PARAMETER ESTIMATION IN MANNEVILLE-POMEAU PROCESSES

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Abstract

In this work we study a class of stochastic processes $\{X_t\}_{t\in\mathbb{N}}$, where $X_t = (\varphi \circ T_s^t)(X_0)$ is obtained from the iterations of the transformation T_s, invariant for an ergodic probability μ_s on $[0,1]$ and a certain constant by part function $\varphi : [0,1] \to \mathbb{R}$. We consider here the family of transformations $T_s : [0,1] \to [0,1]$, indexed by a parameter $s > 0$, known as the Manneville-Pomeau family of transformations. The autocorrelation function of the resulting process decays hyperbolically (or polynomially) and we obtain efficient methods to estimate the parameter s from a finite time series. As a consequence, we also estimate the rate of convergence of the autocorrelation decay of these processes. We compare different estimation methods based on the periodogram function, the smoothed periodogram function, the variance of the partial sum, and the wavelet theory. To obtain our results we analyzed the properties of the spectral density function and the associated Fourier Series.

Key words: Manneville-Pomeau Maps, Long and Not so Long Dependence, Estimation, Autocorrelation Decay, Spectral Density Function.

Mathematics Subject Classification: 62M15, 62M10, 37A05, 37A50 and 37E10.

1 Introduction

The goal of this paper is to estimate the main parameter of some processes obtained from iterations of Manneville-Pomeau maps.

We consider a class of stochastic processes $\{X_t\}_{t\in\mathbb{N}}$, where $X_t = (\varphi \circ T_s^t)(X_0)$ is obtained from the iterations of the transformation T_s, invariant for an ergodic

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probability μ_s on $[0,1]$ and a continuous by part function $\varphi : [0,1] \to \mathbb{R}$. The transformation $T_s : [0,1] \to [0,1]$, $s \in (0,1)$, is considered here as the Manneville-Pomeau map. We analyze the rate of decay of the autocorrelation function for the resulting process. The rate of convergence decays hyperbolically (or polynomially) not exponentially. We obtain efficient methods to estimate the parameter s from a finite time series. As a consequence, we also estimate the rate of convergence of the autocorrelation decay of these processes.

Indeed, given s the decay is known: Young (1999) has shown that the autocorrelation decay of the Manneville-Pomeau processes has an order smaller than $n^{1-\frac{1}{s}}$, for $0 < s$. Other models which have similar properties to the Manneville-Pomeau map are the linear by part approximation of the same map (see Fisher and Lopes, 2001) and the Markov Chain with infinite symbols, described in Lopes (1993).

Models of different phenomena in nature present autocorrelation decay of the form $n^{-\beta}$, also called hyperbolic (or polynomial) decay: the use of the Markov Chain model seems to be appropriate for the analysis of DNA sequences (see Peng et al., 1992 and 1996 and Guharay et al., 2000); cardiac rhythm fluctuations (see Absil et al., 1999 and Peng et al., 1996); turbulence (see Schuster, 1984) and economy (see Mandelbrot, 1997; Lopes et al., 2004 and Lopes, 2007). In most cases, the exact rate of convergence of the autocorrelation function decay is relevant information in the model. Here we are interested in comparing different methods for estimating such β in the case of the Manneville-Pomeau processes.

When $0.5 < s < 1.0$ we have the *long-range dependence regime*. *Fractionally integrated autoregressive moving average* (ARFIMA) models also present such behavior (see Beran et al., 2013; Geweke and Porter-Hudak, 1983; Reisen and Lopes, 1999, Lopes et al., 2004 and Lopes, 2008). The corresponding parameter for the ARFIMA model is $d = 1 - \frac{1}{2s}$. The ARFIMA process has an explicit formula for the spectral density function $f_X(\cdot)$ (see Reisen et al., 2001; Lopes et al., 2002, Olbermann et al., 2006 and Lopes, 2008) but this is not the case for the processes considered here.

When $0 < s < 0.5$ we have the *not so long dependence regime*. The so-called *intermediate dependence regime* happens when $s \in (\frac{1}{3}, \frac{1}{2})$.

Recently several interesting papers appear describing the statistics of time series obtained from dynamical systems: Freitas et al. (2018), Korepanov et al. (2021), Chazottes et al. (1998), Chazottes et al. (2005), Collet et al. (1995), Collet et al. (2004) and Collet (2005). We also refer the reader to the last sections of the book by Collet and Eckmann (2006).

Here we analyze and compare several estimation procedures based on the periodogram function, on the smoothed periodogram function, on the variance of the partial sum and on the wavelet theory.

The paper is organized as follows. In Section 2 we define the Manneville-Pomeau maps and give some definitions, basic properties, and results. Section 3 presents the Manneville-Pomeau processes that will be the setting of the estimation procedures we choose in this work. In Section 4 we consider the estimation procedures for the long dependence case while in Section 6 we present the Monte Carlo simulation study for this regime. In Section 5 we consider the estimation procedures for the not-so-long dependence case while Section 7 presents the Monte Carlo simulation study for this other regime. Section 8 contains a summary of the paper. In appendix A we consider some general properties of the Fourier series which are necessary for the paper. Appendix B contains the theoretical reasoning for some of the estimation procedures proposed in Section 4 of the paper.

2 Manneville-Pomeau Maps

In this section, we present the Manneville-Pomeau maps, some definitions, basic properties, and results.

We first define the Manneville-Pomeau transformation and we give some of its properties.

Definition 2.1: Let $T_s : [0,1] \to [0,1]$ be the *Manneville-Pomeau map* given by

$$T_s(x) = x + x^{1+s} \pmod 1 = \begin{cases} x + x^{1+s}, & \text{if} \quad x + x^{1+s} \leq 1 \\ x + x^{1+s} - 1, & \text{if} \quad x + x^{1+s} > 1, \end{cases} \tag{2.1}$$

where s is a positive constant.

As usual, we shall use the following notation

$$T_s^t \equiv \underbrace{T_s \circ \cdots \circ T_s}_{t-times}.$$

The map T_s (see Figure 2.1 (a)), given by the expression (2.1) has the following properties:

- T_s is a piecewise monotone function with two full branches, that is, there exists $p \in \mathbb{N} - \{0\}$ such that $T_s|_{(0,p)}$ and $T_s|_{(p,1)}$ are strictly monotone, continuous and $T_s((0,p)) = (0,1) = T_s((p,1))$, where $p + p^{1+s} = 1$.

- The branches $T_s|_{(0,p)}$ and $T_s|_{(p,1)}$ are C^2.
- $T_s'(x) > 1$, for all $x > 0$, and $T_s'(x) \geq \lambda > 1$, for $x \in (p, 1)$.
- T_s has a unique indifferent fixed point 0. Therefore, $T_s(0) = 0$ and $|T_s'(0)| = 1$.
- There exists an invariant absolutely continuous ergodic measure μ_s for the Manneville-Pomeau transformation T_s. Thaler (1980), using the properties of the Manneville-Pomeau map, shows that $d\,\mu_s(x) \equiv h_s(x)\,dx$, where $h_s(x) \approx x^{-s}$, for $x \in (0, 1)$, close to 0.

  When $s \geq 1$, the measure μ_s has infinite mass and it is not a probability.

  When $0 < s < 1$, the probability μ_s is *mixing* for $T_s : [0, 1] \to [0, 1]$ (see Young, 1999; and Fisher and Lopes, 2001).

Given a continuous by part function $\varphi : [0, 1] \to \mathbb{R}$, one can consider the random variables $X_t = (\varphi \circ T_s^t)(X_0)$, for $t \in \mathbb{N}$, where X_0 is distributed according to the probability μ_s. The stationary stochastic process $\{X_t\}_{t\in\mathbb{N}}$ is called the *Manneville-Pomeau process*. We will consider here φ as an indicator function of an interval in $[0, 1]$. In this case, the time series obtained from the process $\{X_t\}_{t\in\mathbb{N}}$ will be a binary time series of 0's and 1's only.

It is known that the autocorrelation decay of the Manneville-Pomeau processes, given by the expression (3.1), have an order smaller than $n^{1-\frac{1}{s}}$, for $0 < s < 0.5$ (see Young, 1999). In Fisher and Lopes (2001) it is shown, that for the linear by-part model given by Definition 2.2 below, that these bounds are exact (for the corresponding values).

We refer the reader to Maes et al. (1999) for more details on the dynamics of the system given by (2.1).

Other models which have similar properties to the Manneville-Pomeau map is the linear by-part approximation of the same map (see Definition 2.2 below and Fisher and Lopes, 2001 and Wang, 1989) and the Markov Chain with infinite symbols (see Definition 2.3 below) described in Lopes (1993). The use of the Markov Chain model $\{Y_t\}_{t\in\mathbb{N}}$, defined below, seems to be appropriated for the analysis of DNA sequences (see Peng et al., 1992 and 1996). The same estimation methods, proposed for the Manneville-Pomeau processes in Section 4 can be also applied to these other models.

Definition 2.2: Let $\zeta(\gamma) = \sum_{n\geq 1} n^{-\gamma}$ be the *Riemann zeta* function. Consider the partition in intervals of $[0,1]$ given by

$$M_0 = \left(1 - \frac{1}{\zeta(\gamma)}, 1\right) \text{ and } M_k = \left(1 - \frac{1}{\zeta(\gamma)} \sum_{n=1}^{k-1} n^{-\gamma}, 1 - \frac{1}{\zeta(\gamma)} \sum_{n=1}^{k} n^{-\gamma}\right),$$

for $k \geq 1$. For $\gamma > 2$, we define the following linear by part transformation $T_\gamma : [0,1] \to [0,1]$ such that over the interval M_k, for $k \geq 1$, T_γ has slope $((k+1)k^{-1})^\gamma$ and over the interval M_0 it has slope $\zeta(\gamma)$. We assume that the branches

$$T_\gamma|_{\left(0,1-\frac{1}{\zeta(\gamma)}\right)} \text{ and } T_\gamma|_{M_0}$$

are continuous; under these assumptions the transformation T_γ is uniquely defined (see Figure 2.1 (b)). The transformation T_γ is called *the linear by part approximation of the Manneville-Pomeau map.*

In the same way as before, given a continuous by-part function defined by $\varphi : [0,1] \to \mathbb{R}$, one can consider the random variables $X_t = (\varphi \circ T_\gamma^t)(X_0)$, for $t \in \mathbb{N}$, where X_0 is distributed according to a certain probability μ_γ, invariant for T_γ. The probability μ_γ is absolutely continuous with respect to the Lebesgue measure. We call $\{X_t\}_{t\in\mathbb{N}}$ the *linear by part approximation of the Manneville-Pomeau process*.

Each value of s for the Manneville-Pomeau map corresponds to a value $\gamma = 1 + \frac{1}{s}$ with the same behavior with respect to the autocorrelation decay.

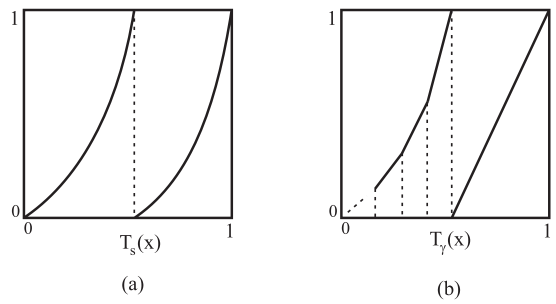

Figure 2.1: (a) Manneville-Pomeau T_s transformation; (b) its linear by part approximation T_γ transformation.

The Manneville-Pomeau map has the advantage of being more suitable than the linear by-part model for computer implementation when one is interested in Monte Carlo simulations. For this reason, in the simulation sections, we will concentrate our analysis on such a model.

Below we define a Markov process with state $\mathbb{N}$ based on a certain transition probability matrix $\mathbf{P}$. The time evolution of such a process will also have similarities with the iteration of Manneville-Pomeau map.

Definition 2.3: Let $\mathbf{P}$ be a Markov chain with infinite transition probability matrix $\mathbf{P}$=$(\mathbb{P}(i,j))_{i,j\in\mathbb{N}}$ (see page 153 in Lopes, 1993; Wang, 1989 and Feller, 1949) with transition probabilities given by

$$\mathbb{P}(n, n-1) = 1, \text{ for all } n \in \mathbb{N} - \{0\},$$

$$\mathbb{P}(n, j) = 0, \text{ for } j \neq n-1,$$

and

$$\mathbb{P}(0, n) = \frac{(n+1)^{-\gamma}}{\zeta(\gamma)},$$

where $\zeta(\gamma)$ is the Riemann zeta function and $\gamma > 2$. There exists an explicit formula for the eigenvector π_0 associated with the eigenvalue 1 (see page 154 in Lopes, 1993).

Let $\{Z_t\}_{t\in\mathbb{N}}$ be the stationary stochastic Markov process obtained from the transition matrix $\mathbf{P}$ above and from the initial stationary distribution π_0. Let $\mathbb{I}_0$ be the indicator function of the set $A = \{0\}$ on $\mathbb{N}$. Let now $\{Y_t\}_{t\in\mathbb{N}}$ be the process $1-\mathbb{I}_0(Z_t)$. In this way, we identify paths $\omega \in \mathbb{N}^{\mathbb{N}}$ with paths $\tilde{\omega} \in \{0,1\}^{\mathbb{N}}$. Then, $\{Y_t\}_{t\in\mathbb{N}}$ is a stochastic process with random variables assuming only the values 0 and 1. For the process $\{Y_t\}_{t\in\mathbb{N}}$ consider the probability induced by the process $\{Z_t\}_{t\in\mathbb{N}}$ by means of the identification of the paths.

To clarify the ideas in the above Definition 2.3, the following example shows the identification paths in $\mathbb{N}^{\mathbb{N}}$ to paths in $\{0,1\}^{\mathbb{N}}$.

Example 2.1: Let $\{Z_t\}_{t\in\mathbb{N}}$ be the process where a sample path $w \in \mathbb{N}^{\mathbb{N}}$, for instance, $w = \{0765432109876543210543210\cdots\}$, is associated with another sample path of the process $\{Y_t\}_{t\in\mathbb{N}}$. The corresponding sample path for the process $\{Y_t\}_{t\in\mathbb{N}}$ is given by

$$\tilde{w} = \{0\underbrace{1111111}_{7}0\underbrace{111111111}_{9}0\underbrace{11111}_{5}0\cdots\}.$$

Hence, we applied the change of coordinates $Z_t \to Y_t$ associating sequences of natural numbers to blocks of 1 intercalated by 0, in such a way that the structure of the process is kept the same.

We say that two different stochastic processes are *equivalent* when there is a bijective change of coordinates acting in the set of paths transferring the probability of one process into the other.

The process $\{Z_t\}_{t\in\mathbb{N}}$ is, by definition, *equivalent* to the process $\{Y_t\}_{t\in\mathbb{N}}$ by the above change of coordinates. One can also show that Y_t is also *equivalent* to $X_t = (\varphi \circ T_\gamma^t)(X_0)$ (see section 4 in Lopes, 1993 with $\varphi \equiv \mathbb{I}_{M_0}$).

The idea of using Markov Chain arguments by linearizing the Manneville-Pomeau maps has been considered by Gaspard and Wang (1988), Lambert et al. (1993), and Bahsoun et al. (2015), but it was used for different purposes other than ours. We are interested in estimating the parameters of this class of maps.

In Gaspard and Wang (1988), the authors were interested in the asymptotic growth of the Kolmogorov algorithmic complexity of a string of symbols S_n, when n goes to infinity. They were able to show results on non-Gaussian fluctuations for the Manneville-Pomeau map based on this linearization.

In Lambert et al. (1993) the purpose was to present a power-law upper bound for the decay of the correlations for Hölder observables, and rates of mixing, when the dynamics are given by the Manneville-Pomeau map.

In Bahsoun et al. (2015) it is presented a numerical procedure (using a Ulam-type discretization scheme) to provide pointwise approximations for the invariant density of a Manneville-Pomeau map. They were able to show the exact rate of convergence based on the mesh size of the approximation.

It is also known that the central limit theorem (converging to a Gaussian distribution) is true for the Manneville-Pomeau stochastic process $\{X_t\}_{t\in\mathbb{N}}$, described in Section 3, when $0 < s < 0.5$ due to the rate of convergence of the autocorrelation decay (see Young, 1999; Lopes, 1993 and pages 1099-1100 in Fisher and Lopes, 2001).

When $0.5 < s < 1.0$ it was conjectured that for the Manneville-Pomeau stochastic process $\{X_t\}_{t\in\mathbb{N}}$ the central limit theorem is true, but it converges to a stable law with parameter $\alpha = s^{-1}$. This was proved by Gouëzel (2004). From Feller (1949) it is known for the corresponding parameter of the Markov Chain model described above (or for the equivalent process $X_t = (\varphi \circ T_\gamma^t)(X_0)$ with $\varphi \equiv \mathbb{I}_{M_0}$ (see Wang, 1989 or section 4 in Lopes, 1993, for more details)).

For the estimation in the *long-range dependence case*, one has to consider larger sample sizes for the time series. In this situation, in general, the computation effort for obtaining good results is very high. This is something that one can not avoid due to the small rate of convergence of decay. The mixing rate is not as good as it happens, for instance, when one considers models with exponential autocorrelation decay. We present here several quite efficient methods to obtain reasonable results. One method is by using the periodogram function described in Sections 4 and 6. The method based on wavelet works fine in several cases and surprisingly can also be applied to estimate s when $s \geq 1.0$ (see Sections 4 and 6).

The paper Lopes and Pinheiro (2009) presents a bias correction for the wavelet estimation in the *long* and *not so long dependence cases.*

3 Manneville-Pomeau Process and Some of its Properties

In this section, we define the Manneville-Pomeau stochastic processes and present some of their properties.

Let $\varphi : [0,1] \to \mathbb{R}$ be a μ_s-integrable function and $T_s(\cdot)$ the Manneville-Pomeau transformation given by the expression (2.1). The *Manneville-Pomeau stochastic process* $\{X_t\}_{t\in\mathbb{N}}$ is given by

$$X_t = (\varphi \circ T_s^t)(X_0) = \varphi(T_s^t(X_0)) = \varphi(T_s(X_{t-1})) = (\varphi \circ T_s)(X_{t-1}), \qquad (3.1)$$

for all $t \in \mathbb{N}$, where X_0 is distributed according to the measure μ_s. In other words, the Manneville-Pomeau process $\{X_t\}_{t\in\mathbb{N}}$ is obtained applying φ to the iterations of T_s, that is, $X_t = \varphi \circ T_s^t$, for s fixed and $t \in \mathbb{N}$.

We shall consider here only the case where φ is the indicator function $\mathbb{I}_A$ of an interval A contained in $[0,1]$ or else $\varphi = \mathbb{I}_A - \mu_s(A)$. Our simulations, shown in Sections 5 and 7, will be done for the case where $A = [0.1, 0.9]$.

We shall denote by $\gamma_X(\cdot)$ the autocovariance function for the process $\{X_t\}_{t\in\mathbb{N}}$, that is,

$$\gamma_X(h) \equiv \mathbb{E}_\mu(X_h X_0) - [\mathbb{E}_\mu(X_0)]^2 = \int \varphi(T^h(x))\varphi(x)d\mu_s(x) - [\int \varphi(x)d\mu_s(x)]^2, \quad (3.2)$$

for $h \in \mathbb{N}$.

We denote by $\rho_X(\cdot)$ the autocorrelation function of the process $\{X_t\}_{t\in\mathbb{N}}$, that is,

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}, \text{ for all } h \in \mathbb{N},$$

where $\gamma_X(0) \equiv \mathbb{E}_\mu(X_0^2) - [\mathbb{E}_\mu(X_0)]^2 = Var_\mu(X_0)$ is the variance of the process.

The spectral density function of the process $\{X_t\}_{t\in\mathbb{N}}$ is given by

$$f_X(\omega) = \frac{1}{2\pi}[\gamma_X(0) + 2\sum_{h=1}^{\infty} \gamma_X(h)\cos(\omega h)], \text{ for } \omega \in [-\pi, \pi]. \qquad (3.3)$$

Now we shall define the *periodogram function* associated with a time series $T_s^t(x_0)$, for $1 \leq t \leq N$, obtained from a x_0 chosen with probability one according to the measure μ_s. The periodogram function is given by

$$I(\omega_h) = f_N(\omega_h)\overline{f_N(\omega_h)}, \qquad (3.4)$$

where

$$f_N(\omega) = \frac{1}{2\pi\sqrt{N}} \sum_{t=1}^{N} \varphi(T_s^t(x_0)) e^{-i\omega t}, \quad \omega \in (0, 2\pi],$$

with $\overline{f_N(\cdot)}$ indicating the complex conjugate of $f_N(\cdot)$ and

$$\omega_h = \frac{2\pi h}{N}, \text{ for } h = 0, 1, \cdots, N, \tag{3.5}$$

the h-th discrete Fourier frequency (see Brockwell and Davis, 1991).

Note that the periodogram function depends on x_0 and N (large). One can obtain a good approximation of the spectral density function $f_X(\cdot)$ by the periodogram function (see Lopes and Lopes, 2002 for a mathematical proof that can be applied to the case we analyze here when $0 < s < 0.5$).

The *periodogram function* is an unbiased estimator for the spectral density function $f_X(\cdot)$, even though it is not consistent (see Brockwell and Davis, 1991).

Another procedure for estimating the parameters which produce good results are by using the wavelet theory. This type of analysis can be also used in the regime $s > 1$ where the spectral density function, defined in the expression (3.4), does not exist since the random process is not associated with a probability.

We shall use the following notation:

- If, for the sequence $\{a_n\}_{n\in\mathbb{N}}$, there exists $u \in \mathbb{R}$ and, for any $\delta > 0$, there exist positive constants c_1 and c_2 such that, for all $n \in \mathbb{N}$,

$$c_1\, n^{u-\delta} \leq |a_n| \leq c_2\, n^{-u+\delta},$$

  then we denote $a_n \approx n^{-u}$. We also say that a_n is *of order* n^{-u}, for $n \to \infty$.

- If, for the real function $g(\cdot)$, there exist $b \in \mathbb{R}$ and $\epsilon > 0$ such that, for any $\delta > 0$, there exist positive constants d_1 and d_2 such that, for all $x \in (0, \epsilon)$,

$$d_1\, x^{b+\delta} \leq |g(x)| \leq d_2\, x^{b-\delta},$$

  then, we denote $g(x) \approx x^b$. We also say that g is *of order* x^b around 0.

If there exist $c_1, c_2 > 0$ such that

$$c_1\, n^{-u} \leq |a_n| \leq c_2\, n^{-u},$$

then, of course, $a_n \approx n^{-u}$.

If there exist $d_1, d_2 > 0$ such that

$$d_1\, x^b \leq |g(x)| \leq d_2\, x^b,$$

then, of course, $g(x) \approx x^b$. We need however, this more general definition because of Theorem A.4 in Appendix A of the present work.

Definition 3.1: Let $\{X_t\}_{t \in \mathbb{N}}$ be a stochastic stationary process with autocovariance function $\gamma_X(\cdot)$ given by the expression (3.2). If there exists $u \in (0, 1)$ such that

$$\gamma_X(h) \approx h^{-u}, \tag{3.6}$$

then we say that $\{X_t\}_{t \in \mathbb{N}}$ is a *stochastic process with long dependence.*

Definition 3.2: Let $\{X_t\}_{t \in \mathbb{N}}$ be a stochastic stationary process with autocovariance function $\gamma_X(\cdot)$ given by the expression (3.2). If there exists $u > 1$ such that

$$\gamma_X(h) \approx h^{-u}, \tag{3.7}$$

then we say that $\{X_t\}_{t \in \mathbb{N}}$ is a *stochastic process with not so long dependence*.

For the Manneville-Pomeau process, it is known that

$$\gamma_X(h) \approx h^{1-\frac{1}{s}}, \tag{3.8}$$

(see Young, 1999 for the upper bound and Fisher and Lopes, 2001 for the lower bound).

When $0.5 < s < 1$ the Manneville-Pomeau process, given by the expression (3.1), has the *long dependence* property and when $0 < s < 0.5$ it has the *not so long dependence* property. We shall consider here different methods for estimating the value of s in both cases.

In the *long dependence regime* there exists a relationship between the velocity of the autocorrelation function decay to zero and the regularity of the function $f_X(\cdot)$. This property follows just from a careful analysis of the Fourier series. We refer the reader to chapter X, section 3 in Bary (1964), pages 1086-1090 in Fisher and Lopes (2001), and also the Appendix A of the present work for a careful description of this relationship. This follows basically from the fact that if $f_X(\lambda) \approx \lambda^{-b}$, with $b > 0$, then $\gamma_X(h) \approx h^{b-1}$. In the case when the coefficients $\gamma_X(h)$ are monotone decreasing in h, then $f_X(\lambda) \approx \lambda^{-b}$, if $\gamma_X(h) \approx h^{b-1}$, for $b > 0$. Fisher and Lopes (2001) show that the autocovariance functions $\gamma_X(h)$ are a monotone function for the linear by part approximation of the Manneville-Pomeau map in the case of a certain φ. These authors also show that $\gamma_X(h) \approx h^{\gamma-3}$, when $2 < \gamma < 3$ (see page 1090).

In the case of Manneville-Pomeau maps with *long dependence*, from the exact asymptotic given by the expression (3.3), one can obtain (by analogy with the linear by-part model) the rate of convergence of the autocorrelation decay to zero from the asymptotic of $f_X(\lambda)$ to infinity when $\lambda \to 0$ and vice versa. It follows from the above considerations and from (3.3) that $f_X(\lambda) \approx \lambda^{\frac{1}{s}-2}$.

The phenomena $f_X(\omega) \approx \omega^{-b}$ is known as $\frac{1}{f}$-noise property (in this case, $\frac{1}{f^b}$-noise would be a more appropriate terminology), where f stands for a frequency (here denoted by ω).

Definition 3.3: The continuous function $g : (-\pi, \pi) \to \mathbb{R}$ is said to be *Hölder of order* a, $0 < a < 1$, if there exists a positive constant K such that

$$|g(x) - g(y)| \leq K|x - y|^a,$$

for any $x, y \in (-\pi, \pi)$. We also call a the *exponent of* g.

Definition 3.4: The continuous function $g : (-\pi, \pi) \to \mathbb{R}$ is said to be exactly *a-Hölder* in the point x_0, for $0 < a < 1$, if for any $\delta > 0$, there exist positive constants c_1 and c_2 such that

$$c_1 \, |x - y|^{a+\delta} \leq |g(x) - g(y)| \leq c_2 \, |x - y|^{a-\delta},$$

for any $y \in (-\pi, \pi)$. We also call a the *exact exponent of* g at x_0.

We will apply this definition for the case $x_0 = 0$.

When one considers the Manneville-Pomeau maps with *not so long dependence*, one can say more about the regularity of $f_X(\cdot)$ (see chapter II, section 3 and chapter X, section 9 in Bary, 1964 and Appendix A of this present work): it is exactly β-Hölder continuous function with exponent $\beta = \frac{1}{s} - 2$. We are using here the notation: a β-Hölder function, with $\beta = n + \alpha$, $0 < \alpha < 1$, is a function such that it is n times differentiable and the n-th derivative is α-Hölder.

The periodogram function $I(\cdot)$ is a useful way to obtain an approximation of $f_X(\cdot)$ (see Lopes and Lopes, 2002). One can obtain an estimation of s from the above considerations and the periodogram function as we will explain in the next section.

4 Estimation in the "Long Dependence" Case

The main goal of this section is to estimate the transformation T_s, or equivalently, to estimate the parameter s, when $0.5 < s < 1$. For this purpose, we consider a finite time series $\{X_t\}_{t=0}^{N-1}$ obtained from the process $\{X_t\}_{t\in\mathbb{N}}$ given by (3.1).

By Monte Carlo simulation, which is given in Section 5, we compare some methods for estimating s with the one presented in Schuster (1984). We are interested in the performance of this method when compared to the others.

The process $\{X_t\}_{t\in\mathbb{N}}$, defined by the expression (3.1), is considered here to be

$$X_t = \mathbb{I}_A \circ T_s^t = \mathbb{I}_{(0.1,0.9)} \circ T_s^t, \tag{4.1}$$

which is stationary and ergodic (see Lopes and Lopes, 1998).

For the *long dependence case* one can express the graph of $f_X(\cdot)$ (or of the periodogram function $I(\cdot)$) in the logarithm scale and this exhibits linear behavior. By ordinary least-squares estimation one can obtain an estimate of the value s.

We now explain more carefully this very useful method for the *long dependence case*: suppose there exists c such that $f_X(\omega) \approx \omega^c$, for ω close to zero. Then, for ω close to zero

$$\frac{\ln(f_X(\omega))}{\ln(\omega)} \approx c.$$

From the estimated value of c we estimate s since $c = \frac{1}{s} - 2$. An estimate of c can be obtained via the periodogram by

$$\frac{\ln(I(\omega))}{\ln(\omega)} \approx \hat{c},$$

with ω chosen very close to 0.

We shall now consider six different methods for estimating the parameter s: the least-squares method proposed in section 4.3 of Schuster (1984); the least-squares method proposed here using the smoothed periodogram function when the Parzen or the "cosine bell" lag window is used to consistently estimate the spectral density function; the one based on the variance of the sample partial sums of the process; the one based on the logarithm of the variance of the sample mean of the process and the one based on the wavelet theory. These methods are described in this section and in Section 5 we present a Monte Carlo simulation study comparing them.

Perio **Estimator**

This method is based on the periodogram function of a time series $\{X_t\}_{t=1}^{N}$ and it is largely used by physicists (see Schuster, 1984).

The estimator of s is obtained from the least squares method based on a linear regression of $y_1, y_2, \cdots, y_{g(N)}$ on $x_1, x_2, \cdots, x_{g(N)}$, where $y_j = \ln(I(\lambda_j))$, $x_j = \ln(j)$ and $g(N) = N^{0.5}$. The $I(\cdot)$ is the periodogram function given by the expression (3.4) and λ_j is the j-th Fourier frequency given by (3.5). Let c be the slope coefficient of the linear regression in the logarithm scale. The coefficient c allows the estimation of s through the equality

$$s = \frac{1}{c+2},$$

since, for $s \in (0,1)$ we know that

$$f_X(\omega) \approx \omega^{\frac{1}{s}-2}, \text{ for } \omega \text{ close to the zero frequency.}$$

Therefore,

$$\hat{c} = \frac{1}{\hat{s}} - 2 \Leftrightarrow \hat{s} = \frac{1}{\hat{c}+2}. \tag{4.2}$$

We shall denote the estimator in (4.2) by *Perio*.

Parzen **Estimator**

This method is also a regression estimator for the parameter s and is obtained by replacing the periodogram function $I(\cdot)$ in the *Perio* method by its smoothed version with the Parzen lag window (see Brockwell and Davis, 1991). It is known that the use of a spectral lag window consistently estimates the spectral density function (see Brockwell and Davis, 1991). This estimator has the same expression as in (4.2), but now $y_j = \ln(f_{sm}(\omega_j))$, where $f_{sm}(\cdot)$ is the smoothed periodogram function. The value of $g(N)$ is chosen as in the *Perio* method. The truncation point in the Parzen lag window is considered to be $m = N^{0.9}$.

Cos **Estimator**

This method is similar to the *Parzen* estimator, where now we use the "cosine bell" spectral lag window (see Brockwell and Davis, 1991). Its expression is given by (4.2), where now the smoothed periodogram function $f_{sm}(\cdot)$ is obtained from the "cosine bell" lag window. Again, by linear regression, we obtain the estimator of s. In this method we considered different limits for $g(N) = N^{\alpha_i}$: we used $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$ and we denote this estimator by $Cos(i)$, $i = 1, 2$.

Remark 4.1: The methods *Perio*, *Parzen* and *Cos*, defined above, are similar to those proposed by Lopes et al. (2004) and Reisen et al. (2001) to estimate the differencing parameter in ARFIMA models. They are also similar to the estimators proposed by Lopes (2007) for the differencing d or the seasonal differencing D parameters in seasonal fractionally integrated SARFIMA$(p, d, q) \times (P, D, Q)_s$ process with sazonality s. Again, we observe that there is no explicit expression for the spectral density function $f_X(\cdot)$ in the case of the Manneville-Pomeau processes.

Varmp **Estimator**

This method, denoted by $Varmp$, is different from the other previous three. To explain this method, we consider a time series of sample size N from the process (4.1) and let M_N be the random variable is given by

$$M_N = \text{ total number of 1's in the time series } \{X_t\}_{t=0}^{N-1} = \sum_{i=0}^{N-1} X_i = S_N. \qquad (4.3)$$

One can show (see Lopes, 1993; Olbermann, 2002 or Wang, 1989) that

$$Var(M_N) \approx N^{4-\gamma} = N^{3-\frac{1}{s}}. \qquad (4.4)$$

We present a proof of this fact in a quite large generality in Appendix B.

The property (4.3) allows one to obtain another estimator for the parameter s. In fact, if one applies the logarithm to that expression one gets

$$Varmp = \frac{1}{3 - \frac{\ln(Var(M_N))}{\ln(N)}} = \hat{s}.$$

Remark 4.2: As in the ARFIMA process (see Beran et al., 2013 and Olbermann, 2002) we observe that this estimator is also very much biased to estimate s in the Manneville-Pomeau processes.

Vpmp **Estimator**

This method is also based on the variance of the random variables M_N. It is proposed by Beran (1994) under the name of *variance plot*. It is obtained from the order of the variance of $\bar{X}_N = \frac{S_N}{N}$ given by

$$\text{Var}(\bar{X}_N) \approx O(N^{2d-1}), \qquad (4.5)$$

where d is the differencing parameter in ARFIMA models.

For the Manneville-Pomeau processes we only need to consider the expression (4.5), the relationship between the random variables M_N and S_N, given by (4.4) and the relationship between the parameters s and d, given by $d = 1 - \frac{1}{2s}$. We shall denote this estimator by $Vpmp$.

Wmp **Estimator**

This method is based on the wavelet estimator proposed by Jensen (1999) to estimate the differencing parameter d in ARFIMA models. To consider this a method to estimate the parameter s in Manneville-Pomeau processes we must consider the relationship between the parameters s and d, given by $d = 1 - \frac{1}{2s}$ and the estimator proposed here, denoted by Wmp.

We refer the reader to Percival and Walden (1993) and Lopes and Pinheiro (2009) for the use of wavelets in several different problems in statistics.

A *wavelet* is any continuous function $\psi(t)$ that decays fast to zero when $|t| \to \infty$ and oscillates in such a way that $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$. The idea is to use dyadic translations and dilations of the function $\psi(\cdot)$ such that they generate the whole $\mathcal{L}^2(\mathbb{R})$. From this, the wavelets considered are of the form

$$\psi_{j,k}(t) = 2^{\frac{j}{2}}\, \psi(2^j\, t - k), \text{ for } j, k \in \mathbb{Z},$$

which constitute an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$ (see Percival and Walden, 1993).

Here we consider only the wavelet bases Haar and Mexican hat, since these bases have easy analytic expressions given by

$$\psi_{j,k}(t) = \begin{cases} 2^{\frac{j}{2}}, & \text{if } 2^{-j}\, k \leq t < 2^{-j}\, (k + \frac{1}{2}) \\ -2^{-j}, & \text{if } 2^{-j}\, (k + \frac{1}{2}) \leq t < 2^{-j}\, (k+1) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\psi_{j,k}(t) = 2^{\frac{j}{2}} [1 - (2^j\, t - k)^2] \exp[-(2^j\, t - k)^2/2],$$

for $j = 0, 1, \cdots, m-1$ and $k = 0, 1, \cdots, 2^j - 1$, where $m \in N$ is such that $N = 2^m$.

Given a time series of the sample size N from the stochastic process (4.1) we define the *wavelet coefficients* as the finite wavelet transform for this time series given by

$$\omega_{j,k} = 2^{\frac{j}{2}} \sum_{t=0}^{N-1} X_t\, \psi(2^j\, t - k),$$

for $j = 0, 1, \cdots, m-1$ and $k = 0, 1, \cdots, 2^j - 1$, where $m \in N$ is such that $N = 2^m$.

To obtain the estimator proposed by Jensen (1999) we define the *variance of the wavelet coefficients* as

$$R(j) = \mathbb{E}[(\omega_{j,k})^2], \text{ for all } j = 0, 1, \cdots, m-1.$$

Considering the relationship between s and d given by $d = 1 - \frac{1}{2s}$, the estimator based on the wavelets is given by

$$Wmp = \frac{\sum_{j=4}^{m-1} x_j^2}{2\left(\sum_{j=4}^{m-1} x_j^2 - \sum_{j=4}^{m-1} x_j \ln(\hat{R}(j))\right)},$$

where x_j is given by

$$x_j = \ln(2^{-2j}) - \frac{1}{m-4}\sum_{j=4}^{m-1} \ln(2^{-2j}),$$

and $\hat{R}(j)$ is the *sample variance of the wavelet coefficients* defined by

$$\hat{R}(j) \equiv \frac{1}{2^j}\sum_{k=0}^{2^j-1} (\omega_{j,k})^2, \quad \text{for all } \ j = 4, 5, \cdots, m-1,$$

with m such that $N = 2^m$.

This method will be also considered for the Manneville-Pomeau processes when $s \geq 1$. This corresponds to the case when the invariant measure μ_s is not a probability measure (see Table 7.1).

5 Monte Carlo Simulation for the "Long Dependence" Case

In this section, we present the Monte Carlo simulation results comparing the six different estimation methods given in Section 4 for the *long dependence case*.

Let $\{X_t\}_{t\in\mathbb{N}}$ be the Manneville-Pomeau process, given by the expression (3.1), where $\varphi = \mathbb{I}_A$ with $A = (0.1, 0.9)$ such that $X_t = \mathbb{I}_A \circ T_s^t$.

One chooses at random a value x_0 of the random variable X_0 according to a uniform distribution (this is the same as choosing x_0 at random according to the probability μ_s). Let $\{X_t\}_{t=0}^{N-1}$ be a time series with N observations from the process $\{X_t\}_{t\in\mathbb{N}}$ obtained from such x_0. Hence, this time series is given by

$$X_t = \mathbb{I}_A(T_s^t(x_0)) = \mathbb{I}_{(0.1,0.9)}(T_s^t(x_0)), \text{ for all } t = 0, \cdots, N-1. \tag{5.1}$$

The simulations presented here are based on such time series.

Figures 5.1 (a) and (b) present the sample autocorrelation and the periodogram functions, respectively, for a time series with a sample size $N = 10,000$ obtained from (5.1) when $s = 0.8$.

The following results were obtained from Monte Carlo simulations in Fortran routines and using the IMSL library. We remark that for the long dependence case, one needs a large number of sets of data requiring high computational time.

For all tables presented here, we calculated the mean (*mean*), the standard deviation (*sd*), and the mean squared error (*mse*) values for all estimators of s. The smallest mean squared error is shown in boldfaced characters in these tables. All simulations are based in 200 replications unless for Tables 5.3 and 5.4 where we use 50 replications. For the estimator Cos we used two different values for the limit $g(N) = N^{\alpha_i}$: $Cos(1)$ means $\alpha_1 = 0.5$ and $Cos(2)$ means $\alpha_2 = 0.7$.

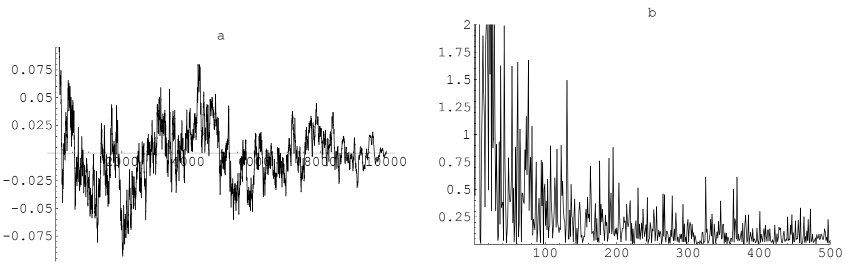

Figure 5.1: (a) Sample Autocorrelation Function; (b) Periodogram Function of a time series with $N = 10,000$ from the process $\{X_t\}_{t\in\mathbb{N}}$ given by (5.1), when $s = 0.8$.

Table 5.1 presents the results for the six estimation methods proposed in Section 4 for the *long dependence case* when $s \in \{0.60, 0.65\}$ and for three different values of $N \in \{10,000; 20,000; 30,000\}$.

From Table 5.1 we observe that the estimators $Varmp$ and $Vpmp$ are very much biased: this was also true for the ARFIMA processes (see Olbermann, 2002). It is natural to state that the best method is the one that minimizes the mean squared error and the absolute bias values. In our simulation study, this will occur when the $Cos(2)$ estimator is used, for both values of s and any sample size considered.

In Table 5.2 we present the results for the case when $s = 0.80$ considering the same sample size $N \in \{10,000; 20,000; 30,000\}$. The best result is for the method $Parzen$, when $N = 10,000$. For the other two values of N, the proposed methods didn't reach the value $s = 0.8$. As s approaches to the value 1, the time series $\{X_t\}_{t=0}^{N-1}$, given by (5.1), stays a long time in zero, resulting in very poor estimates. The methods $Varmp$ and $Vpmp$ are also not recommended in this situation due to their higher bias values when compared to the others methods.

Table 5.1: Estimation results when $s \in \{0.60, 0.65\}$ and $N \in \{10,000; 20,000; 30,000\}$.

s	N	*Method*	*mean*($\hat{s}$)	*sd*($\hat{s}$)	*mse*($\hat{s}$)	s	*Method*	*mean*($\hat{s}$)	*sd*($\hat{s}$)	*mse*($\hat{s}$)
0.60	10,000	*Perio*	0.6545	0.1394	0.0223	0.65	*Perio*	0.7539	0.1518	0.0337
		Parzen	0.6313	0.1125	0.0136		*Parzen*	0.7107	0.0107	0.0151
		Cos(1)	0.5531	0.0572	0.0054		*Cos*(1)	0.6145	0.0614	0.0050
		Cos(2)	**0.5993**	**0.0220**	**0.0005**		*Cos*(2)	**0.6129**	**0.0198**	**0.0017**
		Varmp	0.5309	0.0396	0.0063		*Varmp*	0.5293	0.0332	0.0156
		Vpmp	0.5598	0.0718	0.0067		*Vpmp*	0.5461	0.0763	0.0166
	20,000	*Perio*	0.6364	0.1094	0.0130		*Perio*	0.7113	0.0779	0.0098
		Parzen	0.6147	0.0086	0.0070		*Parzen*	0.6927	0.0706	0.0068
		Cos(1)	0.5488	0.0535	0.0054		*Cos*(1)	0.6035	0.0472	0.0044
		Cos(2)	**0.5979**	**0.0264**	**0.0007**		*Cos*(2)	**0.6076**	**0.0181**	**0.0021**
		Varmp	0.5241	0.0303	0.0067		*Varmp*	0.5251	0.0246	0.0162
		Vpmp	0.5513	0.0583	0.0057		*Vpmp*	0.5257	0.0630	0.0194
	30,000	*Perio*	0.6004	0.1051	0.0110		*Perio*	0.6806	0.0445	0.0029
		Parzen	0.5865	0.0736	0.0056		*Parzen*	0.6910	0.0392	0.0032
		Cos(1)	0.5275	0.0508	0.0078		*Cos*(1)	0.6141	0.0552	0.0043
		Cos(2)	**0.5933**	**0.0316**	**0.0010**		*Cos*(2)	**0.6090**	**0.0147**	**0.0019**
		Varmp	0.5204	0.0264	0.0070		*Varmp*	0.5419	0.0262	0.0123
		Vpmp	0.5144	0.0608	0.0110		*Vpmp*	0.5451	0.0562	0.0141

Table 5.2: Estimation Results when $s = 0.80$ and $N \in \{10,000; 20,000; 30,000\}$.

N	*Method*	*mean*($\hat{s}$)	*sd*($\hat{s}$)	*mse*($\hat{s}$)
10,000	*Perio*	0.7773	0.1648	0.0275
	Parzen	**0.7607**	**0.1444**	**0.0222**
	Cos(1)	0.6286	0.2507	0.0919
	Cos(2)	0.6626	0.0822	0.0256
	Varmp	0.5472	0.0426	0.0657
	Vpmp	0.5781	0.0806	0.0557
20,000	*Perio*	**0.6921**	**0.1220**	**0.0264**
	Parzen	0.6740	0.1127	0.2849
	Cos(1)	0.5731	0.0699	0.0563
	Cos(2)	**0.6434**	**0.0437**	**0.0264**
	Varmp	0.5292	0.0434	0.0752
	Vpmp	0.5416	0.0848	0.0739
30,000	*Perio*	0.6559	0.1164	0.0342
	Parzen	0.6150	0.1044	0.0456
	Cos(1)	0.5335	0.1713	0.1002
	Cos(2)	**0.6382**	**0.0337**	**0.0273**
	Varmp	0.5354	0.0524	0.0727
	Vpmp	0.5434	0.0861	0.0732

The simulations presented in Tables 5.3 and 5.4 are based on 50 replications. Table 5.3 presents the results based only on the wavelet method. We consider both the Haar and Mexican hat bases. We remark that these estimators require a power of two for the sample size. Table 5.3 presents the results when $s \in \{0.65, 0.80\}$ with three different values for $N \in \{8,192; 16,384; 32,768\}$. We observe that the Mexican hat basis has advantages over the Haar basis presenting smaller bias and mean squared error values. We still point out that when $N = 8,192$ the method based on the Haar basis overestimates the mean value when $s \in \{0.65, 0.80\}$. After the analysis of the *long dependence case* we make a few comments about another regime, that is, when $s \geq 1$.

Table 5.3: Estimation results when $s \in \{0.65, 0.80\}$ and $N \in \{8,192; 16,384; 32,768\}$.

s	N	*Wavelet Basis*	$mean(\hat{s})$	$sd(\hat{s})$	$mse(\hat{s})$
0.65	8,192	Haar	0.8531	0.0470	0.0434
		Mexican hat	**0.8022**	**0.0480**	**0.0254**
	16,384	Haar	0.8311	0.0446	0.0347
		Mexican hat	**0.7882**	**0.0472**	**0.0213**
	32,768	Haar	0.8283	0.0619	0.0355
		Mexican hat	**0.7864**	**0.0451**	**0.0206**
0.80	8,192	Haar	0.9839	0.0619	0.0376
		Mexican hat	**0.8873**	**0.0670**	**0.0120**
	16,384	Haar	0.9321	0.0659	0.0217
		Mexican hat	**0.8237**	**0.0675**	**0.0050**
	32,768	Haar	0.8639	0.0915	0.0120
		Mexican hat	**0.7747**	**0.0464**	**0.0027**

In Table 5.4 we present the case when $s \geq 1$ meaning that the invariant measure μ_s does not correspond to a probability measure for the process $\{X_t\}_{t\in\mathbb{N}}$, given by (3.1). This table presents values of $s \in \{1.0, 1.1, 1.2, 1.3\}$ and sample size $N = 32,768$. The best results were for the Haar basis. Notice that when $s \geq 1$ any method based on the periodogram function does not make sense (for the process obtained from the iterations of the Manneville-Pomeau transformation T_s when x_0 is chosen at random).

An interesting question to be investigated: is it true that for any deterministic (such as Manneville-Pomeau, Infinite Markov Chain, etc...) or purely stochastic processes (such as ARFIMA, etc...) depending only on the decay of the rate of convergence of the autocorrelation function, there exists a better wavelet basis (such as Haar, Mexican hat, Shannon, etc ...) to estimate the exponent of decay?

For the "*long dependence*" case, the $Cos(2)$ estimation method is the best estimator procedure when $s \in \{0.60, 0.65, 0.80\}$ for N larger than $10,000$. Only when $N = 10,000$, the Parzen estimation method overcame the $Cos(2)$ method (see Tables 5.1 and 5.2). When we consider the Haar and Mexican hat bases for this case, the best estimation procedure is the one based on the Mexican hat basis when $s \in \{0.65, 0.80\}$ (see Table 5.3) and the Haar basis when $s \geq 1.0$ and $N = 32,768$ (see Table 5.4).

Table 5.4: Estimation results when $s \in \{1.0, 1.1, 1.2, 1.3\}$ and $N = 32,768$.

s	*Wavelet Basis*	*mean*($\hat{s}$)	*sd*($\hat{s}$)	*mse*($\hat{s}$)
1.0	Haar	**0.9461**	**0,1090**	**0.0145**
	Mexican hat	0.8931	0.1148	0.0243
1.1	Haar	**1.0924**	**0.0589**	**0.0034**
	Mexican hat	0.9943	0.0461	0.0132
1.2	Haar	**1.0825**	**0.0729**	**0.0190**
	Mexican hat	0.9642	0.0939	0.0642
1.3	Haar	**1.1422**	**0.0638**	**0.0288**
	Mexican hat	1.0064	0.0703	0.0910

6 Estimation in the "Not So Long Dependence" Case

In the *not so long dependence case* one can estimate the value s using the exactly a-Hölder property in the point $x_0 = 0$ (see Bary, 1964 and Fisher and Lopes, 2001). Suppose

$$a \approx \frac{\ln(|f_X(x_0) - f_X(y)|)}{\ln(|x_0 - y|)}, \text{ for } y \in (-\pi, \pi) \text{ very close to zero,}$$

where $f_X(\cdot)$ is the spectral density function, given in (3.3), of the process $\{X_t\}_{t\in\mathbb{N}}$ given in (3.1). We then define the estimator

$$\hat{s} = \frac{1}{a+2} \quad \text{where} \quad a = \frac{\ln(|I(\omega_0) - I(\omega_j)|)}{\ln(|\omega_0 - \omega_j|)}, \tag{6.1}$$

with $I(\cdot)$ the periodogram function, given by (3.4), with $\omega_0 = 0$ and ω_j is a Fourier frequency, given by (3.5), very close to zero.

The main goal of this section is to describe two different estimation methods to estimate the transformation T_s, or equivalently, to estimate the parameter s, when $s \in (0, \frac{1}{2})$. For this purpose, we consider a finite time series $\{X_t\}_{t=0}^{N-1}$ obtained from the process $\{X_t\}_{t\in\mathbb{N}}$ given by (5.1). The two methods are proposed by (6.1) when the periodogram or its smoothed version by the Parzen lag window functions are used.

These methods are described in this section and in Section 7 we present a Monte Carlo simulation study comparing them.

P Estimator

This estimation method is based on the expression (6.1) where $I(\cdot)$ is the periodogram function given by the expression (3.4). We denote it by P estimator.

SP Estimator

This estimation method is based on the expression (6.1) where the periodogram function $I(\cdot)$ is now replaced by the smoothed periodogram function $f_{sm}(\cdot)$ using the Parzen spectral window. We denote it by SP estimator.

7 Monte Carlo Simulation for the "Not So Long Dependence" Case

In this section, we present the Monte Carlo simulation results comparing the two methods given in Section 6 for the *not so long dependence case*.

Let $\{X_t\}_{t\in\mathbb{N}}$ be the Manneville-Pomeau process, given by the expression (5.1), where $\varphi = \mathbb{I}_A$ with $A = (0.1, 0.9)$ such that $X_t = \mathbb{I}_A \circ T_s^t$.

One chooses at random a value x_0 of the random variable X_0 according to a uniform distribution (this is the same as choosing x_0 at random according to the probability μ_s). Let $\{X_t\}_{t=0}^{N-1}$ be a time series with N observations obtained from (5.1). The simulations presented here are based on such time series and were obtained by Fortran routines with some help from the IMSL library.

In Table 7.1 we present some simulation results for the *not so long dependence case* based on the two methods reported in Section 6. We calculated the mean (*mean*), the standard deviation (*sd*) and the mean squared error (*mse*) values for each method. The smallest mean squared error is shown in the boldfaced character in this table. These simulations are based in 200 replications with $s \in \{0.35, 0.40, 0.45\}$ and two different sample sizes $N \in \{10,000; 30,000\}$. Notice that as we have a better mixing rate of convergence for the *not so long dependence case* the biases here are smaller than in the case of long dependence. However, we have an acceptable estimated mean value only when $s = 0.40$, for both samples sizes N. However, when $s = 0.35$ and $s = 0.45$ both methods overestimate the mean value for both sample sizes N.

For the "*not so long dependence*" case, the best estimation procedure is the SP method, but the P method overcames it when, respectively, $s = 0.35$ and $N = 30,000$, and when $s = 0.45$ and $N = 10,000$ (see Table 7.1).

Table 7.1: Estimation results, based on 200 replications, when $s \in \{0.35, 0.40, 0.45\}$ and $N \in \{10,000; 30,000\}$.

s	N	*Statistics*	P	SP
0.35	10,000	$mean(\hat{s})$	0.4078	**0.3970**
		$sd(\hat{s})$	0.0374	**0.0255**
		$mse(\hat{s})$	0.0047	**0.0028**
	30,000	$mean(\hat{s})$	**0.3870**	0.4136
		$sd(\hat{s})$	**0.0298**	0.0208
		$mse(\hat{s})$	**0.0022**	0.0044
0.40	10,000	$mean(\hat{s})$	0.4210	**0.4024**
		$sd(\hat{s})$	0.0378	**0.0258**
		$mse(\hat{s})$	0.0018	**0.0006**
	30,000	$mean(\hat{s})$	0.4397	**0.4046**
		$sd(\hat{s})$	0.0432	**0.0405**
		$mse(\hat{s})$	0.0034	**0.0016**
0.45	10,000	$mean(\hat{s})$	**0.4652**	0.4359
		$sd(\hat{s})$	**0.0312**	0.0285
		$mse(\hat{s})$	**0.0012**	0.0050
	30,000	$mean(\hat{s})$	0.5218	**0.4808**
		$sd(\hat{s})$	0.0800	**0.0619**
		$mse(\hat{s})$	0.0115	**0.0047**

8 Conclusions

We analyzed the estimation of the parameter s in the Manneville-Pomeau processes in the *long* and *not so long-range dependence* cases.

We described several estimation methods for both situations and we consider that the best estimation procedure is the one with a smaller mean square error value and smaller bias in absolute value. In this direction we compare several estimation procedures with the method called here *Perio*, presented by Schuster (1984), and largely used by physicists.

For the "*long dependence*" case we point out that the $Cos(2)$ estimation method is the best estimator procedure when $s \in \{0.60, 0.65, 0.80\}$ for N larger than $10,000$. Only when $N = 10,000$, the Parzen estimation method overcomes the $Cos(2)$ method (see Tables 5.1 and 5.2). When we consider the Haar and Mexican hat bases for this "*long dependence*" case, the best estimation procedure is the one based on the Mexican hat basis when $s \in \{0.65, 0.80\}$ (see Table 5.3) and the one based on the Haar basis, when $s \geq 1.0$ and $N = 32,768$ (see Table 5.4).

In Tables 5.1 to 5.3 the wavelet method (Wmp) had a better performance than the *Perio* method. One can see this in the case when $s = 0.8$, in which case the estimator Wmp from the Mexican hat wavelet basis gave the best results in terms of smaller mean squared error value and smaller bias in absolute value (see Table 5.3).

The methods $Varmp$ and $Vpmp$ presented the higher biases while the method $Cos(2)$ had the best results for the cases when $s \in \{0.60, 0.65\}$, with the smallest mean squared error for the considered sample size values.

We studied the performance of the method Wmp based on the wavelet theory for the Manneville-Pomeau processes when $s \geq 1$, which corresponds to the situation where the invariant measure μ_s is not a probability measure. In this case, the best results were obtained when the Haar basis was considered.

Among the estimation methods proposed for the "*not so long dependence*" case, the one based on the smoothed periodogram function using the SP Parzen spectral window had the best results with lower bias and mean squared error values. The P method overcomes it only in two situations, respectively, when $s = 0.35$ and $N = 30,000$, and when $s = 0.45$ and $N = 10,000$ (see Table 7.1).

Appendix A

Let $\{X_t\}_{t\in\mathbb{N}}$ be the Manneville-Pomeau process defined in (3.1). Let $\rho_X(\cdot)$ and $f_X(\cdot)$ be, respectively, the autocorrelation and the spectral density functions of this process.

In this appendix, we present some general properties of the Fourier series. In this way, we will explain why the hyperbolic (or polynomial) decay of the autocorrelation function, that is,

$$\rho_X(h) \approx h^{-u}, \ \text{ for } 0 < \mu < 1,$$

corresponds to

$$f_X(\lambda) \approx \lambda^{u-1},$$

for the spectral density function of the process $\{X_t\}_{t\in\mathbb{N}}$ given by (3.1).

First, we will explain the *not so long dependence case*.

If the function g is n-times differentiable and $g^n(\cdot)$ is a-Hölder with $0 < a < 1$, we say that g is $(n + a)$-Hölder.

The relationship of the hyperbolic decay between the autocorrelation function of the Manneville-Pomeau process and its spectral density function is only a question related to the Fourier series (see Bary, 1964).

Theorem A.1: *Suppose that* $b_n \approx n^{-u}$*, for some* u*, and that* $g(\theta) = \sum_{n=1}^{\infty} b_n \cos(n\,\theta)$ *converges to zero, for* $b_n \in \mathbb{R}$*. If* a *is positive and* $g(\cdot)$ *is a Hölder function of order* a*, then there exists a positive constant* c *such that* $b_n < c\,n^{-(1+a)}$*, for all* $n \in \mathbb{N}-\{0\}$*.*

Theorem A.2: *Suppose that* $b_n \approx n^{-u}$*, for some* u*, and that* $g(\theta) = \sum_{n=1}^{\infty} b_n \cos(n\,\theta)$ *decreases monotonously to zero, for* $b_n \in \mathbb{R}$*. If* a *is positive and there exists a positive constant* c *such that* $b_n < c\,n^{-(1+a)}$*, then* $g(\cdot)$ *is a Hölder function of order* a*.*

Theorems A.1 e A.2 (see chapter II, section 3 and chapter X, section 9, respectively, in Bary, 1964) apply to the *not so long dependence case*.

Another interesting result of the Fourier series, that can be applied now for the *long dependence case*, is described in the next theorem.

Theorem A.3 (Riesz): *Suppose that* $g(\theta) = \sum_{n=1}^{\infty} b_n \cos(n\,\theta)$*, for all* $\theta \in (-\pi, \pi)$ *and that* $b_n \in \mathbb{R}$ *is such that the sequence* $\{b_n\}_{n\in\mathbb{N}}$ *decreases monotonously to zero when* $n \to \infty$*. Suppose there exists a positive real constant* u *such that* $b_n \approx n^{-u}$*. Suppose there exists also a positive real constant* $b \in (-1, 0)$ *such that*

$$|g(\theta)| \approx |\theta|^b.$$

(a) *If there exist* $a \in (-1, 0)$, $\epsilon > 0$ *and a positive real constant* k *such that*

$$\left|\frac{g(\theta)}{\theta^a}\right| \leq k, \ \ \textit{for all} \ \ 0 < \theta < \epsilon,$$

then $u \geq 1 + a$. *That is, the decreasing velocity of* $|b_n|$ *is at least of order* $n^{-(1+a)}$, *when* $n \to \infty$.

(b) *If there exist* $a \in (-1, 0)$ *and a positive real constant* v *such that* $|b_n| < v\, n^{-(1+a)}$, *then* $b \leq a$. *That is,* $g(\theta)$ *is at least of order of* $|\theta|^a$, *when* $\theta \to 0$.

Hence, from (a) *and* (b) *above one concludes that* $u = 1 + b$.

Remark A.1: In the general cases, we point out that there exist sequences $\{b_n\}_{n\in\mathbb{N}}$ (not monotonous) such that $c_1 n^{-u} < |b_n| < c_2 n^{-u}$, for some positive constants c_1 and c_2 and u such that $0 < u < 1$, but $g(\theta)$ does not satisfy $c_3|\theta|^b \leq |g(\theta)| \leq c_4|\theta|^b$ for any fixed positive constants c_3, c_4 and b.

Theorem A.3 is a consequence of the following result.

Theorem A.4 (Riesz): *Suppose that* $g(\theta) = \sum_{n=1}^{\infty} b_n \cos(n\,\theta)$, *for all* $\theta \in (-\pi, \pi)$, *and that* $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ *decreases monotonously to zero. Let* $p > 1$ *and* $q > 1$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$.

(a) *If* $g \in \mathcal{L}^p$, *then* $\sum_{n=1}^{\infty} |b_n|^q < \infty$.

(b) *If* $\sum_{n=1}^{\infty} |b_n|^q < \infty$, *then* $g \in \mathcal{L}^p$.

Remark A.2: Theorem A.3 follows from Theorem A.4 making use of

(a) for any continuous function f, on $(0, \pi)$, of order x^α (x close to zero), then $f \in \mathcal{L}^1 \Leftrightarrow \alpha > -1$ and

(b) for any sequence c_n of order $n^{-\beta}$ (n close to infinity) then $\sum_{n=1}^{\infty} |c_n| < \infty \Leftrightarrow \beta > 1$.

Theorem A.4 follows easily from the first theorem of chapter X, section 9 of Bary (1964).

The above results justify the ideas used in the estimation methods *Perio*, *Parzen*, *Cos*(1) and *Cos*(2), given in Section 4.

Appendix B

Considering the rate of convergence to zero of the autocorrelation function one can also get an estimate of the order of magnitude of the variance for the partial sums $S_N = \sum_{i=0}^{N-1} X_i$ from a time-series $\cdots X_{-3}, X_{-2}, X_{-1}, X_0, X_1, \cdots, X_{N-1}$. In Proposition B.1 below we present a proof of the estimated value for the variance of the random variable S_N. In Proposition B.2 we give a precise estimate of the order of growth for the variance of this random process.

We point out that the stationary process stated above and given by

$$X_t = (\varphi \circ T_s^t)(X_0), \text{ for } t \in \mathbb{N},$$

can be considered defined for all $t \in \mathbb{Z}$, via the natural extension transformation (see section 5.3 in Lopes and Lopes, 1998).

Proposition B.1: *Let $\{X_t\}_{t\in\mathbb{Z}}$ be any stationary stochastic process. Let $S_N = \sum_{i=0}^{N-1} X_i$ be the partial sum of a time series $X_0, X_1, \cdots, X_{N-1}$ from this process. Then,*

$$Var(S_N) = 2N \left[\frac{\gamma_X(0)}{2} + \frac{1}{N} \sum_{j=1}^{N-1} (N-j)\, \gamma_X(j) \right],$$

where $\gamma_X(\cdot)$ is the autocovariance function of the process $\{X_t\}_{t\in\mathbb{Z}}$.

Proof: Since the process $\{X_t\}_{t\in\mathbb{Z}}$ is stationary, we observe that

$$\begin{aligned} Var(S_N) &= Var\left(\sum_{i=0}^{N-1} X_i\right) = \sum_{i=0}^{N-1} Var(X_i) + \sum_{j=0}^{N-1}\sum_{\ell=0}^{N-1} cov(X_j, X_\ell) \\ &= NVar(X_0) + \sum_{j=0}^{N-1}\sum_{\ell=0}^{N-1} \left(\mathbb{E}(X_j X_\ell) - [\mathbb{E}(X_0)]^2\right) \\ &= N\,\gamma_X(0) + 2\sum_{\substack{j,l=0 \\ j<\ell}}^{N-1} \gamma_X(j-\ell). \end{aligned} \tag{B.1}$$

It follows from the expression (B.1) that

$$
\begin{aligned}
Var(S_N) &= N\,\gamma_X(0) + 2\sum_{\substack{j,l=0\\ j<\ell}}^{N-1}\gamma_X(j-\ell) \\
&= N\,\gamma_X(0) + 2\Bigg(\underbrace{\gamma_X(-1)+\gamma_X(-2)+\gamma_X(-3)+\cdots+\gamma_X(-N+1)}_{j=0} \\
&+ \underbrace{\gamma_X(-1)+\gamma_X(-2)+\cdots+\gamma_X(1-(N-1))}_{j=1} \\
&+ \underbrace{\gamma_X(-1)+\gamma_X(-2)+\cdots+\gamma_X(2-(N-1))}_{j=2} \\
&+ \underbrace{\gamma_X(-1)+\gamma_X(-2)+\cdots+\gamma_X(3-(N-1))}_{j=3}+\cdots+\underbrace{\gamma_X(-1)}_{j=N-2}\Bigg) \\
&= N\,\gamma_X(0) + 2\left[(N-1)\gamma_X(-1)+(N-2)\gamma_X(-2)+(N-3)\gamma_X(-3)\right. \\
&+ \left.\cdots+3\gamma_X(-(N-3))+2\gamma_X(-(N-2))+\gamma_X(-(N-1))\right] \\
&= N\,\gamma_X(0) + 2\sum_{j=1}^{N-1}(N-j)\,\gamma_X(-j) = N\,\gamma_X(0) + 2\sum_{j=1}^{N-1}(N-j)\,\gamma_X(j).
\end{aligned}
\tag{B.2}
$$

The last equality (B.2) follows from the fact that the process is stationary. This implies that $\gamma_X(j)=\gamma_X(-j)$.

Therefore,

$$
Var(S_N) = N\,\gamma_X(0) + 2\sum_{j=1}^{N-1}(N-j)\,\gamma_X(j),
$$

and this completes the proof of Proposition B.1. □

In the next proposition, we show the order of $Var(S_N)$, with respect to N, for a quite general class of stationary stochastic processes.

Proposition B.2: *Let $\{X_t\}_{t\in\mathbb{Z}}$ be any stationary stochastic process. Let $S_N = \sum_{i=0}^{N-1} X_i$ be the partial sum of a time series $X_0, X_1, \cdots, X_{N-1}$ from the process $\{X_t\}_{t\in\mathbb{Z}}$. If there exists $u\in(0,1)$ such that $\gamma_X(h)\approx h^{-u}$, then*

$$
Var(S_N) \approx N^{2-u}.
$$

Proof: For $u \in (0,1)$, the integral

$$I = \int_0^1 (1-x)x^{-u}dx$$

is finite. Then, for any $N \in \mathbb{N}$, one can consider the Riemann sums associated with the partition

$$\left\{0, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}, 1\right\},$$

obtaining the approximation

$$\sum_{j=1}^{N} \left(1 - \frac{j}{N}\right) \left(\frac{j}{N}\right)^{-u} \frac{1}{N},$$

that converges to I, when $N \to \infty$.

Similar arguments proposed in lemma 8.1 of Fisher and Lopes (2001), consider

$$\begin{aligned} c_N &= \sum_{j=1}^{N} \left(1 - \frac{j}{N}\right) \left(\frac{j}{N}\right)^{-u} = \sum_{j=1}^{N} \left(\frac{N-j}{N}\right) \left(\frac{j}{N}\right)^{-u} \\ &= \sum_{j=1}^{N} (N-j)\, j^{-u} \left(\frac{1}{N}\right)^{1-u}. \end{aligned} \tag{B.3}$$

Given $\varepsilon > 0$, for N sufficiently large, one has that

$$I - \varepsilon \le \frac{1}{N} c_N \le I + \varepsilon.$$

Using the expression (B.3), the above inequality is given by

$$(I - \varepsilon)\, N^{1-u} \le \frac{1}{N} \sum_{j=1}^{N} (N-j)\, j^{-u} \le (I + \varepsilon)\, N^{1-u}, \tag{B.4}$$

for N sufficiently large.

Therefore,

$$\frac{1}{N} \sum_{j=1}^{N} (N-j)\, j^{-u} \text{ is of order } N^{1-u}.$$

From the expressions (B.2) and (B.4) one has

$$Var(S_N) = 2N \left[\frac{\gamma_X(0)}{2} + \frac{1}{N} \sum_{j=1}^{N-1} (N-j)\ \gamma_X(j) \right] \approx N^{-u},$$

and this completes the proof of Proposition B.2. □

The above results justify the ideas used in the estimation methods $Varmp$ and $Vpmp$, given in Section 4.

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