

RUELLE OPERATOR FOR CONTINUOUS POTENTIALS AND DLR-GIBBS MEASURES

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ABSTRACT. In this work we study the Ruelle Operator associated to a continuous potential defined on a countable product of a compact metric space. We prove a generalization of Bowen's criterion for the uniqueness of the eigenmeasures and that one-sided one-dimensional DLR-Gibbs measures associated to a continuous translation invariant specifications are eigenmeasures of the transpose of the Ruelle operator. From the last claim one gets that for a continuous potential the concept of eigenprobability for the transpose of the Ruelle operator is equivalent to the concept of DLR probability.

Bounded extensions of the Ruelle operator to the Lebesgue space of integrable functions, with respect to the eigenmeasures, are studied and the problem of existence of maximal positive eigenfunctions for them is considered. One of our main results in this direction is the existence of such positive eigenfunctions for Bowen's potential in the setting of a compact and metric alphabet. We also present a version of Dobrushin's Theorem in the setting of Thermodynamic Formalism.

1. Introduction. The classical Ruelle operator needs no introduction and nowadays is a key concept of Thermodynamic Formalism. This operator was defined 1968 for infinite dimension by David Ruelle in his seminal paper [28] and since then has had attracted the attention of the Dynamical Systems community. Remarkable applications of this operator to Hyperbolic dynamical systems and Statistical Mechanics were presented by Ruelle, Sinai and Bowen, see [8, 28, 31], and in the presence of a conformal measure, the action of the Ruelle operator and the transfer operator from abstract ergodic theory coincide. Nowadays transfer operators are present in several applications in pure and applied mathematics and are fruitfully

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applied in many areas of active development, see [4] for comprehensive overview of the works before 2000.

In the classical theory, Ruelle's operator is associated with the full shift $\sigma : \Omega \rightarrow \Omega$ on the symbolic space $\Omega \equiv M^{\mathbb{N}}$, for $M = \{1, \dots, n\}$, and acts on the space of Hölder continuous functions. In its classical form, the Ruelle operator \mathcal{L}_f for a given continuous function $f : \Omega \rightarrow \mathbb{R}$ is defined by $\mathcal{L}_f(\varphi) = \psi$, where ψ is given by, for any $x \in \Omega$,

$$\psi(x) = \sum_{y \in \Omega: \sigma(y)=x} e^{f(y)} \varphi(y).$$

This operator is a useful tool for constructing equilibrium states, which are defined as those probability measures which maximize the variational problem

$$\sup_{\mu \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \{h_\mu(\sigma) + \int_\Omega f d\mu\}, \quad (1)$$

as proposed, e.g., by Ruelle in 1967 ([27]) and Walters in 1975 ([34]). In here, $h_\mu(\sigma)$ refers to the Kolmogorov-Sinai entropy of μ and $\mathcal{P}_\sigma(\Omega, \mathcal{F})$ to the set of all σ -invariant Borel probability measures over Ω . Even though the existence of the solution to the variational problem can be easily obtained through abstract theorems of convex analysis, the Ruelle operator approach provides additional informations like uniqueness or decay of correlations (see [4] and [26]). In the slightly more general situation of an expanding map on a compact and metric space, this approach was successfully implemented around the eighties and nineties for Hölder continuous potentials as well as for more general potentials in Walter's class $W(\Omega, \sigma)$ or Bowen's class $B(\Omega, \sigma)$, see [29, 7, 35, 36, 37].

However, the setting in the references above require that the number of preimages of each point is finite, which excludes symbolic spaces like $\Omega = (\mathbb{S}^n)^{\mathbb{N}}$ which are related to several famous models of Statistical Mechanics. For example, $n = 1$ gives us the so-called XY model, for $n = 3$ we obtain the Heisenberg model and for $n = 4$ the toy model for the Higgs sector of the Standard Model, see [3, 5, 18, 19, 20, 21, 32] for more details.

In [5], two of the authors (see also [25]) used the idea of an *a priori* measure $p : \mathcal{B}(\mathbb{S}^1) \rightarrow [0, 1]$ to circumvent the problem of uncountable alphabets and developed the theory of the Ruelle operator for Hölder potentials on $(\mathbb{S}^1)^{\mathbb{N}}$ with the dynamics given by the left shift map. This approach also works for Hölder potentials when the unit circle \mathbb{S}^1 is replaced by a more general compact metric space M , but one has to be careful about the choice of the *a priori* measure, see [23] for details. In this more general setting, the operator is defined as

$$\mathcal{L}_f(\varphi)(x) = \int_M e^{f(ax)} \varphi(ax) dp(a),$$

where $ax := (a, x_1, x_2, \dots)$. A full support condition is imposed on the *a priori* measure in [23] but this is not a strong restriction since in the majority of the applications, there is a natural choice for this measure which always satisfies this condition. For instance, for the classical Ruelle operator \mathcal{L}_f , the metric space is the finite set $M = \{1, 2, \dots, n\}$ and one usually considers p is the counting measure, and if M is a general compact group, the one might consider the Haar measure.

Some of these results have a counterpart when M is not compact. In case of countable alphabets, the applications are motivated through applications to non-uniformly expanding or hyperbolic dynamical systems, see [30] and references therein. On the other hand, the ϕ^4 -model from Statistical Mechanics also motivates

the study of alphabets which are Polish spaces. In this setting, equilibrium states might only exist as positive linear functionals, but for summable Hlder potentials, the Ruelle operator still has a spectral gap (see [13]).

In Statistical Mechanics as well as in Thermodynamic Formalism, existence and multiplicity of DLR-Gibbs Measures play an important role (see [30]), in particular due to Dobrushin's interpretation of phase transitions. Even though there is no universal definition of phase transition, they are nowadays understood as either the existence of more than one DLR-Gibbs measure, more than one equilibrium state or more than one eigenprobability for the dual of the Ruelle operator and so on (see [10, 11, 17] for more details).

The goal of the present paper is to follow the Ruelle operator approach for general continuous potentials defined over the infinite cartesian product of a general compact metric space. A key observation in here is the characterization of extremal DLR-Gibbs measures by their tail sigma-algebras (Theorem 4.4) and that the set of DLR-Gibbs measures coincides with eigenprobabilities of the dual of the Ruelle operator (Theorem 4.6). Moreover, by extending the action of Ruelle's operator to the space of integrable functions in Section 6, Theorems 4.4 and 4.6 establish a further characterization of extremal DLR-Gibbs measures through exactness and, in particular, through Lin's characterization of exactness (see Corollary 5).

From the viewpoint of Dobrushin's interpretation of phase transitions, Theorem 4.6 allows to employ classical ideas from dynamical systems in order to study phase transitions for one-sided one-dimensional translation invariant statistical mechanics systems. Namely, for potentials satisfying a condition similar to the one by Bowen, we obtain in Theorem 5.2 that there exists a unique DLR-Gibbs measure, and that this measure is exact. Furthermore, we also develop a $C(\Omega)$ -perturbation theory for the Ruelle operator and present a constructive approach to solve the classical variational problem, for continuous potentials (see Section 7).

Thereafter, we study in Section 8 the existence of integrable eigenfunctions of the Ruelle operator for potentials beyond Bowen's class. Under a mild hypothesis on the potential and by approximating the potential uniformly by potentials (f_n) , we prove that $\limsup h_{f_n}$ is a non-trivial integrable eigenfunction associated to the maximal eigenvalue. An further relevant result in here is that $\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$ is an eigenfunction of \mathcal{L}_f under fairly general conditions, which implies that a potential in the Bowen class admits an eigenfunction which is bounded from above and below (see Corollary 6).

2. Preliminaries. Here and subsequently (M, d) denotes a compact metric space endowed with a Borel probability measure p which is assumed to be fully supported on M . Let Ω denote the infinite cartesian product $M^{\mathbb{N}}$ and \mathcal{F} be the σ -algebra generated by its cylinder sets. We will consider the dynamics on Ω given by the left shift map $\sigma : \Omega \rightarrow \Omega$ which is defined, as usual, by $\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$. We use the notation $C(\Omega)$ for the space of all real continuous functions on Ω . When convenient we call an element $f \in C(\Omega)$ a potential and unless stated otherwise all the potentials are assumed to be general continuous functions. The Ruelle operator associated to the potential f is a mapping $\mathcal{L}_f : C(\Omega)$ to $C(\Omega)$ that sends $\varphi \mapsto \mathcal{L}_f(\varphi)$ which is defined for each $x \in \Omega$ by

$$\mathcal{L}_f(\varphi)(x) = \int_M \exp(f(ax))\varphi(ax) dp(a), \quad \text{where } ax := (a, x_1, x_2, \dots). \quad (2)$$

Due to compactness of Ω , in the product topology and the Riesz-Markov theorem we have that $C^*(\Omega)$ is isomorphic to $\mathcal{M}_s(\Omega, \mathcal{F})$, the space of all signed Radon measures. Therefore we can define \mathcal{L}_f^* , the dual of the Ruelle operator, as the unique continuous map from $\mathcal{M}_s(\Omega, \mathcal{F})$ to itself satisfying for each $\gamma \in \mathcal{M}_s(\Omega, \mathcal{F})$ the following identity

$$\int_{\Omega} \mathcal{L}_f(\varphi) d\gamma = \int_{\Omega} \varphi d[\mathcal{L}_f^* \gamma] \quad \forall \varphi \in C(\Omega). \quad (3)$$

It follows from the positivity of \mathcal{L}_f that the map $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$ sends the space of all Borel probability measures $\mathcal{P}(\Omega, \mathcal{F})$ to itself. Since $\mathcal{P}(\Omega, \mathcal{F})$ is convex and compact in the weak topology (which is Hausdorff in this case) and the mapping $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$ is continuous, the Schauder-Tychonoff theorem ensures the existence of at least one Borel probability measure ν such that $\mathcal{L}_f^*(\nu) = \mathcal{L}_f^*(\nu)(1) \cdot \nu$. Notice that this eigenvalue $\lambda \equiv \mathcal{L}_f^*(\nu)(1)$ is positive but strictly speaking it could depend on the choice of the fixed point when it is not unique, however any case such eigenvalues trivially satisfies $\exp(-\|f\|_{\infty}) \leq \lambda \leq \exp(\|f\|_{\infty})$ so we can always work with

$$\lambda_f = \sup \left\{ \mathcal{L}_f^*(\nu)(1) : \begin{array}{l} \nu \in \mathcal{P}(\Omega, \mathcal{F}) \text{ and } \nu \text{ is fix point for} \\ \gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1) \end{array} \right\}. \quad (4)$$

Of course, from the compactness of $\mathcal{P}(\Omega, \mathcal{F})$ and continuity of \mathcal{L}_f^* , the supremum is attained and therefore the set defined below is not empty.

Definition 2.1 ($\mathcal{G}^*(f)$). Let f be a continuous potential and λ_f given by (4). We define

$$\mathcal{G}^*(f) = \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^* \nu = \lambda_f \nu\}.$$

To study the eigenfunctions of \mathcal{L}_f , where f is a general continuous potential, we will need the following version of the Ruelle-Perron-Frobenius theorem for Hölder potentials (see [5] and [23] for the proof).

We consider the metric d_{Ω} on Ω given by $d_{\Omega}(x, y) = \sum_{n=1}^{\infty} 2^{-n} d(x_n, y_n)$ and for any fixed $0 < \gamma \leq 1$ we denote by $C^{\gamma}(\Omega)$ the space of all γ -Hölder continuous functions, i.e, the set of all functions $\varphi : \Omega \rightarrow \mathbb{R}$ satisfying

$$\text{Hol}_{\gamma}(\varphi) = \sup_{x, y \in \Omega: x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d_{\Omega}(x, y)^{\gamma}} < +\infty.$$

Theorem 2.2 (Ruelle-Perron-Frobenius). *Let (M, d) be a compact metric space, μ a Borel probability measure of full support on M and f be a potential in $C^{\gamma}(\Omega)$, where $0 < \gamma < 1$. Then $\mathcal{L}_f : C^{\gamma}(\Omega) \rightarrow C^{\gamma}(\Omega)$ has a simple positive eigenvalue of maximal modulus λ_f and there is a strictly positive function h_f satisfying $\mathcal{L}_f(h_f) = \lambda_f h_f$ and a Borel probability measure ν_f for which $\mathcal{L}_f^*(\nu_f) = \lambda_f \nu_f$ and $\mathcal{L}_f^*(\nu_f)(1) = \lambda_f$.*

3. The Pressure of Continuous Potentials. The next proposition is an extension of Corollary 1.3 in [24]. Here M is allowed to be any general compact metric space. It is worth mentioning that Sarig in [30] obtained a similar result for the Gurevich pressure with respect to countable alphabets, but the techniques employed in our proof are much simpler.

Proposition 1. *Let $f \in C(\Omega)$ be a potential and λ_f given by (4). Then, for any $x \in \Omega$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_f^n(1)(\sigma^n x) = \log \lambda_f.$$

Proof. Let $\nu \in \mathcal{G}^*(f)$ a fixed eigenprobability. Without loss of generality we can assume that $\text{diam}(M) = 1$. By the definition of d_Ω for any pair $z, w \in \Omega$ such that $z_i = w_i, \forall i = 1, \dots, N$ we have that $d_\Omega(z, w) \leq 2^{-N}$. From uniform continuity of f given $\varepsilon > 0$, there is $N_0 \in \mathbb{N}$, so that $|f(z) - f(w)| < \varepsilon/2$, whenever $d_\Omega(z, w) < 2^{-N_0}$. If $n > 2N_0$ and $a := (a_1, \dots, a_n)$ we claim that

$$\|S_n(f)(ax) - S_n(f)(ay)\|_\infty \leq (n - N_0)\frac{\varepsilon}{2} + 2\|f\|_\infty N_0, \quad (5)$$

where $S_n(f) \equiv f + f \circ \sigma + \dots + f \circ \sigma^{n-1}$. Indeed, for any $n \geq 2N_0$ we have

$$\begin{aligned} |S_n(f)(ax) - S_n(f)(ay)| &= \left| \sum_{j=0}^{n-1} f(\sigma^j(a_1, \dots, a_n, x)) - \sum_{j=0}^{n-1} f(\sigma^j(a_1, \dots, a_n, y)) \right| \\ &\leq \sum_{j=0}^{n-1-N_0} |f(\sigma^j(a_1, \dots, a_n, x)) - f(\sigma^j(a_1, \dots, a_n, y))| + \\ &\quad \sum_{j=0}^{N_0} |f(\sigma^j(a_{n-N_0}, \dots, a_n, x)) - f(\sigma^j(a_{n-N_0}, \dots, a_n, y))| \\ &\leq (n - N_0)\frac{\varepsilon}{2} + 2N_0\|f\|_\infty. \end{aligned}$$

The last inequality comes from the uniform continuity for the first terms and from the uniform norm of f for the second ones.

We recall that for any probability space $(E, \mathcal{E}, \mathbb{P})$, φ and ψ bounded real \mathcal{E} -measurable functions the following inequality holds

$$\left| \log \int_E e^{\varphi(\omega)} d\mathbb{P}(\omega) - \log \int_E e^{\psi(\omega)} d\mathbb{P}(\omega) \right| \leq \|\varphi - \psi\|_\infty. \quad (6)$$

From the definition of the Ruelle operator, for any $n \in \mathbb{N}$, we have

$$\mathcal{L}_f^n(1)(\sigma^n x) = \int_{M^n} \exp(S_n(f)(a\sigma^n x)) \prod_{i=1}^n dp(a_i)$$

and from (5) and (6) with $\varphi(a) = S_n(f)(a\sigma^n x)$ and $\psi(a) = S_n(f)(ay)$ we get for $n \geq \max\{2N_0, 4\varepsilon^{-1}\|f\|_\infty N_0\}$ the estimate

$$\frac{1}{n} |\log(\mathcal{L}_f^n(1)(\sigma^n x)) - \log(\mathcal{L}_f^n(1)(y))| \leq \frac{1}{n} ((n - N_0)\frac{\varepsilon}{2} + \frac{2\|f\|_\infty N_0}{n}) \leq \varepsilon.$$

By using Fubini's theorem, sum and subtract $\exp(S_n(f)(a\sigma^n y))$, the identity (3) iteratively and the last inequality for $n \geq \max\{2N_0, 4\varepsilon^{-1}\|f\|_\infty N_0\}$ we obtain

$$\begin{aligned} \mathcal{L}_f^n(1)(\sigma^n x) &= \int_{M^n} \exp(S_n(f)(a\sigma^n x)) \prod_{i=1}^n dp(a_i) \\ &= \int_{M^n} \int_\Omega \exp(S_n(f)(a\sigma^n x)) d\nu(y) \prod_{i=1}^n dp(a_i) \\ &\leq \exp((n - N_0)\frac{\varepsilon}{2} + 2\|f\|_\infty N_0) \int_{M^n} \int_\Omega \exp(S_n(f)(ay)) d\nu_f(y) \prod_{i=1}^n dp(a_i) \\ &\leq \exp(n\varepsilon) \int_\Omega \mathcal{L}_f^n(1)(y) d\nu(y) \\ &= \exp(n\varepsilon) \lambda_f^n. \end{aligned}$$

Similarly we obtain the lower bound $\mathcal{L}_f^n(1)(\sigma^n x) \geq \exp(-n\varepsilon)\lambda_f^n$ so the proposition follows. \square

Corollary 1. *Let f be a continuous potential. If ν and $\hat{\nu}$ are fixed points for the map $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$ then $\mathcal{L}_f^*(\nu)(1) = \mathcal{L}_f^*(\hat{\nu})(1) = \lambda_f$.*

Proof. For any $x_0 \in \Omega$, by repeating the same steps of the proof of the previous proposition one shows that $\log(\mathcal{L}_f^*(\nu)(1)) \equiv \log(\lambda_f(\nu)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_f^n(1)(x_0) = \log(\lambda_f(\hat{\nu})) = \log(\mathcal{L}_f^*(\hat{\nu})(1))$. \square

Definition 3.1 (Pressure Functional). The function $p : C(\Omega) \rightarrow \mathbb{R}$ given by $p(f) = \log \lambda_f$ is called pressure functional.

In classical Thermodynamic Formalism, where M is a finite alphabet, the pressure functional usually refers to the function $P : C(\Omega) \rightarrow \mathbb{R}$ given by

$$P(f) \equiv \sup_{\mu \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h_\mu(\sigma) + \int_\Omega f d\mu \right\}.$$

After developing some perturbation theory we will show later that both definitions of the Pressure functional are equivalent for any continuous potential, i.e., $P = p$.

Since Ω is compact and the space of all γ -Hölder continuous functions $C^\gamma(\Omega)$ is an algebra that separate points and contains the constant functions, we can apply the Stone-Weierstrass theorem to conclude that the closure of $C^\gamma(\Omega)$ in the uniform topology is $C(\Omega)$. Therefore, for any arbitrary continuous potential f , there is a sequence $(f_n)_{n \in \mathbb{N}}$ of Hölder continuous potentials such that $\|f_n - f\|_\infty \rightarrow 0$, when $n \rightarrow \infty$. For such uniform convergent sequences we will see that $p(f_n)$ converges to $p(f)$. In fact, a much stronger result holds. That is, the pressure functional is Lipschitz continuous function on $C(\Omega)$.

Proposition 2. *If $f, g : \Omega \rightarrow \mathbb{R}$ are two arbitrary continuous potentials then the following inequality holds $|p(f) - p(g)| \leq \|f - g\|_\infty$.*

Proof. The proof is an immediate consequence of Proposition 1 and inequality (6). \square

Corollary 2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous potentials such that $f_n \rightarrow f$ uniformly, then $p(f_n) \rightarrow p(f)$. In particular, $\lambda_{f_n} \rightarrow \lambda_f$.*

4. DLR-Gibbs Measures and Eigenmeasures. In this section we discuss the concept of specifications in the setting of Thermodynamic Formalism. Some of its elementary properties for finite state spaces are discussed in details within this framework in [11].

For each $n \in \mathbb{N}$, we define the projection on the n -th coordinate $\pi_n : \Omega \rightarrow M$ by $\pi_n(x) = x_n$. We use the notation \mathcal{F}_n to denote the sigma-algebra generated by the projections π_1, \dots, π_n . On the other hand, the notation \mathcal{T}_n stands for the sigma-algebra generated by the collection of projections $\{\pi_k : k \geq n+1\}$. The so-called tail sigma-algebra (or terminal sigma-algebra) is defined as $\mathcal{T} \equiv \bigcap_{n \in \mathbb{N}} \mathcal{T}_n$. A cylinder set in Ω is a set of the form $\pi_1^{-1}(E_1) \cap \dots \cap \pi_k^{-1}(E_k)$, where $E_1, \dots, E_k \in \mathcal{B}(M)$, the Borel sigma-algebra of M .

Let $f \in C(\Omega)$ a potential and for each $n \in \mathbb{N}$, $x \in \Omega$ and $E \in \mathcal{F}$ consider the mapping $K_n : \mathcal{F} \times \Omega \rightarrow [0, 1]$ given by

$$K_n(E, x) \equiv \frac{\mathcal{L}_f^n(1_E)(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))}. \quad (7)$$

For any fixed $x \in \Omega$, the monotone convergence theorem implies that the map $\mathcal{F} \ni E \mapsto K_n(E, x)$ is a probability measure. For any fixed measurable set $E \in \mathcal{F}$ follows from the Fubini theorem that the map $x \mapsto K_n(E, x)$ is \mathcal{T}_n -measurable. So K_n is a **probability kernel** from \mathcal{T}_n to \mathcal{F} .

Notice that, for any $\varphi \in C(\Omega)$, the kernel $K_n(\varphi, x)$ is well defined due to the right hand side of (7). It is easy to see (using the rhs of (7)) that they are proper kernels, meaning that for any bounded \mathcal{T}_n -measurable function φ , we have $K_n(\varphi, x) = \varphi(\sigma^n(x))$. The above probability kernels have the following important property. For any fixed continuous function φ , the map $x \mapsto K_n(\varphi, x)$ is continuous as a consequence of the Lebesgue dominated convergence theorem. We refer to this saying that $(K_n)_{n \in \mathbb{N}}$ has the Feller property.

Definition 4.1. A Gibbsian specification with parameter set \mathbb{N} in the translation invariant setting is an abstract family of probability kernels $K_n : (\mathcal{F}, \Omega) \rightarrow [0, 1]$, $n \in \mathbb{N}$ such that

- a) $\Omega \ni x \rightarrow K_n(E, x)$ is $\sigma^n \mathcal{F}$ -measurable function for any $E \in \mathcal{F}$;
- b) $\mathcal{F} \ni E \mapsto K_n(E, x)$ is a probability measure for any $x \in \Omega$;
- c) for any $n, r \in \mathbb{N}$ and any bounded \mathcal{F} -measurable function $f : \Omega \rightarrow \mathbb{R}$ we have **the compatibility condition**, i.e.,

$$K_{n+r}(f, x) = \int_{\Omega} K_n(f, \cdot) dK_{n+r}(\cdot, x) \equiv K_{n+r}(K_n(f, \cdot), x).$$

Remark 1. The classical definition of specification as given in [17] requires even in our setting a larger family of probability kernels. To be more precise we have to define a probability kernel for any finite subset $\Lambda \subset \mathbb{N}$ and the kernels K_Λ have to satisfy a), b) and a generalization of c). In the translation invariant setting on the lattice \mathbb{N} , the formalism can be simplified and one needs only to consider the family K_n , $n \in \mathbb{N}$, as defined above.

Strictly speaking, in order to use the results in [17], one first has to extend our specifications to any set $\Lambda = \{n_1, \dots, n_r\}$, but this can be consistently done by putting $K_\Lambda \equiv K_{n_r}$. This simplified definition adopted here is further justified by the fact that the DLR-Gibbs measures, compatible with a specification with parameter set \mathbb{N} , are completely determined by the kernels indexed in any cofinal collection of subsets of \mathbb{N} . So here we are taking advantage of this result to define our kernels only on the cofinal collection of subsets of \mathbb{N} of the form $\{1, \dots, n\}$ with $n \in \mathbb{N}$. Therefore, when writing K_n , we are really thinking, in terms of the general definition of specifications, of $K_{\{1, \dots, n\}}$.

The only specifications needed here are the ones given in (7), which are defined in terms of any continuous potential f . Notice that in the translation invariant setting, the construction in (7) for the lattice \mathbb{N} extends the usual construction made in terms of regular interactions. But in any case, the kernels in (7) give us particular constructions of quasilocal specifications which allow us to use some of the results from [17]. We refer the reader to [11] and [36] for results about specifications when the dynamics have the finite pre-images property.

Using $\mathcal{L}_f^n(\varphi \cdot \psi \circ \sigma^n) = \mathcal{L}_f^n(\varphi)\psi$ one easily obtains the following identity for any $r, n \in \mathbb{N}$, $x \in \Omega$ and $\varphi \in C(\Omega)$ (see, e.g., the proof of Theorem 4.3 in [11]).

$$\mathcal{L}_f^{n+r}(\varphi)(\sigma^{n+r}(x)) = \mathcal{L}_f^{n+r}\left(\frac{\mathcal{L}_f^n(\varphi)(\sigma^n(\cdot))}{\mathcal{L}_f^n(1)(\sigma^n(\cdot))}\right)(\sigma^{n+r}(x)). \quad (8)$$

The above identity immediately implies for the Kernels defined by (7) that

$$K_{n+r}(f, x) = \int_{\Omega} K_n(f, \cdot) dK_{n+r}(\cdot, x) \equiv K_{n+r}(K_n(f, \cdot), x). \quad (9)$$

We refer to the above set of identities as compatibility conditions for the family of probability kernels $(K_n)_{n \in \mathbb{N}}$ or simply DLR equations. Similar kernels are also considered in [36], but here we are working with a dynamical system that may have uncountable many elements in the preimage of any point.

Definition 4.2. We say that $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ is a DLR-Gibbs measure for the continuous potential f if for any $n \in \mathbb{N}$ and any continuous function $\varphi : \Omega \rightarrow \mathbb{R}$ we have for μ -almost all x that

$$\mathbb{E}_{\mu}[\varphi | \mathcal{T}_n](x) = \int_{\Omega} \varphi(y) dK_n(y, x).$$

The set of all DLR-Gibbs measures for f is denoted by $\mathcal{G}^{DLR}(f)$.

One very important and elementary result on DLR-Gibbs measure is the equivalence between the two conditions below:

- a) $\mu \in \mathcal{G}^{DLR}(f)$;
- b) for any $n \in \mathbb{N}$ and $E \in \mathcal{F}$ we have that $\mu(E) = \int_{\Omega} K_n(E, \cdot) d\mu$.

We now prove that $\mu \in \mathcal{G}^{DLR}(f)$ is not empty. The result of the next lemma for countable M can also be found in [30]. For these state spaces it is also possible to allow some less regularity than continuity.

Lemma 4.3. *Let $f \in C(\Omega)$ and $(K_n)_{n \in \mathbb{N}}$ as in (7). Then $\mathcal{G}^*(f) \subset \mathcal{G}^{DLR}(f)$.*

Proof. Let ν be such that $\mathcal{L}_f^* \nu = \lambda_f \nu$ and φ a bounded \mathcal{F} -measurable function. Notice that the quotient appearing in the first integral below is \mathcal{T}_n -measurable. Therefore for any bounded \mathcal{F} -measurable ψ the following equality holds.

$$\begin{aligned} \int_{\Omega} (\varphi \circ \sigma^n)(x) \frac{\mathcal{L}_f^n(\psi)(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))} d\nu(x) &= \int_{\Omega} \frac{\mathcal{L}_f^n(\psi(\varphi \circ \sigma^n))(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))} d\nu(x) \\ &= \int_{\Omega} \frac{1}{\lambda^n} \mathcal{L}_f^n \left[\frac{\mathcal{L}_f^n(\psi(\varphi \circ \sigma^n))(\sigma^n(\cdot))}{\mathcal{L}_f^n(1)(\sigma^n(\cdot))} \right] (x) d\nu(x). \end{aligned}$$

By using the equation (8) we see that the rhs above is equal to

$$\int_{\Omega} \frac{1}{\lambda^n} \mathcal{L}_f^n(\psi(\varphi \circ \sigma^n))(x) d\nu(x) = \int_{\Omega} \psi(x) (\varphi \circ \sigma^n)(x) d\nu(x).$$

Since φ is an arbitrary \mathcal{F} -measurable function we can conclude that

$$\nu[E | \mathcal{T}_n](y) = \frac{\mathcal{L}_f^n(I_E)(\sigma^n(y))}{\mathcal{L}_f^n(1)(\sigma^n(y))} \quad \nu - \text{a.s.}$$

so the equation (7) implies that $\nu \in \mathcal{G}^{DLR}(f)$. \square

Corollary 3. *Let $f \in C(\Omega)$ be a potential and $(K_n)_{n \in \mathbb{N}}$ the specification defined by (7). If $\mathcal{G}^{DLR}(f)$ is a singleton, then $\mathcal{G}^*(f) = \mathcal{G}^{DLR}(f)$.*

The next lemma establishes the reverse inclusion between $\mathcal{G}^{DLR}(f)$ and $\mathcal{G}^*(f)$. Before, we state an interesting result from specification theory which essentially says that the tail sigma-algebra \mathcal{T} has to be trivial with respect to an extremal DLR-Gibbs measure.

Theorem 4.4. *Let $(K_n)_{n \in \mathbb{N}}$ be the specification given in (7). Then the following conclusion holds. A probability measure $\mu \in \mathcal{G}^{DLR}(f)$ is extreme in $\mathcal{G}^{DLR}(f)$ if and only if μ is trivial on \mathcal{T} . Consequently, if μ is extreme in $\mathcal{G}^{DLR}(f)$, then every \mathcal{T} -measurable function f is constant μ a.s..*

The proof of the theorem adapted to our setting can be found in the appendix below.

Lemma 4.5. *Let $f \in C(\Omega)$ be a potential and $(K_n)_{n \in \mathbb{N}}$ defined as in (7). If μ is an extreme element in $\mathcal{G}^{DLR}(f)$ then $\mu \in \mathcal{G}^*(f)$.*

Proof. This proof is based on the following algebraic identity

$$K_n(\mathcal{L}_f(\varphi), \sigma(y)) = K_n(\mathcal{L}_f(1), \sigma(y)) K_{n+1}(\varphi, y) \quad (10)$$

which holds for every $n \in \mathbb{N}$, $y \in \Omega$ and $\varphi \in C(\Omega)$. Indeed,

$$\begin{aligned} K_n(\mathcal{L}_f(\varphi), \sigma(y)) &= \frac{\mathcal{L}_f^{n+1}(\varphi)(\sigma^{n+1}(y))}{\mathcal{L}_f^n(1)(\sigma^{n+1}(y))} = \frac{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(y))}{\mathcal{L}_f^n(1)(\sigma^{n+1}(y))} \frac{\mathcal{L}_f^{n+1}(\varphi)(\sigma^{n+1}(y))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(y))} \\ &= K_n(\mathcal{L}_f(1), \sigma(y)) K_{n+1}(\varphi, y). \end{aligned}$$

Let $\mu \in \mathcal{G}^{DLR}(f)$. By definition of a DLR-Gibbs measure and the convergence of backward martingales, we have $\mathbb{E}_\mu[\varphi | \mathcal{T}_n] = K_n(\varphi, \cdot)$ converges almost surely for any $\varphi \in C(\Omega)$. It follows from separability of $C(\Omega)$ that there exists a set of full measure such that $\int_\Omega \varphi d\mu_y := \lim_{n \rightarrow \infty} K_n(\varphi, y)$ for all $\varphi \in C(\Omega)$ and y in this set. By completing the sigma-algebra, we moreover may assume that the set Ω' defined as the set of y such that $\lim_{n \rightarrow \infty} K_n(\varphi, y)$ for all $\varphi \in C(\Omega)$ is measurable. We now show that $\sigma^{-1}(\Omega') = \Omega'$. In order to do so, assume that $y \in \sigma^{-1}(\Omega')$. It then follows from (10) that

$$\begin{aligned} \int_\Omega \mathcal{L}_f(\varphi) d\mu_{\sigma y} &= \lim_{n \rightarrow \infty} K_n(\mathcal{L}_f(\varphi), \sigma(y)) = \lim_{n \rightarrow \infty} K_n(\mathcal{L}_f(1), \sigma(y)) K_{n+1}(\varphi, y) \\ &= \int_\Omega \mathcal{L}_f(1) d\mu_{\sigma y} \lim_{n \rightarrow \infty} K_{n+1}(\varphi, y) = \lambda_{\sigma y} \lim_{n \rightarrow \infty} K_n(\varphi, y), \quad (11) \end{aligned}$$

where $\lambda_{\sigma y} \equiv \int_\Omega \mathcal{L}_f(1) d\mu_{\sigma y}$ exists since $\sigma y \in \Omega'$. Hence, $\lim_{n \rightarrow \infty} K_n(\varphi, y)$ exists and therefore, $\sigma^{-1}(\Omega') \subset \Omega'$. Now assume that $y \in \Omega'$. By substituting φ with $\varphi \circ \sigma / \mathcal{L}_f(1) \circ \sigma$, the same argument implies that

$$\begin{aligned} \int \frac{\varphi \circ \sigma}{\mathcal{L}_f(1) \circ \sigma} d\mu_y &= \lim_{n \rightarrow \infty} \frac{K_n(\mathcal{L}_f(\varphi \circ \sigma / \mathcal{L}_f(1) \circ \sigma), \sigma(y))}{K_n(\mathcal{L}_f(1), \sigma(y))} \\ &= \lim_{n \rightarrow \infty} \frac{K_n(\varphi, \sigma(y))}{K_n(\mathcal{L}_f(1), \sigma(y))}. \end{aligned}$$

Setting $\varphi = 1$, we obtain that $\lim_n K_n(\mathcal{L}_f(1), \sigma(y))$ exists. Hence, by repeating the argument for general φ , it follows that also $\lim_n K_n(\varphi, \sigma(y))$ exists. Hence, $\sigma(y) \in \Omega'$ and consequently $\sigma^{-1}(\Omega') = \Omega'$.

Hence, we have by (11), that

$$\int_{\Omega} \varphi d\mu_y = \frac{1}{\lambda_{\sigma y}} \int_{\Omega} \mathcal{L}_f(\varphi) d\mu_{\sigma y}$$

for μ -almost y . In particular, $\int_{\Omega} \varphi d\mu_y = \int_{\Omega} \varphi d\mu_{\tilde{y}}$ whenever $\sigma(y) = \sigma(\tilde{y})$. In particular, the map $y \mapsto \int_{\Omega} \varphi d\mu_y$ is trivial with respect to \mathcal{T} and therefore, as μ is extremal, constant. Furthermore, assume that $\lambda = \lambda_y$ a.s.. Then, by applying bounded convergence three times,

$$\begin{aligned} \frac{1}{\lambda} \int_{\Omega} \mathcal{L}_f(\varphi) d\mu &= \lim_{n \rightarrow \infty} \frac{1}{\lambda} \int_{\Omega} K_n(\mathcal{L}_f(\varphi))(y) d\mu = \int_{\Omega} \lim_{n \rightarrow \infty} \frac{K_n(\mathcal{L}_f(\varphi))(y)}{K_n(\mathcal{L}_f(1))(y)} d\mu \\ &= \int_{\Omega} \lim_{n \rightarrow \infty} K_n(\varphi)(y) d\mu = \int_{\Omega} \varphi d\mu. \end{aligned}$$

From Corollary 1 it is a simple matter to check that $\lambda = \int_{\Omega} \mathcal{L}_f(1) d\mu$ is the spectral radius of \mathcal{L}_f acting on $C(\Omega)$ and therefore $\mathcal{L}_f^*(\mu) = \lambda_f \mu$. That is, $\mu \in \mathcal{G}^*(f)$. \square

Observe that the combination of Theorem 4.4 with Lemma 4.5 identifies the extreme elements in $\mu \in \mathcal{G}^{DLR}(f)$ with the set of exact, conformal measures, that is those elements in $\mathcal{G}^*(f)$ for which \mathcal{T} is trivial. In particular, after extending the action of \mathcal{L}_f to $L^1(\mu)$, Lin's criterion provides a further characterization of these measure as given in Corollary 5 below. As an immediate corollary of Lemma 4.5, we obtain the main result of this section.

Theorem 4.6. *Let $f \in C(\Omega)$ and $(K_n)_{n \in \mathbb{N}}$ as in (7). Then $\mathcal{G}^{DLR}(f) = \mathcal{G}^*(f)$.*

Proof. By Lemma 4.3, for any continuous potential f , we have $\mathcal{G}^*(f) \subset \mathcal{G}^{DLR}(f)$. On the other hand, Lemma 4.5 ensures $\text{ex}(\mathcal{G}^{DLR}(f)) \subset \mathcal{G}^*(f)$. By compactness and the Krein-Milman theorem it follows that $\mathcal{G}^{DLR}(f) \subset \mathcal{G}^*(f)$, thus proving the theorem. \square

5. Uniqueness Theorem for Eigenprobabilities.

Theorem 5.1. *Let f be continuous potential and $(K_n)_{n \in \mathbb{N}}$ be the specification defined as in (7). Suppose that there is constant $c > 0$ such that for every cylinder set $F \in \mathcal{F}$ there is $n \in \mathbb{N}$ such that*

$$K_n(F, x) \geq cK_n(F, y)$$

for all $x, y \in \Omega$. Then, the set $\mathcal{G}^(f)$ has only one element.*

Proof. Because of Lemma 4.3 it is enough to show that $\mathcal{G}^{DLR}(f)$ is a singleton. Suppose that $\mathcal{G}^{DLR}(f)$ contains two distinct elements μ and ν . Then the convex combination $(1/2)(\mu + \nu) \in \mathcal{G}^{DLR}(f) \setminus \text{ex}(\mathcal{G}^{DLR}(f))$, where $\text{ex}(\mathcal{G}^{DLR}(f))$ denotes the set of extreme measures of $\mathcal{G}^{DLR}(f)$. Therefore it is sufficient to show that $\mathcal{G}^{DLR}(f) \subset \text{ex}(\mathcal{G}^{DLR}(f))$.

Let $\mu \in \mathcal{G}^{DLR}(f)$, $E_0 \in \mathcal{T}$ and suppose that $\mu(E_0) > 0$. The existence of such set is ensured by the Theorem 7.7 item (c) in [17], which says that any element $\mu \in \mathcal{G}^{DLR}(f)$ is uniquely determined by its restriction to the tail sigma-algebra \mathcal{T} (see Corollary 9 in the Appendix). Since $\mu(E_0) > 0$ the probability measure $\nu \equiv \mu(\cdot|E_0) \in \mathcal{G}^{DLR}(f)$, see Theorem 7.7 (b) in [17] (or, see Proposition 8 and Corollary 8 in the Appendix).

We now prove that for all $E \in \mathcal{F}$, we have $\nu(E) \geq c\mu(E)$. Fix a cylinder set $F \in \mathcal{F}$. Then, for n sufficiently large, it follows from the characterization of the DLR-Gibbs measures and from the hypothesis that

$$\begin{aligned} \nu(F) &= \int_{\Omega} K_n(F, x) d\nu(x) = \int_{\Omega} \left[\int_{\Omega} K_n(F, x) d\nu(x) \right] d\mu(y) \\ &\geq c \int_{\Omega} \left[\int_{\Omega} K_n(F, y) d\nu(x) \right] d\mu(y) \\ &= c \int_{\Omega} \left[\int_{\Omega} K_n(F, y) d\mu(y) \right] d\nu(x) \\ &= c\mu(F). \end{aligned}$$

Using the monotone class theorem we may conclude that for all $E \in \mathcal{F}$ we have $\nu(E) \geq c\mu(E)$. In particular, $0 = \nu(\Omega \setminus E_0) \geq c\mu(\Omega \setminus E_0)$ therefore $\mu(E_0) = 1$. Consequently μ is trivial on $\bigcap_{j \in \mathbb{N}} \mathcal{F}_j$. Hence another application of Theorem 7.7 (a) of [17] (or, see Corollary 8) ensures that μ is extreme. \square

A consequence of this theorem we prove the uniqueness of the eigenmeasures for the dual of Ruelle operator associated to a potential $f : \Omega \rightarrow \mathbb{R}$ satisfying the following conditions:

- (Walters) $\lim_{d(x,y) \rightarrow 0} \sup_{n \in \mathbb{N}} \sup_{a \in M^n} |S_n(f)(ax) - S_n(f)(ay)| = 0;$ (12)

- (Bowen) $D \equiv \sup_{n \in \mathbb{N}} \sup_{\substack{x, y \in \Omega; \\ x_i = y_i, i=1, \dots, n}} |S_n(f)(x) - S_n(f)(y)| < \infty.$ (13)

Of course, a potential f satisfying the Walters condition satisfies the Bowen condition. What we are calling here Bowen's condition is actually a generalization to uncountable alphabets of classical Bowen's condition, see [36].

Theorem 5.2. *Let f be a continuous potential satisfying*

$$D \equiv \sup_{n \in \mathbb{N}} \sup_{\substack{x, y \in \Omega; \\ x_i = y_i, i=1, \dots, n}} |S_n(f)(x) - S_n(f)(y)| < \infty$$

then the set $\mathcal{G}^(f)$ is a singleton and $\mathcal{G}^*(f) = \mathcal{G}^{DLR}(f)$.*

Proof. Let D be the constant as in the above theorem and C a cylinder such that its basis is contained in the set $\{1, \dots, p\}$, i.e., for every $n \geq p$ we have $1_C(x_1 \dots x_n \sigma^n(z)) = 1_C(x_1 \dots x_n \sigma^n(y))$ for all $y, z \in \Omega$ and $x_1, \dots, x_n \in M$. We claim that for any choice of $y, z \in \Omega$ and for all $n \geq p$, we have

$$e^{-2D} K_n(C, z) \leq K_n(C, y) \leq e^{2D} K_n(C, z).$$

By definition of D we have, uniformly in $n \in \mathbb{N}$, $x, y, z \in \Omega$, the following inequality $-D \leq S_n(f)(x_1 \dots x_n \sigma^n(z)) - S_n(f)(x_1 \dots x_n \sigma^n(y)) \leq D$ which immediately imply the inequalities $\exp(-D) \exp(-S_n(f)(x_1 \dots x_n \sigma^n(z))) \leq \exp(-S_n(f)(x_1 \dots x_n \sigma^n(y)))$ and $\exp(-S_n(f)(x_1 \dots x_n \sigma^n(y))) \leq \exp(D) \exp(-S_n(f)(x_1 \dots x_n \sigma^n(z)))$. Using these two previous inequalities we get that

$$e^{-D} \mathcal{L}_f^n(1)(\sigma^n(z)) \leq \mathcal{L}_f^n(1)(\sigma^n(y)) \leq e^D \mathcal{L}_f^n(1)(\sigma^n(z)) \quad (14)$$

and also

$$K_n(C, y) = \frac{\mathcal{L}_f^n(1_C)(\sigma^n(y))}{\mathcal{L}_f^n(1)(\sigma^n(y))} \leq \frac{e^D \mathcal{L}_f^n(1_C)(\sigma^n(z))}{e^{-D} \mathcal{L}_f^n(1)(\sigma^n(z))} = e^{2D} K_n(C, z).$$

Analogously we obtain $e^{-2D}K_n(C, z) \leq K_n(C, y)$ and so the claim is proved.

Let μ and ν be distinct extreme measures in $\mathcal{G}^{DLR}(f)$. Since we are assuming that M is compact, it follows from Theorem 7.12 of [17] that there exist $y, z \in \Omega$ such that both measures μ and ν are thermodynamic limits of $K_n(\cdot, y)$ and $K_n(\cdot, z)$, respectively, when $n \rightarrow \infty$. Given an open cylinder set C such that its basis is contained in the set $\{1, \dots, p\}$ there is an increasing sequence of closed cylinders $C_1 \subset C_2 \subset \dots$ such that for all $k \in \mathbb{N}$ the basis of C_k is contained in the set $\{1, \dots, p\}$, and $\bigcup_{k \in \mathbb{N}} C_k = C$. By Urysohn's lemma for each $k \in \mathbb{N}$ there is a continuous function $\varphi_k : \Omega \rightarrow [0, 1]$ such that $1_{C_k} \leq \varphi_k \leq 1_C$ and $\varphi_k \rightarrow 1_C$ pointwise. Since C and $(C_k)_{k \in \mathbb{N}}$ have their basis contained in $\{1, \dots, p\}$, then the function φ_k can be chosen as a continuous function depending only on its first p coordinates.

By using the claim and a standard approximation arguments we get, for any fixed k , the inequality $K_n(\varphi_k, y) \leq e^{2D}K_n(\varphi_k, z)$ for all $n \geq p$. By taking the limits, when n goes to infinity and next when k goes to infinity we get $\mu(C) \leq e^{2D}\nu(C)$. Clearly the collection $\mathcal{D} = \{E \in \mathcal{F} : \mu(E) \leq e^{2D}\nu(E)\}$ is a monotone class. Since it contains the open cylinder sets, which is stable under intersections, we have that $\mathcal{D} = \mathcal{F}$. Therefore $\mu \leq e^{2D}\nu$, in particular $\mu \ll \nu$. This contradicts the fact that two distinct extreme DLR-Gibbs measures are mutually singular, therefore $\mathcal{G}^{DLR}(f)$ is a singleton and by Lemma 4.3 we are done. \square

This result generalizes two conditions for uniqueness presented in two recent works by the authors when general compact state space M is considered, see [12] and [23]. In fact, the above theorem generalizes the Hölder, Walters (weak and stronger as introduced in [12]) and Bowen conditions because it can be applied for potentials defined on $\Omega = M^{\mathbb{N}}$, where the state space M is any general compact metric space.

5.1. Dobrushin Uniqueness Theorem. In this section we prove an uniqueness theorem in the high temperature regime (β small) for potentials not satisfying Bowen's condition in (13). This result applies for a very large class of potentials which live outside the Hölder, Walters and Bowen spaces. Its proof is based on the Dobrushin Uniqueness Theorem suitably adapted to our setting.

For each positive integer n let $\Lambda_n \subset \mathbb{N}$ be a finite set such that $1 \in \Lambda_n$. Denote by $\pi_{\Lambda_n} : \Omega \rightarrow M^{\Lambda_n}$ the natural projection from Ω onto M^{Λ_n} . For each $n \geq 1$ let $f_n : M^{|\Lambda_n|} \rightarrow \mathbb{R}$ be a continuous function and suppose that $\sum_{n=1}^{\infty} \|f_n\|_{\infty} < +\infty$. Consider the continuous potential $f : \Omega \rightarrow \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} f_n \circ \pi_{\Lambda_n}(x).$$

The next theorem is a version of Dobrushin's Theorem for Thermodynamic Formalism. We point out that in [11] it was described a natural way to connect the classical setting of Thermodynamic with interactions, specifications, etc... (which is more close to the classical setting of Statistical Mechanics). We will follow such point of view here.

Theorem 5.3. *Let f be as above and suppose that $\sum_{n \geq 1} |\Lambda_n| \|f_n\|_{\infty} < +\infty$. Then there exists $\beta_D \in (0, \infty)$ such that for any $\beta < \beta_D$, the set $\mathcal{G}^*(\beta f)$ is a singleton.*

Proof. Consider the interaction $\Phi \equiv (\Phi_{\Lambda})_{\Lambda \subset \mathbb{N}}$ given by: $\Phi_{\Lambda} \equiv 0$ if $\Lambda \neq k + \Lambda_n$ for some $k, n \in \mathbb{N}$; $\Phi_{\Lambda_n}(x) = f_n \circ \pi_{\Lambda_n}(x)$ and $\Phi_{k+\Lambda_n}(x) = f_n \circ \pi_{k+\Lambda_n}(x)$.

Note that

$$H_{\{1, \dots, n\}}^\Phi(x) = \sum_{\substack{\Lambda \cap \{1, \dots, n\} \neq \emptyset \\ 0 < |\Lambda| < +\infty}} \Phi_\Lambda(x) = \sum_{i=1}^n \sum_{\substack{\Lambda \ni i \\ 0 < |\Lambda| < +\infty}} \Phi_\Lambda(x) = S_n(f)(x).$$

Let γ^Φ be the specification determined by H_Λ^Φ . Since the DLR Gibbs measures are completely determined in a cofinal collection of volumes, we have that $\mathcal{G}(\gamma^{\beta\Phi}) = \mathcal{G}^{DLR}(\beta f)$.

From the construction of Φ we have

$$\sup_{i \in \mathbb{N}} \sum_{i \ni \Lambda} (|\Lambda| - 1) \|\Phi_\Lambda\|_\infty = \sum_{n \geq 1} (|\Lambda_n| - 1) \|f_n\|_\infty < +\infty$$

and it therefore follows from the Dobrushin Uniqueness Theorem (see [17], Theorem 8.7 and Proposition 8.8) that $\mathcal{G}(\gamma^{\beta\Phi})$ is a singleton whenever

$$\beta < \beta_D \equiv \frac{2}{\sum_{n \geq 1} (|\Lambda_n| - 1) \|f_n\|_\infty}.$$

Since Lemma 4.3 ensures that for any continuous potential f we have $\mathcal{G}^*(\beta f) \subset \mathcal{G}^{DLR}(\beta f)$ the result follows. \square

We apply the above theorem in the following case. We take $E = \{-1, 1\}$, the a priori measure p as the normalized counting measure and fix $0 < \varepsilon < 1$. Consider the potential

$$f(x) = \sum_{n=2}^{\infty} \frac{x_1 x_n}{n^{1+\varepsilon}}.$$

In the literature this potential is sometimes called Dyson potential (see [11]). In this case, for any $\beta > 0$ we have

$$D \equiv \sup_{n \in \mathbb{N}} \sup_{\substack{x, y \in \Omega; \\ x_i = y_i, i=1, \dots, n}} |S_n(\beta f)(x) - S_n(\beta f)(y)| = \infty.$$

The above equality implies that f is not in Hölder, Walters and Bowen spaces. Of course, this potential can be rewritten as $f(x) = \sum_{n=1}^{\infty} f_n \circ \pi_{\Lambda_n}(x)$, by taking $\Lambda_n = \{1, n\}$ and $f_n \circ \pi_{\Lambda_n}(x) = x_1 x_n / n^{1+\varepsilon}$. Then

$$\sum_{n \geq 1} (|\Lambda_n| - 1) \|f_n\|_\infty = \sum_{n \geq 1} \frac{1}{n^{1+\varepsilon}} = \zeta(1 + \varepsilon).$$

Now applying the above theorem (taking $|\Lambda_n| = 2$) we get that $\mathcal{G}^*(\beta f)$ is a singleton for any choice of $\beta < 2\zeta(1 + \varepsilon)^{-1}$.

By taking $\Lambda_n = \{1, n, n + 1\}$, $n \geq 2$, one can get results for potentials of the form $f(x) = \sum_{n=2}^{\infty} x_1 x_n x_{n+1} / n^{1+\varepsilon}$ or more generally for potentials of the form $f(x) = \sum_{n=2}^{\infty} a_n x_1 x_n x_{n+1}$, where $\sum a_n$ is absolutely convergent. In this case, to estimate the critical temperature, one considers the expression $\sum_{n \geq 1} 3 \|f_n\|_\infty$.

6. The Extension of the Ruelle Operator to the Lebesgue Space $L^1(\Omega, \mathcal{F}, \nu_f)$.

Let f be a fixed continuous potential and ν_f the Borel probability measure obtained above. In this section we show how to construct a bounded linear extension of the operator $\mathcal{L}_f : C(\Omega) \rightarrow C(\Omega)$ acting on $L^1(\Omega, \mathcal{F}, \nu_f)$, by abusing notation also called \mathcal{L}_f , and under suitable assumptions prove the existence of an almost surely non-negative eigenfunction $\varphi_f \in L^1(\Omega, \mathcal{F}, \nu_f)$ associated to the eigenvalue λ_f constructed in the previous section.

Proposition 3. *Fix a continuous potential f and let λ_f and ν_f be the eigenvalue and eigenmeasure of \mathcal{L}^* , respectively. Then the Ruelle operator $\mathcal{L}_f : C(\Omega) \rightarrow C(\Omega)$ can be uniquely extended to a bounded linear operator $\mathcal{L}_f : L^1(\Omega, \mathcal{F}, \nu_f) \rightarrow L^1(\Omega, \mathcal{F}, \nu_f)$. Moreover, this extension has operator norm $\|\mathcal{L}_f\| = \lambda_f$.*

Proof. If $\varphi \in C(\Omega)$ then $\varphi^\pm \equiv \max\{0, \pm\varphi\} \in C(\Omega)$. Therefore, it follows from the positivity of the Ruelle operator and (3) that

$$\begin{aligned} \|\mathcal{L}_f(\varphi)\|_{L^1} &= \int_{\Omega} |\mathcal{L}_f(\varphi^+ - \varphi^-)| d\nu_f \leq \int_{\Omega} |\mathcal{L}_f(\varphi^+)| + |\mathcal{L}_f(\varphi^-)| d\nu_f \\ &= \int_{\Omega} \mathcal{L}_f(\varphi^+) + \mathcal{L}_f(\varphi^-) d\nu_f = \int_{\Omega} (\varphi^+ + \varphi^-) d(\mathcal{L}_f^* \nu_f) \\ &= \lambda_f \int_{\Omega} (\varphi^+ + \varphi^-) d\nu_f = \lambda_f \int_{\Omega} |\varphi| d\nu_f \\ &= \lambda_f \|\varphi\|_{L^1}. \end{aligned}$$

Since Ω is a compact Hausdorff space we have

$$\overline{C(\Omega, \mathbb{R})}^{L^1(\Omega, \mathcal{F}, \nu_f)} = L^1(\Omega, \mathcal{F}, \nu_f),$$

and \mathcal{L}_f therefore admits a unique continuous extension to $L^1(\Omega, \mathcal{F}, \nu_f)$. By taking $\varphi \equiv 1$ it is easy to see that $\|\mathcal{L}_f\| = \lambda_f$. \square

Proposition 4. *For any fixed potential $f \in C(\Omega)$ we have that*

$$L^1(\Omega, \mathcal{F}, \nu_f) = \Xi(f) \equiv \left\{ \varphi \in L^1(\Omega, \mathcal{F}, \nu_f) : \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f = \lambda_f \int_{\Omega} \varphi d\nu_f \right\}.$$

Proof. From (3) it follows that $C(\Omega) \subset \Xi(f)$. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence in $C(\Omega)$ such that $\varphi_n \rightarrow \varphi$ in $L^1(\Omega, \mathcal{F}, \nu_f)$. Then

$$\left| \int_{\Omega} \varphi_n d\nu_f - \int_{\Omega} \varphi d\nu_f \right| \leq \int_{\Omega} |\varphi_n - \varphi| d\nu_f \xrightarrow{n \rightarrow \infty} 0$$

and using the boundedness of \mathcal{L}_f , we can also conclude that

$$\left| \int_{\Omega} \mathcal{L}_f(\varphi_n) d\nu_f - \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f \right| \leq \int_{\Omega} |\mathcal{L}_f(\varphi_n - \varphi)| d\nu_f \leq \lambda_f \|\varphi_n - \varphi\|_{L^1} \xrightarrow{n \rightarrow \infty} 0.$$

By using the above convergences and the triangular inequality we can see that $\Xi(f)$ is closed subset of $L^1(\Omega, \mathcal{F}, \nu_f)$. Indeed,

$$\begin{aligned} \left| \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f - \lambda_f \int_{\Omega} \varphi d\nu_f \right| &\leq \\ &\left| \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f - \int_{\Omega} \mathcal{L}_f(\varphi_n) d\nu_f + \int_{\Omega} \mathcal{L}_f(\varphi_n) d\nu_f - \lambda_f \int_{\Omega} \varphi d\nu_f \right| \end{aligned}$$

and the right hand side goes to zero when $n \rightarrow \infty$ therefore $\varphi \in \Xi(f)$. Since $C(\Omega, \mathbb{R}) \subset \Xi(f)$ and $\Xi(f)$ is closed in $L^1(\Omega, \mathcal{F}, \nu_f)$ we have that

$$L^1(\Omega, \mathcal{F}, \nu_f) = \overline{C(\Omega, \mathbb{R})}^{L^1(\Omega, \mathcal{F}, \nu_f)} \subset \overline{\Xi(f)}^{L^1(\Omega, \mathcal{F}, \nu_f)} = \Xi(f) \subset L^1(\Omega, \mathcal{F}, \nu_f). \quad \square$$

The above proposition also implies that the extension of \mathcal{L}_f/λ_f to $L^1(\Omega, \mathcal{F}, \nu_f)$ can be identified with the transfer operator associated to a non-singular measure as formulated in the following corollary.

Corollary 4. For any $\varphi \in L^1(\Omega, \mathcal{F}, \nu_f)$ and $\psi \in L^\infty(\Omega, \mathcal{F}, \nu_f)$,

$$\int \lambda_f^{-1} \mathcal{L}_f(\varphi) \psi d\nu_f = \int \varphi \psi \circ \sigma d\nu_f.$$

Proof. Observe that $\mathcal{L}_f(\varphi)\psi = \mathcal{L}_f(\varphi \cdot \psi \circ \sigma)$ for $\varphi, \psi \in C(\Omega)$. In order to extend this property to the situation of the Corollary that (ψ_n) is a sequence of uniformly bounded functions in $C(\Omega)$ which converges almost surely to $\psi \in L^\infty(\Omega, \mathcal{F}, \nu_f)$. Then, for $\varphi \in C(\Omega)$, it follows from bounded convergence and Proposition 4 that

$$\begin{aligned} \int \mathcal{L}_f(\varphi) \psi d\nu_f &= \lim_{n \rightarrow \infty} \int \mathcal{L}_f(\varphi) \psi_n d\nu_f = \lim_{n \rightarrow \infty} \int \mathcal{L}_f(\varphi \psi_n \circ \sigma) d\nu_f \\ &= \lim_{n \rightarrow \infty} \lambda_f \int \varphi \psi_n \circ \sigma d\nu_f = \lambda_f \int \varphi \psi \circ \sigma d\nu_f \\ &= \int \mathcal{L}_f(\varphi \psi \circ \sigma) d\nu_f \end{aligned}$$

The assertion then follows as in the proof of Prop. 4 by approximation of φ by continuous functions in $L^1(\Omega, \mathcal{F}, \nu_f)$. \square

As a corollary of a theorem by Lin (Th. 4.1 in [22], or Th. 1.3.3 in [1]), the extremal DLR-measures hence can be characterized through the convergence of Ruelle's operator.

Corollary 5. The DLR-Gibbs measure ν_f is extremal if and only if for any φ in the space $L^1(\Omega, \mathcal{F}, \nu_f)$ with $\int_\Omega \varphi d\nu_f = 0$, we have that

$$\lim_{n \rightarrow \infty} \|\lambda_f^{-n} \mathcal{L}_f^n(\varphi)\|_1 = 0.$$

7. Strong Convergence of Ruelle Operators.

Proposition 5. For any fixed potential $f \in C(\Omega)$ there is a sequence $(f_n)_{n \in \mathbb{N}}$ contained in $C^\gamma(\Omega)$ such that $\|f_n - f\|_\infty \rightarrow 0$. Moreover, for any eigenmeasure ν_f associated to the eigenvalue λ_f we have that \mathcal{L}_{f_n} has a unique continuous extension to an operator defined on $L^1(\Omega, \mathcal{F}, \nu_f)$ and moreover, in the uniform operator norm, $\|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} \rightarrow 0$, when $n \rightarrow \infty$.

Proof. The first statement is a direct consequence of the Stone-Weierstrass Theorem.

For any $\varphi \in L^1(\Omega, \mathcal{F}, \nu_f)$ the extension of \mathcal{L}_{f_n} is given by $\mathcal{L}_{f_n}(\varphi) \equiv \mathcal{L}_f(\exp(f_n - f)\varphi)$ which is well-defined due to Proposition 3. From this proposition we also obtain the inequality

$$\begin{aligned} \int_\Omega |\mathcal{L}_{f_n}(\varphi)| d\nu_f &= \int_\Omega |\mathcal{L}_f(\exp(f_n - f)\varphi)| d\nu_f \\ &\leq \lambda_f \|\exp(f_n - f)\|_\infty \|\varphi\|_{L^1(\Omega, \mathcal{F}, \nu_f)} < \infty. \end{aligned}$$

Since the distance in the uniform operator norm between \mathcal{L}_{f_n} and \mathcal{L}_f can be bounded from above by

$$\begin{aligned} \|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} &= \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\mathcal{L}_{f_n}(\varphi) - \mathcal{L}_f(\varphi)| d\nu_f \\ &\leq \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\mathcal{L}_f(\exp(f_n - f)\varphi) - \mathcal{L}_f(\varphi)| d\nu_f \\ &\leq \lambda_f \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\varphi| |(\exp(f_n - f) - 1)| d\nu_f \\ &\leq \lambda_f |\exp(\|f_n - f\|_{\infty}) - 1| \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\varphi| d\nu_f, \end{aligned}$$

we can conclude that $\|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} \rightarrow 0$ as $n \rightarrow \infty$. \square

8. Existence of the Eigenfunctions. We point out that for a given continuous potential f there always exists eigenprobabilities ν_f whereas in some situations, there is no positive and continuous eigenfunction (see, for instance, [10]). We now will show the existence of a non-trivial eigenfunction of \mathcal{L}_f in $L^1(\Omega, \mathcal{F}, \nu_f)$ for potentials f satisfying Bowen's condition. This partially extends a result of Walters (see [36]) to our case with a possibly uncountable alphabet.

In this section we consider sequences of Borel probability measures $(\mu_{f_n})_{n \in \mathbb{N}}$ defined by

$$\mathcal{F} \ni E \mapsto \mu_{f_n}(E) \equiv \int_E h_{f_n} d\nu_f, \quad (15)$$

where $f_n \in C^{\gamma}(\Omega)$ satisfies $\|f_n - f\|_{\infty} \rightarrow 0$, and h_{f_n} is the unique eigenfunction of \mathcal{L}_{f_n} , which is assumed to have $L^1(\Omega, \mathcal{F}, \nu_f)$ norm one. Since Ω is compact we can also assume, up to subsequence, that $\mu_{f_n} \rightarrow \mu \in \mathcal{P}(\Omega, \mathcal{F})$.

From the definition of μ_{f_n} we immediately have that $\mu_{f_n} \ll \nu_f$. Notice that, in such generality, it is **not** possible to guarantee that $\mu \ll \nu_f$. When this is true the Radon-Nikodym theorem ensures the existence of a non-negative function $d\mu/d\nu_f \in L^1(\Omega, \mathcal{F}, \nu_f)$ such that for all $E \in \mathcal{F}$ we have

$$\mu(E) = \int_E \frac{d\mu}{d\nu_f} d\nu_f. \quad (16)$$

In what follows we give sufficient conditions for this Radon-Nikodym derivative to be an eigenfunction of \mathcal{L}_f .

Theorem 8.1. *Let (f_n) be a sequence of Hlder continuous functions which converges uniformly to f , and μ_{f_n} defined as in (15). If $(h_{f_n})_{n \in \mathbb{N}}$ is a relatively compact subset of $L^1(\Omega, \mathcal{F}, \nu_f)$ then up to subsequence $\mu_{f_n} \rightarrow \mu$, $\mu \ll \nu_f$ and $\mathcal{L}_f(d\mu/d\nu_f) = \lambda_f d\mu/d\nu_f$.*

Proof. Without loss of generality we can assume that h_{f_n} converges to some non-negative function $h_f \in L^1(\Omega, \mathcal{F}, \nu_f)$. This convergence implies

$$\left| \int_{\Omega} \varphi h_{f_n} d\nu_f - \int_{\Omega} \varphi h_f d\nu_f \right| \rightarrow 0, \quad \forall \varphi \in C(\Omega).$$

Therefore $\mu_{f_n} \rightarrow \mu$ with $\mu \ll \nu_f$ and $d\mu/d\nu_f = h_f$ almost surely.

Let us show that this Radon-Nikodym derivative is an non-negative eigenfunction for the Ruelle operator \mathcal{L}_f . It follows from the triangular inequality that

$$\|\mathcal{L}_f(h_f) - \lambda_f h_f\|_{L^1(\nu_f)} \leq \|\mathcal{L}_f(h_f) - \mathcal{L}_{f_n}(h_f)\|_{L^1(\nu_f)} + \|\mathcal{L}_{f_n}(h_f) - \lambda_f h_f\|_{L^1(\nu_f)}.$$

Proposition 5 implies that the first term tends to zero when n tends to infinity, whereas the second term can be estimated as follows.

$$\begin{aligned} \|\mathcal{L}_{f_n}(h_f) - \lambda_f h_f\|_{L^1(\nu_f)} &\leq \|\mathcal{L}_{f_n}(h_f - h_{f_n} + h_{f_n}) - \lambda_f h_f\|_{L^1(\nu_f)} \\ &\leq \|\mathcal{L}_{f_n}(h_f - h_{f_n}) + \lambda_{f_n} h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)} \\ &\leq \|\mathcal{L}_{f_n}\|_{L^1(\nu)} \cdot \|h_f - h_{f_n}\|_{L^1(\nu)} + \|\lambda_{f_n} h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)}. \end{aligned}$$

Since $\sup_{n \in \mathbb{N}} \|\mathcal{L}_{f_n}\|_{L^1(\nu)} < +\infty$ and $\|h_f - h_{f_n}\|_{L^1(\Omega, \mathcal{F}, \nu_f)} \rightarrow 0$ as $n \rightarrow \infty$, we have that the first term on the right hand side also goes to zero when n tends to infinity. The second term on the right hand side above is bounded by

$$\begin{aligned} \|\lambda_{f_n} h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)} &\leq \|\lambda_{f_n} h_{f_n} - \lambda_f h_{f_n}\|_{L^1(\nu_f)} + \|\lambda_f h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)} \\ &= |\lambda_{f_n} - \lambda_f| \cdot \|h_{f_n}\|_{L^1(\nu_f)} + |\lambda_f| \cdot \|h_{f_n} - h_f\|_{L^1(\nu_f)}. \end{aligned}$$

From Corollary 2 and our assumption follows that the lhs above can be made small if n is big enough. Piecing together all these estimates we conclude that $\|\mathcal{L}_f(h_f) - \lambda_f h_f\|_{L^1(\nu_f)} = 0$ and therefore $\mathcal{L}_f(h_f) = \lambda_f h_f$, ν_f a.s.. \square

Theorem 8.2. *Let (f_n) be a sequence of Hlder continuous functions which converges uniformly to f , μ_{f_n} as in (15) and suppose that that $\mu_{f_n} \rightarrow \mu$. If $\mu \ll \nu_f$ and $h_{f_n}(x) \rightarrow d\mu/d\nu_f$ ν_f -a.s. then $\mathcal{L}_f(d\mu/d\nu_f) = \lambda_f d\mu/d\nu_f$.*

Proof. Notice that

$$\int_{\Omega} |h_{f_n}| d\nu_f = 1 = \int_{\Omega} \left| \frac{d\mu}{d\nu_f} \right| d\nu_f \quad \text{and} \quad h_{f_n}(x) \rightarrow d\mu/d\nu_f \quad \nu_f - \text{a.s.}$$

Scheffé's lemma implies that h_{f_n} converges to $d\mu/d\nu_f$ in the $L^1(\Omega, \mathcal{F}, \nu_f)$ norm. To finish the proof it is enough to apply the previous theorem. \square

We now construct an eigenfunction for \mathcal{L}_f without assuming converge of h_{f_n} neither in $L^1(\Omega, \mathcal{F}, \nu_f)$ nor almost surely. We should remark that the next theorem applies even when no convergent subsequence of $(h_{f_n})_{n \in \mathbb{N}}$ exists.

Theorem 8.3. *Let $(h_{f_n})_{n \in \mathbb{N}}$ be a sequence of eigenfunctions in the unit sphere of the Lebesgue space $L^1(\Omega, \mathcal{F}, \nu_f)$, where $(f_n)_{n \in \mathbb{N}}$ is a sequence of Hölder potentials converging uniformly to f . If $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_{\infty} < +\infty$, then $\limsup h_{f_n} \in L^1(\Omega, \mathcal{F}, \nu_f) \setminus \{0\}$ and moreover $\mathcal{L}_f(\limsup h_{f_n}) = \lambda_f \limsup h_{f_n}$.*

Proof. Since we are assuming that $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_{\infty} < +\infty$, it follows that $\limsup h_{f_n} \in L^1(\Omega, \mathcal{F}, \nu_f)$. For any fixed $x \in \Omega$ follows from this uniform bound that the mapping

$$M \ni a \mapsto \limsup_{n \rightarrow \infty} h_{f_n}(ax)$$

is uniformly bounded and therefore integrable with respect to the a-priori measure ν so can apply the lim sup version of the Fatou's lemma to get the inequality

$$\begin{aligned}
\mathcal{L}_f(\limsup_{n \rightarrow \infty} h_{f_n}) &= \int_M \exp(f(ax)) \limsup_{n \rightarrow \infty} h_{f_n}(ax) dp(a) \\
&= \int_M \lim_{n \rightarrow \infty} \exp(f_n(ax)) \limsup_{n \rightarrow \infty} h_{f_n}(ax) dp(a) \\
&= \int_M \limsup_{n \rightarrow \infty} (\exp(f_n(ax)) h_{f_n}(ax)) dp(a) \\
&\geq \limsup_{n \rightarrow \infty} \int_M \exp(f_n(ax)) h_{f_n}(ax) dp(a) \\
&= \limsup_{n \rightarrow \infty} \lambda_{f_n} h_{f_n} \\
&= \lambda_f \limsup_{n \rightarrow \infty} h_{f_n}.
\end{aligned}$$

These inequalities implies that $\limsup h_{f_n}$ is a super solution to the eigenvalue problem. On the other hand, we have proved that $\|\mathcal{L}_f\|_{L^1(\nu_f)} = \lambda_f$. This fact together with the previous inequality implies that

$$\mathcal{L}_f(\limsup_{n \rightarrow \infty} h_{f_n}) = \lambda_f \limsup_{n \rightarrow \infty} h_{f_n}$$

ν_f -almost surely. It remains to prove that $\limsup h_{f_n}$ is non trivial. Since $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_\infty < +\infty$ we can ensure that $\mu_{f_n} \rightharpoonup \mu \ll \nu_f$. Indeed, for any open set $A \subset \Omega$, weak convergence and the Portmanteau Theorem imply that

$$\mu(A) \leq \liminf_{n \rightarrow \infty} \int_\Omega 1_A h_{f_n} d\nu_f.$$

Since ν_f is outer regular we have for any $B \in \mathcal{F}$ that $\nu_f(B) = \inf\{\nu_f(A) : A \supset B, A \text{ open}\}$. From the previous inequality and uniform limitation of h_{f_n} we get for any $B \subset A$ (A open set) that $\mu(B) \leq \mu(A) \leq \sup_{n \in \mathbb{N}} \|h_{f_n}\|_\infty \nu_f(A)$. Taking the infimum over $A \supset B$, A open, we have $\mu(B) \leq \sup_{n \in \mathbb{N}} \|h_{f_n}\|_\infty \nu_f(B)$ and thus $\mu \ll \nu_f$. By applying again the limit sup version of the Fatou Lemma we get that

$$1 = \int_\Omega \frac{d\mu}{d\nu_f} d\nu_f = \lim_{n \rightarrow \infty} \int_\Omega h_{f_n} d\nu_f = \limsup_{n \rightarrow \infty} \int_\Omega h_{f_n} d\nu_f \leq \int_\Omega \limsup_{n \rightarrow \infty} h_{f_n} d\nu_f,$$

where the second equality comes from the definition of the weak convergence. \square

We point out that the conditions $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_\infty < +\infty$ and (f_n) Hlder, $f_n \rightarrow f$ uniformly is not satisfied for Hofbauer potentials.

All the previous theorems of this section required informations about the eigenfunction. Now we present an existence result that one can check by using only the potential (via \mathcal{L}_f^n) and some estimates on the maximal eigenvalue. We remark that a similar approach can be found in [14] in the setting of random shift spaces with countable alphabets. In there, it was shown that the limes inferior defines a random eigenfunction.

Theorem 8.4. *Let f be a continuous potential and λ_f the eigenvalue of \mathcal{L}_f^* provided by Proposition 1. If*

$$\sup_{n \in \mathbb{N}} \|\mathcal{L}_f^n(1)/\lambda_f^n\|_\infty < +\infty,$$

then, $\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$ is a non-trivial eigenfunction of \mathcal{L}_f in $L^1(\Omega, \mathcal{F}, \nu_f)$ associated to λ_f .

Proof. The key idea is to prove that $\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$ is a super solution for the eigenvalue problem, since it belongs to $L^1(\Omega, \mathcal{F}, \nu_f)$ it has to be a sub solution and then it is in fact a solution. Its non-triviality is based on the arguments given in the previous proof and the weak convergence of suitable sequence of probability measures.

The super solution part of the argument is again based on the reverse Fatou Lemma as follows

$$\begin{aligned} \mathcal{L}_f(\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n) &= \int_M \exp(f(ax)) \limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)(ax)/\lambda_f^n dp(a) \\ &\geq \limsup_{n \rightarrow \infty} \int_M \exp(f(ax)) \mathcal{L}_f^n(1)(ax)/\lambda_f^n dp(a) \\ &= \limsup_{n \rightarrow \infty} \lambda_f \mathcal{L}_f^{n+1}(1)(x)/\lambda_f^{n+1} \\ &= \lambda_f \limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)(x)/\lambda_f^n. \end{aligned}$$

The next step is to prove the non-triviality of this limsup. From the definition of ν_f we can say that the following sequence of probability measures is contained in $\mathcal{P}(\Omega, \mathcal{F})$

$$\mathcal{F} \ni E \mapsto \int_E \frac{\mathcal{L}_f^n(1)}{\lambda_f^n} d\nu_f.$$

Similarly, to the previous theorem we can ensure that all its cluster points in the weak topology are absolutely continuous with respect to ν_f . Up to subsequence, we can get from another application of the Fatou Lemma that

$$1 = \int_{\Omega} \frac{d\mu}{d\nu_f} d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\mathcal{L}_f^n(1)}{\lambda_f^n} d\nu_f \leq \int_{\Omega} \limsup_{n \rightarrow \infty} \frac{\mathcal{L}_f^n(1)}{\lambda_f^n} d\nu_f.$$

□

Corollary 6. *Let f be a potential satisfying Bowen's condition and let D as in (13). Then,*

$$h_f \equiv \limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$$

is a non trivial $L^1(\Omega, \mathcal{F}, \nu_f)$ eigenfunction of \mathcal{L}_f associated to λ_f and $e^{-D} \leq h_f \leq e^D$.

Proof. Since we are assuming that the potential f satisfies Bowen's condition and D is given by (13) it follows from (14) that uniformly in $n \in \mathbb{N}$ and $z, y \in \Omega$ we have $e^{-D} \mathcal{L}_f^n(1)(\sigma^n(z)) \leq \mathcal{L}_f^n(1)(\sigma^n(y)) \leq e^D \mathcal{L}_f^n(1)(\sigma^n(z))$. Replacing in this inequality z by $a_1 \dots a_n z$ and similarly y by $a_1 \dots a_n y$, where $(a_1, \dots, a_n) \in M^n$ is a fixed we obtain the following estimate which holds for all $n \geq 1$ and $y, z \in \Omega$

$$e^{-D} \mathcal{L}_f^n(1)(z) \leq \mathcal{L}_f^n(1)(y) \leq e^D \mathcal{L}_f^n(1)(z).$$

By integrating the above inequality in z , with respect to the eigenmeasure, we get

$$e^{-D} \leq \frac{\mathcal{L}_f^n(1)(y)}{\lambda_f^n} \leq e^D.$$

The conclusions then follow from the last inequality and Theorem 8.4. □

Remark 2. It is not possible to conclude from the above argument whether h_f is a continuous function. Similarly to the case of finite alphabet considered in [36] the best information we have so far about its regularity is that this eigenfunction is at least $L^\infty(\Omega, \mathcal{F}, \nu_f)$. In the context of Markov maps, this result for example in [2]. However, the continuity of this eigenfunction as far as we know remains open, even in the finite alphabet setting.

9. Applications.

Weak Convergence of Eigenprobabilities. In this section we consider a continuous potential $f : \Omega \rightarrow \mathbb{R}$ or an element of $C^\gamma(\Omega)$ for some $0 \leq \gamma < 1$. We would like to get results for continuous potentials via limits of Hölder potentials.

We choose a point in the state space M and for simplicity call it 0. We denote by $(f_n)_{n \in \mathbb{N}} \subset C^\gamma(\Omega)$ the sequence given by $f_n(x) = f(x_1, \dots, x_n, 0, 0, \dots)$. Keeping the notation of the previous sections, the eigenprobabilities of \mathcal{L}_{f_n} and \mathcal{L}_f are also denoted by ν_{f_n} and ν_f , respectively. Notice that $\|f - f_n\|_\infty \rightarrow 0$, when $n \rightarrow \infty$ and, moreover, if f is Hölder then this convergence is exponentially fast. We denote by $\mathcal{L}(C(\Omega))$ the space of all bounded operators from $C(\Omega)$ to itself and for each $T \in \mathcal{L}(C(\Omega))$ we use the notation $\|T\|_{C(\Omega)}$ for its operator norm. The next lemma is inspired by Proposition 5.

Lemma 9.1. *The sequence $(\mathcal{L}_{f_n})_{n \in \mathbb{N}}$ converges in the operator norm to the Ruelle operator \mathcal{L}_f , i.e., $\|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{C(\Omega)} \rightarrow 0$, when $n \rightarrow \infty$.*

Proof. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} \|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{C(\Omega)} &= \sup_{0 < \|\varphi\|_\infty \leq 1} \sup_{x \in \Omega} |\mathcal{L}_{f_n}(\varphi)(x) - \mathcal{L}_f(\varphi)(x)| \\ &\leq \sup_{0 < \|\varphi\|_\infty \leq 1} \sup_{x \in \Omega} |\mathcal{L}_f(\exp(f_n - f)\varphi)(x) - \mathcal{L}_f(\varphi)(x)| \\ &\leq \|\mathcal{L}_f\|_{C(\Omega)} \sup_{0 < \|\varphi\|_\infty \leq 1} \|\varphi\|_\infty \|(\exp(f_n - f) - 1)\|_\infty \\ &\leq \|\mathcal{L}_f\|_{C(\Omega)} \|(\exp(f_n - f) - 1)\|_\infty. \end{aligned}$$

□

Proposition 6. *Any cluster point, in the weak topology, of the sequence $(\nu_{f_n})_{n \in \mathbb{N}}$ belongs to the set $\mathcal{G}^*(f)$.*

Proof. By the previous lemma for any given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ we have for all $\varphi \in C(\Omega)$ and for all $x \in \Omega$ that $\mathcal{L}_{f_n}(\varphi)(x) - \varepsilon < \mathcal{L}_f(\varphi)(x) < \mathcal{L}_{f_n}(\varphi)(x) + \varepsilon$. From the duality relation of the Ruelle operator and the weak convergence and Corollary 2 we have that

$$\begin{aligned} \int_{\Omega} \varphi d(\mathcal{L}_f^* \nu) &= \int_{\Omega} \mathcal{L}_f(\varphi) d\nu = \lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_{f_n} \\ &< \lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{L}_{f_n}(\varphi) d\nu_{f_n} + \varepsilon \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \varphi d(\mathcal{L}_{f_n} \nu_{f_n}) + \varepsilon \\ &= \lim_{n \rightarrow \infty} \lambda_{f_n} \int_{\Omega} \varphi d\nu_{f_n} + \varepsilon \\ &= \lambda_f \int_{\Omega} \varphi d\nu + \varepsilon. \end{aligned}$$

We obtain analogous lower bound, with $-\varepsilon$ instead. Since $\varepsilon > 0$ is arbitrary follows for any $\varphi \in C(\Omega)$ that

$$\int_{\Omega} \varphi d(\mathcal{L}_f^* \nu) = \lambda_f \int_{\Omega} \varphi d\nu$$

and therefore $\mathcal{L}_f^* \nu = \lambda_f \nu$. \square

Remark 3. The above proposition for $f \in C^\gamma(\Omega)$ says that up to subsequence $\nu_{f_n} \rightharpoonup \nu_f$, which is the unique eigenprobability of \mathcal{L}_f^* . Therefore the eigenprobability ν_f inherits all the properties of the sequence ν_{f_n} that are preserved by weak limits.

Constructive Approach for Equilibrium States for General Continuous Potentials. Before proceed we should mention that Sarig in [30] has also presented a construction of equilibrium measures for topologically mixing TMS.

Lemma 9.2. *For each $n \in \mathbb{N}$, let f_n be the potential defined above and h_{f_n} the main eigenfunction of \mathcal{L}_{f_n} associated to λ_{f_n} , normalized so that $\|h_{f_n}\|_{L^1(\nu_{f_n})} = 1$, where ν_{f_n} is the unique eigenprobability of $\mathcal{L}_{f_n}^*$. Then there exist a σ -invariant Borel probability measure μ_f such that, up to taking subsequences,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varphi h_{f_n} d\nu_{f_n} = \int_{\Omega} \varphi d\mu_f, \quad \forall \varphi \in C(\Omega)$$

Proof. It is well known that $h_{f_n} d\nu_{f_n}$ defines a σ -invariant Borel probability measure and therefore any of its cluster point, in the weak topology is a shift invariant probability measure. \square

As observed in [23], when M is uncountable, the Kolmogorov-Sinai entropy is not suitable in the formulation of the variational problem. In what follows we use the concept of entropy introduced in [23]. This entropy is defined for each probability measure μ by

$$h_\mu(\sigma) \equiv \inf_{g \in C^\alpha(\Omega, \mathbb{R})} \left\{ - \int_{\Omega} g d\mu + \log \lambda_g \right\}. \quad (17)$$

Note that this entropy depends on the choice of the a priori measure and similar ideas are employed in Statistical Mechanics when studying translation invariant Gibbs measures of continuous spin systems on the lattice, see [17, 33] and references therein.

Theorem 9.3 (Equilibrium States). *Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous potential and $(f_n)_{n \in \mathbb{N}}$ a sequence of Hölder potentials such that $\|f_n - f\|_\infty \rightarrow 0$, when $n \rightarrow \infty$. Then any probability measure μ_f as constructed in the Lemma 9.2 is an equilibrium state for f .*

Proof. Given any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then $-\varepsilon < f - f_n < \varepsilon$. We know that the equilibrium measure μ_{f_n} for f_n is given by $\mu_{f_n} = h_{f_n} \nu_{f_n}$ and therefore, we have that

$$\begin{aligned} \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f d\rho \right\} &= \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} (f - f_n) d\rho + \int_{\Omega} f_n d\rho \right\} \\ &< \varepsilon + \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f_n d\rho \right\} \\ &= \varepsilon + h(\mu_{f_n}) + \int_{\Omega} f_n d\mu_{f_n}. \end{aligned}$$

Since the entropy defined by (17) is upper semi-continuous and $\mu_{f_n} \rightharpoonup \mu_f$ it follows that for some $n_1 \in \mathbb{N}$ and $n \geq n_1$ we have

$$h(\mu_{f_n}) < h(\mu_f) + \varepsilon.$$

Using again the uniform convergence of f_n to f and the weak convergence of μ_{f_n} to μ_f , for some $n_2 \in \mathbb{N}$ and $n \geq n_2$ we get

$$\int_{\Omega} f_n d\mu_{f_n} = \int_{\Omega} f_n - f d\mu_{f_n} + \int_{\Omega} f d\mu_{f_n} < 2\varepsilon + \int_{\Omega} f d\mu_f.$$

Using the previous three inequalities we get for $n \geq \max\{n_0, n_1, n_2\}$

$$\sup_{\rho \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f d\rho \right\} < 4\varepsilon + h(\mu_f) + \int_{\Omega} f d\mu_f.$$

Since $\varepsilon > 0$ is arbitrary follows from the definition of the supremum and above inequality that

$$\sup_{\rho \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f d\rho \right\} = h(\mu_f) + \int_{\Omega} f d\mu_f$$

and therefore μ_f constructed in Lemma 9.2 is an equilibrium state. \square

Corollary 7. *For any continuous potential $f : \Omega \rightarrow \mathbb{R}$ we have that*

$$\log \lambda_f = \sup_{\rho \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f d\rho \right\}.$$

Proof. Consider the Hölder approximations $(f_n)_{n \in \mathbb{N}}$ of f as above. Then for any given $\varepsilon > 0$ and n large enough we have

$$\begin{aligned} \log \lambda_{f_n} - \varepsilon &= h(\mu_{f_n}) + \int_{\Omega} f_n d\mu_{f_n} - \varepsilon \\ &< \sup_{\rho \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f d\rho \right\} \\ &< \varepsilon + h(\mu_{f_n}) + \int_{\Omega} f_n d\mu_{f_n} \\ &= \varepsilon + \log \lambda_{f_n}. \end{aligned}$$

Since $\lambda_{f_n} \rightarrow \lambda_f$ it follows from the above inequality that

$$\sup_{\rho \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_{\Omega} f d\rho \right\} = \log \lambda_f. \quad \square$$

Necessary and Sufficient Conditions for the Existence of L^1 Eigenfunctions.

Theorem 9.4. *Let $\nu \in \mathcal{G}^*(f)$. The Ruelle operator has a non-negative eigenfunction $h \in L^1(\nu)$ if, and only if, there exists $\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})$ such that $\mu \ll \nu$.*

Proof. We first assume that there is $\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})$ so that $\mu \ll \nu$. In this case we claim that

$$\mathcal{L}_f \left(\frac{d\mu}{d\nu} \right) = \lambda_f \frac{d\mu}{d\nu}.$$

Indeed, for any continuous function φ we have

$$\begin{aligned} \int_{\Omega} \varphi \mathcal{L}_f \left(\frac{d\mu}{d\nu} \right) d\nu &= \int_{\Omega} \mathcal{L}_f \left(\varphi \circ \sigma \cdot \frac{d\mu}{d\nu} \right) d\nu \\ &= \lambda_f \int_{\Omega} \varphi \circ \sigma \cdot \frac{d\mu}{d\nu} d\nu = \lambda_f \int_{\Omega} \varphi \circ \sigma \cdot d\mu \\ &= \lambda_f \int_{\Omega} \varphi d\mu = \lambda_f \int_{\Omega} \varphi \cdot \frac{d\mu}{d\nu} d\nu. \end{aligned}$$

Conversely, suppose that $h \in L^1(\nu)$ is a non-negative eigenfunction for the Ruelle operator associated to the main eigenvalue and normalized so that $\int_{\Omega} h d\nu = 1$. Define the probability measure $\mu = h d\nu$. Then for any $\varphi \in C(\Omega)$ we have

$$\begin{aligned} \lambda_f \int_{\Omega} \varphi d\mu &= \lambda_f \int_{\Omega} \varphi h d\nu = \int_{\Omega} \varphi \mathcal{L}_f h d\nu \\ &= \int_{\Omega} \mathcal{L}_f(\varphi \circ \sigma \cdot h) d\nu = \lambda_f \int_{\Omega} \varphi \circ \sigma \cdot h d\nu \\ &= \lambda_f \int_{\Omega} \varphi \circ \sigma d\mu \end{aligned}$$

and therefore $\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})$ and $\mu \ll \nu$. \square

Continuous Potentials not Having Continuous Eigenfunctions. Following the results of [9] now we assume that the state space $M = \{-1, 1\}$ and the a priori measure is the uniform measure, which we denote by κ . Let ρ be the infinite product measure $\rho = \prod_{i \in \mathbb{N}} \kappa$. Consider the continuous potential f given by $f(x) = \sum_{n \geq 1} (x_n/n^\gamma)$, where $3/2 < \gamma \leq 2$. For each $n \in \mathbb{N}$ set $\alpha_n = \zeta(\gamma) - \sum_{j=1}^n n^{-\gamma}$. The main eigenvalue for \mathcal{L}_f is $\lambda_f = 2 \cosh(\zeta(\gamma))$ and there is a \mathcal{F} -measurable set $\Omega_0 \subset \Omega$ satisfying $\rho(\Omega_0) = 1$ and such that for all $x \in \Omega_0$ the following function

$$x \mapsto h_f(x) \equiv \exp(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_n x_n + \dots)$$

is well defined. One can show that h_f is the unique eigenfunction associated to λ_f and it is not an element of $L^\infty(\Omega, \mathcal{F}, \rho)$ which implies that $h_f \notin C(\Omega)$. We can also prove that $h_f \in L^1(\Omega, \mathcal{F}, \nu_f)$.

10. Appendix. On this appendix we adapt some results from the reference [17] to the present setting. Let \mathcal{L}_f be the Ruelle operator of a continuous potential f and for each $n \in \mathbb{N}$, $x \in \Omega$ and $E \in \mathcal{F}$, consider the probability kernel $K_n : \mathcal{F} \times \Omega \rightarrow [0, 1]$ given by the expression

$$K_n(E, x) \equiv \frac{\mathcal{L}_f^n(1_E)(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))}.$$

Proposition 7. *Suppose $\mu \in \mathcal{G}^{DLR}(f)$, then for all $n \in \mathbb{N}$*

$$\mathcal{A}_n(\mu) = \{E \in \mathcal{F} : K_n(E, \omega) = 1_E(\omega) \mu \text{ a.s.}\} \text{ is a } \sigma\text{-algebra.}$$

Proof. Since $K_n(\Omega, \omega) = 1 = 1_\Omega(\omega)$ we get that $\Omega \in \mathcal{A}_n(\mu)$. For the empty set the proof is trivial.

Now suppose that $(E_j)_{j \in \mathbb{N}}$ is a disjoint collection of elements of $\mathcal{A}_n(\mu)$. Then, for all ω we get $K_n(\cup_{j \in \mathbb{N}} E_j, \omega) = \sum_{j \in \mathbb{N}} K_n(E_j, \omega)$. Note that μ -a.e. $K_n(E_j, \omega) = 1_{E_j}(\omega)$ for all $j \in \mathbb{N}$, because $E_j \in \mathcal{A}_n(\mu)$. Clearly, $1_{\cup_{j \in \mathbb{N}} E_j}(\omega) = \sum_{j \in \mathbb{N}} 1_{E_j}(\omega)$, then by using that the intersection of sets of measure one has measure one, we get that $K_n(\cup_{j \in \mathbb{N}} E_j, \omega) = 1_{\cup_{j \in \mathbb{N}} E_j}(\omega)$, μ -a.e..

Note that $\mathcal{A}_n(\mu)$ is closed by the complement operation. Indeed, for all $\omega \in \Omega$ and $E \in \mathcal{A}_n(\mu)$ we have that $K_n(E^c, \omega) = 1 - K_n(E, \omega) = 1 - 1_E(\omega) = 1_{E^c}(\omega)$.

Since we have shown that $\mathcal{A}_n(\mu)$ is closed under denumerable disjoint unions then the remaining task is to show that $\mathcal{A}_n(\mu)$ is closed under finite intersections. Then it will follow that $\mathcal{A}_n(\mu)$ is closed under any denumerable union. Suppose that $E, F \in \mathcal{A}_n(\mu)$. By the monotonicity of the measure we have μ -a.e that

$$\begin{aligned} K_n(E \cap F, \omega) &\leq \min\{K_n(E, \omega), K_n(F, \omega)\} \\ &= \min\{1_E(\omega), 1_F(\omega)\} \\ &= 1_{E \cap F}(\omega). \end{aligned}$$

By using the hypothesis we get that

$$\begin{aligned} \int_{\Omega} [1_{E \cap F} - K_n(E \cap F, \cdot)] d\mu &= \int_{\Omega} 1_{E \cap F} d\mu - \int_{\Omega} K_n(E \cap F, \cdot) d\mu \\ &= \mu(E \cap F) - \int_{\Omega} K_n(E \cap F, \cdot) d\mu \\ &= \mu(E \cap F) - \mu(E \cap F) \\ &= 0. \end{aligned}$$

From the previous inequality we know that the integrand in the left hand side of the above is non-negative. So it has to be zero μ -a.e.. Therefore, $K_n(E \cap F, \omega) = 1_{E \cap F}(\omega)$, μ -a.e.. and finally we get that $\mathcal{A}_n(\mu)$ is closed for finite intersections. Therefore, $\mathcal{A}_n(\mu)$ is a σ -algebra. \square

Proposition 8. *Given a function $g : \Omega \rightarrow [0, \infty)$ we get the equivalence:*

- 1- $\int_{\Omega} K_n(E, \cdot) g d\mu = \int_{\Omega} 1_E g d\mu$ for all $E \in \mathcal{F}$,
- 2- The function g is measurable with respect to the sigma-algebra $\mathcal{A}_n(\mu)$.

Remark 4. In [17] the condition 1 is denoted by $(g\mu)K_n = g\mu$, where $g\mu$ is the measure defined by $E \mapsto \int_{\Omega} 1_E g d\mu$. This condition is equivalent to say that $g\mu$ is compatible with K_n .

Proof. First we will prove that $1 \implies 2$. This follows from the following claim: for all $g : \Omega \rightarrow [0, \infty)$ for which the condition 1 holds, we have $\{g \geq c\} \in \mathcal{A}_n(\mu)$, for any $c \in \mathbb{R}$. Indeed, the identity $1_{\{g < c\}} = 1 - 1_{\{g \geq c\}}$ implies

$$\begin{aligned} \int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega) \\ = \int_{\Omega} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega) - \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega). \end{aligned}$$

By using the condition 1 in the first expression of rhs we get

$$\begin{aligned} \int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega) \\ = \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) d\mu(\omega) - \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \\ = \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) (1 - K_n(1_{\{g \geq c\}}, \omega)) d\mu(\omega). \end{aligned}$$

Now, we will use the two inequalities $1_{\{g \geq c\}}(\omega)g(\omega) \geq c \cdot 1_{\{g \geq c\}}(\omega)$ and $K_n(1_{\{g \geq c\}}, \omega) \leq 1$, in the above expression, to get

$$\begin{aligned}
& \int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega)g(\omega) d\mu(\omega) \\
&= \int_{\Omega} 1_{\{g \geq c\}}(\omega)g(\omega)(1 - K_n(1_{\{g \geq c\}}, \omega)) d\mu(\omega) \\
&\geq c \int_{\Omega} 1_{\{g \geq c\}}(\omega)(1 - K_n(1_{\{g \geq c\}}, \omega)) d\mu(\omega) \\
&= c \int_{\Omega} 1_{\{g \geq c\}}(\omega) d\mu(\omega) - c \int_{\Omega} 1_{\{g \geq c\}}(\omega)K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \\
&\stackrel{(cond 1)}{=} c \int_{\Omega} K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) - c \int_{\Omega} 1_{\{g \geq c\}}(\omega)K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \\
&\stackrel{(1_{\{g < c\}} = 1 - 1_{\{g \leq c\}})}{=} c \int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega).
\end{aligned}$$

Now, the two extremes of the above inequality give us

$$\int_{\{g < c\}} (g - c) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \geq 0.$$

Therefore, $1_{\{g < c\}}(\omega)K_n(1_{\{g \geq c\}}, \omega) = 0$ μ -a.e.. From this follows that

$$\begin{aligned}
K_n(1_{\{g \geq c\}}, \omega) &= 1_{\{g \geq c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) + 1_{\{g < c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) \\
&= 1_{\{g \geq c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) \\
&\leq 1_{\{g \geq c\}}(\omega).
\end{aligned}$$

By another application of the condition 1 we get

$$\int_{\Omega} 1_{\{g \geq c\}}(\omega) - K_n(1_{\{g \geq c\}}, \omega) d\mu = 0$$

and, then from the last inequality we obtain the μ -a.e. equality $1_{\{g \geq c\}}(\omega) = K_n(1_{\{g \geq c\}}, \omega)$. This means that $\{g \geq c\} \in \mathcal{A}_n(\mu)$ and so g is $\mathcal{A}_n(\mu)$ -measurable.

Now we will show that $2 \implies 1$. Suppose g is $\mathcal{A}_n(\mu)$ -measurable. First we will show that $2 \implies 1$ holds when $g = 1_F$, for some $F \in \mathcal{A}_n(\mu)$. To prove this claim, it only remains to verify that

$$\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu = \int_{\Omega} 1_F \cdot 1_E d\mu, \quad \forall E \in \mathcal{F}. \quad (18)$$

Note that for any $E \in \mathcal{F}$ we have

$$\begin{aligned}
\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu &= \int_{\Omega} 1_F \cdot K_n(E \cap F, \cdot) d\mu + \int_{\Omega} 1_F \cdot K_n(E \cap F^c, \cdot) d\mu \\
&\leq \int_{\Omega} K_n(E \cap F, \cdot) d\mu + \int_{\Omega} 1_F \cdot K_n(F^c, \cdot) d\mu \\
&\stackrel{(\text{Hip. on } K_n)}{=} \int_{\Omega} 1_{E \cap F} d\mu + \int_{\Omega} 1_F \cdot K_n(F^c, \cdot) d\mu \\
&\stackrel{(F \in \mathcal{A}_n(\mu))}{=} \int_{\Omega} 1_{E \cap F} d\mu + \int_{\Omega} 1_F \cdot 1_{F^c} d\mu \\
&= \int_{\Omega} 1_E \cdot 1_F d\mu.
\end{aligned}$$

By a similar argument we can show that

$$\int_{\Omega} 1_F \cdot K_n(E^c, \cdot) d\mu \leq \int_{\Omega} 1_{E^c} \cdot 1_F d\mu.$$

Since

$$\begin{aligned}
\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu + \int_{\Omega} 1_F \cdot K_n(E^c, \cdot) d\mu &= \mu(F) \\
&= \int_{\Omega} 1_F \cdot 1_E d\mu + \int_{\Omega} 1_F \cdot 1_{E^c} d\mu
\end{aligned}$$

it follows from the two last inequalities that

$$\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu = \int_{\Omega} 1_F \cdot 1_E d\mu, \quad \forall E \in \mathcal{F}.$$

The above identity extends by linearity for simple functions. By taking a sequence of simple functions $\varphi_k \uparrow f$, and using the monotone convergence theorem we get for any $\mathcal{A}_n(\mu)$ -measurable function g that

$$\int_{\Omega} g \cdot K_n(E, \cdot) d\mu = \int_{\Omega} g \cdot 1_E d\mu, \quad \forall E \in \mathcal{F}.$$

□

It follows from last proposition that if g is measurable with respect to the sigma-algebra $\mathcal{A}_n(\mu)$ for all $n \in \mathbb{N}$, and $\mu \in \mathcal{G}^{DLR}(f)$, f continuous, then $g\mu$ is also in $\mathcal{G}^{DLR}(f)$.

Corollary 8. *Given $\mu \in \mathcal{G}^{DLR}(f)$ define $\mathcal{A}(\mu) \equiv \bigcap_{n \in \mathbb{N}} \mathcal{A}_n(\mu)$. Then, μ is extreme in $\mathcal{G}^{DLR}(f)$, if and only if, μ is trivial on $\mathcal{A}(\mu)$.*

Proof. Suppose that there exists $F \in \mathcal{A}(\mu)$ such that $0 < \mu(F) < 1$ and consider the following probability measures

$$\begin{aligned}
\mathcal{F} \ni E \mapsto \nu(E) &= \mu(E|F) = \int_{\Omega} \frac{1}{\mu(F)} 1_E 1_F d\mu, \\
\mathcal{F} \ni E \mapsto \gamma(E) &= \mu(E|F^c) = \int_{\Omega} \frac{1}{\mu(F^c)} 1_E 1_{F^c} d\mu.
\end{aligned}$$

Clearly $\nu \neq \gamma$ and moreover

$$\mu = \mu(F)\nu + (1 - \mu(F))\gamma. \tag{19}$$

The last proposition guarantees that both ν and γ belong to $\mathcal{G}^{DLR}(f)$. Indeed, in last proposition take f as $(1/\mu(F)) \cdot 1_F$ and $(1/\mu(F^c)) \cdot 1_{F^c}$, respectively (these functions are $\mathcal{A}_n(\mu)$ -measurable for $n \in \mathbb{N}$). However the existence of the non trivial convex combination (19), of two elements in $\mathcal{G}^{DLR}(f)$, is a contradiction. Therefore, any set $F \in \mathcal{A}(\mu)$ has the μ measure zero or one.

Conversely, suppose that μ is trivial on $\mathcal{A}(\mu)$ and at same time expressible as $\mu = \lambda\nu + (1 - \lambda)\gamma$, with $0 < \lambda < 1$ and $\nu, \gamma \in \mathcal{G}(f)$.

Note that $\nu \ll \mu$ and then from Radon-Nikodym Theorem we get that $\nu(E) = \int_{\Omega} 1_E f d\mu$ for some measurable function $f \geq 0$. Once more by the equivalence $1 \iff 2$ we get that f is $\mathcal{A}_n(\mu)$ -measurable for all $n \in \mathbb{N}$ (recall that $\nu \in \mathcal{G}^{DLR}(f)$). Since we assumed that μ is trivial on $\mathcal{A}(\mu)$ we get that both integrals below are always equals to each other being zero or one

$$\int_{\Omega} 1_F f d\mu = \int_{\Omega} 1_F d\mu, \quad \forall F \in \mathcal{A}(\mu).$$

As the equality is valid for all $F \in \mathcal{A}(\mu)$ and f is $\mathcal{A}_n(\mu)$ -measurable we can conclude that $f = 1$ μ -a.e.. Therefore, $\mu = \nu$ and consequently $\gamma = \mu$. So μ is a extreme point of $\mathcal{G}^{DLR}(f)$. \square

It follows from last corollary that if $\mathcal{G}^{DLR}(f)$ has only one element μ , then μ is trivial on $\mathcal{A}(\mu)$. If there is phase transition, in the sense that the cardinality of $\mathcal{G}^{DLR}(f)$ is bigger than one, then any extreme probability measure μ in $\mathcal{G}^{DLR}(f)$ is trivial on $\mathcal{A}(\mu)$.

In the next proposition we show the relationship between $\mathcal{A}(\mu)$ and $\mathcal{T} \equiv \bigcap_{n \in \mathbb{N}} \mathcal{T}_n$, for $\mu \in \mathcal{G}^{DLR}$.

Corollary 9. *If $\mu \in \mathcal{G}^{DLR}(f)$ then $\mathcal{A}(\mu)$ is a μ completion of \mathcal{T} . In particular, it follows from last corollary that if $\mu \in \mathcal{G}^{DLR}(f)$ is extreme, then, it is trivial on \mathcal{T} .*

Proof. For all $n \in \mathbb{N}$ we have that K_n is a proper kernel. Therefore, for any set $F \in \mathcal{T}$ we get that $K_n(F, \omega) = 1_F(\omega)$.

On the other hand, if $F \in \{E \in \mathcal{F} : K_n(E, \omega) = 1_E(\omega), \forall n \in \mathbb{N}, \forall \omega \in \Omega\}$, then, $F = \{\omega \in \Omega : K_n(F, \omega) = 1\} \in \mathcal{T}_n$. Therefore, $F \in \mathcal{T}$. Consider $\mu \in \mathcal{G}^{DLR}(f)$ and let $F \in \mathcal{A}(\mu)$, then,

$$B = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \{\omega \in \Omega : K_m(F, \omega) = 1\}$$

is an element on the sigma algebra \mathcal{T} and moreover, $\mu(F \Delta B) = 0$, because

$$1_B = \limsup_{n \rightarrow \infty} 1_{\{\omega \in \Omega : K_n(F, \omega) = 1\}} = \limsup_{n \rightarrow \infty} 1_F = 1_F \quad \mu \text{ a.e..}$$

\square

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