

A THERMODYNAMIC FORMALISM FOR DENSITY MATRICES IN QUANTUM INFORMATION

A. BARAVIERA, C. F. LARDIZABAL, A. O. LOPES, AND M. TERRA CUNHA

ABSTRACT. We consider new concepts of entropy and pressure for stationary systems acting on density matrices which generalize the usual ones in Ergodic Theory. Part of our work is to justify why the definitions and results we describe here are natural generalizations of the classical concepts of Thermodynamic Formalism (in the sense of R. Bowen, Y. Sinai and D. Ruelle). It is well-known that the concept of density operator should replace the concept of measure for the cases in which we consider a quantum formalism.

We consider the operator Λ acting on the space of density matrices \mathcal{M}_N over a finite N -dimensional complex Hilbert space

$$\Lambda(\rho) := \sum_{i=1}^k \text{tr}(W_i \rho W_i^*) \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)},$$

where W_i and V_i , $i = 1, 2, \dots, k$ are linear operators in this Hilbert space. In some sense this operator is a version of an Iterated Function System (IFS). Namely, the $V_i(\cdot)V_i^* =: F_i(\cdot)$, $i = 1, 2, \dots, k$, play the role of the inverse branches (i.e., the dynamics on the configuration space of density matrices) and the W_i play the role of the weights one can consider on the IFS. In this way a family $W := \{W_i\}_{i=1, \dots, k}$ determines a Quantum Iterated Function System (QIFS).

We also present some estimates related to the Holevo bound.

1. INTRODUCTION

In this work we investigate a generalization of the classical Thermodynamic Formalism (in the sense of Bowen, Sinai and Ruelle) for the setting of density matrices. We consider the operator Λ acting on the space of density matrices \mathcal{M}_N over a finite N -dimensional complex Hilbert space

$$(1) \quad \Lambda(\rho) := \sum_{i=1}^k \text{tr}(W_i \rho W_i^*) \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)},$$

where W_i and V_i , $i = 1, 2, \dots, k$ are linear operators in this Hilbert space. Note that Λ is not a linear operator. This operator can be seen as a version of an Iterated Function System (IFS). Namely, the $V_i(\cdot)V_i^* =: F_i(\cdot)$, $i = 1, 2, \dots, k$, play the role of the inverse branches (i.e., the dynamics on the configuration space of density matrices ρ) and the W_i play the role of the weights one can consider on the IFS. We suppose that for all ρ we have that $\sum_{i=1}^k \text{tr}(W_i \rho W_i^*) = 1$. Note that such trace preserving condition, for any normalized operator ρ (that is, with $\text{tr}(\rho) = 1$), is

Supported in part by CAPES and CNPq.

equivalent to the explicit condition $\sum_i W_i^* W_i = I$. We say that Λ is a normalized operator.

A family $W := \{W_i\}_{i=1,\dots,k}$ determines a Quantum Iterated Function System (QIFS) \mathcal{F}_W ,

$$\mathcal{F}_W = \{\mathcal{M}_N, F_i, W_i\}_{i=1,\dots,k}$$

Basic references on QIFS are [13] and [16]. We want to consider a new concept of entropy for stationary systems acting on density matrices which generalizes the usual one in Ergodic Theory. In our setting the V_i , $i = 1, 2, \dots, k$ are fixed (i.e. the dynamics of the inverse branches is fixed in the beginning) and we consider the different families W_i , $i = 1, 2, \dots, k$, (also with the attached corresponding eigendensity matrix ρ_W) as possible Jacobians of stationary probabilities.

Given a normalized family W_i , $i = 1, 2, \dots, k$, a natural definition of entropy is given by

$$(2) \quad h_V(W) = - \sum_{i=1}^k \frac{\text{tr}(W_i \rho_W W_i^*)}{\text{tr}(V_i \rho_W V_i^*)} \sum_{j=1}^k \text{tr}(W_j V_i \rho_W V_i^* W_j^*) \log \left(\frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)} \right)$$

where ρ_W denotes the barycenter of the unique invariant, attractive measure for the Markov operator \mathcal{V} associated to \mathcal{F}_W . We show that this generalizes the entropy of a Markov System.

We also want to present a concept of pressure for stationary systems acting on density matrices which generalizes the usual one in Ergodic Theory. In addition to the dynamics obtained by the V_i , which are fixed, a family of potentials H_i , $i = 1, 2, \dots, k$ induces a kind of Ruelle operator given by

$$(3) \quad \mathcal{L}_H(\rho) := \sum_{i=1}^k \text{tr}(H_i \rho H_i^*) V_i \rho V_i^*$$

We show that such operator admits an eigenvalue β and an associated eigenstate ρ_β , that is, one satisfying $\mathcal{L}_H(\rho_\beta) = \beta \rho_\beta$.

The natural generalization of the concept of pressure for a family H_i , $i = 1, 2, \dots, k$ is the problem of maximizing, on the possible normalized families W_i , $i = 1, 2, \dots, k$, the expression

$$(4) \quad h_V(W) + \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \text{tr}(W_j \rho_W W_j^*)$$

We show a relation between the eigendensity matrix ρ_β for the Ruelle operator and the set of W_i , $i = 1, 2, \dots, k$, which maximizes pressure. In the particular case that each of the V_i is unitary, $i = 1, 2, \dots, k$, the maximum value is $\log \beta$.

Our work is inspired by the results presented in [16] and [21]. We would like to thank these authors for supplying us with the corresponding references.

It is well-known that completely positive mappings (operators) acting on density matrices are of great importance in Quantum Computing. These operators can be written in the Stinespring-Kraus form (see section 12). Also a nice exposition on the interplay of Ergodic Theory and Quantum Information is presented in [4].

The initial part of our work aims to present some of the definitions and concepts that are not very well-known (at least for the general audience of people in Dynamical Systems), in a systematic way. We present the main basic definitions which are necessary to understand the theory. However, we do not have the intention of exhausting what is already known. We believe that the theoretical results presented here can be useful as a general tool to understand problems in Quantum Computing.

Several examples are presented in the text. We believe that this will help the reader to understand some of the main issues of the theory. In order to simplify the notation we will present most of our results for the case of matrices of order 2.

In sections 2 and 3 we present some basic definitions, examples and we show some preliminary relations of our setting to the classical Thermodynamic Formalism. In section 4 we present an eigenvalue problem for non-normalized Ruelle operators which will be required later. Some properties and concepts about density matrices and Ruelle operators are presented in sections 6 and 7. Sections 8 and 9 are dedicated to the introduction of some different kinds of entropy that were already known but do not have a stationary character. In section 10 we introduce the concept of stationary entropy for *measures* defined on the set of density matrices. In section 11 we compare this definition with the usual one for Markov Chains. Section 12 is dedicated to motivate the interest on pressure and the capacity-cost function. Section 13, 14, 15 and 16 are dedicated to the presentation of our main results on pressure, important inequalities, examples and its relation with the classical theory of Thermodynamic Formalism.

In [1] we present a general exposition (describing the setting we consider here) where we omit proofs, but provide many examples. We believe that paper will help to complement the present paper for the reader which is a newcomer in the area. We also present there some basic results concerning the discrete Wigner measure.

In [2] we propose a different concept of entropy which is also a generalization of the classical one. We also describe some properties of the Quantum Stochastic Process associated to the Quantum Iterated Function System.

This work is part of the thesis dissertation of C. F. Lardizabal in Prog. Pos-Grad. Mat. UFRGS (Brazil).

2. BASIC DEFINITIONS

Let $M_N(\mathbb{C})$ the set of complex matrices of order n . If $\rho \in M_N(\mathbb{C})$ then ρ^* denotes the transpose conjugate of ρ . A state (or vector) in \mathbb{C}^n will be denoted by ψ or $|\psi\rangle$, and the associated projection will be written $|\psi\rangle\langle\psi|$. Define

$$\mathcal{H}_N := \{\rho \in M_N(\mathbb{C}) : \rho^* = \rho\}$$

$$\mathcal{PH}_N := \{\rho \in \mathcal{H}_N : \langle\rho\psi, \psi\rangle \geq 0, \forall\psi \in \mathbb{C}^N\}$$

$$\mathcal{M}_N := \{\rho \in \mathcal{PH}_N : tr(\rho) = 1\}$$

$$\mathcal{P}_N := \{\rho \in \mathcal{H}_N : \rho = |\psi\rangle\langle\psi|, \psi \in \mathbb{C}^N, \langle\psi|\psi\rangle = 1\},$$

the space of hermitian, positive, density operators and pure states, respectively. Density operators are also called mixed states. If a quantum system can be in one of the states $\{\psi_1, \dots, \psi_k\}$ then a mixed state ρ will be written as

$$(5) \quad \rho = \sum_{i=1}^k p_i |\psi_i\rangle\langle\psi_i|$$

where the p_i are positive numbers with $\sum_i p_i = 1$.

Definition 1. Let $F_i : \mathcal{M}_N \rightarrow \mathcal{M}_N$, $p_i : \mathcal{M}_N \rightarrow [0, 1]$, $i = 1, \dots, k$ and such that $\sum_i p_i(\rho) = 1$. We call

$$(6) \quad \mathcal{F}_N = \{\mathcal{M}_N, F_i, p_i : i = 1, \dots, k\}$$

a **Quantum Iterated Function System (QIFS)**.

Definition 2. A QIFS is **homogeneous** if p_i and $F_i p_i$ are affine mappings, $i = 1, \dots, k$.

Suppose that the QIFS considered is such that there are V_i and W_i linear maps, $i = 1, \dots, k$, with $\sum_{i=1}^k W_i^* W_i = I$ such that

$$(7) \quad F_i(\rho) = \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)}$$

and

$$(8) \quad p_i(\rho) = \text{tr}(W_i \rho W_i^*)$$

Then we have that a QIFS is homogeneous if $V_i = W_i$, $i = 1, \dots, k$. Now we can define a Markov operator $\mathcal{V} : \mathcal{M}^1(\mathcal{M}_N) \rightarrow \mathcal{M}^1(\mathcal{M}_N)$,

$$(9) \quad (\mathcal{V}\mu)(B) = \sum_{i=1}^k \int_{F_i^{-1}(B)} p_i(\rho) d\mu(\rho),$$

where $\mathcal{M}^1(\mathcal{M}_N)$ denotes the space of probability measures over \mathcal{M}_N . We also define $\Lambda : \mathcal{M}_N \rightarrow \mathcal{M}_N$,

$$(10) \quad \Lambda(\rho) := \sum_{i=1}^k p_i(\rho) F_i(\rho)$$

The operator defined above has no counterpart in the classical Thermodynamic Formalism. We will also consider the operator defined on the space of density matrices ρ ,

$$(11) \quad \mathcal{L}(\rho) = \sum_{i=1}^k q_i(\rho) V_i \rho V_i^*.$$

If for all ρ we have $\sum_{i=1}^k q_i(\rho) = 1$, we say the operator is normalized. We are also interested in the non-normalized case. If the QIFS is homogeneous, then

$$(12) \quad \Lambda(\rho) = \sum_i V_i \rho V_i^*$$

Theorem 1. [21] *A mixed state $\hat{\rho}$ is Λ -invariant if and only if*

$$(13) \quad \hat{\rho} = \int_{\mathcal{M}_N} \rho d\mu(\rho),$$

for some \mathcal{V} -invariant measure μ .

We recall the definition of the integral above in section 5.

In order to define hyperbolic QIFS, one has to define a distance on the space of mixed states. For instance, we could choose one of the following:

$$(14) \quad D_1(\rho_1, \rho_2) = \sqrt{\text{tr}[(\rho_1 - \rho_2)^2]}$$

$$(15) \quad D_2(\rho_1, \rho_2) = \text{tr} \sqrt{(\rho_1 - \rho_2)^2}$$

$$(16) \quad D_3(\rho_1, \rho_2) = \sqrt{2\{1 - \text{tr}[(\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2}]\}},$$

the Hilbert-Schmidt, trace, and Bures distances, respectively. Such metrics generate the same topology on \mathcal{M}_N . Considering the space of mixed states with one of those metrics we can use a definition of hyperbolicity similar to the one used for IFS. That is, we say a QIFS is **hyperbolic** if the quantum maps F_i are contractions with respect to one of the distances on \mathcal{M}_N and if the maps p_i are Hölder-continuous and positive, see for instance [16].

Proposition 1. *If a QIFS (6) is homogeneous and hyperbolic then the associated Markov operator admits a unique invariant measure μ . Such invariant measure determines a unique Λ -invariant state $\rho \in \mathcal{M}_N$, given by (13).*

See [16], [21] for the proof.

3. EXAMPLES OF QIFS

Example 1. $\Omega = \mathcal{M}_N$, $k = 2$, $p_1 = p_2 = 1/2$, $G_1(\rho) = U_1 \rho U_1^*$, $G_2(\rho) = U_2 \rho U_2^*$. The normalized identity matrix $\rho_* = I/N$ is Λ -invariant, for any choice of unitary U_1 and U_2 . Note that we can write

$$(17) \quad \rho_* = \int_{\mathcal{M}_N} \rho d\mu(\rho)$$

where the measure μ , uniformly distributed over \mathcal{P}_N (the Fubini-Study metric), is \mathcal{V} -invariant.

◇

We recall that a mapping Λ is **completely positive** (CP) if $\Lambda \otimes I$ is positive for any extension of the Hilbert space considered $\mathcal{H}_N \rightarrow \mathcal{H}_N \otimes \mathcal{H}_E$. We know that every CP mapping which is trace-preserving can be represented (in a nonunique way) in the Stinespring-Kraus form

$$(18) \quad \Lambda(\rho) = \sum_{j=1}^k V_j \rho V_j^*, \quad \sum_{j=1}^k V_j^* V_j = 1,$$

where the V_i are linear operators. Moreover if we have $\sum_{j=1}^k V_j V_j^* = I$, then $\Lambda(I/N) = I/N$. This is the case if each of the V_i are normal.

We call a unitary trace-preserving CP map a **bistochastic map**. An example of such a mapping is

$$(19) \quad \Lambda_U(\rho) = \sum_{i=1}^k p_i U_i \rho U_i^*,$$

where the U_i are unitary operators and $\sum_i p_i = 1$. Note that if we write $F_i(\rho) = U_i \rho U_i^*$, then example 1 is part of this class of operators. For such operators we have that ρ_* is an invariant state for Λ_U and also that δ_{ρ_*} is invariant for the Markov operator P_U induced by this QIFS.

We will present a simple example of the kind of problems we are interested here, namely eigenvalues and eigendensity matrices. Let \mathcal{H}_N be a Hilbert space of dimension N . As before, let \mathcal{M}_N be the space of density operators on \mathcal{H}_N . A natural problem is to find fixed points for $\Lambda : \mathcal{M}_N \rightarrow \mathcal{M}_N$,

$$(20) \quad \Lambda(\rho) = \sum_{i=1}^k V_i \rho V_i^*$$

In order to simplify our notation we fix $N = 2$ and $k = 2$. Let

$$V_1 = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, \quad V_2 = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_2 & \rho_4 \end{pmatrix},$$

where V_1 and V_2 are invertible and ρ is a density operator. We would like to find ρ such that

$$(21) \quad V_1 \rho V_1^* + V_2 \rho V_2^* = \rho.$$

Example 2. Let

$$V_1 = e^{ik} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix}, \quad V_2 = e^{il} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & -\sqrt{1-p} \end{pmatrix},$$

where $k, l \in \mathbb{R}$, $p \in (0, 1)$. Then $V_1^* V_1 + V_2^* V_2 = I$. A simple calculation shows that $\rho_2 = 0$, and then

$$\rho = \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix}$$

is invariant to $\Lambda(\rho) = V_1 \rho V_1^* + V_2 \rho V_2^*$, for $q \in (0, 1)$.

◇

Now we make a few considerations about the Ruelle operator \mathcal{L} defined before. In particular, we show that Perron's classic eigenvalue problem is a particular case of the problem for the operator \mathcal{L} acting on matrices. Let

$$V_1 = \begin{pmatrix} p_{00} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & p_{01} \\ 0 & 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 & 0 \\ p_{10} & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 \\ 0 & p_{11} \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$$

Define

$$(22) \quad \mathcal{L}(\rho) = \sum_{i=1}^4 q_i(\rho) V_i \rho V_i^*$$

We have that $\mathcal{L}(\rho) = \rho$ implies $\rho_2 = 0$ and

$$(23) \quad a\rho_1 + b\rho_4 = \rho_1$$

$$(24) \quad c\rho_1 + d\rho_4 = \rho_4$$

where

$$a = q_1 p_{00}^2, \quad b = q_2 p_{01}^2, \quad c = q_3 p_{10}^2, \quad d = q_4 p_{11}^2$$

Solving (23) and (24) in terms of ρ_1 gives

$$\rho_1 = \frac{b}{1-a} \rho_4, \quad \rho_1 = \frac{1-d}{c} \rho_4$$

that is,

$$(25) \quad \frac{b}{1-a} = \frac{1-d}{c}$$

which is a restriction over the q_i . For simplicity we assume here that the q_i are constant. One can show that

$$(26) \quad \rho = \begin{pmatrix} \frac{q_2 p_{01}^2}{q_2 p_{01}^2 - q_1 p_{00}^2 + 1} & 0 \\ 0 & \frac{1 - q_1 p_{00}^2}{q_2 p_{01}^2 - q_1 p_{00}^2 + 1} \end{pmatrix} = \begin{pmatrix} \frac{1 - q_4 p_{11}^2}{1 - q_4 p_{11}^2 + q_3 p_{10}^2} & 0 \\ 0 & \frac{q_3 p_{10}^2}{1 - q_4 p_{11}^2 + q_3 p_{10}^2} \end{pmatrix}$$

Now let

$$P = \sum_i V_i = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix},$$

be a column-stochastic matrix. Let $\pi = (\pi_1, \pi_2)$ such that $P\pi = \pi$. Then

$$(27) \quad \pi = \left(\frac{p_{01}}{p_{01} - p_{00} + 1}, \frac{1 - p_{00}}{p_{01} - p_{00} + 1} \right)$$

Comparing (27) and (26) suggests that we should fix

$$(28) \quad q_1 = \frac{1}{p_{00}}, \quad q_2 = \frac{1}{p_{01}}, \quad q_3 = \frac{1}{p_{10}}, \quad q_4 = \frac{1}{p_{11}}$$

Then the nonzero entries of ρ are equal to the entries of π and therefore we associate the fixed point of P to the fixed point of some \mathcal{L} in a natural way. But note that such a choice of q_i is not unique, because

$$(29) \quad q_2 = \frac{1 - q_1 p_{00}^2}{p_{01} p_{10}}, \quad q_4 = \frac{1 - q_3 p_{10} p_{01}}{p_{11}^2},$$

for any q_1, q_3 also produces ρ with nonzero coordinates equal to the coordinates of π . We also note that the above calculations can be made by taking the V_i matrices with nonzero entries equal to $\sqrt{p_{ij}}$ instead of p_{ij} .

Now we consider the following problem. Let

$$V_1 = \begin{pmatrix} h_{00} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & h_{01} \\ 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 \\ h_{10} & 0 \end{pmatrix}$$

$$V_4 = \begin{pmatrix} 0 & 0 \\ 0 & h_{11} \end{pmatrix}, \quad H = \sum_i V_i, \quad \rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$$

Define

$$(30) \quad \mathcal{L}(\rho) = \sum_{i=1}^4 q_i V_i \rho V_i^*,$$

where $q_i \in \mathbb{R}$. Assume that $h_{ij} \in \mathbb{R}$, so we want to obtain λ such that $\mathcal{L}(\rho) = \lambda\rho$, $\lambda \neq 0$, and λ is the largest eigenvalue. With a few calculations we obtain $\rho_2 = \rho_3 = 0$,

$$q_1 h_{00}^2 \rho_1 + q_2 h_{01}^2 \rho_4 = \lambda \rho_1$$

$$q_3 h_{10}^2 \rho_1 + q_4 h_{11}^2 \rho_4 = \lambda \rho_4$$

that is,

$$(31) \quad a\rho_1 + b\rho_4 = \lambda\rho_1$$

$$(32) \quad c\rho_1 + d\rho_4 = \lambda\rho_4,$$

with

$$a = q_1 h_{00}^2, \quad b = q_2 h_{01}^2, \quad c = q_3 h_{10}^2, \quad d = q_4 h_{11}^2$$

Therefore

$$\rho = \begin{pmatrix} \frac{\lambda-d}{c}\rho_4 & 0 \\ 0 & \rho_4 \end{pmatrix} = \begin{pmatrix} \frac{b}{\lambda-a}\rho_4 & 0 \\ 0 & \rho_4 \end{pmatrix}$$

and

$$\frac{\lambda-d}{c} = \frac{b}{\lambda-a}$$

Solving for λ , we obtain the eigenvalues

$$\lambda = \frac{a+d}{2} \pm \frac{\zeta}{2} = \frac{a+d}{2} \pm \frac{\sqrt{(d-a)^2 + 4bc}}{2}$$

$$= \frac{1}{2} \left(q_1 h_{00}^2 + q_4 h_{11}^2 \pm \sqrt{(q_4 h_{11}^2 - q_1 h_{00}^2)^2 + 4q_2 q_3 h_{01}^2 h_{10}^2} \right),$$

where

$$\zeta = \sqrt{(d-a)^2 + 4bc} = \sqrt{(q_4 h_{11}^2 - q_1 h_{00}^2)^2 + 4q_2 q_3 h_{01}^2 h_{10}^2}$$

and the associated eigenfunctions

$$\rho = \begin{pmatrix} \frac{a-d\pm\zeta}{2c}\rho_4 & 0 \\ 0 & \rho_4 \end{pmatrix} = \begin{pmatrix} \frac{2b}{d-a\pm\zeta}\rho_4 & 0 \\ 0 & \rho_4 \end{pmatrix}$$

But $\rho_1 + \rho_4 = 1$ so we obtain

$$\rho = \begin{pmatrix} \frac{a-d\pm\zeta}{a-d\pm\zeta+2c} & 0 \\ 0 & \frac{2c}{a-d\pm\zeta+2c} \end{pmatrix}$$

$$(33) \quad = \begin{pmatrix} \frac{q_1 h_{00}^2 - q_4 h_{11}^2 \pm \zeta}{q_1 h_{00}^2 - q_4 h_{11}^2 \pm \zeta + 2q_3 h_{10}^2} & 0 \\ 0 & \frac{2q_3 h_{10}^2}{q_1 h_{00}^2 - q_4 h_{11}^2 \pm \zeta + 2q_3 h_{10}^2} \end{pmatrix}$$

that is,

$$(34) \quad \rho = \begin{pmatrix} \frac{-2b}{a-2b-d\mp\zeta} & 0 \\ 0 & \frac{a-d\mp\zeta}{a-2b-d\mp\zeta} \end{pmatrix} \\ = \begin{pmatrix} \frac{-2q_2 h_{01}^2}{q_1 h_{00}^2 - 2q_2 h_{01}^2 - q_4 h_{11}^2 \mp \zeta} & 0 \\ 0 & \frac{q_1 h_{00}^2 - q_4 h_{11}^2 \mp \zeta}{q_1 h_{00}^2 - 2q_2 h_{01}^2 - q_4 h_{11}^2 \mp \zeta} \end{pmatrix}$$

Therefore we obtained that $\rho_1, \rho_4, q_1, \dots, q_4, \lambda$ are implicit solutions for the set of equations (31)-(32). Recall that in this case we obtained $\rho_2 = \rho_3 = 0$.

Now we consider the problem of finding the eigenvector associated to the dominant eigenvalue of H . The eigenvalues are

$$\lambda = \frac{1}{2} \left(h_{00} + h_{11} \pm \sqrt{(h_{00} - h_{11})^2 + 4h_{01}h_{10}} \right)$$

Then we can find v such that $Hv = \lambda v$ from the set of equations

$$(35) \quad h_{00}v_1 + h_{01}v_2 = \lambda v_1$$

$$(36) \quad h_{10}v_1 + h_{11}v_2 = \lambda v_2$$

which determine v_1, v_2, λ implicitly. Note that if we set

$$(37) \quad q_1 = \frac{1}{p_{00}}, \quad q_2 = \frac{1}{p_{01}}, \quad q_3 = \frac{1}{p_{10}}, \quad q_4 = \frac{1}{p_{11}}$$

we have that the set of equations (31)-(32) and (35)-(36) are the same. Hence we conclude that Perron's classic eigenvalue problem is a particular case of the problem for \mathcal{L} acting on matrices. ◇

A different analysis in the quantum setting which is related to Perron's theorem is presented in [6].

4. A THEOREM ON EIGENVALUES FOR THE RUELLE OPERATOR

The following proposition is inspired in [18]. We say that a hermitian operator $P : V \rightarrow V$ on a Hilbert space $(V, \langle \cdot, \cdot \rangle)$ is **positive** if $\langle Pv, v \rangle \geq 0$, for all $v \in V$, denoted $P \geq 0$. Consider the positive operator $\mathcal{L}_{W,V} : \mathcal{PH}_N \rightarrow \mathcal{PH}_N$,

$$(38) \quad \mathcal{L}_{W,V}(\rho) := \sum_{i=1}^k \text{tr}(W_i \rho W_i^*) V_i \rho V_i^*.$$

We point out that this operator is completely general. In an analogy with the classical case we can say it corresponds to the general Perron Theorem for positive matrices (having positive eigenvalues which can be bigger or smaller than one), by the other hand the setting described in [16], [21] "basically" considers the analogous case of the Perron Theorem for stochastic matrices.

We need a result in this form in order to better understand the Pressure problem which will be described later.

Proposition 2. *There exists $\rho \in \mathcal{M}_N$ and $\beta > 0$ such that $\mathcal{L}_{W,V}(\rho) = \beta\rho$. The value β is obtained explicitly: $\beta = \text{tr}(\mathcal{L}_{W,V}(\rho))$.*

Proof Define $\mathcal{L}_n : \mathcal{M}_N \rightarrow \mathcal{M}_N$,

$$\mathcal{L}_n(\rho) := \frac{\mathcal{L}_{W,V}(\rho + \frac{I}{n})}{\text{tr}(\mathcal{L}_{W,V}(\rho + \frac{I}{n}))}, \quad n \geq 1$$

The operator above is well defined. In fact, note that $\mathcal{L}_{W,V}(\rho)$, $W_j W_j^*$, $V_j V_j^*$ are positive for all j . Then

$$\begin{aligned} \text{tr} \left[\sum_i \text{tr} \left(W_i \left(\rho + \frac{I}{n} \right) W_i^* \right) V_i \left(\rho + \frac{I}{n} \right) V_i^* \right] &= \sum_i \text{tr} \left(W_i \left(\rho + \frac{I}{n} \right) W_i^* \right) \text{tr} \left(V_i \left(\rho + \frac{I}{n} \right) V_i^* \right) \\ &= \sum_i \text{tr} \left(W_i \rho W_i^* + \frac{1}{n} W_i W_i^* \right) \text{tr} \left(V_i \rho V_i^* + \frac{1}{n} V_i V_i^* \right) \geq \\ &\geq \sum_i \text{tr} \left(W_i \rho W_i^* \right) \text{tr} \left(V_i \rho V_i^* \right) = \text{tr}(\mathcal{L}_{W,V}) \end{aligned}$$

We know that for any positive operator $P \neq 0$, if $\{v_1, \dots, v_N\}$ is a orthonormal base for \mathcal{H}_N , then

$$\text{tr}(P) = \sum_{i=1}^N \langle P v_i, v_i \rangle > 0$$

Therefore, $\text{tr}(\mathcal{L}_{W,V}(\rho + \frac{I}{n})) > 0$, $n \geq 1$. Hence $\mathcal{L}_n(\rho)$ is well defined.

We know that \mathcal{M}_N is compact and convex, so we can apply Schauder's theorem for each of the mappings \mathcal{L}_n , $n \geq 1$ and get $\rho_n \in \mathcal{M}_N$ such that

$$\mathcal{L}_n(\rho_n) = \rho_n \quad \Rightarrow \quad \mathcal{L}_{W,V}(\rho_n + \frac{I}{n}) = \beta_n \rho_n, \quad n \geq 1$$

where

$$\beta_n := \text{tr}(\mathcal{L}_{W,V}(\rho_n + \frac{I}{n}))$$

By the compacity of \mathcal{M}_N , we can choose a point $\rho \in \mathcal{M}_N$ which is limit of the sequence $\{\rho_n\}_{n=1}^\infty$ and then, by continuity, $\mathcal{L}_{W,V}(\rho) = \beta\rho$, where $\beta = \text{tr}(\mathcal{L}_{W,V}(\rho))$. Also, note that $\beta \geq 0$, because if $\{v_1, \dots, v_N\}$ is a orthonormal base of \mathcal{H}_N ,

$$\text{tr}(\mathcal{L}_{W,V}(\rho)) = \sum_{i=1}^N \langle \mathcal{L}_{W,V}(\rho) v_i, v_i \rangle \geq 0,$$

since $\mathcal{L}_{W,V}(\rho)$ is positive, and the inequality will be equal to zero if and only if $\mathcal{L}_{W,V}(\rho)$ is the zero operator. Hence, we proved that there exists $\rho \in \mathcal{M}_N$ and $\beta > 0$ such that $\mathcal{L}_{W,V}(\rho) = \beta\rho$. □

5. VECTOR INTEGRALS AND BARYCENTERS

We recall here a few basic definitions. For more details, see [16] and [21]. Let X be a metric space. Let $(V, +, \cdot)$ be a real vector space, and τ a topology on V . We say that $(V, +, \cdot; \tau)$ is a topological vector space if it is Hausdorff and if the operations $+$ and \cdot are continuous. For instance, in the context of density matrices, we will consider V as the space of hermitian operators \mathcal{H}_N and X will be the space of density matrices \mathcal{M}_N .

Definition 3. *Let (X, Σ) be a measurable space, let $\mu \in M(X)$, let $(V, +, \cdot; \tau)$ be a locally convex space and let $f : X \rightarrow V$. We say that $x \in V$ is the **integral** of f in X , denoted by*

$$(39) \quad x := \int_X f d\mu$$

if

$$(40) \quad \Psi(x) = \int_X \Psi \circ f d\mu,$$

for all $\Psi \in V^*$.

It is known that if we have a compact metric space X , V is a locally convex space and $f : X \rightarrow V$ is a continuous function such that $\overline{\text{co}}f(X)$ is compact then the integral of f in X exists and belongs to $\overline{\text{co}}f(X)$. We will also use the following well-known result, the barycentric formula:

Proposition 3. [22] *Let V be a locally convex space, let $E \subset V$ be a complete, convex and bounded set, and $\mu \in M^1(E)$. Then there is a unique $x \in E$ such that*

$$l(x) = \int_E l d\mu,$$

for all $l \in V^*$.

In the context of QIFS, we can take $V = E = \mathcal{M}_N$.

6. EXAMPLE: DENSITY MATRICES

In this section we briefly review how the constructions of the previous section adjust to the case of density matrices. Define $V := \mathcal{H}_N$, $V^+ := \mathcal{PH}_N$ (note that such space is a convex cone), and let the partial order \leq on \mathcal{PH}_N be $\rho \leq \psi$ if and only if $\psi - \rho \geq 0$, i.e., if $\psi - \rho$ is positive. Then

$$(V, V^+, e) = (\mathcal{H}_N, \mathcal{PH}_N, \text{tr}),$$

is a regular state space [21]. Also, the set B of unity trace in V^+ is, of course, the space of density matrices, so $B = \mathcal{M}_N$.

Let $Z \subset V^*$ be a nonempty vector subspace of V^* . The smallest topology in V such that every functional defined in Z is continuous on that topology, denoted by $\sigma(V, Z)$, turns V into a locally convex space. In particular, $\sigma(V, V^*)$ is the weak topology in V . If $(V, \|\cdot\|)$ is a normed space, then $\sigma(V^*, V)$ is called a

weak* topology in V^* (we identify V with a subspace of V^{**}). We also have that $(C, \tau) = (\mathcal{PH}_N, \tau)$, where τ is the weak* topology (and which is equal to the Euclidean, see [21]) is a metrizable compact structure. In this case we have that $B_C = B \cap C = \mathcal{M}_N$.

Definition 4. A Markov operator for probability measures is an operator $P : M^1(X) \rightarrow M^1(X)$ such that

$$(41) \quad P(\lambda\mu_1 + (1 - \lambda)\mu_2) = \lambda P\mu_1 + (1 - \lambda)P\mu_2,$$

for $\mu_1, \mu_2 \in M^1(X)$, $\lambda \in (0, 1)$.

An example of such operator is the one which we have defined before and we denote it by $\mathcal{V} : M^1(\mathcal{M}_N) \rightarrow M^1(\mathcal{M}_N)$,

$$(42) \quad (\mathcal{V}\nu)(B) = \sum_{i=1}^k \int_{F_i^{-1}(B)} p_i d\nu$$

We call it the Markov operator induced by the QIFS $\mathcal{F} = \{\mathcal{M}_N, F_i, p_i\}_{i=1, \dots, k}$. Define

$$m_b(X) := \{f : X \rightarrow \mathbb{R} : f \text{ is bounded, measurable}\}$$

Then define $\mathcal{U} : m_b(X) \rightarrow m_b(X)$,

$$(43) \quad (\mathcal{U}f)(x) := \sum_{i=1}^k p_i(x) f(F_i(x))$$

Proposition 4. [21] Let $f \in m_b(X)$ and $\mu \in M^1(X)$, then

$$(44) \quad \langle f, \mathcal{V}\mu \rangle = \langle \mathcal{U}f, \mu \rangle = \sum_{i=1}^k \int p_i(f \circ F_i) d\mu,$$

where $\langle f, \mu \rangle$ denotes the integral of f with respect to μ .

Definition 5. An operator $Q : V^+ \rightarrow V^+$ is **submarkovian** if

- (1) $Q(x + y) = Q(x) + Q(y)$
- (2) $Q(\alpha x) = \alpha Q(x)$
- (3) $\|Q(x)\| \leq \|x\|$,

for all $x, y \in V^+$, $\alpha > 0$.

Every submarkovian operator $Q : V^+ \rightarrow V^+$ can be extended in a unique way to a positive linear contraction on V , see [21].

Definition 6. Let $P : V^+ \rightarrow V^+$ a Markov operator and let $P_i : V^+ \rightarrow V^+$, $i = 1, \dots, k$ be submarkovian operators such that $P = \sum_i P_i$. We say that $(P, \{P_i\}_{i=1}^k)$ is a **Markov pair**.

From [21], we know that there is a 1-1 correspondence between homogeneous IFS and Markov pairs.

7. SOME LEMMAS FOR IFS

We want to understand the structure of $\Lambda : \mathcal{M}_N \rightarrow \mathcal{M}_N$,

$$(45) \quad \Lambda(\rho) := \sum_{i=1}^k p_i F_i = \sum_{i=1}^k \text{tr}(W_i \rho W_i^*) \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)},$$

where V_i, W_i are linear, $\sum_i W_i^* W_i = I$. Such operator is associated in a natural way to an IFS which is not homogeneous. In this section we state a few useful properties which are relevant for our study. The following lemmas hold for any IFS, except for lemma 3, where the proof presented here is valid only for homogeneous IFS.

Lemma 1. *Let $\{X, F_i, p_i\}_{i=1, \dots, k}$ be an IFS, Ψ a linear functional on X . Then $\mathcal{U} \circ \Psi = \Psi \circ \Lambda$, where \mathcal{U} is given by (43).*

Proof We have

$$(\mathcal{U}\Psi)(x) = \sum_i p_i(x) \Psi(F_i(x)) = \Psi\left(\sum_i p_i(x) F_i(x)\right) = \Psi(\Lambda(x))$$

□

Corollary 1. *Let $\mathcal{F} = (X, F_i, p_i)_{i=1, \dots, k}$ be an IFS and let $\rho_0 \in X$. Then $\Lambda(\rho_0) = \rho_0$ if and only if $\mathcal{U}(\Psi(\rho_0)) = \Psi(\rho_0)$, for all Ψ linear functional.*

Proof Suppose that $\mathcal{L}(\rho_0) = \rho_0$. Then

$$\mathcal{U}(\Psi(\rho_0)) = \sum_i p_i(\rho_0) \Psi(F_i(\rho_0)) = \Psi\left(\sum_i p_i(\rho_0) F_i(\rho_0)\right) = \Psi(\Lambda(\rho_0)) = \Psi(\rho_0)$$

Conversely, if $\mathcal{U}(\Psi(\rho_0)) = \Psi(\rho_0)$, then

$$\Psi(\Lambda(\rho_0)) = \mathcal{U}(\Psi(\rho_0)) = \Psi(\rho_0)$$

□

Lemma 2. *Let $\mathcal{F} = \{X, F_i, p_i\}_{i=1, \dots, k}$ be an IFS.*

- (1) *Let $\rho_0 \in X$ such that $F_i(\rho_0) = \rho_0$, $i = 1, \dots, k$. Then $\mathcal{V}\delta_{\rho_0} = \delta_{\rho_0}$.*
- (2) *Let $\rho_0 \in X$ such that $\mathcal{V}\delta_{\rho_0} = \delta_{\rho_0}$, then $\Lambda(\rho_0) = \rho_0$.*

Proof 1. We have

$$\begin{aligned} \mathcal{V}\delta_{\rho_0}(B) &= \sum_{i=1}^k \int_{F_i^{-1}(B)} p_i d\delta_{\rho_0} = \sum_{i=1}^k \int p_i(\rho) 1_B(F_i(\rho)) d\delta_{\rho_0} \\ &= \sum_{i=1}^k p_i(\rho_0) 1_B(F_i(\rho_0)) = \sum_{i=1}^k p_i(\rho_0) 1_B(\rho_0) = \delta_{\rho_0}(B) \end{aligned}$$

2. Let Ψ be a linear functional. Then

$$\begin{aligned} \Psi(\Lambda(\rho_0)) &= \mathcal{U}(\Psi(\rho_0)) = \int \mathcal{U}(\Psi(\rho)) d\delta_{\rho_0} = \int \Psi(\rho) d\mathcal{V}\delta_{\rho_0} \\ &= \int \Psi(\rho) d\delta_{\rho_0} = \Psi(\rho_0) \end{aligned}$$

□

Lemma 3. *Let $\{X, F_i, p_i\}_{i=1, \dots, k}$ be a homogeneous IFS, $\Lambda = \sum_i p_i F_i$.*

- (1) Let ρ_ν be the barycenter of a probability measure ν . Then $\Lambda(\rho_\nu)$ is the barycenter of $\mathcal{V}\nu$, where \mathcal{V} is the associated Markov operator.
- (2) Let μ be an invariant probability measure for \mathcal{V} . Then the barycenter of μ , denoted by ρ_μ , is a fixed point of Λ .

Proof 1. We have, for Ψ linear functional,

$$\Psi(\Lambda(\rho_\nu)) = \int \Psi(\Lambda(\rho))d\nu = \int \mathcal{U} \circ \Psi d\nu = \int \Psi d\mathcal{V}\nu$$

2. By lemma (1), we have

$$\Psi(\Lambda(\rho_\mu)) = \mathcal{U} \circ \Psi(\rho_\mu) = \int \mathcal{U} \circ \Psi d\mu = \int \Psi d\mathcal{V}\mu = \int \Psi d\mu = \Psi(\rho_\mu),$$

where the fact that $\mathcal{U} \circ \Psi$ is linear follows from the homogeneity of \mathcal{F} . □

In order to prove uniqueness in item (2) above it would be necessary to assume hyperbolicity [20]. It is known that without this hypothesis even in the classical case (for transformations for instance) it can happen the phenomena of phase transition (two or more probabilities which are solutions) [23] [15]. The present setting contains the classical case and therefore in general there is no uniqueness.

Example 3. Let $k = N = 2$,

$$V_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & -\frac{3\sqrt{2}}{4} \\ -\frac{3\sqrt{2}}{2} & 0 \end{pmatrix},$$

$W_1 = (1/2)I$, $W_2 = (\sqrt{3}/2)I$. Then

$$\begin{aligned} \Lambda(\rho) &= \sum_i p_i(\rho)F_i(\rho) = \sum_i \text{tr}(W_i\rho W_i^*) \frac{V_i\rho V_i^*}{\text{tr}(V_i\rho V_i^*)} \\ &= \frac{1}{4}V_1\rho V_1^* + \frac{3}{4} \frac{V_2\rho V_2^*}{\text{tr}(V_2\rho V_2^*)} = \frac{1}{4}V_1\rho V_1^* + \frac{3}{4} \frac{V_2\rho V_2^*}{(\frac{9}{8} + \frac{27}{8}\rho_1)} \end{aligned}$$

induces an IFS and it is such that $\rho_0 = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$ is a fixed point, with $F_1(\rho_0) = F_2(\rho_0) = \rho_0$. We can apply lemma 2 and conclude that δ_{ρ_0} is an invariant measure for the Markov operator \mathcal{V} associated to the IFS determined by p_i and F_i . ◇

The following lemma, a simple variation from results seen in [21], specifies a condition we need in order to obtain a fixed point for Λ from a certain measure which is invariant for the Markov operator \mathcal{V} .

Lemma 4. Let $\{\mathcal{M}_N, F_i, p_i\}_{i=1, \dots, k}$ be an IFS which admits an attractive invariant measure μ for \mathcal{V} . Then $\lim_{n \rightarrow \infty} \Lambda^n(\rho_0) = \rho_\mu$, for every $\rho_0 \in \mathcal{M}_N$, where ρ_μ is the barycenter of μ .

Proof Let $\rho_0 \in \mathcal{M}_N$. Then

$$\Psi(\Lambda^n(\rho_0)) = \mathcal{U}^n(\Psi(\rho_0)) = \int \mathcal{U}^n(\Psi(\rho)) d\delta_{\rho_0} = \int \Psi(\rho) d\mathcal{V}^n \delta_{\rho_0}$$

so $\Psi(\Lambda^n(\rho_0)) \rightarrow \int \Psi(\rho) d\mu = \Psi(\rho_\mu)$, as $n \rightarrow \infty$, for all Ψ linear functional. Hence, $\Lambda^n(\rho_0) \rightarrow \rho_\mu$ as $n \rightarrow \infty$, for all $\rho_0 \in \mathcal{M}_N$. \square

In lemma 4, we have a general QIFS and an attractive invariant μ , then μ is the unique invariant measure, an easy consequence of attractivity [21]. In general, we will be interested in QIFS which has an attractive invariant measure. This will follow if we assume hyperbolicity.

8. INTEGRAL FORMULAE FOR THE ENTROPY OF IFS

Part of the results we present here in this section are variations of results presented in [21]. Let (X, d) be a complete separable metric space. Let (V, V^+, e) be a complete state space, $B = \{x \in V^+ : e(x) = 1\}$ and $\mathcal{F} = (X, F_i, p_i)_{i=1, \dots, k}$ the homogeneous IFS induced by the Markov pair $(\Lambda, \{\Lambda_i\}_{i=1}^k)$. Now define $I_k := \{1, \dots, k\}$. Let $n \in \mathbb{N}$, $\iota \in I_k^n$, $i \in I_k$. Define $F_{i\iota} := F_i \circ F_\iota$ and

$$(46) \quad p_{i\iota}(x) = \begin{cases} p_i(F_\iota x) p_\iota(x) & \text{if } p_\iota(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 5. *Let $n \in \mathbb{N}$, $f \in m_b(X)$, $x \in X$. Then*

$$(\mathcal{U}^n f)(x) = \sum_{\iota \in I_k^n} p_\iota(x) f(F_\iota(x))$$

Proposition 6. *Let $x \in B$, $n \in \mathbb{N}$. Then*

$$\Lambda^n(x) = \sum_{\iota \in I_k^n} p_\iota(x) F_\iota(x).$$

Proposition 7. *Let \mathcal{F} be an IFS and let $g : B \rightarrow \mathbb{R}$. Then for $n \in \mathbb{N}$,*

- (1) *If g is concave (resp. convex, affine) then $\mathcal{U}^n g \leq g \circ \Lambda^n$ (resp. $\mathcal{U}^n g \geq g \circ \Lambda^n$, $\mathcal{U}^n g = g \circ \Lambda^n$).*
- (2) *If \bar{x} is a fixed point for Λ then the sequence $(\mathcal{U}^n g)(\bar{x})_{n \in \mathbb{N}}$ is decreasing (resp. increasing, constant) if g is concave (resp. convex, affine).*

Also suppose that \mathcal{F} is homogeneous. Then

- (3) *If g is concave (resp. convex, affine), then $\mathcal{U}g$ is concave (resp. convex, affine).*

We recall some well-known definitions and results. Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$(47) \quad \eta(x) = \begin{cases} -x \log x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then the Shannon-Boltzmann entropy function is $h : X \rightarrow \mathbb{R}^+$,

$$(48) \quad h(x) := \sum_{i=1}^k \eta(p_i(x))$$

Let $n \in \mathbb{N}$. Define the **partial entropy** $H_n : X \rightarrow \mathbb{R}^+$ as

$$(49) \quad H_n(x) := \sum_{\iota \in I_k^n} \eta(p_\iota(x)),$$

for $n \geq 1$ and $H_0(x) := 0$, $x \in X$. Define, for $x \in X$,

$$(50) \quad \bar{\mathcal{H}}(x) := \limsup_{n \rightarrow \infty} \frac{1}{n} H_n(x), \quad \underline{\mathcal{H}}(x) := \liminf_{n \rightarrow \infty} \frac{1}{n} H_n(x),$$

the **upper** and **lower** entropy on x . If such limits are equal, we call its common value the **entropy on x** , denoted by $\mathcal{H}(x)$.

Denote by $M^\mathcal{V}(X)$ the set of \mathcal{V} -invariant probability measures on X . Let $\mu \in M^\mathcal{V}(X)$. The **partial entropy of the measure** μ is defined by

$$(51) \quad H_n(\mu) := \sum_{\iota \in I_k^n} \eta(\langle p_\iota, \mu \rangle),$$

for $n \geq 1$ and $H_0(\mu) := 0$.

Proposition 8. *Let $\mu \in M^\mathcal{V}(X)$. Then the sequences $(\frac{1}{n} H_n(\mu))_{n \in \mathbb{N}}$ and $(H_{n+1}(\mu) - H_n(\mu))_{n \in \mathbb{N}}$ are nonnegative, decreasing, and have the same limit.*

We denote the common limit of the sequences mentioned in the proposition above as $\mathcal{H}(\mu)$ and we call it the **entropy of the measure** μ , i.e.,

$$(52) \quad \mathcal{H}(\mu) := \lim_{n \rightarrow \infty} \frac{1}{n} H_n(\mu) = \lim_{n \rightarrow \infty} (H_{n+1}(\mu) - H_n(\mu))$$

The following result gives us an integral formula for entropy, and also a relation between the entropies defined before. We write $S(\mu) := M^\mathcal{V}(X) \cap \text{Lim}(\mathcal{V}^n \mu)_{n \in \mathbb{N}}$, where $\text{Lim}(\mathcal{V}^n \mu)_{n \in \mathbb{N}}$ is the convex hull of the set of accumulation points of $(\mathcal{V}^n \mu)_{n \in \mathbb{N}}$, and $S_{\mathcal{F}}(\mu)$ is the set $S(\mu)$ associated to the Markov operator induced by the IFS \mathcal{F} . For the definition of compact structure and (C, τ) -continuity, see [21].

Theorem 2. [21] *(Integral formula for entropy of homogeneous IFS, compact case).* *Let (C, τ) be a metrizable compact structure (V, V^+, e) such that $(\Lambda, \{\Lambda_i\}_{i=1}^k)$ is (C, τ) -continuous. Assume that $\rho_0 \in B_C := B \cap C$ is such that $\Lambda(\rho_0) = \rho_0$. Then*

$$\mathcal{H}(\rho_0) = \mathcal{H}(\nu) = \int_X h d\nu$$

for each $\nu \in S_{\mathcal{F}_C}(\delta_{\rho_0})$, where \mathcal{F}_C is the IFS \mathcal{F} restricted to (B_C, τ) .

The analogous result for hyperbolic IFS is the following.

Theorem 3. [21] *Let $\mathcal{F} = (X, F_i, p_i)_{i=1, \dots, k}$ be a hyperbolic IFS, $x \in X$, $\mu \in M^1(X)$ an attractive invariant measure for \mathcal{F} . Then*

$$\mathcal{H}(x) = \lim_{n \rightarrow \infty} (H_{n+1}(x) - H_n(x))$$

and

$$\mathcal{H}(x) = \mathcal{H}(\mu) = \int_X h d\mu.$$

9. SOME CALCULATIONS ON ENTROPY

Let U be a unitary matrix of order mn acting on $\mathcal{H}_m \otimes \mathcal{H}_n$. Its Schmidt decomposition is

$$U = \sum_{i=1}^K \sqrt{q_i} V_i^A \otimes V_i^B, \quad K = \min\{m^2, n^2\}$$

The operators V_i^A and V_i^B act on certain Hilbert spaces \mathcal{H}_m and \mathcal{H}_n , respectively. We also have that $\sum_{i=1}^K q_i = 1$. Let $\sigma = \rho_A \otimes \rho_*^B = \rho_A \otimes I_n/n$ and define

$$\Lambda(\rho_A) := \text{tr}_B(U\sigma U^*) = \sum_{i=1}^K q_i V_i^A \rho_A V_i^{A*}$$

Above, recall that the partial trace is

$$\text{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) := |a_1\rangle\langle a_2| \text{tr}(|b_1\rangle\langle b_2|)$$

where $|a_1\rangle$ and $|a_2\rangle$ are vectors on the state space of A and $|b_1\rangle$ and $|b_2\rangle$ are vectors on the state space of B . The trace on the right side is the usual trace on B . A calculation shows that if $\rho_*^A = I_m/m$, then $\Lambda(\rho_*^A) = \rho_*^A$ and so Λ is such that $\Lambda(I_m/m) = I_m/m$ and Λ is trace preserving.

Let \mathcal{F} be the homogeneous IFS associated to the V_i^A , that is, $p_i(\rho) = \text{tr}(q_i V_i^A \rho V_i^{A*})$, $F_i(\rho) = (q_i V_i^A \rho V_i^{A*}) / \text{tr}(q_i V_i^A \rho V_i^{A*})$ and let ρ_0 be a fixed point of $\Lambda = \sum_i p_i F_i$. Following [21], we have that ρ_0 is the barycenter of $\mathcal{V}^n \delta_{\rho_0}$, $n \in \mathbb{N}$. By theorem 2, we can calculate the entropy of such IFS. In this case we have

$$(53) \quad \mathcal{H}(\rho_0) = \mathcal{H}(\nu) = \int_{\mathcal{M}_N} h d\nu,$$

where $\nu \in M^{\mathcal{V}}(X) \cap \text{Lim}(\mathcal{V}^n \delta_{\rho_0})_{n \in \mathbb{N}}$.

◇

Let $\mathcal{F} = (\mathcal{M}_N, F_i, p_i)_{i=1, \dots, k}$ be an IFS, $\Lambda(\rho) = \sum_i p_i F_i$. Let \mathcal{U} be the conjugate of \mathcal{V} . By proposition 5,

$$(\mathcal{U}^n h)(\rho) = \sum_{\iota \in I_k^n(\rho)} p_\iota(\rho) h(F_\iota(\rho))$$

and since $h(\rho) = \sum_{j=1}^k \eta(p_j(\rho))$, we have, for $\iota = (i_1, \dots, i_n)$, and every $\rho_0 \in \mathcal{M}_N$,

$$(54) \quad \int_{\mathcal{M}_N} h d\mathcal{V}^n \delta_{\rho_0} = \int_{\mathcal{M}_N} \mathcal{U}^n h d\delta_{\rho_0}$$

$$(55) \quad = - \int_{\mathcal{M}_N} \sum_{\iota \in I_k^n} p_\iota(\rho) \sum_{j=1}^k p_j(F_\iota(\rho)) \log p_j(F_\iota(\rho)) d\delta_\rho$$

$$(56) \quad = - \sum_{\iota \in I_k^n} p_\iota(\rho_0) \sum_{j=1}^k p_j(F_\iota(\rho_0)) \log p_j(F_\iota(\rho_0))$$

$$(57) \quad = - \sum_{\iota \in I_k^n} p_{i_1}(\rho_0) p_{i_2}(F_{i_1}\rho_0) \cdots p_{i_n}(F_{i_{n-1}}(F_{i_{n-2}}(\cdots(F_{i_1}\rho_0)))) \times$$

$$(58) \quad \times \sum_{j=1}^k p_j(F_{i_n}(F_{i_{n-1}}(\cdots(F_{i_1}\rho_0)))) \log p_j(F_{i_n}(F_{i_{n-1}}(\cdots(F_{i_1}\rho_0)))) = (\mathcal{U}^n h)(\rho_0)$$

Suppose $\Lambda(\rho_0) = \rho_0$. We have by proposition 7, since h is concave, that $(\mathcal{U}^n h)_{n \in \mathbb{N}}$ is decreasing, $\mathcal{U}^n h \leq h \circ \Lambda^n$ and so

$$(59) \quad \int_{\mathcal{M}_N} h d\mathcal{V}^n \delta_{\rho_0} \leq h(\Lambda^n(\rho_0)) = h(\rho_0),$$

for every n .

10. AN EXPRESSION FOR A STATIONARY ENTROPY

In this section we present a definition of entropy which captures a stationary behavior. Let H be a hermitian operator and V_i , $i = 1, \dots, k$ linear operators. We can define the dynamics $F_i : \mathcal{M}_N \rightarrow \mathcal{M}_N$:

$$(60) \quad F_i(\rho) := \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)}$$

Let W_i , $i = 1, \dots, k$ be linear and such that $\sum_{i=1}^k W_i^* W_i = I$. This determines functions $p_i : \mathcal{M}_N \rightarrow \mathbb{R}$,

$$(61) \quad p_i(\rho) := \text{tr}(W_i \rho W_i^*)$$

Then we have $\sum_{i=1}^k p_i(\rho) = 1$, for every ρ . Therefore a family $W := \{W_i\}_{i=1, \dots, k}$ determines a QIFS $\mathcal{F}_W = \{\mathcal{M}_N, F_i, p_i\}_{i=1, \dots, k}$, with F_i, p_i given by (60) and (61). We introduce the following definition.

Definition 7. Let \mathcal{F}_W be a QIFS such that there is a unique attractive invariant measure for the associated Markov operator \mathcal{V} . Let ρ_W be the barycenter of such measure. Define the **QIFS entropy**:

$$(62) \quad h_V(W) := - \sum_{i=1}^k \text{tr}(W_i \rho_W W_i^*) \sum_{j=1}^k \text{tr} \left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)} \right) \log \text{tr} \left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)} \right)$$

Remember that by lemma 4, we have that ρ_W is a fixed point for

$$(63) \quad \Lambda(\rho) = \Lambda_{\mathcal{F}_W}(\rho) := \sum_{i=1}^k p_i(\rho) F_i(\rho) = \sum_{i=1}^k \text{tr}(W_i \rho W_i^*) \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)}$$

Lemma 5. $h_V(W) \geq 0$, for every family W_i of linear operators satisfying $\sum_i W_i^* W_i = I$.

Proof Note that, by definition,

$$h_V(W) = (\mathcal{U}h)(\rho_W) = \int_{\mathcal{M}_N} h d\mathcal{V}\delta_{\rho_W}$$

and the function h (Shannon-Boltzmann entropy) is ≥ 0 . This proves the lemma. Another elementary proof is the following. Since ρ_W is positive, we have that $\langle W_i \rho_W W_i^* v, v \rangle = \langle \rho_W W_i^* v, W_i^* v \rangle \geq 0$, $v \in \mathcal{H}_N$. So for $\{v_l\}_{l=1, \dots, N}$ an orthonormal base for \mathcal{H}_N ,

$$\text{tr}(W_i \rho_W W_i^*) = \sum_{l=1}^N \langle W_i \rho_W W_i^* v_l, v_l \rangle > 0$$

Analogously the expression above holds for the $V_i \rho_W V_i^*$, and therefore also for $W_j V_i \rho_W V_i^* W_j^*$, because

$$\langle W_j V_i \rho_W V_i^* W_j^* v, v \rangle = \langle V_i \rho_W V_i^* W_j^* v, W_j^* v \rangle \geq 0$$

To conclude that $h_V(W) \geq 0$, we have to show that $\text{tr}(W_j V_i \rho_W V_i^* W_j^*) \leq \text{tr}(V_i \rho_W V_i^*)$. From $\sum_{i=1}^k W_i^* W_i = I$, we get

$$\begin{aligned} \text{tr}(W_j V_i \rho_W V_i^* W_j^*) &= \text{tr}(W_j^* W_j V_i \rho_W V_i^*) \leq \sum_{j=1}^k \text{tr}(W_j^* W_j V_i \rho_W V_i^*) \\ &= \text{tr}\left(\sum_{j=1}^k W_j^* W_j V_i \rho_W V_i^*\right) = \text{tr}(V_i \rho_W V_i^*) \end{aligned}$$

□

Remark For any fixed dynamics V , if we have that $W_m^* W_m = I$ for some m then the remaining p_i must be zero, because of the condition $\sum_i W_i^* W_i = I$. In this case we have $h_V(W) = 0$. We also have that $h_V(W) \leq \log k$ and for any given dynamics V , $h_V(W)$ attains the maximum if we choose $W_i = 1/\sqrt{k}I$, for each i , where I denotes the identity operator.

◇

Note that by the calculations made in section 9, we have $h_V(W) = \mathcal{U}h(\rho_W)$, where $\mathcal{U}h(\rho) = \sum_i p_i(\rho)h(F_i(\rho))$.

Lemma 6. Let $\mathcal{F} = (\mathcal{M}_N, F_i, p_i)$ be a QIFS, with F_i, p_i in the form (60) and (61). Suppose there is $\rho_0 \in \mathcal{M}_N$ such that δ_{ρ_0} is the unique \mathcal{V} -invariant measure. Then $\Lambda_{\mathcal{F}}(\rho_0) = \rho_0$ ($\Lambda_{\mathcal{F}}$ is the operator associated to \mathcal{F}) and

$$\int \mathcal{U}^n h d\delta_{\rho_0} = \mathcal{U}^n h(\rho_0) = h(\rho_0),$$

for all $n \in \mathbb{N}$. Besides, $\mathcal{U}^n h(\rho_0) = \mathcal{U}h(\rho_0)$ and so

$$h_V(W) = \mathcal{U}^n h(\rho_0),$$

for all $n \in \mathbb{N}$.

Proof The fact that $\Lambda(\rho_0) = \rho_0$ follows from lemma 2, item 2. Also,

$$\mathcal{U}^n h(\rho_0) = \int \mathcal{U}^n h d\delta_{\rho_0} = \int h d\mathcal{V}^n \delta_{\rho_0} = \int h d\delta_{\rho_0} = h(\rho_0)$$

and

$$\mathcal{U}^n h(\rho_0) = \int \mathcal{U}^n h d\delta_{\rho_0} = \int h d\mathcal{V}^n \delta_{\rho_0} = \int h d\mathcal{V} \delta_{\rho_0} = \int \mathcal{U} h d\delta_{\rho_0} = \mathcal{U} h(\rho_0)$$

□

Lemma 7. *Let μ be a \mathcal{V} -invariant attractive measure. Then if ρ_μ is the barycenter of μ we have, for any ρ ,*

$$(64) \quad \lim_{n \rightarrow \infty} \mathcal{U}^n h(\rho) = \int \mathcal{U} h d\mu = \int h d\mu \leq h(\rho_\mu)$$

Proof The inequality follows from [21], proposition 1.15. Also, by proposition 4 we have

$$\lim_{n \rightarrow \infty} \mathcal{U}^n h(\rho) = \lim_{n \rightarrow \infty} \int \mathcal{U}^n h d\delta_\rho = \lim_{n \rightarrow \infty} \int \mathcal{U} h d\mathcal{V}^{n-1} \delta_\rho = \int \mathcal{U} h d\mu,$$

the last equality being true because of the weak convergence of $(\mathcal{V}^n \delta_\rho)_{n \in \mathbb{N}}$. This proves the first equality in (64). Since $\int \mathcal{U} h d\mu = \int h d\mathcal{V} \mu = \int h d\mu$, we obtain the second equality.

□

Lemma 8. *Let $\mathcal{F} = (\mathcal{M}_N, F_i, p_i)$ be a QIFS, with F_i, p_i in the form (60) and (61). Suppose that ρ is the unique point such that $\Lambda_{\mathcal{F}}(\rho) = \rho$. Suppose that $F_i(\rho) = \rho$, $i = 1, \dots, k$. Then*

$$\mathcal{U}^n h(\rho) = h(\rho),$$

$n = 1, 2, \dots$, and therefore $h_V(W)$ does not depend on n .

Proof The proof follows by induction. Let $n = 1$. We have:

$$\mathcal{U} h(\rho) = \sum_i p_i(\rho) h(F_i(\rho)) = h(\rho) \sum_i p_i(\rho) = h(\rho)$$

And note that $\mathcal{U}^n h(\rho) = \mathcal{U}(\mathcal{U}^{n-1} h)(\rho)$, which concludes the proof.

□

11. ENTROPY AND MARKOV CHAINS

Let V_i, W_i be linear operators, $i = 1, \dots, k$, $\sum_{i=1}^k W_i^* W_i = I$. Suppose the V_i are fixed and that they determine a dynamics given by $F_i : \mathcal{M}_N \rightarrow \mathcal{M}_N$, $i = 1, \dots, k$. Define

$$(65) \quad P := \{(p_1, \dots, p_k) : p_i : \mathcal{M}_N \rightarrow \mathbb{R}^+, i = 1, \dots, k, \sum_{i=1}^k p_i(\rho) = 1, \forall \rho \in \mathcal{M}_N\}$$

$$P' := P \cap \{(p_1, \dots, p_k) : \exists W_i, i = 1, \dots, k : p_i(\rho) = \text{tr}(W_i \rho W_i^*),$$

$$(66) \quad W_i \text{ linear}, \sum_i W_i^* W_i = I\}$$

$$(67) \quad \mathcal{M}_F := \{\mu \in M^1(\mathcal{M}_N) : \exists p \in P' \text{ such that } \mathcal{V}_p \mu = \mu\},$$

where $\mathcal{V}_p : M^1(\mathcal{M}_N) \rightarrow M^1(\mathcal{M}_N)$,

$$(68) \quad \mathcal{V}_p(\mu)(B) := \sum_{i=1}^k \int_{F_i^{-1}(B)} p_i d\mu$$

Note that a family $W := \{W_i\}_{i=1,\dots,k}$ determines a QIFS \mathcal{F}_W ,

$$\mathcal{F}_W = \{\mathcal{M}_N, F_i, p_i\}_{i=1,\dots,k}$$

Let $P = (p_{ij})_{i,j=1,\dots,N}$ be a stochastic, irreducible matrix. Let p be the stationary vector of P . The entropy of P is defined as

$$(69) \quad H(P) := - \sum_{i,j=1}^N p_i p_{ij} \log p_{ij}$$

We consider a few examples which will be useful later in this work.

Example 4. (Homogeneous case, 4 matrices). Let $N = 2$, $k = 4$ and

$$V_1 = \begin{pmatrix} \sqrt{p_{00}} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \sqrt{p_{01}} \\ 0 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & 0 \\ \sqrt{p_{10}} & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p_{11}} \end{pmatrix}$$

Note that

$$\sum_i V_i^* V_i = \begin{pmatrix} p_{00} + p_{10} & 0 \\ 0 & p_{01} + p_{11} \end{pmatrix}$$

and so $\sum_i V_i^* V_i = I$ if we suppose that

$$P := \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$$

is column-stochastic. We have

$$V_1 \rho V_1^* = \begin{pmatrix} p_{00} \rho_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 \rho V_2^* = \begin{pmatrix} p_{01} \rho_4 & 0 \\ 0 & 0 \end{pmatrix}$$

$$V_3 \rho V_3^* = \begin{pmatrix} 0 & 0 \\ 0 & p_{10} \rho_1 \end{pmatrix}, \quad V_4 \rho V_4^* = \begin{pmatrix} 0 & 0 \\ 0 & p_{11} \rho_4 \end{pmatrix}$$

so

$$\begin{aligned} \text{tr}(V_1 \rho V_1^*) &= p_{00} \rho_1, & \text{tr}(V_2 \rho V_2^*) &= p_{01} \rho_4 \\ \text{tr}(V_3 \rho V_3^*) &= p_{10} \rho_1, & \text{tr}(V_4 \rho V_4^*) &= p_{11} \rho_4 \end{aligned}$$

The fixed point of $\Lambda(\rho) = \sum_i V_i \rho V_i^*$ is

$$\rho_V = \begin{pmatrix} \frac{p_{01}}{1-p_{00}+p_{01}} & 0 \\ 0 & \frac{1-p_{00}}{1-p_{00}+p_{01}} \end{pmatrix}$$

Let $\pi = (\pi_1, \pi_2)$ such that $P\pi = \pi$. We know that

$$(70) \quad \pi = \left(\frac{p_{01}}{1 - p_{00} + p_{01}}, \frac{1 - p_{00}}{1 - p_{00} + p_{01}} \right)$$

Then the nonzero entries of ρ_V are the entries of π and so we associate the fixed point of P to the fixed point of a certain Λ in a natural way. Let us calculate $h_V(W)$. Note that Λ defined above is associated to a homogeneous IFS. Then $W_i = V_i$, $i = 1, \dots, k$ and

$$(71) \quad \begin{aligned} h_V(W) &= h_V(V) \\ &= - \sum_{i=1}^k \frac{\text{tr}(W_i \rho_V W_i^*)}{\text{tr}(V_i \rho_V V_i^*)} \sum_{j=1}^k \text{tr}(W_j V_i \rho_V V_i^* W_j^*) \log \left(\frac{\text{tr}(W_j V_i \rho_V V_i^* W_j^*)}{\text{tr}(V_i \rho_V V_i^*)} \right) \\ &= - \sum_{i,j} \text{tr}(V_j V_i \rho_V V_i^* V_j^*) \log \left(\frac{\text{tr}(V_j V_i \rho_V V_i^* V_j^*)}{\text{tr}(V_i \rho_V V_i^*)} \right) \end{aligned}$$

A simple calculation yields $H(P) = h_V(V)$, where $H(P)$ is the entropy of P , given by (69). This shows that the entropy of Markov chains is a particular case of the QIFS entropy. ◇

Example 5. (Nonhomogeneous case, 4 matrices). Let $N = 2$, $k = 4$ and

$$\begin{aligned} V_1 &= \begin{pmatrix} \sqrt{p_{00}} & 0 \\ 0 & 0 \end{pmatrix}, & V_2 &= \begin{pmatrix} 0 & \sqrt{p_{01}} \\ 0 & 0 \end{pmatrix} \\ V_3 &= \begin{pmatrix} 0 & 0 \\ \sqrt{p_{10}} & 0 \end{pmatrix}, & V_4 &= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p_{11}} \end{pmatrix} \\ W_1 &= \begin{pmatrix} \sqrt{q_{00}} & 0 \\ 0 & 0 \end{pmatrix}, & W_2 &= \begin{pmatrix} 0 & \sqrt{q_{01}} \\ 0 & 0 \end{pmatrix} \\ W_3 &= \begin{pmatrix} 0 & 0 \\ \sqrt{q_{10}} & 0 \end{pmatrix}, & W_4 &= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{q_{11}} \end{pmatrix} \end{aligned}$$

Note that

$$\sum_i V_i^* V_i = \begin{pmatrix} p_{00} + p_{10} & 0 \\ 0 & p_{01} + p_{11} \end{pmatrix}, \quad \sum_i W_i^* W_i = \begin{pmatrix} q_{00} + q_{10} & 0 \\ 0 & q_{01} + q_{11} \end{pmatrix}$$

and so $\sum_i V_i^* V_i = \sum_i W_i^* W_i = I$ if we suppose that

$$P := \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}, \quad Q := \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$$

are column-stochastic. Then

$$\begin{aligned} \text{tr}(V_1 \rho V_1^*) &= p_{00} \rho_1, & \text{tr}(V_2 \rho V_2^*) &= p_{01} \rho_4 \\ \text{tr}(V_3 \rho V_3^*) &= p_{10} \rho_1, & \text{tr}(V_4 \rho V_4^*) &= p_{11} \rho_4 \\ \text{tr}(W_1 \rho W_1^*) &= q_{00} \rho_1, & \text{tr}(W_2 \rho W_2^*) &= q_{01} \rho_4 \\ \text{tr}(W_3 \rho W_3^*) &= q_{10} \rho_1, & \text{tr}(W_4 \rho W_4^*) &= q_{11} \rho_4 \end{aligned}$$

We want the fixed point of $\Lambda(\rho) = \sum_i p_i(\rho) F_i(\rho)$. This leads us to

$$\frac{q_{00}}{p_{00}} \begin{pmatrix} p_{00}\rho_1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{q_{01}}{p_{01}} \begin{pmatrix} p_{01}\rho_4 & 0 \\ 0 & 0 \end{pmatrix} + \frac{q_{10}}{p_{10}} \begin{pmatrix} 0 & 0 \\ 0 & p_{10}\rho_1 \end{pmatrix} + \frac{q_{11}}{p_{11}} \begin{pmatrix} 0 & 0 \\ 0 & p_{11}\rho_4 \end{pmatrix} = \rho$$

Note that the p_{ij} cancel and so we obtain a calculation which is the same as the one obtained in the previous example. Hence

$$\rho_W = \begin{pmatrix} \frac{q_{01}}{1-q_{00}+q_{01}} & 0 \\ 0 & \frac{1-q_{00}}{1-q_{00}+q_{01}} \end{pmatrix},$$

and its nonzero entries are the entries of the fixed point for the stochastic matrix Q . Calculating $h_V(W)$ gives

$$\begin{aligned} h_V(W) &= - \sum_{i=1}^k \frac{\text{tr}(W_i \rho_W W_i^*)}{\text{tr}(V_i \rho_W V_i^*)} \sum_{j=1}^k \text{tr}(W_j V_i \rho_W V_i^* W_j^*) \log \left(\frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)} \right) \\ (72) \quad &= - \frac{q_{01}}{q_{01} + q_{10}} (q_{00} \log q_{00} + q_{10} \log q_{10}) - \frac{q_{10}}{q_{01} + q_{10}} (q_{01} \log q_{01} + q_{11} \log q_{11}) = H(Q) \end{aligned}$$

So we have obtained a calculation which is analogous to the one for the homogeneous case. This result generalizes what we have seen in the previous example.

◇

Example 6. (Homogeneous case, 2 matrices). Let $N = 2$, $k = 2$ and

$$V_1 = \begin{pmatrix} \sqrt{p_{00}} & 0 \\ \sqrt{p_{10}} & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \sqrt{p_{01}} \\ 0 & \sqrt{p_{11}} \end{pmatrix},$$

Note that, just as in the previous examples

$$\sum_i V_i^* V_i = \begin{pmatrix} p_{00} + p_{10} & 0 \\ 0 & p_{01} + p_{11} \end{pmatrix}$$

and so $\sum_i V_i^* V_i = I$ if we suppose

$$P := \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$$

is column-stochastic. The fixed point for Λ is

$$\rho_V = \begin{pmatrix} \frac{p_{01}}{p_{01}+p_{10}} & \frac{p_{00}p_{10}p_{01}}{p_{01}+p_{10}} + \frac{p_{01}p_{11}p_{10}}{p_{01}+p_{10}} \\ \frac{p_{00}p_{10}p_{01}}{p_{01}+p_{10}} + \frac{p_{01}p_{11}p_{10}}{p_{01}+p_{10}} & \frac{p_{10}}{p_{01}+p_{10}} \end{pmatrix}$$

The entries of the main diagonal of ρ_V correspond to the entries of the fixed point of P . The entries of the secondary diagonal are a linear combination of the ones in the main diagonal. Then for the V_i chosen we have

$$(73) \quad h_V(W) = h_V(V) = - \sum_{i,j} \text{tr}(V_j V_i \rho_V V_i^* V_j^*) \log \left(\frac{\text{tr}(V_j V_i \rho_V V_i^* V_j^*)}{\text{tr}(V_i \rho_V V_i^*)} \right) = H(P)$$

by an identical calculation made for the equation (72) from the previous example. In other words, the fact that the fixed point of Λ is not diagonal does not change the calculations for the entropy.

◇

Example 7. (Nonhomogeneous case, 2 matrices). Let $N = 2$, $k = 2$,

$$V_1 = \begin{pmatrix} \sqrt{p_{00}} & 0 \\ \sqrt{p_{10}} & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \sqrt{p_{01}} \\ 0 & \sqrt{p_{11}} \end{pmatrix}$$

$$W_1 = \begin{pmatrix} \sqrt{q_{00}} & 0 \\ \sqrt{q_{10}} & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & \sqrt{q_{01}} \\ 0 & \sqrt{q_{11}} \end{pmatrix}$$

As in the other examples, $\sum_i V_i^* V_i = \sum_i W_i^* W_i = I$ if we suppose

$$P := \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}, \quad Q := \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$$

is column-stochastic. From

$$\begin{aligned} \text{tr}(V_1 \rho V_1^*) &= \rho_1, & \text{tr}(V_2 \rho V_2^*) &= \rho_4 \\ \text{tr}(W_1 \rho W_1^*) &= \rho_1, & \text{tr}(W_2 \rho W_2^*) &= \rho_4 \\ \text{tr}(W_1 V_1 \rho V_1^* W_1^*) &= p_{00} \rho_1, & \text{tr}(W_2 V_1 \rho V_1^* W_2^*) &= p_{10} \rho_1 \\ \text{tr}(W_1 V_2 \rho V_2^* W_1^*) &= p_{01} \rho_4, & \text{tr}(W_2 V_2 \rho V_2^* W_2^*) &= p_{11} \rho_4 \end{aligned}$$

and a simple calculation, we get $h_V(W) = H(P)$.

◇

Lemma 9. Let V_{ij} be matrices of order n ,

$$V_{ij} = \sqrt{p_{ij}} |i\rangle\langle j|$$

for $i, j = 1, \dots, n$. Let

$$\Lambda_P(\rho) := \sum_{i,j} V_{ij} \rho V_{ij}^*$$

where $P = (p_{ij})_{i,j=1,\dots,n}$. Then for all n , $\Lambda_P^n(\rho) = \Lambda_{P^n}(\rho)$.

Proof Note that

$$(74) \quad V_{kl} V_{ij} = \sqrt{p_{kl}} \sqrt{p_{ij}} \delta_{li} |k\rangle\langle j|$$

so

$$\begin{aligned} \Lambda_P^2(\rho) &= \Lambda_P\left(\sum_{i,j} V_{ij} \rho V_{ij}^*\right) = \sum_{k,l,i,j} V_{kl} V_{ij} \rho (V_{kl} V_{ij})^* \\ &= \sum_{k,j} \sum_i p_{ki} p_{ij} |k\rangle\langle j| \rho |j\rangle\langle k| = \sum_{k,j} p_{kj}^2 |k\rangle\langle j| \rho |j\rangle\langle k| = \Lambda_{P^2}(\rho) \end{aligned}$$

The general case follows by iterating the above calculation.

□

Corollary 2. Under the lemma hypothesis, we have $\lim_{n \rightarrow \infty} \Lambda_P^n(\rho) = \Lambda_\pi(\rho)$, where $\pi = \lim_{n \rightarrow \infty} P^n$ is the stochastic matrix which has all columns equal to the stationary vector for P .

12. CAPACITY-COST FUNCTION AND PRESSURE

Recall that every trace preserving, completely positive (CP) mapping can be written in the Stinespring-Kraus form,

$$\Lambda(\rho) = \sum_{i=1}^k V_i \rho V_i^*, \quad \sum_{i=1}^k V_i^* V_i = I,$$

for V_i linear operators. These mappings are also called **quantum channels**. This is one of the main motivations for considering the class of operators (a generalization of the above ones) described in the present paper. These are natural objects in the analysis of certain problems in quantum computing.

Definition 8. *The Holevo capacity for sending classic information via a quantum channel Λ is defined as*

$$(75) \quad C_\Lambda := \max_{\substack{p_i \in [0,1] \\ \rho_i \in \mathcal{M}_N}} S\left(\sum_{i=1}^n p_i \Lambda(\rho_i)\right) - \sum_{i=1}^n p_i S\left(\Lambda(\rho_i)\right)$$

where $S(\rho) = -\text{tr}(\rho \log \rho)$ is the von Neumann entropy. The maximum is, therefore, over all choices of p_i , $i = 1, \dots, n$ and density operators ρ_i , for some $n \in \mathbb{N}$. The Holevo capacity establishes an upper bound on the amount of information that a quantum system contains [17].

Definition 9. *Let Λ be a quantum channel. Define the minimum output entropy as*

$$H^{\min}(\Lambda) := \min_{|\psi\rangle} S(\Lambda(|\psi\rangle\langle\psi|))$$

Additivity conjecture We have that

$$C_{\Lambda_1 \otimes \Lambda_2} = C_{\Lambda_1} + C_{\Lambda_2}$$

Minimum output entropy conjecture For any channels Λ_1 and Λ_2 ,

$$H^{\min}(\Lambda_1 \otimes \Lambda_2) = H^{\min}(\Lambda_1) + H^{\min}(\Lambda_2)$$

In [19], it is shown that the additivity conjecture is equivalent to the minimum output entropy conjecture, and in [10] a counterexample is obtained for this last conjecture.

Remark Concerning QIFS, our interest in capacity is motivated by the following observation. Considering expression (75), note that the term

$$(76) \quad \sum_{i=1}^n p_i S(\Lambda(\rho_i))$$

is a convex combination of von Neumann entropies, in the same way as the QIFS entropy. So we see that given a QIFS, we can consider capacity functions, and the QIFS entropy arises in a natural way. For an example, we perform the following calculation. If λ_i are the eigenvalues of ρ then we can write

$$(77) \quad S(\rho) = - \sum_i \lambda_i \log \lambda_i$$

Then write the QIFS entropy as

$$(78) \quad h_V(W) = - \sum_{i,j=1}^k \text{tr}(W_i \rho_W W_i^*) a_{ij}(\rho_W) \log a_{ij}(\rho_W)$$

where

$$(79) \quad a_{ij}(\rho) := \frac{\text{tr}(W_j V_i \rho V_i^* W_j^*)}{\text{tr}(V_i \rho V_i^*)}$$

We see that for $\rho_W \in \mathcal{M}_N$ and i fixed, we have $\sum_i a_{ij}(\rho_W) = 1$. Define for each i the density operator

$$(80) \quad \rho_i := \sum_j a_{ij}(\rho_W) |j\rangle\langle j|$$

Then by (77),

$$(81) \quad S(\rho_i) = - \sum_j a_{ij}(\rho_W) \log a_{ij}(\rho_W)$$

By (78), we can write

$$(82) \quad h_V(W)(\rho_W) = \sum_{i=1}^k \text{tr}(W_i \rho_W W_i^*) S(\rho_i)$$

A **Positive Operator-Valued Measurement** (POVM) is described by a set of positive operators P_i (POVM elements) such that $\sum_i P_i = I$. If the measurement is performed on a system described by the state vector $|\psi\rangle$, then the probability of obtaining i as the outcome is given by

$$(83) \quad p_i = \langle \psi | F_i | \psi \rangle$$

Note that a QIFS \mathcal{F} induced by linear V_i and W_i , contains a POVM by taking $W_i^* W_i$ as POVM elements. If X is a random variable that takes values p_1, \dots, p_k then the Shannon entropy is $H(X) = - \sum_i p_i \log p_i$ and the joint entropy of variables X and Y is

$$(84) \quad H(X, Y) := - \sum_{x,y} p(x, y) \log p(x, y)$$

where $p(x, y)$ is the probability that $X = x$ and $Y = y$. The mutual information $I(X : Y)$ is defined by $I(X : Y) := H(X) + H(Y) - H(X, Y)$. Then, considering the QIFS entropy we can state the Holevo bound in the following way: first consider a QIFS \mathcal{F} such that there is a unique attractive measure which is invariant for the Markov operator \mathcal{V} associated to \mathcal{F} . Let ρ_W be the barycenter of such measure.

Theorem 4. (Holevo bound for QIFS) *Suppose \mathcal{F} is induced by linear operators V_i and W_i with $\sum_i W_i^* W_i = I$ and for each $i = 1, \dots, k$ write $p_i = \text{tr}(W_i \rho_W W_i^*)$ and $\rho_i = \sum_j a_{ij}(\rho_W) |j\rangle\langle j|$, where a_{ij} is given by (79). Suppose Alice prepares a mixed state ρ_X chosen from the ensemble $\{\rho_1, \dots, \rho_k\}$ with probabilities $\{p_1, \dots, p_k\}$ (that is, we assume ρ_X is a state determined by a random variable X such that it*

assumes the value ρ_i with probability p_i). Suppose Bob performs a POVM measurement on that state with POVM elements $\{P_i\}_{i=1,\dots,m}$ and measurement outcome described by a random variable Y . Then, by writing $\rho = \sum_i p_i \rho_i$, we have

$$(85) \quad I(X : Y) \leq S(\rho) - h_V(W)(\rho_W) =: \xi(\mathcal{E})$$

The number $\xi(\mathcal{E})$ is the Holevo information of the ensemble given by $\mathcal{E} = \{\rho_i; p_i\}_{i=1,\dots,k}$. We see that (85) holds by applying the Holevo bound for the von Neumann entropy (see [17]) together with (80) and (82).

◇

We are also interested in a different class of problems which concern maximization (and not minimization) of entropy plus a given potential (a cost) [9],[11],[12].

Definition 10. Let \mathcal{M}_F be the set of invariant measures defined in the section 11 and let H be a hermitian operator. For $\mu \in \mathcal{M}_F$ let ρ_μ be its barycenter. Define the capacity-cost function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as

$$(86) \quad C(a) := \max_{\mu \in \mathcal{M}_F} \{h_{W,V}(\rho_\mu) : \text{tr}(H\rho_\mu) \leq a\}$$

The following analysis is inspired in [8]. There is a relation between the cost-capacity function and the variational problem for pressure. In fact, let $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function given by

$$(87) \quad F(\lambda) := \sup_{\mu \in \mathcal{M}_F} \{h_{W,V}(\rho_\mu) - \lambda \text{tr}(H\rho_\mu)\}$$

We have the following fact. There is a unique probability measure $\nu_0 \in \mathcal{M}_F$ such that

$$F(\lambda) = h_{W,V}(\rho_{\nu_0}) - \lambda \text{tr}(H\rho_{\nu_0})$$

Also, we have the following lemma:

Lemma 10. Let $\lambda \leq 0$, and $\hat{a} = \text{tr}(H\rho_{\nu_0})$. Then

$$(88) \quad C(\hat{a}) = h_{W,V}(\rho_{\nu_0})$$

Proof Let $\nu \in \mathcal{M}_F$, $\nu \neq \nu_0$, with $\text{tr}(H\rho_\nu) \leq \hat{a} = \text{tr}(H\rho_{\nu_0})$. Then

$$h_{W,V}(\rho_\nu) - \lambda \text{tr}(H\rho_\nu) < h_{W,V}(\rho_{\nu_0}) - \lambda \text{tr}(H\rho_{\nu_0})$$

so

$$h_{W,V}(\rho_\nu) < h_{W,V}(\rho_{\nu_0})$$

Hence

$$h_{W,V}(\rho_{\nu_0}) = \sup_{\mu \in \mathcal{M}_F} \{h_{W,V}(\rho_\mu) : \text{tr}(H\rho_\mu) \leq \hat{a}\} = C(\hat{a})$$

□

13. ANALYSIS OF THE PRESSURE PROBLEM

Let V_i, W_i, H_i be linear operators, $i = 1, \dots, k$, with $\sum_i W_i^* W_i = I$ and let

$$(89) \quad H\rho := \sum_{i=1}^k H_i \rho H_i^*$$

a hermitian operator. We are interested in obtaining a version of the variational principle of pressure for our context. We will see that the pressure will be a maximum whenever we have a certain relation between the potential H and the probability distribution considered (represented here by the W_i). We begin by fixing a dynamics, given by the V_i . From the reasoning described below, it will be natural to consider as definition of pressure the maximization among the possible stationary W_i of the expression

$$h_V(W) + \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \text{tr}(W_j \rho_W W_j^*)$$

where ρ_β is the eigenstate of a certain Ruelle operator, described below. We begin our analysis by using the following elementary lemma.

Lemma 11. [18] *If r_1, \dots, r_k and q_1, \dots, q_k are two probability distributions over $1, \dots, k$, such that $r_j > 0$, $j = 1, \dots, k$, then*

$$(90) \quad -\sum_{j=1}^k q_j \log q_j + \sum_{j=1}^k q_j \log r_j \leq 0$$

and equality holds if and only if $r_j = q_j$, $j = 1, \dots, k$.

The potential given by (89), together with the V_i , induces an operator given by

$$(91) \quad \mathcal{L}_H(\rho) := \sum_{i=1}^k \text{tr}(H_i \rho H_i^*) V_i \rho V_i^*$$

By proposition 2 we know that such operator admits an eigenvalue β with its associated eigenstate ρ_β . Then $\mathcal{L}_H(\rho_\beta) = \beta \rho_\beta$ implies

$$(92) \quad \sum_{i=1}^k \text{tr}(H_i \rho_\beta H_i^*) V_i \rho_\beta V_i^* = \beta \rho_\beta$$

In coordinates, (92) can be written as

$$(93) \quad \sum_{i=1}^k \text{tr}(H_i \rho_\beta H_i^*) (V_i \rho_\beta V_i^*)_{jl} = \beta (\rho_\beta)_{jl}$$

Remark Comparing the above calculation with the problem of finding an eigenvalue λ of a matrix $A = (a_{ij})$, we have that equation (92) can be seen as the analogous of the expression

$$(94) \quad lE^A = \lambda l$$

Above, the matrix A plays the role of a potential, E^A denotes the matrix with entries $e^{a_{ij}}$ and l_j denotes the j -th coordinate of the left eigenvector l associated to the eigenvalue λ . In coordinates,

$$(95) \quad \sum_i l_i e^{a_{ij}} = \lambda l_j, \quad i, j = 1, \dots, k$$

◇

From this point we can perform two calculations. First, considering (92) we will take the trace of such equation in order to obtain a scalar equation. In spite of the fact that taking the trace makes us lose part of the information given by the eigenvector equation, we are still able to obtain a version of what we will call the **basic inequality**, which can be seen as a QIFS version of the variational principle of pressure. However, there is an algebraic drawback to this approach, namely, that we will not be able to recover the classic variational problem as a particular case of such inequality (such disadvantage is a consequence of taking the trace, clearly). The second calculation begins at equation (93), the coordinate equations associated to the matrix equation for the eigenvectors. In this case we also obtain a basic inequality, but then we will have the classic variational problem of pressure as a particular case.

An important question which is of our interest, regarding both calculations mentioned above, is to ask whether it is possible for a given system to attain its maximum pressure. It is not clear that given any dynamics, we can obtain a measure reaching such a maximum. With respect to our context, we will remark a natural condition on the dynamics which allows us to determine expressions for the measure which maximizes the pressure. Now we perform the calculations mentioned above.

Based on (92), define

$$(96) \quad r_j = \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*)$$

So we have $\sum_j r_j = 1$. Let

$$(97) \quad q_j^i := \text{tr} \left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)} \right)$$

where, as before, ρ_W is the fixed point associated to the operator $\Lambda_{\mathcal{F}_W}$

$$(98) \quad \Lambda_{\mathcal{F}_W}(\rho) := \sum_{i=1}^k p_i(\rho) F_i(\rho)$$

induced by the QIFS $(\mathcal{M}_N, F_i, p_i)_{i=1, \dots, k}$,

$$F_i(\rho) = \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)}$$

and

$$p_i(\rho) = \text{tr}(W_i \rho W_i^*)$$

Note that we have

$$\begin{aligned} \sum_{j=1}^k q_j^i &= \frac{1}{\text{tr}(V_i \rho_W V_i^*)} \sum_{j=1}^k \text{tr}(W_j^* W_j V_i \rho_W V_i^*) \\ &= \frac{1}{\text{tr}(V_i \rho_W V_i^*)} \text{tr}\left(\sum_{j=1}^k W_j^* W_j V_i \rho_W V_i^*\right) = 1 \end{aligned}$$

Then we can apply lemma 11 for $r_j, q_j^i, j = 1, \dots, k$, with i fixed, to obtain

$$(99) \quad - \sum_j \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \log \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \\ + \sum_j \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \log\left(\frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*)\right) \leq 0$$

and equality holds if and only if for all i, j ,

$$(100) \quad \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

Then

$$\begin{aligned} & - \sum_j \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \log \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \\ & + \sum_j \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \log\left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*)\right) \\ & \leq \sum_j \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \log \beta \end{aligned}$$

which is equivalent to

$$(101) \quad - \sum_j \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \log \text{tr}\left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\text{tr}(V_i \rho_W V_i^*)}\right) \\ + \sum_j \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)} \log\left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*)\right) \leq \log \beta$$

Multiplying by $\text{tr}(W_i \rho_W W_i^*)$ and summing over the i index, we have

$$(102) \quad h_V(W) + \sum_j \log\left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*)\right) \sum_i \frac{\text{tr}(W_i \rho_W W_i^*)}{\text{tr}(V_i \rho_W V_i^*)} \text{tr}(W_j V_i \rho_W V_i^* W_j^*) \\ \leq \sum_i \text{tr}(W_i \rho_W W_i^*) \log \beta = \log \beta$$

and equality holds if and only if for all i, j ,

$$(103) \quad \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

Let us rewrite inequality (102). First we use the fact that ρ_W is a fixed point of $\Lambda_{\mathcal{F}_W}$,

$$(104) \quad \sum_{i=1}^k \text{tr}(W_i \rho_W W_i^*) \frac{V_i \rho_W V_i^*}{\text{tr}(V_i \rho_W V_i^*)} = \rho_W$$

Now we compose both sides of the equality above with the operator

$$(105) \quad \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) W_j^* W_j$$

and then we obtain

$$(106) \quad \begin{aligned} & \sum_{i=1}^k \text{tr}(W_i \rho_W W_i^*) \frac{V_i \rho_W V_i^*}{\text{tr}(V_i \rho_W V_i^*)} \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) W_j^* W_j \\ &= \rho_W \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) W_j^* W_j \end{aligned}$$

Reordering terms we get

$$(107) \quad \begin{aligned} & \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \sum_{i=1}^k \frac{\text{tr}(W_i \rho_W W_i^*)}{\text{tr}(V_i \rho_W V_i^*)} V_i \rho_W V_i^* W_j^* W_j \\ &= \rho_W \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) W_j^* W_j \end{aligned}$$

Taking the trace on both sides we get

$$(108) \quad \begin{aligned} & \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \sum_{i=1}^k \frac{\text{tr}(W_i \rho_W W_i^*)}{\text{tr}(V_i \rho_W V_i^*)} \text{tr}(W_j V_i \rho_W V_i^* W_j^*) \\ &= \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \text{tr}(\rho_W W_j^* W_j) \end{aligned}$$

Note that the left hand side of (108) is one of the sums appearing in (102). Therefore replacing (108) into (102) gives our main result.

Theorem 5. *Let \mathcal{F}_W be a QIFS such that there is a unique attractive invariant measure for the associated Markov operator \mathcal{V} . Let ρ_W be the barycenter of such measure and let ρ_β be an eigenstate of $\mathcal{L}_H(\rho)$ with eigenvalue β . Then*

$$(109) \quad h_{\mathcal{V}}(W) + \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \text{tr}(W_j \rho_W W_j^*) \leq \log \beta$$

and equality holds if and only if for all i, j ,

$$(110) \quad \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

In section 16 we make some considerations about certain cases in which we can reach an equality in (109). ◇

For the calculations regarding expression (93), define

$$(111) \quad r_{jlm} = \frac{1}{\beta} \operatorname{tr}(H_j \rho_\beta H_j^*) \frac{(V_j \rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}}$$

Then we have $\sum_j r_{jlm} = 1$. Let

$$(112) \quad q_{ij} := \operatorname{tr} \left(\frac{W_j V_i \rho_W V_i^* W_j^*}{\operatorname{tr}(V_i \rho_W V_i^*)} \right)$$

A calculation similar to the one we have made for (109) gives us

$$(113) \quad \begin{aligned} & h_V(W) + \sum_{j=1}^k \operatorname{tr}(W_j \rho_W W_j^*) \log \operatorname{tr}(H_j \rho_\beta H_j^*) \\ & + \sum_{j=1}^k \operatorname{tr}(W_j \rho_W W_j^*) \log \left(\frac{(V_j \rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}} \right) \leq \log \beta \end{aligned}$$

and equality holds if and only if for all i, j, l, m ,

$$(114) \quad \frac{1}{\beta} \operatorname{tr}(H_j \rho_\beta H_j^*) \frac{(V_j \rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}} = \frac{\operatorname{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\operatorname{tr}(V_i \rho_W V_i^*)}$$
◇

14. REVISITING THE EIGENVALUE PROBLEM

Consider the operator

$$(115) \quad \mathcal{L}_H(\rho) = \sum_{i=1}^k \operatorname{tr}(H_i \rho H_i^*) V_i \rho V_i^*$$

induced by a fixed dynamics V_i $i = 1, \dots, k$, V_i linear, and by $H\rho := \sum_i H_i \rho H_i^*$, H_i linear. The eigenvalues equation for \mathcal{L}_H written in coordinates gives us the following system, for $k = 2$:

$$(116) \quad \begin{aligned} & \operatorname{tr}(H_1 \rho_\beta H_1^*) (v_{11}^2 \rho_{11} + 2v_{11}v_{12}\rho_{12} + v_{12}^2 \rho_{22}) \\ & + \operatorname{tr}(H_2 \rho_\beta H_2^*) (w_{11}^2 \rho_{11} + 2w_{11}w_{12}\rho_{12} + w_{12}^2 \rho_{22}) = \beta \rho_{11} \end{aligned}$$

$$(117) \quad \begin{aligned} & \operatorname{tr}(H_1 \rho_\beta H_1^*) (v_{21}v_{11}\rho_{11} + (v_{21}v_{12} + v_{22}v_{11})\rho_{12} + v_{22}v_{12}\rho_{22}) \\ & + \operatorname{tr}(H_2 \rho_\beta H_2^*) (w_{21}w_{11}\rho_{11} + (w_{21}w_{12} + w_{22}w_{11})\rho_{12} + w_{22}w_{12}\rho_{22}) = \beta \rho_{12} \end{aligned}$$

$$(118) \quad \begin{aligned} & \operatorname{tr}(H_1 \rho_\beta H_1^*) (v_{21}^2 \rho_{11} + 2v_{21}v_{22}\rho_{12} + v_{22}^2 \rho_{22}) \\ & + \operatorname{tr}(H_2 \rho_\beta H_2^*) (w_{21}^2 \rho_{11} + 2w_{21}w_{22}\rho_{12} + w_{22}^2 \rho_{22}) = \beta \rho_{22} \end{aligned}$$

And we can also write, for $i = 1, 2$,

$$(119) \quad tr(H_i \rho H_i^*) = ((h_{11}^i)^2 + (h_{12}^i)^2) \rho_{11} + 2(h_{11}^i h_{12}^i + h_{12}^i h_{22}^i) \rho_{12} + ((h_{12}^i)^2 + (h_{22}^i)^2) \rho_{22}$$

◇

Fix H_1, H_2 , let V_1, V_2 be defined by

$$(120) \quad V_1 = \begin{pmatrix} v_{11} & v_{12} \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ w_{21} & w_{22} \end{pmatrix}$$

then we have, by (116)-(118) that $\rho_{12} = 0$ and

$$(121) \quad tr(H_1 \rho_\beta H_1^*) (v_{11}^2 \rho_{11} + v_{12}^2 \rho_{22}) = \beta \rho_{11}$$

$$(122) \quad tr(H_2 \rho_\beta H_2^*) (w_{21}^2 \rho_{11} + w_{22}^2 \rho_{22}) = \beta \rho_{22}$$

that is,

$$(123) \quad [((h_{11}^1)^2 + (h_{12}^1)^2) \rho_{11} + ((h_{12}^1)^2 + (h_{22}^1)^2) \rho_{22}] (v_{11}^2 \rho_{11} + v_{12}^2 \rho_{22}) = \beta \rho_{11}$$

$$(124) \quad [((h_{11}^2)^2 + (h_{12}^2)^2) \rho_{11} + ((h_{12}^2)^2 + (h_{22}^2)^2) \rho_{22}] (w_{21}^2 \rho_{11} + w_{22}^2 \rho_{22}) = \beta \rho_{22}$$

Also, suppose that

$$(125) \quad v_{11} = v_{12} = w_{21} = w_{22} = 1$$

Then we get

$$(126) \quad ((h_{11}^1)^2 + (h_{12}^1)^2) \rho_{11} + ((h_{12}^1)^2 + (h_{22}^1)^2) \rho_{22} = \beta \rho_{11}$$

$$(127) \quad ((h_{11}^2)^2 + (h_{12}^2)^2) \rho_{11} + ((h_{12}^2)^2 + (h_{22}^2)^2) \rho_{22} = \beta \rho_{22}$$

Let $A = (a_{ij})$ be a matrix with positive entries and consider the problem of finding its eigenvalues and eigenvectors. Then from

$$(128) \quad a_{11} v_1 + a_{12} v_2 = \beta v_1$$

$$(129) \quad a_{21} v_1 + a_{22} v_2 = \beta v_2$$

we see that the systems (126)-(127) and (128)-(129) are the same if we choose

$$(130) \quad a_{11} = (h_{11}^1)^2 + (h_{12}^1)^2, \quad a_{12} = (h_{12}^1)^2 + (h_{22}^1)^2$$

$$(131) \quad a_{21} = (h_{11}^2)^2 + (h_{12}^2)^2, \quad a_{22} = (h_{12}^2)^2 + (h_{22}^2)^2$$

We conclude that Perron's classic eigenvalue problem is a particular case of the problem associated to \mathcal{L}_H acting on matrices. In fact, if we fix

$$(132) \quad V_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

and given A a matrix with positive entries, choose

$$(133) \quad H_1 = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ 0 & \sqrt{a_{12}} \end{pmatrix}, \quad H_2 = \begin{pmatrix} \sqrt{a_{21}} & 0 \\ 0 & \sqrt{a_{22}} \end{pmatrix}$$

Then the operator \mathcal{L}_H has a diagonal eigenstate

$$(134) \quad \rho_\beta = \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{22} \end{pmatrix}$$

associated to the eigenvalue β , and we have that, defining $v = (\rho_{11}, \rho_{22})$, we get $Av = \beta v$.

Example 8. Let

$$(135) \quad V_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 4 \\ 3 & \frac{1}{2} \end{pmatrix}$$

Then $Av = \beta v$ leads us to

$$(136) \quad v_1 + 4v_2 = \beta v_1$$

$$(137) \quad 3v_1 + \frac{1}{2}v_2 = \beta v_2$$

The eigenvalues are

$$\frac{3}{4} \pm \frac{1}{4}\sqrt{193}$$

with eigenvectors

$$\frac{1}{1 \pm \frac{1}{12} + \frac{1}{12}\sqrt{193}} \left(\frac{1}{12} \pm \frac{1}{12}\sqrt{193}, 1 \right)$$

Then we have $\beta = \frac{3}{4} + \frac{1}{4}\sqrt{193}$, $v = \frac{1}{1 + \frac{1}{12} + \frac{1}{12}\sqrt{193}} \left(\frac{1}{12} + \frac{1}{12}\sqrt{193}, 1 \right)$ such that $Av = \beta v$.

Let

$$(138) \quad H_1 = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ 0 & \sqrt{a_{12}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \sqrt{a_{21}} & 0 \\ 0 & \sqrt{a_{22}} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Then solving $\mathcal{L}_H(\rho) = \beta\rho$ gives us $\rho_{12} = 0$ and

$$(139) \quad \rho_{11} + 4\rho_{22} = \beta\rho_{11}$$

$$(140) \quad 3\rho_{11} + \frac{1}{2}\rho_{22} = \beta\rho_{22}$$

which is the same system as (136)-(137). So $\beta = \frac{3}{4} + \frac{1}{4}\sqrt{193}$ and the corresponding eigenstate, since $\rho_{12} = 0$, is

$$(141) \quad \rho = \begin{pmatrix} \frac{\frac{1}{12} + \frac{1}{12}\sqrt{193}}{1 + \frac{1}{12} + \frac{1}{12}\sqrt{193}} & 0 \\ 0 & \frac{1}{1 + \frac{1}{12} + \frac{1}{12}\sqrt{193}} \end{pmatrix}$$

◇

15. SOME CLASSIC INEQUALITY CALCULATIONS

A natural question is to ask whether the maximum among normalized W_i , $i = 1, \dots, k$, for the pressure problem associated to a given potential is realized as the logarithm of the main eigenvalue of a certain Ruelle operator associated to the potential H_i , $i = 1, \dots, k$. This problem will be considered in this section and also in the next one.

We begin by recalling a classic inequality. Consider

$$(142) \quad -\sum_{j=1}^k q_j \log q_j + \sum_{j=1}^k q_j \log r_j \leq 0$$

given by lemma 11. Let A be a matrix. If v denotes the left eigenvector of matrix E^A (such that each entry is $e^{a_{ij}}$), then $vE^A = \beta v$ can be written as

$$(143) \quad \sum_i v_i e^{a_{ij}} = \beta v_j, \quad \forall j$$

Define

$$(144) \quad r_{ij} := \frac{e^{a_{ij}} v_i}{\beta v_j}$$

So $\sum_i r_{ij} = 1$. Let $q_{ij} > 0$ such that $\sum_i q_{ij} = 1$. By (142), we have

$$(145) \quad -\sum_{i=1}^k q_{ij} \log q_{ij} + \sum_{i=1}^k q_{ij} \log \frac{e^{a_{ij}} v_i}{\beta v_j} \leq 0$$

That is,

$$(146) \quad -\sum_{i=1}^k q_{ij} \log q_{ij} + \sum_{i=1}^k q_{ij} a_{ij} + \sum_{i=1}^k q_{ij} (\log v_i - \log v_j) \leq \log \beta$$

Let Q be a matrix with entries q_{ij} , let $\pi = (\pi_1, \dots, \pi_k)$ be the stationary vector associated to Q . Since $\sum_i q_{ij} = 1$, Q is column-stochastic so we write $Q\pi = \pi$. Multiplying the above inequality by π_j and summing the j index, we get

$$(147) \quad -\sum_j \pi_j \sum_i q_{ij} \log q_{ij} + \sum_j \pi_j \sum_i q_{ij} a_{ij} + \sum_j \pi_j \sum_i q_{ij} (\log v_i - \log v_j) \leq \log \beta$$

In coordinates, $Q\pi = \pi$ is $\sum_j q_{ij} \pi_j = \pi_i$, for all i . Then

$$(148) \quad -\sum_j \pi_j \sum_i q_{ij} \log q_{ij} + \sum_j \pi_j \sum_i q_{ij} a_{ij} + \sum_j \pi_j \sum_i q_{ij} \log v_i - \sum_j \pi_j \sum_i q_{ij} \log v_j \leq \log \beta$$

These calculations are well-known and gives us the following inequality:

$$(149) \quad -\sum_j \pi_j \sum_i q_{ij} \log q_{ij} + \sum_j \pi_j \sum_i q_{ij} a_{ij} \leq \log \beta$$

Definition 11. We call inequality (149) the **classic inequality** associated to the matrix A with positive entries, and stochastic matrix Q .

Definition 12. For fixed k , and $l, m = 1, \dots, k$ we call the inequality

$$(150) \quad h_V(W) + \sum_{j=1}^k \text{tr}(W_j \rho_W W_j^*) \log \text{tr}(H_j \rho_\beta H_j^*) \\ + \sum_{j=1}^k \text{tr}(W_j \rho_W W_j^*) \log \left(\frac{(V_j \rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}} \right) \leq \log \beta,$$

the **basic inequality** associated to the potential $H\rho = \sum_i H_i \rho H_i^*$ and to the QIFS determined by $V_i, W_i, i = 1, \dots, k$. Equality holds if for all i, j, l, m ,

$$(151) \quad \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \frac{(V_j \rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}} = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

◇

As before, ρ_β is an eigenstate of $\mathcal{L}_H(\rho)$ and ρ_W is the barycenter of the unique attractive, invariant measure for the Markov operator \mathcal{V} associated to the QIFS \mathcal{F}_W . Given the classic inequality (149) we want to compare it to the basic inequality (150). More precisely, we would like to obtain operators V_i that satisfy the following: given a matrix A with positive entries and a stochastic matrix Q , there are H_i and W_i such that inequality (150) becomes inequality (149). We have the following proposition.

Proposition 9. Define

$$(152) \quad V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $A = (a_{ij})$ be a matrix with positive entries and $Q = (q_{ij})$ a two-dimensional column-stochastic matrix. Define

$$(153) \quad H_{11} = \begin{pmatrix} \sqrt{e^{a_{11}}} & \sqrt{e^{a_{11}}} \\ 0 & 0 \end{pmatrix}, \quad H_{12} = \begin{pmatrix} \sqrt{e^{a_{12}}} & \sqrt{e^{a_{12}}} \\ 0 & 0 \end{pmatrix}$$

$$(154) \quad H_{21} = \begin{pmatrix} 0 & 0 \\ \sqrt{e^{a_{21}}} & \sqrt{e^{a_{21}}} \end{pmatrix}, \quad H_{22} = \begin{pmatrix} 0 & 0 \\ \sqrt{e^{a_{22}}} & \sqrt{e^{a_{22}}} \end{pmatrix}$$

and also

$$(155) \quad W_1 = \begin{pmatrix} \sqrt{q_{11}} & 0 \\ 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & \sqrt{q_{12}} \\ 0 & 0 \end{pmatrix}$$

$$(156) \quad W_3 = \begin{pmatrix} 0 & 0 \\ \sqrt{q_{21}} & 0 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{q_{22}} \end{pmatrix}$$

Then the basic inequality associated to $W_i, V_i, H_i, i = 1, \dots, 4, l = m = 1$ or $l = m = 2$, is equivalent to the classic inequality associated to A and Q .

We use the following lemma.

Lemma 12. For V_i given by

$$(157) \quad V_1 = \begin{pmatrix} \sqrt{v_{11}} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \sqrt{v_{12}} \\ 0 & 0 \end{pmatrix}$$

$$(158) \quad V_3 = \begin{pmatrix} 0 & 0 \\ \sqrt{v_{21}} & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{v_{22}} \end{pmatrix}$$

where $v_{ij} > 0$, we have that the associated QIFS is such that ρ_W and ρ_β are diagonal density operators, for any choice of W_i and H_i , $i = 1, \dots, 4$.

Proof of Lemma 12 We have that ρ_W is a fixed point of

$$\Lambda(\rho) = \sum_i \text{tr}(W_i \rho W_i^*) \frac{V_i \rho V_i^*}{\text{tr}(V_i \rho V_i^*)}$$

Writing

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{pmatrix},$$

we have that $\Lambda(\rho) = \rho$ leads us to

$$\begin{aligned} & \frac{\text{tr}(W_1 \rho W_1^*)}{\text{tr}(V_1 \rho V_1^*)} \begin{pmatrix} v_{11} \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} + \frac{\text{tr}(W_2 \rho W_2^*)}{\text{tr}(V_2 \rho V_2^*)} \begin{pmatrix} v_{12} \rho_{22} & 0 \\ 0 & 0 \end{pmatrix} \\ & + \frac{\text{tr}(W_3 \rho W_3^*)}{\text{tr}(V_3 \rho V_3^*)} \begin{pmatrix} 0 & 0 \\ 0 & v_{21} \rho_{11} \end{pmatrix} + \frac{\text{tr}(W_4 \rho W_4^*)}{\text{tr}(V_4 \rho V_4^*)} \begin{pmatrix} 0 & 0 \\ 0 & v_{22} \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{pmatrix} \end{aligned}$$

Then $\rho_{12} = 0$ and so ρ_W is diagonal. In a similar way we prove ρ_β is diagonal. \square

Proof of Proposition 9 Let V_i, W_i , $i = 1, \dots, 4$ and H_{ij} , $i, j = 1, 2$ as in the statement of the proposition. A simple calculation shows that

$$(159) \quad \text{tr}(H_{ij} \rho_\beta H_{ij}^*) = e^{a_{ij}}$$

(since ρ_β is diagonal, by lemma 12). By example 5, the choice of V_i and W_i we made is such that the entropy $h_V(W)$ reduces to the Markov chain entropy. Then a calculation yields

$$(160) \quad \frac{(V_i \rho_\beta V_i^*)_{11}}{(\rho_\beta)_{11}} = \frac{(\rho_\beta)_{11}}{(\rho_\beta)_{11}} = 1$$

In a similar way,

$$(161) \quad \frac{(V_i \rho_\beta V_i^*)_{22}}{(\rho_\beta)_{22}} = \frac{(\rho_\beta)_{22}}{(\rho_\beta)_{22}} = 1$$

Then from the basic inequality with $l = m = 1$ or $l = m = 2$ we get

$$(162) \quad h_V(W) + \sum_j \text{tr}(W_j \rho_W W_j^*) \sum_i \frac{\text{tr}(W_i V_j \rho_W V_j^* W_i^*)}{\text{tr}(V_j \rho_W V_j^*)} \log \text{tr}(H_i \rho_\beta H_i^*) \leq \log \beta$$

Finally, since $\text{tr}(H_{ij} \rho_\beta H_{ij}^*) = e^{a_{ij}}$ and $Q\pi = \pi$, we conclude that (162) becomes (149). \square

Example 9. Let

$$H_1 = \begin{pmatrix} 2i & 2i \\ 0 & 0 \end{pmatrix}, \quad H_2 = I, \quad H_3 = \begin{pmatrix} i\sqrt{2} & i\sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad H_4 = I$$

Then

$$H_1^* = \begin{pmatrix} -2i & 0 \\ -2i & 0 \end{pmatrix}, \quad H_2^* = I, \quad H_3^* = \begin{pmatrix} -i\sqrt{2} & 0 \\ -i\sqrt{2} & 0 \end{pmatrix}, \quad H_4^* = I$$

If we suppose the V_i are the same as from proposition 9, we have that ρ_β is diagonal, so

$$\text{tr}(H_1\rho_\beta H_1^*) = 4, \quad \text{tr}(H_2\rho_\beta H_2^*) = 1, \quad \text{tr}(H_3\rho_\beta H_3^*) = 2, \quad \text{tr}(H_4\rho_\beta H_4^*) = 1$$

Then $\mathcal{L}_H(\rho) = \beta\rho$ leads us to

$$4\rho_{11} + \rho_{22} = \beta\rho_{11}$$

$$2\rho_{11} + \rho_{22} = \beta\rho_{22}$$

A simple calculation gives

$$\beta = \frac{5 + \sqrt{17}}{2}$$

with eigenstate

$$\rho_\beta = \frac{4}{7 + \sqrt{17}} \begin{pmatrix} \frac{3 + \sqrt{17}}{4} & 0 \\ 0 & 1 \end{pmatrix}$$

◇

We want to calculate the W_i which maximize the basic inequality (150). Recall that from proposition 9, the choice of V_i we made is such that

$$\frac{(V_j\rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}} = 1,$$

So

$$(163) \quad h_V(W) + \sum_{j=1}^k \text{tr}(W_j\rho_W W_j^*) \log \text{tr}(H_j\rho_\beta H_j^*) \leq \log \beta$$

and equality holds if and only if, for all i, j, l, m ,

$$(164) \quad \frac{1}{\beta} \text{tr}(H_j\rho_\beta H_j^*) \frac{(V_j\rho_\beta V_j^*)_{lm}}{(\rho_\beta)_{lm}} = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

Choose, for instance, $l = m = 1$. Then condition (164) becomes

$$(165) \quad \frac{1}{\beta} \text{tr}(H_j\rho_\beta H_j^*) = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

To simplify calculations, write $\widehat{W}_i = W_i^* W_i$ and $\widehat{W}_i = (w_{ij}^i)$. Then we get

$$(166) \quad \frac{\text{tr}(H_i\rho_\beta H_i^*)}{\beta} = w_{11}^i = w_{22}^i, \quad i = 1, \dots, 4$$

So we conclude

$$(167) \quad W_i = \frac{1}{\sqrt{\beta}} \begin{pmatrix} \sqrt{\text{tr}(H_i\rho_\beta H_i^*)} & 0 \\ 0 & \sqrt{\text{tr}(H_i\rho_\beta H_i^*)} \end{pmatrix}, \quad i = 1, \dots, 4$$

That is,

$$(168) \quad W_1 = \frac{2}{\sqrt{\beta}}I, \quad W_2 = \frac{1}{\sqrt{\beta}}I, \quad W_3 = \frac{\sqrt{2}}{\sqrt{\beta}}I, \quad W_4 = \frac{1}{\sqrt{\beta}}I$$

Note that

$$\sum_i W_i^* W_i = \frac{4 + \sqrt{2}}{\sqrt{\beta}}I \neq I$$

To solve that, we renormalize the potential. Define

$$(169) \quad \tilde{H}_i := \sqrt{\alpha}H_i, \quad \alpha := \frac{\sqrt{\beta}}{4 + \sqrt{2}}$$

Then a calculation shows that $\mathcal{L}_{\tilde{H}}(\rho) = \tilde{\beta}\rho$ gives us the same eigenstate as before, that is $\rho_{\tilde{\beta}} = \rho_{\beta}$. But note that the associated eigenvalue becomes $\tilde{\beta} = \alpha\beta$. Now, note that it is possible to renormalize the W_i in such a way that we obtain \tilde{W}_i with $\sum_i \tilde{W}_i^* \tilde{W}_i = I$, and that these maximize the basic inequality for the H_i initially fixed. In fact, given the renormalized \tilde{H}_i , define

$$(170) \quad \tilde{W}_i = \sqrt{\alpha}W_i, \quad i = 1, \dots, 4$$

Note that $\sum_i \tilde{W}_i^* \tilde{W}_i = I$. Also we obtain

$$(171) \quad h_V(\tilde{W}) + \sum_{j=1}^k \text{tr}(\tilde{W}_j \rho_{\tilde{W}} \tilde{W}_j^*) \log \text{tr}(\sqrt{\alpha}H_j \rho_{\beta} \sqrt{\alpha}H_j^*) \leq \log \alpha \beta$$

which is equivalent to

$$(172) \quad h_V(\tilde{W}) + \sum_{j=1}^k \text{tr}(\tilde{W}_j \rho_{\tilde{W}} \tilde{W}_j^*) \log(\alpha \text{tr}(H_j \rho_{\beta} H_j^*)) \leq \log \alpha + \log \beta$$

That is

$$(173) \quad h_V(\tilde{W}) + \sum_{j=1}^k \text{tr}(\tilde{W}_j \rho_{\tilde{W}} \tilde{W}_j^*) \log \alpha + \sum_{j=1}^k \text{tr}(\tilde{W}_j \rho_{\tilde{W}} \tilde{W}_j^*) \log \text{tr}(H_j \rho_{\beta} H_j^*) \leq \log \alpha + \log \beta,$$

and if we cancel $\log \alpha$ on both sides, we get the same inequality as for the non-renormalized H_i . As we have seen before, such \tilde{W}_i gives us equality. Hence

$$(174) \quad h_V(\tilde{W}) + \sum_{j=1}^k \text{tr}(\tilde{W}_j \rho_{\tilde{W}} \tilde{W}_j^*) \log \text{tr}(H_j \rho_{\beta} H_j^*) = \log \beta$$

◇

16. REMARKS ON THE PROBLEM OF PRESSURE AND QUANTUM MECHANICS

One of the questions we are interested in is to understand how to formulate a variational principle for pressure in the context of quantum information theory. An appropriate combination of such theories could have as a starting point a relation between the inequality for positive numbers

$$-\sum_i q_i \log q_i + \sum_i q_i \log p_i \leq 0,$$

(lemma 11, seen in certain proofs of the variational principle of pressure), and the QIFS entropy. We have carried out such a plan and then we have obtained the basic inequality, which can be written as

$$(175) \quad h_V(W) + \sum_{j=1}^k \log \left(\text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) \right) \text{tr}(W_j \rho_W W_j^*) \leq \log \beta$$

where equality holds if and only if for all i, j ,

$$(176) \quad \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) \text{tr}(V_j \rho_\beta V_j^*) = \frac{\text{tr}(W_j V_i \rho_W V_i^* W_j^*)}{\text{tr}(V_i \rho_W V_i^*)}$$

As we have discussed before, it is not clear that given any dynamics, we can obtain a measure such that we can reach the maximum value $\log \beta$. Considering particular cases we can suppose, for instance, that the V_i are unitary. In this way we combine in a natural way a problem of classic thermodynamics, with an evolution which has a quantum character. In this particular setting, we have for each i that $V_i V_i^* = V_i^* V_i = I$ and then the basic inequality becomes

$$(177) \quad h_V(W) + \sum_{j=1}^k \text{tr}(W_j \rho_W W_j^*) \log \text{tr}(H_j \rho_\beta H_j^*) \leq \log \beta$$

and equality holds if and only if for all i, j ,

$$(178) \quad \frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*) = \text{tr}(W_j V_i \rho_W V_i^* W_j^*)$$

We have the following:

Lemma 13. *Given a QIFS with a unitary dynamics (i.e., V_i is unitary for each i), there are \hat{W}_i which maximize (175), i.e., such that*

$$(179) \quad h_V(\hat{W}) + \sum_{j=1}^k \text{tr}(\hat{W}_j \rho_{\hat{W}} \hat{W}_j^*) \log \text{tr}(H_j \rho_\beta H_j^*) = \log \beta$$

Proof Define, for each j ,

$$(180) \quad \hat{W}_j := \sqrt{\frac{1}{\beta} \text{tr}(H_j \rho_\beta H_j^*)} I$$

where I is the identity. The equality condition (178) is satisfied by such \hat{W}_j , so the lemma follows. □

Remark The above lemma also holds for the basic inequality in coordinates, given by (150). Also, it is immediate to obtain a similar version of the above lemma for any QIFS such that the V_i are multiples of the identity, and also for QIFS such that ρ_W fixes each branch of the QIFS, that is, satisfying, for each i ,

$$\frac{V_i \rho_W V_i^*}{\text{tr}(V_i \rho_W V_i^*)} = \rho_W$$

◇

17. CONCLUDING REMARKS

Considering the QIFS setting, we defined a concept of entropy and a Ruelle operator in such a way that we are able to get some analogous results to the classical Thermodynamic Formalism. Such Ruelle operator admits a positive eigenvalue, which gives us an upper bound for the pressure (entropy plus a potential) associated to the QIFS. Our configuration space is the set of density matrices. We did not consider the usual space of symbols or a shift operator, as it is assumed in the Ruelle-Perron-Frobenius theory. We have replaced the dynamics given by the shift with the one given by the inverse branches of the iterated functions (which are defined by a set of operators).

The references [16] and [21] are of fundamental importance in our investigation.

A starting point for further investigation could be to study more properties of the QIFS entropy, such as convexity and subadditivity. Also, a natural question is to ask whether it is possible to consider a QIFS acting in an infinite tensor product of finite Hilbert spaces which would be the analogous of considering the full Bernoulli space.

In a forthcoming paper we are going to consider relative entropies and quantum conditional expectations.

REFERENCES

- [1] Baraviera, A., Lardizabal, C. F., Lopes, A. O., Terra Cunha, M. A dynamical point of view of Quantum Information: entropy, pressure and Wigner measures. arXiv:0911.0193. To appear on "Dynamics, Games and Science in honour of Mauricio Peixoto and David Rand".
- [2] Baraviera, A., Lardizabal, C. F., Lopes, A. O., Terra Cunha, M. Quantum Stochastic Processes, Quantum Iterated Function Systems and Entropy. arXiv:0911.0182. To appear on São Paulo Journal of Mathematical Sciences.
- [3] Bengtsson, I., Życzkowski, K. Geometry of Quantum States. Cambridge University Press, 2006.
- [4] Benatti, F. Dynamics, Information and Complexity in Quantum Systems. Springer Verlag, 2009
- [5] Benenti, G., Casati, G. Strini, G. Principles of Quantum Computation and Information, Vol I and II, World scientific, 2007.
- [6] Bruzda W., Cappellini, V., Sommers, H.-J., Życzkowski, K. Random Quantum Operations. Phys. Lett. A 373, 320-324 (2009).
- [7] Busch, P., Ruch, E. The measure cone: irreversibility as a geometrical phenomenon, Int. J. Quantum Chemistry. 41, 163-185, 1992.
- [8] Craizer, M, Lopes, A. O. The capacity-cost function of a hard-constrained channel. Int. Journal of Appl. Math. Vol 2, N 10 pp 1165-1180 (2000).
- [9] Gray, R. M. Entropy and information theory. Springer-Verlag, New York, 1990.

- [10] Hastings, M. B. A counterexample to additivity of minimum output entropy. arXiv:0809.3972v3 [quant-ph], 2008.
- [11] Hayashi, M. Capacity with energy constraint in coherent state channel. arXiv:0904.0307v1 [quant-ph], 2009.
- [12] Hayashi, M., Nagaoka, H. General formulas for capacity of classical-quantum channels. IEEE Transactions on Information Theory, 7, v. 49, 2003.
- [13] Jadczyk, A. On Quantum Iterated Function Systems, CEJP 2, 492-503 (2004).
- [14] Lopes, A., Oliveira, E. Entropy and variational principles for holonomic probabilities of IFS. Discrete and Continuous Dynamical Systems Vol. 23, N, 3, 937-955 (2009) Series A.
- [15] Fisher, A., Lopes, A. O. Exact bounds for the polynomial decay of correlation, 1/f noise and the central limit theorem for a non-Holder Potential. Nonlinearity, Vol 14, Number 5, pp 1071-1104 (2001).
- [16] Lozinski, A., Życzkowski, K., Słomczyński, W. Quantum iterated function systems, Physical Review E, Volume 68, 04610, 2003.
- [17] Nielsen, M., Chuang, I. Quantum computation and quantum information. Cambridge University Press, 2000.
- [18] Parry, W., Pollicott, M. Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics. Société Mathématique de France. 187-188, Astérisque, 1990.
- [19] Shor, P. W. Equivalence of additivity question in quantum information theory. Comm. Math. Phys. 246, 453-472 (2004).
- [20] Słomczyński, W., Życzkowski, K. Quantum Chaos: an entropy approach. J. Math. Physics, 32 (1), 1994, p. 5674-5700.
- [21] Słomczyński, W., Dynamical Entropy, Markov Operators and Iterated Function Systems. Jagiellonian University Press, 2003.
- [22] Winkler, G., Choquet Order and Simplices. Lecture notes in Mathematics 1145. Springer-Verlag, Berlin, 1985.
- [23] Young, L-S. Recurrence times and rates of mixing. Israel J. Math. 110 pp 153-188, 1999.