longrightarrow M$, $\pi(p, v) = p$ will denote the canonical projection. $(\tilde{M}, \tilde{g})$ will be the universal covering of M endowed with the pullback of the metric g by the covering map $\Pi : \tilde{M} \longrightarrow M$; and (T_1M, g_S) will be the unit tangent bundle of M endowed with the Sasaki metric induced by g. We shall use the notation $d(p, q)$ for the distance between two points in M (or $\tilde{M}$) and $d_S(\theta, \eta)$ for the distance between two points in T_1M (or $T_1\tilde{M}$). All the geodesics will be parametrized by arc length, and given $\theta = (p, v) \in T_1M$, the geodesic γ_θ will denote the geodesic such that $\gamma_\theta(0) = p$, $\gamma'_\theta(0) = v$. A very special property of manifolds with non-positive curvature (and manifolds without conjugate points in general) is the existence of the so-called *Busemann functions*: given $\theta = (p, v) \in T_1\tilde{M}$ the *Busemann function* $b^\theta : \tilde{M} \longrightarrow R$ associated to θ is defined by

$$b^\theta(x) = lim_{t \to +\infty}(d(x, \gamma_\theta(t)) - t)$$

The level sets of b^θ are the *horospheres* $H_\theta(t)$ where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$. We have that $\gamma_\theta(t)$ intersects each level set of b^θ perpendicularly at only one point in $H_\theta(t)$, and that $b^\theta(H_\theta(t)) = -t$ for every $t \in R$. Next, we list some basic properties of horospheres and Busemann functions that will be needed in the forthcoming sections (see [1], [9], [2] for instance, for details).

Lemma 1.1. *Let (M, g) be a C^3, compact Riemannian manifold without conjugate points.*

1. *b^θ is a C^1 function for every θ, and if the curvature of (M, g) is non-positive the Busemann functions are C^2.*

2. *The gradient ∇b^θ has norm equal to one at every point.*

3. *Every horosphere is a C^{1+K}, embedded submanifold of dimension $n-1$ (C^{1+K} means K-Lipschitz normal vector field), where K is a constant depending on curvature bounds. In the case of non-positive curvature, the horospheres are convex, C^2 submanifolds.*

4. *The orbits of the integral flow of $-\nabla b^{\theta}$, $\psi_t^{\theta} : \tilde{M} \longrightarrow \tilde{M}$, are geodesics which are everywhere perpendicular to the horospheres H_{θ}. In particular, the geodesic γ_{θ} is an orbit of this flow and we have that*

$$\psi_t^{\theta}(H_{\theta}(s)) \;=\; H_{\theta}(s+t)$$

for every $t, s \in R$.

The integral orbits of the gradient $-\nabla b^{\theta}$ are called *Busemann asymptotes* of γ_{θ}. Recall that the usual notion of *asymptoticity* is the following: a geodesic β is *asymptotic* to a geodesic γ in $\tilde{M}$ if there exists a constant $C > 0$ such that $d(\beta(t), \gamma(t)) \leq C$ for every $t \geq 0$. If $d(\beta(t), \gamma(t)) \to 0$ as $t \to +\infty$ we say that β is strongly asymptotic to γ. It is well known that the orbits of the flow ψ_t^{θ} are all asymptotic to each other if (M, g) has non-positive curvature.

The canonical lift in $T_1\tilde{M}$ of $H_{\theta}(0)$ is the set

$$\tilde{W}^s(\theta) = \{(p, -\nabla_p b^{\theta}), p \in H_{\theta}(0)\},$$

and the canonical lift $W^s(\bar{\pi}(\theta))$ of $H_{\theta}(0)$ in T_1M is the projection of $\tilde{W}^s(\theta)$ in T_1M by the natural covering map $\bar{\Pi} : T_1\tilde{M} \longrightarrow T_1M$ induced by the covering map $\Pi : \tilde{M} \longrightarrow M$. The canonical lifts $W^s(\theta)$, $\tilde{W}^s(\eta)$ are often called *stable horospheres* of $\theta \in T_1M$, $\eta \in T_1\tilde{M}$ respectively, although the behavior of Busemann asymptotes might be very different from the hyperbolic behavior. The set

$$\tilde{W}^u((p,v)) = \{(p, \nabla_p b^{(p,-v)}), p \in H_{(p,-v)}(0)\},$$

is called the *unstable horosphere* of $\theta = (p, v) \in T_1\tilde{M}$, and its projection $W^u(\bar{\pi}(\theta))$ in T_1M by the map $\bar{\pi}$ is called the *unstable horosphere* of $\bar{\pi}(\theta) \in T_1M$. We shall often use the notations

$$H^s(\theta) = H_{\theta}(0),$$

$$H^u(p,v) = H_{(p,-v)}(0),$$

for every $(p, v) \in T_1\tilde{M}$, because it is more suggestive of the underlying (topological) hyperbolic dynamics of the invariant sets of expansive flows.

We shall focus on the study of expansive, non Anosov geodesic flows. Given a C^{∞} Riemannian manifold (N, g), a differentiable flow $f_t : N \longrightarrow N$ without singularities is said to be *expansive* if there exists $\epsilon > 0$ such that the following holds: let $p \in N$, and suppose that there exist $q \in N$, and a continuous, surjective reparametrization $\rho : R \longrightarrow R$, with $\rho(0) = 0$, of the orbit of q such that $d(f_t(p), f_{\rho(t)}(q)) \leq \epsilon$ for every $t \in R$; then q belongs to the orbit of p.

The topological dynamics of expansive geodesic flows in compact manifolds without conjugate points is well understood. We give next a survey of results contained in [21], [25], [27], which show essentially that the topological dynamics of such flows is practically the same as in the case of Anosov geodesic flows.

Theorem 1.2. *Let (M, g) be a compact Riemannian manifold without conjugate points. Then the geodesic flow is expansive if and only if for every pair of geodesics γ, β in $(\tilde{M}, g)$ with $d(\gamma, \beta) \leq D$ we have that $\gamma = \beta$. Moreover, two geodesics are Busemann asymptotic in $\tilde{M}$ if and only if they are asymptotic, and two horospheres $H_{(p,v)}(t)$, $H_{(p,-v)}(s)$ have points of tangency if and only if $s = -t$ and the only point of tangency is $\gamma_{(p,v)}(t)$. The sets $W^s(\theta)$, $W^u(\theta)$, are the stable and unstable sets of $\theta \in T_1M$ according to the usual notion: if $\psi \in W^s(\theta)$ then*

$$\lim_{t \to +\infty} d(\phi_t(\theta), \phi_t(\psi)) = 0,$$

and if $\psi \in W^u(\theta)$ then

$$\lim_{t \to -\infty} d(\phi_t(\theta), \phi_t(\psi)) = 0.$$

Moreover, expansive geodesic flows in compact manifolds without conjugate points have a local product structure, such flows are topologically transitive and the collection of periodic orbits is dense.

In the case of surfaces, the works of M. Paternain [21] and E. Ghys [11] imply that

Theorem 1.3. *Let (M, g) be a compact surface whose geodesic flow is expansive. Then the geodesic flow is C^0 conjugated to the geodesic flow of a hyperbolic metric in M.*

One of the main consequences of the above Theorems is the so-called shadowing property or pseudo-orbit tracing property for expansive geodesic flows (proved in [26] for dimension greater than 2). From the shadowing lemma we get the next two statements:

Theorem 1.4. *(Anosov closing lemma for expansive geodesic flows)*

Let (M, g) be a compact Riemannian manifold without conjugate points whose geodesic flow is expansive. There exist $\epsilon > 0$, $D > 0$, $T(\epsilon) > 0$ with the following property: given $\theta \in T_1M$ such that $d_S(\theta, \phi_T(\theta)) < \epsilon$ for some $T \geq T(\epsilon)$, there exists a periodic point θ_0 of period $\sigma_T > 0$, and a continuous, increasing function $\rho : [0, T] \longrightarrow [0, \sigma_T]$ with $\rho(0) = 0$, $\rho(T) = \sigma_T$, such that

$$d_S(\phi_t(\theta), \phi_{\rho(t)}(\theta_0)) \leq D d_S(\theta, \phi_T(\theta)) \leq D\epsilon,$$

for every $t \in [0, T]$.

Theorem 1.4 is proved in [26]. The following result, known as the shadowing lemma with delay, can be obtained from the proof of the shadowing lemma as in [4].

Lemma 1.5. *Let (M, g) be a compact Riemannian manifold without conjugate points whose geodesic flow is expansive. Given $\epsilon > 0$, there exists a constant $K > 0$ such that given any $\theta \in T_1M$ and $T > 0$, there exists a periodic point $\eta = \eta(\theta)$ with period $0 \leq T(\eta) \leq T + K$, such that*

$$d_S(\phi_t(\theta), \phi_{\rho(t)}(\eta)) \leq \epsilon,$$

for every $t \in [0, T]$, where $\rho : [0, T] \to [0, \tau_T]$ is a continuous, increasing function such that $\rho(0) = 0$ and $\rho(T) = \tau_T$.

Proof. We just give a sketch of proof for the sake of completeness, since the argument follows the same line of the proof in the Anosov case. Let $\epsilon > 0$ be such that the shadowing of pseudo-orbits holds: there exists $\delta = \delta(\epsilon)$ such that every δ-pseudo-orbit can be ϵ-traced by a connected piece of a single orbit of the dynamics. Consider a Markov partition of T_1M (exists by Theorems 1.2 and 1.3) whose elements have diameter less than δ. By Theorem 1.3 or Theorem 1.2, the geodesic flow is transitive, so there exists a dense orbit $O(\theta_0)$ in T_1M. Since the number of elements of the partition is finite, there exists $L > 0$ and θ_1 in the orbit $O(\theta_0)$ with the following properties:

1. There exists $s > 0$, $s \leq L$, such that the piece of orbit $\phi_t(\theta_1)$, $t \in [0, s]$ meets all the elements of the Markov partition,
2. $d_S(\theta_1, \phi_s(\theta_1)) \leq \delta$.

This yields that given $\theta \in T_1M$, $T > 0$, there exists $t_1 < t_2 \in [0, s]$ such that

$$d_S(\phi_T(\theta), \phi_{t_1}(\theta_1)) \leq \delta,\ d_S(\theta), \phi_{t_2}(\theta_1)) \leq \delta.$$

So the curve $C = \phi_{[0,T]}(\theta) \cup \phi_{[t_1,t_2]}(\theta_1)$ is a δ-pseudo-orbit and hence there exists a piece of orbit $\phi_{[0,T(\eta)]}(\eta)$ and an increasing continuous function $\rho : [0, T + |t_2 - t_1|] \to [0, T(\eta)]$ with $\rho(0) = 0$ such that

1. $d_S(\phi_t(\theta), \phi_{\rho(t)}(\eta)) \leq \epsilon$ for every $t \in [0, T]$,

2. $d_S(\phi_t(\theta_1), \phi_{\rho(t)}(\eta)) \leq \epsilon$ for every $t \in [t_1, t_2]$.

Items (1) and (2) above imply that $d_S(\eta, \phi_{T(\eta)}(\eta)) \leq 2\epsilon$, and since the geodesic flow is expansive, we can choose the constant ϵ small enough such that the piece of orbit $\phi_{[0,T(\eta)]}(\eta)$ is in fact a closed orbit, like in the Anosov closing lemma. This finishes the proof of the lemma. □

2 An accurate Anosov closing lemma for surfaces whose ideal triangles have finite area

In this section we begin to study cohomological properties of expansive geodesic flows which are not Anosov, with the additional assumption of finiteness of the area of ideal triangles. We shall consider the family of surfaces of type $\beta > 1$ defined in the introduction, which includes the family of examples introduced in [7]. Let us recall briefly for the sake of completeness the main result of [7].

Theorem 2.1. *Given $\beta > 1$ there exist $\alpha > 0$, $C, D > 0$ and a $C^{2+\alpha}$ compact surface M having negative curvature at all points but along a simple closed geodesic $\gamma(t)$-where the curvature is zero at every point-such that:*

1. *The geodesic flow of M is expansive.*

2. *The stable geodesics $\gamma_s(t)$ of $\gamma(t)$ satisfy*

$$d(\gamma(t), \gamma_s(t)) \leq \frac{C}{(1+t)^\beta} d(\gamma(0), \gamma_s(0))$$

   for every $t \geq 0$.

3. *The area of every ideal triangle in $\tilde{M}$ is bounded above by D.*

The equation relating α and β in Theorem 2.1 is $\alpha = \frac{2}{\beta}$, so the family of surfaces given in this Theorem cannot be of class C^4 for any $\beta > 1$. Moreover, the surfaces are of class C^3 if and only if $\beta < 2$. To begin with the study of the cohomological features of the geodesic flows of such surfaces, we shall follow the same line of reasoning of the Anosov case, where an accurate closing lemma estimating the distance between a piece of recurrent orbit and a periodic one tracing this piece of orbit is of fundamental importance. In the Anosov case, the estimate of the distance between a piece of recurrent orbit and its periodic "shadow" follows from the hyperbolic behavior of the local stable and unstable sets of the dynamics. In our case, there is no hyperbolic behavior of the invariant sets in a neighborhood of the closed orbit where the curvature vanishes. Outside of a neighborhood of the flat closed orbit we have non-uniform hyperbolic behavior of the invariant sets and there is a measurable version of the Anosov closing lemma for hyperbolic measures due to Katok [8] which applies to our expansive systems. However, Katok's result is not enough for our purposes, because we want to solve cohomology and subcohomology equations in the whole manifold and not just in a subset of positive measure. Instead of using the ideas of the classical Anosov closing lemma (i.e., shadowing lemma with accurate estimates of distances) to deal with the considered expansive flows, we shall use some global, simpler arguments of non-positive curvature geometry. For simplicity, we shall use the notation $[p,q](t)$ to designate the geodesic arc $[p,q]:[0,T]\longrightarrow \tilde{M}$ with $[p,q](0)=p$, $[p,q](T)=q$ (t the arc length parameter), and we shall denote $f_{\beta}(t)=\frac{1}{(1+t)^{\beta}}$.

Proposition 2.2. *Let (M,g) be a C^2 compact surface with non-positive curvature of type $\beta>1$. There exist $\epsilon_0>0$, $T(\epsilon_0)>0$, $D>0$ such that for any given $\theta\in T_1M$ with $d_S(\theta,\phi_T(\theta))<\epsilon$, where $\epsilon\leq\epsilon_0$ and some $T\geq T(\epsilon_0)$, there exists a periodic point θ_0 of period $\sigma_T>0$ satisfying the following conditions:*

1. $|T-\sigma_T|\leq 2sup\{d_S(\theta,\theta_0),d_S(\phi_T(\theta),\theta_0)\}\leq 2\epsilon$,
2. $d_S(\phi_t(\theta),\phi_t(\theta_0))\leq 2D\sup\{d_S(\theta,\theta_0),d_S(\phi_T(\theta),\phi_T(\theta_0))\}f_{\beta}(t)$ *for every* $t\in[0,\frac{T}{2}]$,
3. *There exists* $\delta=\delta(\epsilon,T)>0$ *such that*

   $$d_S(\phi_t(\theta),\phi_{t+\sigma_T-T-\delta}(\theta_0))\leq 2D\sup\{d_S(\theta,\theta_0),d_S(\phi_T(\theta),\phi_T(\theta_0))\}f_{\beta}(T-t)$$

   for every $t\in[\frac{T}{2},T]$.
4. *The number* $\delta(\epsilon,T)$ *satisfies* $\lim_{\epsilon\to 0}\delta(\epsilon,T)=0$.

We subdivide the proof in three steps. Recall that $S_T(q)$ is the sphere of radius T centered at q, and $B_r(q)$ is the open ball of radius r centered at q.

Lemma 2.3. *Let (M,g) be a C^2 compact surface with non-positive curvature of type $\beta>1$. Given $\epsilon>0$, there exists $\bar{A}>0$, such that for every $\theta=(x,v)\in T_1\tilde{M}$, for every $T>0$, and every $p\in H_{\theta}(0))$ such that $d(p,\gamma_{\theta}(0))\leq\epsilon$, we get that*

1. *There exists a parametrization* $[p,\gamma_\theta(T)] : [-\epsilon_{p,T}, T] \longrightarrow \tilde{M}$ *of* $[p,\gamma_\theta(T)]$, *where* $\epsilon_{p,T} > 0$, *such that*

$$d(\gamma_\theta(t), [p,\gamma_\theta(T)](t)) \leq \bar{A} d(p,\gamma_\theta(0)) f_\beta(t)$$

for every $t \in [0,T]$.

2. *The number* $\epsilon_{p,T}$ *satisfies*

$$\lim_{T\to+\infty} \epsilon_{p,T} = 0.$$

Proof. Let $\theta = (p_0, v_0) \in T_1\tilde{M}$ and let $p \in H_\theta(0)) \cap B_\epsilon(\gamma_\theta(0))$. Let $[p,\gamma_\theta(T)] : [-\epsilon_{p,T}, T] \longrightarrow \tilde{M}$ be the arc length parametrization of the geodesic $[p,\gamma_\theta(T)]$ such that $[p,\gamma_\theta(T)](T) = \gamma_\theta(T)$, and $[p,\gamma_\theta(T)](-\epsilon_{p,T}) = p$. Notice that $T + \epsilon_{p,T}$ is the distance from $\gamma_\theta(T)$ to p which must be greater than T. Moreover, we have that

$$[p,\gamma_\theta(T)](t) = [p,\gamma_\theta(T)](T) \cap S_{T-t}(\gamma_\theta(T)),$$

for every $t \in [0,T]$. Let $v_p = -\nabla_p b^\theta$, and let $\theta_p = (p, v_p)$. By the assumptions on (M,g), we have that

$$d(\gamma_\theta(t), \gamma_{\theta_p}(t)) \leq A d(\gamma_\theta(0), p) f_\beta(t).$$

Since the curvature is non-positive, the horospheres, the spheres and the distances between geodesics in $\tilde{M}$ are convex. Moreover, the point $[p,\gamma_\theta(T)](t)$ is contained in the strip bounded by $\gamma_\theta[t,\infty)$, $\gamma_{\theta_p}[t,\infty)$, and the horosphere $H_\theta(t)$ for every $t \geq 0$. Let

$$[p,\gamma_\theta(T)](s(t)) = [p,\gamma_\theta(T)] \cap H_\theta(t),$$

for $t \in [0,T]$. The above remarks imply that $s(t) < t$, and that

$$d([p,\gamma_\theta(T)](s(t)), \gamma_\theta(t)) \leq d(\gamma_{\theta_p}(t), \gamma_\theta(t)),$$

for every $t \in [0,T]$. Since the distance $d([p,\gamma_\theta(T)\,]\,(t)\ ,\ \gamma_\theta)$ decreases with t, we have that

$$d([p,\gamma_\theta(T)\,]\,(t)\,,\,\gamma_\theta) < d([p,\gamma_\theta(T)](s(t)), \gamma_\theta),$$

for every $t \in [0,T]$. Hence, we get

$$d([p,\gamma_\theta(T)](t), \gamma_\theta) < d(\gamma_{\theta_p}(t), \gamma_\theta(t)) \leq A d(\gamma_\theta(0), p) f_\beta(t),$$

for every $t \in [0,T]$. Finally, it is easy to show that there is a constant $D = D(\epsilon) > 0$ such that

$$d([p,\gamma_\theta(T)](t),\gamma_\theta(t)) \leq Dd([p,\gamma_\theta(T)](t),\gamma_\theta),$$

for every $t \in [0,T]$. So we conclude that

$$d([p,\gamma_\theta(T)](t),\gamma_\theta(t)) < \bar{A}d(\gamma_\theta(0),p)f_\beta(t),$$

for every $t \in [0,T]$, where $\bar{A} = \frac{A}{D}$, as we wished to prove. □

Lemma 2.4. *Let (M,g) be a C^2 compact surface with non-positive curvature of type β. Given $\epsilon > 0$ there is a number $T(\epsilon) > 0$ such that for every $T \geq T(\epsilon)$ we have the following: let $\theta = (x,v) \in T_1\tilde{M}$, let $p \in H_\theta(0) = H^s(\theta)$, $q \in H^u(\phi_T(\theta))$ such that $d(p,x) \leq \epsilon$, $d(q,\gamma_\theta(T)) \leq \epsilon$, and let $[p,q] : [0,T'] \longrightarrow \tilde{M}$ be the arc length parametrization of $[p,q]$. Then,*

1. $|T - T'| \leq 2\epsilon$,
2. $d(\gamma_\theta(t),[p,q](t)) \leq 2C \sup\{d(x,p),d(\gamma_\theta(T),q)\}f_\beta(t)$ *for every* $t \in [0,\frac{T}{2}]$,
3. $d(\gamma_\theta(T-t),[p,q](T'-t)) \leq 2C \sup\{d(x,p),d(\gamma_\theta(T),q)\}f_\beta(t)$ *for every* $t \in [0,\frac{T}{2}]$. *Or equivalently,*

   $$d(\gamma_\theta(t),[p,q](t+T'-T)) \leq 2C \sup\{d(x,p),d(\gamma_\theta(T),q)\}f_\beta(T-t)$$

   for every $t \in [\frac{T}{2},T]$

Proof. The first item is a straightforward consequence of the triangular inequality, since the arc length parameter is the distance between points in the geodesics of $\tilde{M}$. Following the notation in the proof of the previous lemma, let $\theta_p = (p,v_p)$, where γ_{θ_p} is asymptotic to γ_θ. Let $\theta_q = (q,[p,q]'(T'))$ be the tangent vector of the geodesic $[p,q]$ at q, and let $T_p > 0$ be defined by

$$\gamma_{\theta_p}(T_p) = \gamma_{\theta_p} \cap H^u(\theta_q).$$

T_p is unique by convexity, and $T_p > T'$. Since horospheres vary continuously in the compact open topology, given $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ such that if $sup\{d(p,\gamma_\theta(0)),d(q,\gamma_\theta(T))\} \leq \epsilon$ we have that $|T' - T_p| \leq \delta$. By Lemma 2.3 we get that

$$d([p,q](T'-t),\gamma_{\theta_p}(T'-t)) \leq \bar{A}d(q,\gamma_{\theta_p}(T_p))f_\beta(t)$$

for every $t \in [0,T']$. Making $s = T' - t$ we obtain

$$d([p,q](s),\gamma_{\theta_p}(s)) \leq \bar{A}d(q,\gamma_{\theta_p}(T_p))f_\beta(T'-s)$$

for every $s \in [0, T']$. Since (M, g) is a surface of type β we have

$$d(\gamma_\theta(t)\,,\,\gamma_{\theta_p}(t)) \leq \bar{A} d(p\,,\,\gamma_\theta(0)) f_\beta(t)$$

for every $t \in [0, T]$. Therefore, take $T(\epsilon) > 0$, such that $T' > T/2$, if $T > T(\epsilon)$, then we have by the triangular inequality that

$$d([p, q](t), \gamma_\theta(t)) \leq 2\bar{A}\,\sup\{d(p, \gamma_\theta(0)), d(q, \gamma_{\theta_p}(T_p))\} f_\beta(t)$$

for every $t \in [0, \frac{T}{2}]$.

Note that,

$$d(q, \gamma_{\theta_p}(T_p)) \leq d(q, \gamma_\theta(T)) + d(\gamma_\theta(T), \gamma_{\theta_p}(T_p)) \leq$$

$$d(q, \gamma_\theta(T)) + d(p, x) + |T - T_p|.$$

Finally, from the way T_p was defined, (namely, the only number such that

$$\gamma_{\theta_p}(T_p) = \gamma_{\theta_p} \cap H^u(\theta_q),$$

we have that there exists $C_1 > 0$ such that

$$|T - T_p| \leq C_1\,\sup\{d(q, \gamma_\theta(T)), d(p, x)\}.$$

This shows item (2) in the lemma. Item (3) follows from a similar reasoning. □

Proof of Proposition 2.2.

By hypothesis, the surface (M, g) is of type $\beta > 1$, which means that the area of ideal triangles in the universal covering is finite. Since the curvature is non-positive, this yields that the geodesic flow is expansive, and hence we can apply the results of Section 1, in particular, the geodesic flow of (M, g) has an Anosov closing Lemma (Proposition 1.4). So given $\epsilon > 0$, there exist $T(\epsilon) > 0$, $D > 0$, such that given a recurrent point $\theta \in T_1M$ with $d_S(\theta, \phi_T(\theta)) < \epsilon$ for some $T \geq T(\epsilon)$, there exists a periodic point θ_0 of period $\sigma_T > 0$, and a continuous, increasing function $\rho : [0, T] \longrightarrow [0, \sigma_T]$ such that

$$d_S(\phi_t(\theta), \phi_{\rho(t)}(\theta_0)) \leq D\epsilon.$$

Let us recall that the distance $d_S(\theta, \eta)$ between points in T_1M ($T_1\tilde{M}$) is the Sasaki metric. Given a Riemannian manifold (N, h), let us denote by $\parallel V \parallel_h$ the norm of a vector V in the metric h. Let us consider the geodesics

$$\gamma_\theta(t) = \pi(\phi_t(\theta)),\ \gamma_{\theta_0}(t) = \pi(\phi_t(\theta_0)),$$

where $\pi : T_1M \longrightarrow M$ is the canonical projection. Since the canonical projection is a Riemannian submersion with respect to the Sasaki metric, namely,

$$\| D_\theta \pi(W) \|_g \leq \| W \|_{g_S}$$

for every $\theta \in T_1M$ and $W \in T_\theta T_1 M$), we have that

$$d(\gamma_\theta(t), \gamma_{\theta_0}(\rho(t))) \leq D\epsilon,$$

for every $t \in [0, T]$. In particular the distance between the endpoints of the above geodesics is less than $D\epsilon$. Thus, we can apply Lemma 2.4 to nearby lifts of these geodesics in $\tilde{M}$ to get an estimate of their distance in M. Now, it is easy to show that there exists $K = K(\epsilon, D, T(\epsilon)) > 0$ such that if two geodesics γ_η, γ_τ satisfy $d(\gamma_\eta(t), \gamma_\tau(t)) \leq D\epsilon$ for every $t \in [0, T]$, $T \geq T(\epsilon)$, then

$$d_S(\phi_t(\theta), \phi_t(\tau)) \leq K d(\gamma_\eta(t), \gamma_\tau(t))$$

for every $t \in [0, T]$, i.e., the Sasaki distance and the distance in M are locally equivalent in the above hypothesis on the geodesics γ_η and γ_τ. The above assertion allows us to obtain an estimate of the distance between the orbits $\phi_t(\theta)$ and $\phi_t(\theta_0)$ that is similar to the estimate for the distance between the underlying geodesics, we leave the details to the reader. This finishes the proof of Proposition 2.2.

3 Livsic Theorem for expansive non-Anosov geodesic flows in surfaces of type β

The goal of this section is to show a version for expansive, non-Anosov systems of the classical Livsic Theorem for Anosov systems.

Theorem 3.1. *Let (M, g) be a compact C^2 surface with non-positive curvature of type $\beta > 1$. Let $f : T_1M \longrightarrow \mathbb{R}$ be a Hölder continuous function, with Hölder exponent $\alpha > \frac{1}{\beta}$, such that for every periodic orbit $\phi_t(\tau)$ of period $T(\tau)$ we have $\int_0^{T(\tau)} f(\phi_t(\tau))dt = 0$. Then there exists a Hölder continuous function $F : T_1M \longrightarrow \mathbb{R}$ such that*

$$F(\phi_t(\theta)) = F(\theta) + \int_0^t f(\phi_s(\theta))ds$$

for every $\theta \in T_1M$ and $t > 0$. Moreover, F is unique up to an additive constant and its Hölder exponent is α. If the function f is Lipschitz, then the potential F is also Lipschitz.

Proof. We follow the line of reasoning of the proof in the Anosov case. Since the geodesic flow of (M, g) is expansive, there exists a dense orbit $O(\theta) = \{\phi_t(\theta), t \in \mathbb{R}\}$ by Theorem 1.2. Let us first define the potential $F : T_1M \longrightarrow \mathbb{R}$ in the dense orbit in the usual way:

$$F(\phi_t(\theta)) = \int_0^t f(\phi_s(\theta))ds.$$

Let us show that F is Hölder continuous with the same Hölder exponent $\alpha > \frac{1}{\beta}$ of f along the dense orbit. Let ϵ_0, $T(\epsilon_0)$ be as in Proposition 2.2, and let $\phi_{t_1}(\theta)$, $\phi_{t_2}(\theta)$ satisfy

1. $t_2 - t_1 > T(\epsilon_0)$,
2. $d(\phi_{t_1}(\theta), \phi_{t_2}(\theta)) \leq \epsilon$, for some $\epsilon < \epsilon_0$.

Consider a periodic orbit $O(\bar{\theta})$ with period σ_0 shadowing the arc $\phi_{[t_1,t_2]}(\theta)$ as in Proposition 2.2. Let $\eta = \phi_{t_1}(\theta)$, $T = t_2 - t_1$, and $\theta_0 = \phi_{t_1}(\bar{\theta})$, so

$$F(\phi_{t_2}(\theta)) - F(\phi_{t_1}(\theta)) = F(\phi_T(\eta)) - F(\eta).$$

Then we have:

$$\begin{aligned} F(\phi_T(\eta)) - F(\eta) &= \int_{t_1}^{t_2} f(\phi_s(\theta))ds \\ &= \int_0^T f(\phi_s(\eta))ds \\ &= \int_0^T f(\phi_s(\eta))ds - \int_0^{\sigma_0} f(\phi_s(\theta_0))ds \end{aligned}$$

where in the last equation we used the hypothesis on f. By Proposition 2.2 we have that

1. $|T - \sigma_0| \leq 2sup\{d_S(\eta, \theta_0), d_S(\phi_T(\eta), \theta_0)\} \leq 2\epsilon$,
2. $d_S(\phi_t(\eta), \phi_t(\theta_0)) \leq 2Dsup\{d_S(\eta, \theta_0), d_S(\phi_T(\eta), \phi_T(\theta_0)\}f_\beta(t)$ for every $t \in [0, \frac{T}{2}]$,
3. There exists $\delta = \delta(\epsilon, T) > 0$ such that

   $$d_S(\phi_t(\eta), \phi_{t+\sigma_0-T-\delta}(\theta_0)) \leq 2Dsup\{d_S(\eta, \theta_0), d_S(\phi_T(\eta), \phi_T(\theta_0)\}f_\beta(T-t)$$

   for every $t \in [\frac{T}{2}, T]$.
4. The number $\delta(\epsilon, T)$ satisfies $\lim_{\epsilon\to 0} \delta(\epsilon, T) = 0$.

So let us split the above integrals in the following way:

$$\int_0^T f(\phi_s(\eta))ds - \int_0^{\sigma_0} f(\phi_s(\theta_0))ds = \int_0^T f(\phi_s(\eta)) - f(\phi_s(\theta_0))ds + \int_T^{\sigma_0} f(\phi_s(\theta_0))ds.$$

Let C be the maximum of f in T_1M. The second integral can be bounded above by

$$\begin{aligned}\int_T^{\sigma_0} f(\phi_s(\theta_0))ds &\leq C|T-\sigma_0| \\ &\leq 2C \sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\theta_0)\} \leq 2C\epsilon.\end{aligned}$$

The first integral admits the estimate

$$|\int_0^T f(\phi_s(\eta)) - f(\phi_s(\theta_0))ds| \leq 2\,(2D)^\alpha (\sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\phi_T(\theta_0))\})^\alpha \int_0^{T/2} f_\beta^\alpha(t)dt.$$

To shorten notation, let $c(\eta,T) = \sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\phi_T(\theta_0))\}$. ¿From Proposition 2.2 this gives

$$|\int_0^T f(\phi_s(\eta)) - f(\phi_s(\theta_0))ds| \leq 2\,(2D)^\alpha (c(\eta,T))^\alpha \int_0^{T/2} \frac{1}{(1+t)^{\alpha\beta}}dt$$

and hence

$$|\int_0^T f(\phi_s(\eta)) - f(\phi_s(\theta_0))ds| \leq 2\,(2D)^\alpha (c(\eta,T))^\alpha I(\alpha,\beta),$$

where $I(\alpha,\beta) = \int_0^{+\infty} \frac{1}{(1+t)^{\alpha\beta}}dt$ is finite since $\alpha\beta > 1$. Joining the estimates of the two integrals we get

$$|F(\phi_T(\eta)) - F(\eta)| \leq 2\,(2D)^\alpha (c(\eta,T))^\alpha I(\alpha,\beta) + 2C \sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\theta_0)\}.$$

To estimate $c(\eta,T)$ in terms of $\sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\theta_0)\}$ we first observe that

$$\begin{aligned}d_S(\phi_T(\eta),\phi_T(\theta_0)) &\leq d_S(\phi_T(\eta),\theta_0) + d_S(\theta_0,\phi_T(\theta_0)) \\ &\leq d_S(\phi_T(\eta),\theta_0) + d_S(\phi_{\sigma_0}(\theta_0),\phi_T(\theta_0)) \\ &\leq d_S(\phi_T(\eta),\theta_0) + |T-\sigma_0| \\ &\leq d_S(\phi_T(\eta),\theta_0) + 2\sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\theta_0)\}.\end{aligned}$$

So we get

$$c(\eta,T) \leq 3\sup\{d_S(\eta,\theta_0), d_S(\phi_T(\eta),\theta_0)\} \leq 3\,D\,d_S(\phi_T(\eta),\eta),$$

where D is the constant in the Anosov closing Lemma (Theorem 1.4). This yields

$$\begin{aligned}|F(\phi_T(\eta)) - F(\eta)| &\leq 2\,(2D)^\alpha(3\,Dd_S(\phi_T(\eta),\eta))^\alpha I(\alpha,\beta) + 3\,CDd_S(\phi_T(\eta),\eta)\\ &\leq Md_S(\phi_T(\eta),\eta)^\alpha,\end{aligned}$$

where M is a constant depending on D, C, α, β. Therefore, we have shown that the function F is uniformly continuous in the dense orbit $O(\theta)$, and by elementary analysis F admits a continuous extension to T_1M. Moreover, if f is Hölder continuous with exponent α then F is uniformly Hölder continuous with exponent α in the dense orbit $O(\theta)$. Similarly, if f is Lipschitz, the above calculation proceeds just assuming that the exponent α is equal to 1, so the function F is Lipschitz in $O(\theta)$. By elementary analysis we get that the continuous extension of F has the same regularity of f. The formula

$$F(\phi_t(\tau)) = F(\tau) + \int_0^t f(\phi_s(\tau))ds$$

for every $\tau \in T_1M$ follows like in the Anosov case by approaching τ with points in the dense orbit $O(\theta)$.

The function $F : T_1M \longrightarrow \mathbb{R}$ is unique up to an additive constant because the difference between two potentials of f is constant along the orbits. In particular, such a difference must be constant along a dense orbit and since the potential are continuous the difference must be constant in T_1M. This finishes the proof of Theorem 3.1. □

We would like to point out that the Livsic Theorem can be proved in a slightly more general setting, just assuming that the ideal triangles of the surface have finite area regardless of the decay of the area in ideal cones. The point is that we have to assume that the function f satisfying the zero cohomological condition is Lipschitz. We shall state this result without proof for the sake of completeness.

Proposition 3.2. *Let (M, g) be a compact C^2 surface with non-positive curvature whose ideal triangles have finite area: there exists an upper bound $A > 0$ for the area of all ideal triangles in $(\tilde{M}, \tilde{g})$. Let $f : T_1M \longrightarrow \mathbb{R}$ be a Lipschitz function, such that for every periodic orbit $\phi_t(\tau)$ of period $T(\tau)$ we have $\int_0^{T(\tau)} f(\phi_t(\tau))dt = 0$. Then there exists a continuous function $F : T_1M \longrightarrow \mathbb{R}$ such that*

$$F(\phi_t(\theta)) = F(\theta) + \int_0^t f(\phi_s(\theta))ds$$

for every $\theta \in T_1M$ and $t > 0$. Moreover, F is unique up to an additive constant.

The proof follows the same line of reasoning of the proof of Theorem 3.1: we can show the Anosov closing lemma for geodesic flows in compact surfaces whose ideal triangles have finite area, and then the estimates of the proof of Theorem 3.1 proceed just taking the Hölder exponent α equal to 1.

4 Subcohomological equation for expansive non-Anosov geodesic flows in surfaces of type β: the continuous case

The goal of this section is to show that given a certain Hölder function $f : T_1M \longrightarrow \mathbb{R}$ there exists continuous function F satisfying a subcohomological equation, where (M, g) is a compact surface of type β. The function F is often called a subaction function. The main result of the section is the following:

Theorem 4.1. *Let (M, g) be a compact C^2 surface with non-positive curvature of type $\beta > 1$. Let $f : T_1M \longrightarrow \mathbb{R}$ be a Hölder continuous function, with Hölder exponent $\alpha > \frac{1}{\beta}$. Denote by $m(f) = \sup\{\int f d\nu \,|\, \nu$ invariant probability for the geodesic flow $\}$. Then there exists a continuous function $F : T_1M \longrightarrow \mathbb{R}$ such that*

$$F(\phi_t(\theta)) \geq F(\theta) + \int_0^t [f(\phi_s(\theta)) - m(f)]\, ds$$

for every $\theta \in T_1M$ and $t > 0$.

Proof. The idea of the proof is to use a version for continuous time of the definition of subaction functions for Anosov diffeomorphisms given in [16] (Definition 8 in [16]). Given $\theta \in T_1M$, let

$$W^s_\epsilon(\theta) = \{(p, -\nabla_p b^\theta), p \in H_\theta(0), d_{H_\theta(0)}(p, \gamma_\theta(0)) < \epsilon\}.$$

Here, $d_{H_\theta(0)}$ is the distance induced by the restriction of $\tilde{g}$ to $H_\theta(0)$. Similarly, let us define,

$$W^u_\epsilon(\theta) = \{(p, \nabla_p b^{-\theta}), p \in H_{-\theta}(0), d_{H_{-\theta}(0)}(p, \gamma_\theta(0)) < \epsilon\},$$

where $\theta = (x, v)$ and $-\theta$ is just a notation for $-\theta = (x, -v)$. Given $L > 0$, consider the function $F : T_1M \longrightarrow \mathbb{R}$ given by

$$F(\theta) = \sup\{\int_0^t [f(\phi_s(\phi_{-t}(\eta))) - m(f)]\, ds +$$

$$\int_0^\infty [\, f(\phi_s(\eta)) \,-\, f(\phi_s(\theta))\,]\, ds \;\mid\; \eta \in W^s_L(\theta), t \geq 0\,\}$$

According to [16], F satisfies the subcohomological equation in the statement of Theorem 4.1 provided that $W^s(\theta)$ is uniformly contracted by ϕ_t, for $t > 0$ and every $\theta \in T_1M$; $W^u(\theta)$ is uniformly contracted by ϕ_t, for $t < 0$ and every $\theta \in T_1M$; and F is finite. Let us explain briefly the kind of uniform contraction enjoyed by our expansive geodesic flows. Let us recall that by Theorem 1.2, the distance between any two orbits starting at $W^s(\theta)$ goes to zero as $t \to +\infty$. By the compactness of T_1M and the continuity of stable leaves (Theorems 1.2, 1.3) we have that given $0 < \epsilon_1 < \epsilon_2$, there exists $T = T(\epsilon_1, \epsilon_2) > 0$ such that for every $\theta_1, \theta_2 \in W^s(\theta)$ with $d_{W^s}(\theta_1, \theta_2) \leq \epsilon_2$ we get

$$d_{W^s}(\phi_t(\theta_1), \phi_t(\theta_2)) \leq \epsilon_1$$

for every $t \geq T$. Here, the notation $d_{W^s}(\ ,\ )$ means the restriction of the Sasaki distance to stable sets. So stable sets are uniformly contracted in this sense. An analogous property is satisfied by negative iterates of unstable sets.

Since the uniform contraction of invariant sets in the above sense already imply that F satisfies the subcohomological equation once we know that F is well defined, we are left to show that $F(\theta)$ is finite for every $\theta \in T_1M$.

To see that $F(\theta)$ is well defined for any $\theta \in T_1M$, let us show that the set of numbers

$$A_\theta = \bigcup_{\eta \in W^s_L(\theta)} \{ \int_0^t [f(\phi_s(\phi_{-t}(\eta))) - m(f)]\, ds +$$

$$\int_0^\infty [\, f(\phi_s(\eta)) \,-\, f(\phi_s(\theta))\,]\, ds \,,\, t \geq 0\,\},$$

is bounded above by a constant K. Since (M, g) is a surface of type $\beta > 1$, the distance between asymptotic orbits is integrable, and hence

$$\sup\{ \int_0^\infty [\, f(\phi_s(\eta)) \,-\, f(\phi_s(\theta))\,]\, ds \;\mid\; \eta \in W^s_L(\theta)\} \,\leq\, D = D(\alpha, \beta, L, f),$$

for every $\theta \in T_1M$.

Here we use the fact that f is α-Hölder and $\alpha\,\beta > 1$.

Suppose, by contradiction, that there exists a sequence $\eta_n \in W^s_L(\theta)$ and $t_n \geq 0$ such that $\int_0^{t_n} [f(\phi_s(\phi_{-t_n}(\eta_n)))ds - m(f)]$ goes to ∞.

Without lost of generality one can assume that $\phi_{-t_n}(\eta_n) \to y$ and $\eta_n \to x$, when $n \to \infty$.

¿From the shadowing property with delay (Lemma 1.5), given $\epsilon > 0$, there is a number $K > 0$, such that for any $n \in \mathbb{N}$, there exists a periodic orbit $\gamma_{n,\epsilon}$ defined on an time interval $[0, t_n + k]$, with $0 \leq k \leq K$, such that

$$d(\phi_s(\phi_{-t_n}(\eta_n))), \gamma_{n,\epsilon}(s)) \leq \epsilon,$$

for $s \in [0, t_n]$. By the accurate closing lemma (Lemma 2.4) and the definition of the Sasaki metric we get, as in the proof of Proposition 2.1, that there exists a number $t'_n > 0$ with $|t_n - t'_n| \leq 2\epsilon$ such that

$$d_S(\phi_t(\phi_{-t_n}(\eta_n)), \gamma_{n,\epsilon}(t)) \leq 2C \sup\{d_S(\phi_{-t_n}(\eta_n), \gamma_{n,\epsilon}(0)), d_S(\eta_n, \gamma_{n,\epsilon}(t_n))\} f_\beta(t)$$

for every $t \in [0, \frac{t_n}{2}]$ and,

$$d_S(\phi_t(\phi_{-t_n}(\eta_n)), \gamma_{n,\epsilon}(t+t'_n-t_n)) \leq 2C \sup\{d_S(\phi_{-t_n}(\eta_n), \gamma_{n,\epsilon}(0)), d_S(\eta_n, \gamma_{n,\epsilon}(t_n))\} f_\beta(t_n-t)$$

for every $t \in [\frac{t_n}{2}, t_n]$.

Now, as f is Hölder with exponent $\alpha > \frac{1}{\beta}$, the same integral estimates made in the proof of Theorem 3.1 apply to this case in order to give a constant $C > 0$ depending on α, β, ϵ such that

$$\left| \int_0^{t_n} f(\gamma_{n,\epsilon}(s))\, ds - \int_0^{t_n} f(\phi_s(\phi_{-t_n}(\eta_n))))\, ds \right| < C.$$

Since the normalized arc length along the closed geodesic $\gamma_{n,\epsilon}$ gives an invariant probability measure, we get by the definition of $m(f)$,

$$\int_0^{t_n+k} [f(\gamma_{n,\epsilon}(s)) - m(f)]\, ds \leq 0.$$

The number K does not depend on $\eta \in H_\theta(0)$, $\theta \in T_1M$, therefore

$$\int_{t_n}^{t_n+k} f(\phi_s(\phi_{-t_n}(\eta_n))))\, ds$$

is also bounded because $|k| \leq K$. But this implies that

$$\int_0^{t_n} [f(\phi_s(\phi_{-t_n}(\eta_n)))ds - m(f)]$$

is bounded and we get a contradiction.

The proof of the continuity of F follows from a natural extension of the continuity argument employed in Lemma 9 in [16] to deal with Anosov diffeomorphisms. Since we are dealing with flows and some technical difficulties arise from this fact, we give next an outline of the proof in four steps for the sake of completeness.

Step One: There exists an uniform ϵ^* and $C = C(f) > 1 > 0$, such that if θ and ϑ are two points in the same stable leave and up to a distance ϵ^*, then for any $t > 0$

$$| \int_0^t f(\phi_s(\theta))ds - \int_0^t f(\phi_s(\vartheta))ds| < C^\alpha \, d(\theta, \vartheta)^\alpha.$$

This follows from f being α-Hölder, with $\alpha \, \beta > 1$, and the analytical expression

$$d(\phi_s(\theta), \phi_s(\vartheta)) \leq \frac{C_1}{(1+s)^\beta} d(\theta, \vartheta),$$

for every $s \geq 0$.

Step two: There exists $K = K(f) > 0$ and $C > 0$ such that for every $t > 0$, $x, x' \in T_1M$, if $d(\phi_s(x), \phi_s(x')) < C \, \epsilon^*$, for all $s \in [0, t]$, then

$$| \int_0^t f(\phi_s(x))ds - \int_0^t f(\phi_s(x'))ds| < K \, \max \{ \, d(\theta, \vartheta)^\alpha, \, d(x, x')^\alpha \, \},$$

where $\phi_t(x) = \theta$, $\phi_t(x') = \vartheta$.

This follows from the estimate of the integral in Theorem 3.1.

Step three: F is well defined. This was proved in the first part of the Theorem.

Step four: We suppose bellow that L is small enough. For a given $\vartheta \in T_1M$ and δ small enough, consider $Z_{L,\delta}(\vartheta)$ the two dimensional surface on the unitary bundle given by

$$Z_{L,\delta} \, (\vartheta) = \cup_{-\delta < r < \delta} \cup_{\eta_2 \in W^s_L(\vartheta)} \;\; \varphi_r(\eta_2).$$

Given θ and $\eta_1 \in W^s_L(\theta)$ let us define for each $\vartheta \in T_1M$ (close enough to θ) the point $\eta = \eta_{\theta, \eta_1, \vartheta} \in W^s_L(\vartheta)$ given by the following property: there exists $y' \in Z_{L,\delta} \, (\vartheta) \cap W^u(\eta_1)$ and $s_2 \in (-\delta, \delta)$ such that, $\varphi_{s_2}(\eta) = y'$. We shall employ the notation $\eta = [\vartheta, \eta_1]$, clearly inspired by the standard notation used for the local product structure of invariant sets.

Such y' (and η)) exists by the local product structure of expansive systems (Theorem 1.1). The point η depends just continuously on the variables θ, η_1, ϑ, because the geodesic flow is not Anosov.

Note that $\int_0^{s_2} f(\varphi_r(\eta))dr$ is small if θ and ϑ are close enough and L is small.

For a given t and N (to be defined bellow) we want to estimate

$$|\,(\int_0^t f(\phi_s(\phi_{-t}(\eta_1)))\,ds + \int_0^\infty [f(\phi_s(\eta_1)) \,-\, f(\phi_s(\theta))\,]\,ds\,) -$$

$$(\int_0^t f(\phi_s(\phi_{-t}(\eta)))\,ds + \int_0^\infty [\,f(\phi_s(\eta)) \,-\, f(\phi_s(\vartheta))\,]\,ds\,)\,| \,\leq$$

$$\int_0^{t+N} |\,f(\phi_s(\phi_{-t}(\eta_1))) - f(\phi_s(\phi_{-t}(\eta)))\,|\,ds + \qquad (=\Sigma_1)$$

$$\int_0^N |\,f(\phi_s(\theta)) - f(\phi_s((\vartheta))\,|\,ds + \qquad (=\Sigma_2)$$

$$|\int_0^\infty [\,f(\phi_{s+N}(\eta_1)) \,-\, f(\phi_{s+N}(\theta))\,]\,ds| + \qquad (=\Sigma_3)$$

$$|\int_0^\infty [\,f(\phi_{s+N}(\eta)) \,-\, f(\phi_{s+N}(\vartheta))\,]\,ds\,| \qquad (=\Sigma_4)$$

We now show that each Σ_i is small if $d(\theta,\vartheta)$ is small.

Indeed, we just use step one for Σ_3 (for $\varphi_N(\theta)$ and $\varphi_N(\eta_1)$) and Σ_4 (for $\varphi_N(\vartheta)$ and $\varphi_N(\eta)$).

In order to show the claim for Σ_1 and Σ_2 we have to be more careful.

Since the geodesic flow is expansive and the curvature is non-positive, we have a uniform rate of forward expansion in strong unstable sets, and a uniform rate of backward expansion in strong stable sets. So for each θ,ϑ such that $d(\theta,\vartheta) < \epsilon^*$ and $\vartheta \notin W^s_L(\theta)$, there exist a first $N = N(\theta,\vartheta) > 0$ such that $d(\varphi_N(\theta),\varphi_N(\vartheta)) = C\,\epsilon^*$.

Note that there exists C_1 such that

$$\Sigma_1 \leq C_1 \max\{\, d(\varphi_{-t}(\eta_1),\varphi_{-t}(\eta))^\alpha,\, d(\varphi_N(\eta_1),\varphi_N(\eta))^\alpha\,\},$$

and

$$\Sigma_2 \leq C_1 \max\{\, d(\theta,\vartheta)^\alpha,\, d(\varphi_N(\theta),\varphi_N(\vartheta))^\alpha\,\}.$$

It follows at once that $\Sigma_2 < (C\epsilon^*)^\alpha$.

As $\eta \in W^u(\eta_1)$ then $d(\varphi_{-t}(\eta_1),\varphi_{-t}(\eta))^\alpha \leq (\epsilon^*)^\alpha$.

Note that $\eta_1 \in W^s_L(\theta)$, $\eta \in W^s_L(\vartheta)$. Moreover, if $d(\theta,\vartheta)$ is small, then $d(\eta,\vartheta)$ and $d(\eta_1,\theta)$ are small.

As $d(\varphi_N(\theta),\varphi_N(\vartheta)) \leq C\epsilon^*$ and $\eta_1 \in W^s - L(\theta)$, $\eta \in W^s_L(\vartheta)$, then $d(\varphi_N(\eta_1),\varphi_N(\eta))^\alpha$ is small.

This shows that F is continuous.

□

As a final remark, we would like to stress that the continuity of the subaction function F obtained in Theorem 4.1 might not be improved to Hölder continuity in general. This is due essentially to the lack of transversality of the invariant sets in the case of expansive, non-hyperbolic systems. However, in the specific case of the type β surfaces studied in [7], it is possible to show that the invariant sets have Hölder continuous transversal holonomies with Hölder exponent depending on β, and hence the argument of the proof of Theorem 4.1 yields the Hölder continuity of the subaction function associated to a Hölder observable. We shall not prove this fact in this paper because the proof is quite technical and just holds for the examples in [7]. We do not know if the Hölder continuity of the subaction function associated to a Hölder observable can be obtained for any surface of type β.

5 Appendix: Density of Dirac measures and rigidity

The purpose of this section is to give a further, simple application of the shadowing lemma for expansive geodesic flows to a problem of rigidity of manifolds without conjugate points. Our starting point is a paper by H.-R. Fanai [10] whose main result is the following:

Theorem: Let (M, g) be a differentiable Riemannian manifold without conjugate points and finite volume, such that the convex closure of Dirac measures in the unit tangent bundle is dense in the set of probabilities which are invariant under the geodesic flow. Then every manifold without conjugate points and finite volume which is conformally equivalent to (M, g) and has the same marked length spectrum of (M, g) is isometric to (M, g).

A Dirac measure in T_1M is an invariant probability supported in a closed orbit of the geodesic flow. The above result holds for Anosov geodesic flows in compact manifolds, and Fanai remarks in [10] (without proving) that the same property should hold for expansive geodesic flows in manifolds without conjugate points, based on the density of closed orbits in T_1M proved in [27]. We give in this section a complete proof of the former remark.

Theorem 5.1. *Let (M, g) be a compact Riemannian manifold without conjugate points whose geodesic flow is expansive. Then the convex closure of the Dirac measures in the unit tangent bundle is dense in the set of probabilities which are invariant under the geodesic flow.*

Basically, Theorem 5.1 is a consequence of the following version of the shadowing lemma for expansive geodesic flows (Theorem 1.4).

Lemma 5.2. *Let (M,g) be a compact Riemannian manifold without conjugate points whose geodesic flow is expansive. There exist $\epsilon > 0$, $D > 0$, $T(\epsilon) > 0$ with the following property: given $\theta \in T_1M$ such that $d_S(\theta, \phi_T(\theta)) < \epsilon$ for some $T \geq T(\epsilon)$, there exists a periodic point θ_0 of period $\sigma_T > 0$, and a continuous, increasing function $\rho : [0,T] \longrightarrow [0,\sigma_T]$ such that*

1. $d_S(\phi_t(\theta), \phi_{\rho(t)}(\theta_0)) \leq D d_S(\theta, \phi_T(\theta)) \leq D\epsilon$, *for every* $t \in [0,T]$,
2. $|t - \rho(t)| \leq 2D d_S(\theta, \phi_T(\theta))$, *for every* $t \in [0,T]$.

Proof. Lemma 5.2 item (1) is just Theorem 1.4. Item (2) is a straightforward consequence of the triangular inequality. In fact, the parameter t is the arc length parameter of (M,g), as well as the distance from $\beta(0)$ and $\beta(t)$ for every geodesic $\beta \subset (\tilde{M}, \tilde{g})$. Since we have that the geodesic $\gamma_{\theta_0}(\rho(t)) = \pi(\phi_{\rho(t)}(\theta_0))$ stays within distance $D d_S(\theta, \phi_T(\theta))$ from the geodesic $\gamma_\theta(t)$ for every $t \in [0,T]$, these estimate remains true if we lift the two geodesics to $(\tilde{M}, \tilde{g})$. ☐

Let $\mathcal{M}$ be the set of probability measures in T_1M which are invariant by the geodesic flow of (M,g), endowed with the weak topology. A basis for the weak topology is the following collection of neighborhoods

$$V_{f,\epsilon}(\mu) = \{\nu \in \mathcal{M} \ s.t. \ |\int_{T_1M} f d\mu - \int_{T_1M} f d\nu| < \epsilon\},$$

for $\epsilon > 0$ and $f : T_1M \longrightarrow \mathbb{R}$ a continuous function. It is well known that the set of $\theta \in T_1M$ such that the orbit of θ supports an invariant, ergodic probability measure has total Liouville measure in T_1M.

Lemma 5.3. *Let (M,g) be a compact Riemannian manifold without conjugate points whose geodesic flow is expansive. Let $\theta \in T_1M$ such that there exists an ergodic probability measure μ supported in the orbit of θ. Then there exists a sequence of Dirac measures μ_n such that*

$$\lim_{n \to +\infty} \mu_n = \mu,$$

where the limit is taken in the weak topology.

Proof. We want to show that there exists a sequence of Dirac measures μ_n such that given any continuous function $f : T_1M \longrightarrow \mathbb{R}$ and $\epsilon > 0$, there exists $n_0 > 0$ such that $\mu_n \in V_{f,\epsilon}(\mu)$ for every $n \geq n_0$. By Birkhoff's Theorem,

$$\mu(f) = \int_{T_1M} f d\mu = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\phi_t(\theta)) dt.$$

Moreover, there exists a sequence $T_n \to +\infty$ such that

1. $d_S(\theta, \phi_{T_n}(\theta)) < \frac{1}{n}$,

2. $\mu(h) = \int_{T_1 M} h d\mu = \lim_{n\to+\infty} \frac{1}{T_n} \int_0^{T_n} h(\phi_t(\theta))dt$, for every continuous function h.

By the shadowing lemma for expansive geodesic flows (Lemma 5.2), there exists a sequence of periodic points $\theta_n \in T_1 M$ with periods σ_n, and continuous surjective functions $\rho_n : [0, \sigma_n] \longrightarrow [0, T_n]$, with $\rho_n(0) = 0$, such that

1. $d_S(\phi_t(\theta), \phi_{\rho(t)}(\theta_0)) \leq D d_S(\theta, \phi_{T_n}(\theta)) \leq \frac{D}{n}$, for every $t \in [0, \sigma_n]$,

2. $|t - \rho_n(t)| \leq \frac{2\,D}{n}$, for every $t \in [0, \sigma_n]$, $n \in \mathbb{N}$.

Since f is continuous, given $\delta > 0$ there exists $n_\delta > 0$ such that $|f(x) - f(y)| \leq \delta$ if $d_S(x, y) \leq \frac{2D}{n_\delta}$.

Since $d_S(\phi_{\rho_n(t)}(\theta_n), \phi_t(\theta_n)) \leq |1 - \rho_n(t)|$, if $n \geq n_\delta$, we have that

$$\begin{aligned} |f(\phi_t(\theta)) - f(\phi_t(\theta_n))| &\leq |f(\phi_t(\theta)) - f(\phi_{\rho_n(t)}(\theta_n))| \\ &+ |f(\phi_{\rho_n(t)}(\theta_n)) - f(\phi_t(\theta_n))| \;\leq\; \delta + \delta, \end{aligned}$$

for every $t \in [0, T_n]$.

This implies

$$|\frac{1}{T_n} \int_0^{T_n} f(\phi_t(\theta))dt - \frac{1}{T_n} \int_0^{T_n} f(\phi_t(\theta_n))dt| \leq \frac{1}{T_n} T_n(2\,\delta) = 2\,\delta.$$

Therefore we obtain

$$\mu(f) = \lim_{n\to+\infty} \frac{1}{T_n} \int_0^{T_n} f(\phi_t(\theta_n))dt.$$

Finally, observe that

$$\frac{1}{\sigma_n} \int_0^{\sigma_n} f(\phi_t(\theta_n))dt = \frac{T_n}{\sigma_n} \frac{1}{T_n} (\int_0^{T_n} f(\phi_t(\theta_n))dt + \int_{T_n}^{\sigma_n} f(\phi_t(\theta_n))dt),$$

which implies that

$$|\mu_n(f) - \frac{1}{T_n} \int_0^{T_n} f(\phi_t(\theta_n))dt| \leq \frac{4D}{n\,\sigma_n} \parallel f \parallel_\infty .$$

Joining the two last statements we deduce

$$\mu(f) = \lim_{n\to+\infty} \mu_n(f),$$

thus finishing the proof of the lemma. □

Proof of Theorem 5.1:

We know that the set of invariant probability measures $\mathcal{M}$ is a compact, convex set whose extremal points are precisely the ergodic measures. In particular, $\mathcal{M}$ is the convex closure of the ergodic measures, and since every ergodic measure is approached by Dirac measures, we conclude that the convex closure of Dirac measures is dense in the set of invariant probability measures, as we wished to show.

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