

Directional derivatives and the central limit theorem on compact general one-dimensional lattices

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Abstract

We will show the central limit theorem for the case of a general one-dimensional lattice where the space of symbols is a compact metric space. We consider the CLT for Lipschitz-Gibbs probabilities and in the proof we use several properties of the Ruelle operator, like existence of eigenfunctions, eigenprobabilities, asymptotic variance, etc.... For defining the Ruelle operator is required to fix an *a priori probability*. An important issue in the proof of the CLT is the existence the second-order derivative for the pressure, and this will follow from associated analytic properties. As additional results of independent interest, we will also describe some explicit estimates of the first and second directional derivatives of some dynamical entities like entropy and pressure. For example: given a fixed normalized Hölder potential f , and a variable observable η on the Kernel of the Ruelle operator $\mathcal{L}_f$, we consider the equilibrium probability $\mu_{f+t\eta}$ for $f + t\eta$. We estimate the values $\frac{d}{dt}h(\mu_{f+t\eta})|_{t=0}$ and $\frac{d^2}{dt^2}h(\mu_{f+t\eta})|_{t=0}$, where $h(\mu_{f+t\eta})$ is the entropy of $\mu_{f+t\eta}$. For fixed f we can find conditions that can indicate the η attaining the maximal possible value of $\frac{d}{dt}h(\mu_{f+t\eta})|_{t=0}$ (up to a natural normalization of η), entirely in terms of elements on the kernel of $\mathcal{L}_f$; these vector spaces play the role of tangent spaces in the infinite-dimensional manifold of equilibrium measures for Hölder normalized potentials f .

Keywords: Asymptotic Variance, Central Limit Theorem, Directional Derivatives, Lipschitz-Gibbs Probability, RPF Theorem, Ruelle Operator.

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1 Introduction

Our main goal in this paper is to show the central limit theorem on a class of compact general one-dimensional lattices (see Section 4), which are defined

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as symbolic spaces where the set of symbols is a compact metric space M and the dynamics is played by the shift map $\sigma : M^{\mathbb{N}} \rightarrow M^{\mathbb{N}}$; this kind of models is sometimes called in the mathematical literature *generalized one-dimensional XY models* (see [2] and [22]). We obtain the CLT for Lipschitz-Gibbs probabilities on $\Omega := M^{\mathbb{N}}$ from the spectral properties of a Ruelle operator (to be defined in Section 2). This will require exploring properties of the so-called Ruelle operator as presented in [22], where it is necessary to consider an *a priori probability* on M .

In the classical case (where $M = \{1, 2, \dots, d\}$), well-known results in this direction have been obtained by diverse authors on several classes of dynamical contexts (for instance in [19]). In [21] it was proven a CLT via martingale differences in the context of a bijective dynamical system satisfying suitable conditions, even more, it was shown a proof characterizing the CLT through properties of decay of correlations when the dynamics is surjective (see also [34] for an interesting approach). In [13] central limit theorems were proven for a wide class of dynamical systems, among them, geodesic flows, piece-wise expanding maps on the interval, and irrational rotations. In [14] the authors proved a CLT using as a main tool the so-called transfer operators defined on a suitable skew product.

A key element in the proof of the CLT that we present in this paper is the existence of a certain second-order derivative and this will follow from the analytic properties of the Ruelle operator that will be properly described later (see Section 3). As additional results of independent interest, in Theorems 19, 20, Proposition 22 and Theorems 26, 27, 28, we will also describe some explicit estimates of the first and second derivatives for some dynamical entities which are somehow related to the problem.

The characterization of the Ruelle operator in the context of the so-called generalized one-dimensional XY models has been widely studied in different works. For instance, in [22] questions related to the spectral properties of this kind of operator were addressed; it was also shown the existence of equilibrium states, a variational principle, and the existence of calibrated sub-actions. In [32] the authors studied the mentioned problems on a model where the allowed sequences are given in terms of a continuous map of admissibility (a kind of generalization of the classical shifts of finite type). The spectral gap (and the analyticity following this property) obtained in item (4) in Theorem 1 in [32] contemplates our setting here as a particular case.

The Central Limit Theorem is a classical topic in Ergodic Theory. In [5] the author aims to apply the result for Anosov systems via Markov partitions. In [13] specific mixing or dependence conditions are assumed, and a general result follows from assuming the appropriate rate of mixing. The proof in [31] uses the analytic dependence of parameters for the case of equilibrium probabilities for Hlder potentials in the case of finite alphabets. [10] proves CLT for shifts of finite type. In [4] the author improves the classical Gordin's method in order to guarantee the result via ergodic martingales; relaxes independence assumptions to establish that martingale difference sequences converge to a normal distribution. With different methods, but still using martingales, [35] achieves quite

general results on the topic. We chose the analytic framework because it is more appropriate for handling the technical demands of the case where the alphabet is a compact metric space.

This paper is organized as follows: in Section 2 are presented the main definitions to be used throughout the paper, among them, the one for the Ruelle operator which we are interested in this context. In Section 3 we state some well-known properties about the analyticity of some observables derived from the Ruelle operator and how it depends on the so-called spectral gap property. In Section 4 we present the proofs of the main results of the paper, among them a generalized version of the CLT in the context of the so-called *general one-dimensional lattices*. At the end of the paper, in Section 5, we present explicit calculations for the directional derivatives of entropy and pressure in terms of elements on the Kernel of the Ruelle operator.

We can take advantage of knowing differentiability in the present setting to investigate questions related to the first and second-directional derivatives of variations on time of dynamical entities related to pressure and entropy. As an example, considering a fixed potential f and a variable observable g , denote by μ_{f+tg} the Lipschitz-Gibbs probability (which is also equilibrium for the pressure of $f+tg$) for the perturbed potential $f+tg$, when t is close to zero. It is natural to ask about the values

$$\frac{d}{dt}h(\mu_{f+tg})|_{t=0} \quad \text{and} \quad \frac{d^2}{dt^2}h(\mu_{f+tg})|_{t=0} \quad (1)$$

where $h(\mu_{f+tg})$ is the entropy of μ_{f+tg} (see Proposition 23 and Corollary 24).

Throughout the text we will use results from [17] that considers finite alphabets, but the proofs presented there basically depend on the expansiveness of the shift, and thus apply to the case where the alphabet is a compact metric space, as we are considering here.

We can ask about the direction g where the value $\frac{d}{dt}h(\mu_{f+tg})|_{t=0}$ is the highest possible and how to get analytic estimates of such g ; all of that in terms of elements in the Kernel of the Ruelle operator $\mathcal{L}_f$. This can indicate the direction g of the maximal increasing of entropy. Questions of this nature and others are the main topics of Section 5. The estimation of $\frac{d}{dt}h(\mu_{f+tg})|_{t=0}$ was already considered in [17], but for other results we consider here it is appropriate to prove things in a different way.

We also consider directional derivatives associated with the pressure via expressions based entirely on elements on the kernel of the Ruelle operator (see Theorem 25). For the explicit expression (66) we take advantage of the existence of a certain orthogonal family on the Kernel of the Ruelle operator (see [23]).

2 Preliminaries

Consider a *compact metric space* (M, d_M) satisfying $\text{diam}(M) = 1$ and define $\Omega := M^{\mathbb{N}}$ as the set of sequences taking values on M with the *shift map* given by $T((x_n)_{n=1}^{\infty}) := (x_{n+1})_{n=1}^{\infty}$ acting on it. It is well-known that Ω equipped

with the product topology results in a compact metric space and, even more, it is metrizable with $d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_M(x_n, y_n)$.

Throughout the paper, we denote by $\mathcal{B}_\Omega$, resp. $\mathcal{C}(\Omega)$, resp. $\text{Lip}(\Omega)$, resp. $\mathcal{M}(\Omega) := \mathcal{C}(\Omega)^*$ resp. $\mathcal{M}_1(\Omega)$, resp. $\mathcal{M}_T(\Omega)$; the *Borel sigma-algebra* on Ω , resp. the space of *continuous functions* from Ω into $\mathbb{R}$, resp. *Lipschitz continuous functions* from Ω into $\mathbb{R}$, resp. *Borel finite measures* on Ω , resp. *Borel probability measures* on Ω , resp. *Borel T -invariant probability measures* on Ω .

In this work, we are interested in the statistical behavior of probability measures obtained as the fixed points of the dual of a Ruelle operator which we call Lipschitz-Gibbs probabilities. The *Ruelle operator* for a potential $f \in \mathcal{C}(\Omega)$ is defined as the linear one assigning to each $w \in \text{Lip}(\Omega)$ the map $\mathcal{L}_f(w) \in \text{Lip}(\Omega)$, given by the expression

$$\mathcal{L}_f(w)(x) := \int_M e^{f(ax)} w(ax) d\nu(a) , \quad (2)$$

where $\nu \in \mathcal{M}_1(M)$ is an *a priori measure* (with support equal M), and each sequence $ax := (a, x_1, x_2, \dots) \in \Omega$ satisfies $T(ax) = x$.

By the *RPF theorem* (see [22], or [25], for details), for each $f \in \text{Lip}(\Omega)$, there are $\lambda_f > 0$, $w_f \in \text{Lip}(\Omega)^+$ and $\rho_f \in \mathcal{M}_1(\Omega)$, such that,

$$\mathcal{L}_f(w_f) = \lambda_f w_f \text{ and } \mathcal{L}_f^*(\rho_f) = \lambda_f \rho_f , \quad (3)$$

and, even more, for any $w \in \text{Lip}(\Omega)$ the following limit holds

$$\lim_{n \rightarrow \infty} \left\| \frac{\mathcal{L}_f^n(w)}{\lambda_f^n} - w_f \int_\Omega w d\rho_f \right\|_\infty = 0 . \quad (4)$$

A straightforward argument shows that the limit in (4) implies the spectral gap property, i.e., that the main eigenvalue λ_f is maximal and isolated into the spectrum of the operator (see for details [32]).

A potential $f \in \text{Lip}(\Omega)$ is called *normalized* when $\mathcal{L}_f(1) = 1$, we denote the set of normalized potentials by $\mathcal{N}(\Omega)$. It is easy to check that for $f \in \mathcal{N}(\Omega)$ the operator $\mathcal{L}_f^*$ preserves $\mathcal{M}_1(\Omega)$ and it is well-known that for arbitrary $f \in \text{Lip}(\Omega)$, the *normalization* $N(f) := f + \log(w_f) - \log(w_f \circ \sigma) - \log(\lambda_f)$ belongs to $\mathcal{N}(\Omega)$.

We call here $P(f) := \log \lambda_f$ the *pressure* of $f : \Omega \rightarrow \mathbb{R}$ (which due to the classical variational formula, as in [31] or [22], will play a role only in Section 5).

We denote by $\mu_f \in \mathcal{M}_1(\Omega)$ the *Lipschitz-Gibbs probability* for f , which is the unique fixed point for the dual Ruelle operator associated to the normalization $N(f)$, i.e., $\mathcal{L}_{N(f)}^*(\mu_f) = \mu_f$. Moreover, it is easy to check that $\mu_f \in \mathcal{M}_T(\Omega)$ and, even more, choosing the eigenfunction and eigenprobability in (3) satisfying $\int_\Omega w_f d\rho_f = 1$, we have $\frac{d\mu_f}{d\rho_f} = w_f$ (see [22] for details). Besides that, under suitable assumptions, the identification map $f \mapsto \mu_f$ is bijective and analytic (see [3]). In this way, the set $\mathcal{N}(\Omega)$ can be also identified as the set of Lipschitz-Gibbs probabilities associated with all possible Lipschitz potentials, which we also denote by $\mathcal{N}(\Omega)$.

In the following sections we elaborate on the analyticity of the pressure map and later we use that to prove a version of the central limit theorem. For considering the meaning of equilibrium state for a given Lipschitz continuous potential, it is necessary the concept of entropy and pressure for T -invariant probabilities on the XY model (see [22]). Actually, in the mentioned work, it is shown that the so-called Ruelle operator (taking into account an *a priori probability*) has several interesting properties allowing to characterize the dynamical and statistical behavior of the map T .

For the benefit of the reader, in the first part of the next section we will state some properties of Thermodynamic Formalism for the generalized XY model, among them analyticity issues, that we will need later. Our focus is on Lipschitz-Gibbs probabilities.

3 Analyticity of the Ruelle Operator

Our main goal in the next section is to prove a central limit theorem on the setting of one-dimensional lattices defined on compact metric spaces, using as a main tool the analyticity of the map $f \mapsto P(f)$, more specifically, using the existence of first and second derivatives of the function $t \mapsto P(f + tg)$, with $t \in \mathbb{R}$.

In order to do that, first we introduce some important analytic properties of functions related to the Ruelle operator defined in Section 2 and its corresponding dual, besides their eigenfunctions and eigenprobabilities, the above as a function of the potential f . We will state some useful properties of the Ruelle operator given by the expression in (2).

It is known that for any $f \in \text{Lip}(\Omega)$ and each $n \in \mathbb{N}$ the map $f \mapsto \mathcal{L}_f^n$ is analytic (see [3, 31] for the classical case, and [32, 33] for the XY model case). Even more, the expression for the Fréchet derivative of $\mathcal{L}_{(\cdot)}^n$ at the point f in the direction of g is given by

$$D(\mathcal{L}_{(\cdot)}^n)_f(g) = \sum_{i=1}^n \mathcal{L}_f^i(g \mathcal{L}_f^{n-i}(g)) . \quad (5)$$

The former expression also helps to guarantee that both of the maps $f \mapsto w_f$ and $f \mapsto \lambda_f = e^{P(f)}$ are analytic. In order to guarantee the former claim, we need to introduce a strong result requiring advanced technical tools in Functional Analysis and the so-called approximation theory (for instance the ones presented in Chapter VII Sections 3-6 in [12], see also Section 5 in [28] and Section IV in [26]).

A detailed study of the spectrum of the Ruelle operator appears in [15] and [1] under the assumption that $f \in \text{Lip}(\Omega)$. In the mentioned works it is shown that the space of eigenfunctions associated with the main eigenvalue has dimension 1 and, even more, by a general result in [22] the eigenvalue λ_f results isolated. Using the same reasoning, one can take advantage of the following propositions (see [15], [27] and [28]) to get similar results in our setting.

Proposition 1. [3, 7, 28, 26] Given $f \in \text{Lip}(\Omega)$, not necessarily normalized, consider a small circle curve $\gamma : [0, 1] \rightarrow \mathbb{C}$, $\gamma(t) = \epsilon(e^{2\pi it} + \lambda_f)$ around the eigenvalue λ_f (by the spectral gap property no other elements of the spectrum of $\mathcal{L}_f$ belong to the interior of the curve). Then, the eigenfunction w_g , where the map $g \in \text{Lip}(\Omega)$ is close by f , can be obtained as the line integral

$$w_g = \frac{1}{2\pi i} \left(\int_{\gamma} (zI - \mathcal{L}_g)^{-1} dz \right) (1). \quad (6)$$

From the former expression, it follows that the map $g \mapsto w_g$ is analytic on a neighborhood of f contained into the space $\text{Lip}(\Omega)$.

Note that the eigenfunction w_g for $\mathcal{L}_g$ described by (6) is not necessarily normalized (in the sense that $\int_{\Omega} w_g d\rho_g = 1$). In fact, the projection operator $\pi_1(\mathcal{L}_g)$ given by the expression

$$\pi_1(\mathcal{L}_g) := \frac{1}{2\pi i} \left(\int_{\gamma} (zI - \mathcal{L}_g)^{-1} dz \right), \quad (7)$$

is describing the projection of the space $\text{Lip}(\Omega)$ on the one-dimensional space of eigenfunctions for the operator $\mathcal{L}_g$ (see for instance [12]). Furthermore, the function $g \mapsto \pi_1(\mathcal{L}_g)$ is complex analytic and any $w \in \text{Lip}(\Omega)$ satisfies

$$\mathcal{L}_g(\pi_1(\mathcal{L}_g)(w)) = \lambda_g \pi_1(\mathcal{L}_g)(w).$$

In the next proposition we use the operator in (7) to guarantee analyticity of the maps $f \mapsto \lambda_f$ and $f \mapsto P(f)$.

Proposition 2. Let $f \in \text{Lip}(\Omega)$ be a not normalized potential. Then, the map $g \mapsto \lambda_g$ is analytic on the variable g varying on a neighborhood of f . By the above, it follows that also the pressure map $g \mapsto P(g) = \log(\lambda_g)$ is analytic on the same neighborhood of f .

Proof. Note that $\pi_1(\mathcal{L}_g)(w_f)$ is an eigenfunction for the operator $\mathcal{L}_g$ associated to λ_g , and varies analytically on g . Consider $\rho \in \mathcal{M}_1(\Omega)$ supported on Ω , such that, $\int_{\Omega} w_f d\rho \neq 0$. Then,

$$\lambda_g = \frac{\int_{\Omega} \mathcal{L}_g(\pi_1(\mathcal{L}_g)(w_f)) d\rho}{\int_{\Omega} \pi_1(\mathcal{L}_g)(w_f) d\rho}. \quad (8)$$

Note that the denominator is not zero for g close to f , which guarantees that the former expression is well-defined and a quotient of analytic functions. Therefore, the function $g \mapsto \lambda_g$ results in analytic on the variable g . $\square$

Consider θ acting on $\text{Lip}(\Omega)$ given by $\theta(f) := \mathcal{L}_f$. It is well-known that the map θ is analytic on all the space $\text{Lip}(\Omega)$ (see [8, 27, 33] for details).

More precisely, considering a fixed $w \in \text{Lip}(\Omega)$, we have that the Fréchet derivative of θ at f in direction of g satisfies the expression

$$\theta(f + g)(w) - \theta(f)(w) = \sum_{j=1}^{\infty} \frac{1}{j!} \theta(f)(w g^j)$$

$$= \sum_{j=1}^{\infty} \frac{1}{j!} \mathcal{L}_f(w g^j) = \sum_{j=1}^{\infty} \frac{1}{j!} D^j \theta(f) (\underbrace{g, g, \dots, g}_j).$$

In [33] the authors show analytic properties of the Ruelle operator defined in the expression (2), even more, from Theorem 3.5 in [33], we have that for $f, g, w \in \text{Lip}(\Omega)$ the following analytic expression holds

$$\mathcal{L}_{f+g}(w)(x) = \sum_{j=1}^{\infty} \frac{1}{j!} \int_M e^{f(ax)} w(ax) g(ax)^j d\nu(a). \quad (9)$$

In particular, a second-order expression for Taylor's polynomial in our setting is given by

$$\mathcal{L}_{f+g}(w)(x) = \int_M e^{f(ax)} w(ax) \left(1 + g(ax) + \frac{g(ax)^2}{2}\right) d\nu(a) + o_3(g). \quad (10)$$

It is important to point out that the proof of the former expression was obtained without using the spectral gap property of the Ruelle operator (well-known in the case where M is a finite set as described in [28] and [31]).

Above it is considered the strong norm of operators and that $\mathcal{M}(\Omega) := C(\Omega)^*$ is the Banach space of continuous linear functionals with domain on $C(\Omega)$. We remark that on top page 190 in [20] the author mention the fact that the linear set of measures on Ω - with the total variation norm - does not carry a natural analytic structure.

It follows that if we fix $f \in \text{Lip}(\Omega)$ and consider a family of potentials tf , where $t \in \mathbb{R}$, then, the maps $t \mapsto w_{tf}$ and $t \mapsto \lambda_{tf}$ are analytic on the variable t . For the corresponding result for the map $t \mapsto \rho_{tf}$, as mentioned in [20], given a normalized potential f , and a fixed Hölder function g , the function $t \mapsto \int g \rho_{tf}$ is analytic. In the next lemma, we present an explicit expression for the Gateaux derivative for the map $f \mapsto \lambda_f$.

Lemma 3. [3, 17] Consider $f, g \in \text{Lip}(\Omega)$, $t \in \mathbb{R}$, and assume that $f \in \mathcal{N}(\Omega)$. Then,

$$\frac{d}{dt} \lambda_{f+tg}|_{t=0} = \frac{d}{dt} e^{P(f+tg)}|_{t=0} = \int_{\Omega} g d\mu_f. \quad (11)$$

If the potential f is not normalized. Then,

$$\frac{d}{dt} \lambda_{f+tg}|_{t=0} = \frac{d}{dt} e^{P(f+tg)}|_{s=0} = \lambda_f \int_{\Omega} g d\mu_f. \quad (12)$$

The former lemma implies directly the following result characterizing the Gateaux derivative for the function $f \mapsto P(f)$.

Lemma 4. [3, 17] Given $f, g \in \text{Lip}(\Omega)$ and $t \in \mathbb{R}$, denote by $p(t) := P(f + tg)$ the pressure of the potential $f + tg$. Then, we have

$$p'(0) = \frac{d}{dt} P(f + tg)|_{t=0} = \int_{\Omega} g d\mu_f. \quad (13)$$

Remark 5. Note that the possible values of $\frac{d}{dt}e^{p(t)}|_{t=s}$ are on a finite subinterval $(a, b) \subset \mathbb{R}$. Moreover, we have

$$p'(s) = \frac{d}{dt}p(t)|_{t=s} = \int_{\Omega} g d\mu_{f+sg} . \quad (14)$$

Actually, it is only necessary to take $r := t - s$, $\psi := f + sg$ and later calculate $\frac{d}{dr}p(r)|_{r=0}$ such as appears in Lemma 4.

The next theorem is the main purpose of [17] and provides a structure of the Riemann manifold for $\mathcal{N}(\Omega)$.

Theorem 6. [17] The set $\mathcal{N}(\Omega)$ is an analytic infinite dimensional manifold, and the tangent space of $\mathcal{N}(\Omega)$ at f satisfies $T_f(\mathcal{N}(\Omega)) = \text{Ker}(\mathcal{L}_f)$, with $\mathcal{L}_f$ acting on the space $\text{Lip}(\Omega)$. Furthermore, the inner product $\langle v, w \rangle := \int_{\Omega} v w d\mu_f$, where $v, w \in T_f(\mathcal{N}(\Omega))$, provides a Riemann structure on the manifold $\mathcal{N}(\Omega)$. In particular, the norm of an element $\eta \in \text{Ker}(\mathcal{L}_f)$ is given by $\|\eta\| = \sqrt{\int_{\Omega} \eta^2 d\mu_f}$.

We point out that the results of [17] are also applied to the present case, where we consider the action of the shift on Ω and when the space of symbols M is a compact metric space. This is so because the main ingredient was the use of Ruelle operators (in the same way that can be handled here) plus analyticity (which is true in our setting).

Remark 7. The calculation of the Gauss curvature for $v, w \in T_f(\mathcal{N}(\Omega))$ appears in [23] and the existence of geodesics in this context is the topic of [24].

The Riemannian metric described in Theorem 6 is related (but somehow different) to the so-called Pressure Riemannian metric (see [6] and [29]).

In Section 5 we will be interested, among other topics, in estimating directional derivatives in the direction $\eta \in \text{Ker}(\mathcal{L}_f)$ related to the *pressure problem* $P(\varphi)$ for a potential $\varphi \in \text{Lip}(\Omega)$. In this case, it is also natural to consider variations of the form $f + tg$, with $g \in \text{Lip}(\Omega)$ not necessarily on the Kernel of $\mathcal{L}_f$, and the corresponding directional derivative

$$\frac{d}{dt} \left(h(\mu_{f+tg}) + \int_{\Omega} \varphi d\mu_{f+tg} \right) |_{t=0}, \quad (15)$$

where $f \in \mathcal{N}(\Omega)$ and μ_{f+tg} is the Lipschitz-Gibbs probability for $f + tg$, with t close to zero. We want to get explicit expressions for such derivatives and this will be presented in (55).

For fixed φ , we can ask about the maximal value of (15) for $\eta \in \text{Ker}(\mathcal{L}_f)$, when η satisfies $\int_{\Omega} \eta^2 = 1$.

4 Central Limit Theorem

Throughout this section, we prove the central limit theorem for our setting. We will show that one can adapt some ideas of the classical case to the XY

model. Given the probability space $(\Omega, \mathcal{B}_\Omega, P)$, i.e., $P \in \mathcal{M}_1(\Omega)$, we say that the measurable function $X : \Omega \rightarrow \mathbb{R}$ has a *Gaussian distribution* with mean α and variance $\sigma^2 > 0$, when for any interval $(a, b) \subset \mathbb{R}$

$$P(\{x \in \Omega | X(x) \in (a, b)\}) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_a^b e^{-\frac{1}{2} \frac{(t-\alpha)^2}{\sigma^2}} dt =: \int_a^b d\phi_{\alpha, \sigma^2}(t) .$$

The former expression is equivalent to saying that

$$\int_\Omega I_{(a,b)}(X) dP = \int_\Omega I_{X^{-1}(a,b)} dP = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} I_{(a,b)}(t) e^{-\frac{1}{2} \frac{(t-\alpha)^2}{\sigma^2}} dt ,$$

which implies (via approximation by simple functions) that for any $\varphi \in C(\mathbb{R})$ the following expression holds

$$\int_\Omega \varphi(X) dP = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} \varphi(t) e^{-\frac{1}{2} \frac{(t-\alpha)^2}{\sigma^2}} dt = \int_{\mathbb{R}} \varphi(t) d\phi_{\alpha, \sigma^2}(t) .$$

Then, in this case, the *mean* is given by

$$\alpha := \int_\Omega X dP = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} t e^{-\frac{1}{2} \frac{(t-\alpha)^2}{\sigma^2}} dt = \int_{\mathbb{R}} t d\phi_{\alpha, \sigma^2}(t) .$$

and the *variance* satisfies the expression

$$\begin{aligned} \sigma^2 &:= \int_\Omega (X - \int_\Omega X dP)^2 dP \\ &= \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} (t - \alpha)^2 e^{-\frac{1}{2} \frac{(t-\alpha)^2}{\sigma^2}} dt = \int_{\mathbb{R}} (t - \alpha)^2 d\phi_{\alpha, \sigma^2}(t) . \end{aligned}$$

Theorem 8. Denote $\beta(t) := P(f + tg) - P(f)$, where f and g belong to $\text{Lip}(\Omega)$ and $t \in \mathbb{R}$. Given $c \in \mathbb{R}$, the Central Limit Theorem claims that

$$\lim_{n \rightarrow \infty} \mu_f \left(\left\{ x \in \Omega \mid \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (g(\sigma^j(x)) - \beta'(0)) < c \right\} \right) = \int_{-\infty}^c d\phi_{0, \beta''(0)}(t) . \quad (16)$$

The right-hand side of expression (16) describes the Gaussian distribution with mean $\alpha = 0$ and variance $\sigma^2 = \beta''(0)$.

Below, we will present a general formulation of the above claim, for the specific proof of the CLT see Theorem 14. Before that, we will present some technical results that allow us to prove one of the main theorems of this paper:

Lemma 9. Denote by $\beta(t) := P(f + tg) - P(f) = p(t) - p(0)$, where $f, g \in \text{Lip}(\Omega)$, $\int_\Omega g d\mu_f = 0$, and consider $s \in \mathbb{R}$. Then,

$$\begin{aligned} \frac{d^2}{dt^2} p(t) |_{t=s} &= \beta''(s) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_\Omega \left(\sum_{j=0}^{n-1} (g \circ \sigma^j - \beta'(s)) \right)^2 d\mu_{f+sg} =: \sigma_{f+sg}^2(g) . \end{aligned} \quad (17)$$

The last expression is known as the asymptotic variance $\sigma_{f+sg}^2(g)$ for the Lipschitz-Gibbs probability μ_{f+sg} .

Proof. In order to simplify the proof we consider the case where $f \in \mathcal{N}(\Omega)$, i.e., when $\mathcal{L}_f(1) = 1$. The above implies $w_f \equiv 1$, $P(f) = 0$ and so $\beta(t) = p(t)$. Observe that $\beta'(0) = p'(0) = \int_{\Omega} g d\mu_f = 0$; we point out that the hypothesis $\int_{\Omega} g d\mu_f = 0$ is satisfied by tangent vectors to the infinite-dimensional manifold of Lipschitz-Gibbs probabilities, at the point μ_f (see Theorem 6)).

In this case, we have to show that

$$\beta''(0) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right)^2 d\mu_f. \quad (18)$$

Indeed, in the same way, such as appears in the last result, we assume

$$\mathcal{L}_{f+tg}(w_{f+tg}) = e^{p(t)} w_{f+tg}, \quad (19)$$

and $\beta'(0) = p'(0) = \int g d\mu_f = 0$.

First, note that the derivative at the left side in (19) satisfies

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_{f+tg}(w_{f+tg})(x)|_{t=0} &= \int_M e^{f(ax)} g(ax) d\nu(a) \\ &\quad + \int_M e^{f(ax)} \left(\frac{d}{dt} w_{f+tg}(ax) \right) |_{t=0} d\nu(a). \end{aligned}$$

In order to estimate $\beta''(0)$, for a fixed $x \in \Omega$, we have to take the derivative

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{L}_{f+tg}(w_{f+tg})(x)|_{t=0} &= \frac{d}{dt} \left(\int_M e^{(f+tg)(ax)} g(ax) w_{f+tg}(ax) d\nu(a) \right. \\ &\quad \left. + \int_M e^{(f+tg)(ax)} \frac{d}{dt} w_{f+tg}(ax) d\nu(a) \right) |_{t=0}. \end{aligned} \quad (20)$$

We consider initially the first term of (20), for a fixed $x \in \Omega$

$$\begin{aligned} &\frac{d}{dt} \left(\int_M e^{(f+tg)(ax)} g(ax) w_{f+tg}(ax) d\nu(a) \right) |_{t=0} \\ &= \int_M e^{(f+tg)(ax)} g(ax)^2 w_{f+tg}(ax) d\nu(a) |_{t=0} \\ &\quad + \int_M e^{(f+tg)(ax)} g(ax) \frac{d}{dt} w_{f+tg}(ax) d\nu(a) |_{t=0} \\ &= \int_M e^{f(ax)} g(ax)^2 d\nu(a) + \int_M e^{f(ax)} g(ax) \left(\frac{d}{dt} w_{f+tg}(ax) \right) |_{t=0} d\nu(a) \\ &= \mathcal{L}_f(g^2)(x) + \mathcal{L}_f(g \frac{d}{dt} w_{f+tg}|_{t=0})(x). \end{aligned}$$

Then, integrating w.r.t. μ_f , we have

$$\begin{aligned} &\int_{\Omega} \frac{d}{dt} \left(\int_M e^{(f+tg)(ax)} g(ax) w_{f+tg}(ax) d\nu(a) \right) |_{t=0} d\mu_f(x) \\ &= \int_{\Omega} \mathcal{L}_f(g^2) d\mu_f + \int_{\Omega} \mathcal{L}_f(g \frac{d}{dt} w_{f+tg}|_{t=0}) d\mu_f \\ &= \int_{\Omega} g^2 d\mu_f + \int_{\Omega} g \cdot \left(\frac{d}{dt} w_{f+tg} \right) |_{t=0} d\mu_f. \end{aligned} \quad (21)$$

Now we estimate the second term of (20), also for a fixed $x \in \Omega$

$$\begin{aligned}
& \frac{d}{dt} \left(\int_M e^{(f+tg)(ax)} \frac{d}{dt} w_{f+tg}(ax) d\nu(a) \right) \Big|_{t=0} \\
&= \int_M e^{(f+tg)(ax)} g(ax) \frac{d}{dt} w_{f+tg}(ax) d\nu(a) \Big|_{t=0} \\
&\quad + \int_M e^{(f+tg)(ax)} \frac{d^2}{dt^2} w_{f+tg}(ax) d\nu(a) \Big|_{t=0} \\
&= \mathcal{L}_f \left(g \frac{d}{dt} w_{f+tg} \Big|_{t=0} \right)(x) + \mathcal{L}_f \left(\frac{d^2}{dt^2} w_{f+tg} \Big|_{t=0} \right)(x) .
\end{aligned}$$

Therefore, integrating w.r.t. the measure μ_f , it follows that

$$\begin{aligned}
& \int_\Omega \frac{d}{dt} \left(\int_M e^{(f+tg)(ax)} \frac{d}{dt} w_{f+tg}(ax) d\nu(a) \right) \Big|_{t=0} d\mu_f(x) \\
&= \int_\Omega \mathcal{L}_f \left(g \frac{d}{dt} w_{f+tg} \Big|_{t=0} \right) d\mu_f + \int_\Omega \mathcal{L}_f \left(\frac{d^2}{dt^2} w_{f+tg} \Big|_{t=0} \right) d\mu_f \quad (22) \\
&= \int_\Omega g \cdot \left(\frac{d}{dt} w_{f+tg} \right) \Big|_{t=0} d\mu_f + \int_\Omega \frac{d^2}{dt^2} w_{f+tg} \Big|_{t=0} d\mu_f .
\end{aligned}$$

Therefore, from (21) and (22) we get

$$\begin{aligned}
\int_\Omega \frac{d^2}{dt^2} \mathcal{L}_{f+tg}(w_{f+tg}) \Big|_{t=0} d\mu_f &= \int_\Omega g^2 d\mu_f + 2 \int_\Omega g \cdot \left(\frac{d}{dt} w_{f+tg} \right) \Big|_{t=0} d\mu_f \\
&\quad + \int_\Omega \frac{d^2}{dt^2} w_{f+tg} \Big|_{t=0} d\mu_f . \quad (23)
\end{aligned}$$

On the other hand, since we are assuming $\beta'(0) = p'(0) = \int g d\mu_f = 0$, for each x the following holds

$$\begin{aligned}
& \frac{d^2}{dt^2} (e^{p(t)} w_{f+tg}(x)) \Big|_{t=0} = \frac{d}{dt} \left(\frac{d}{dt} (e^{p(t)} w_{f+tg}(x)) + e^{p(t)} \frac{d}{dt} w_{f+tg}(x) \right) \Big|_{t=0} \\
&= \frac{d^2}{dt^2} e^{p(t)} \Big|_{t=0} w_f(x) + 2 \frac{d}{dt} e^{p(t)} \Big|_{t=0} \frac{d}{dt} w_{f+tg}(x) \Big|_{t=0} + e^{p(0)} \frac{d^2}{dt^2} w_{f+tg}(x) \Big|_{t=0} \\
&= \frac{d^2}{dt^2} e^{p(t)} \Big|_{t=0} + 2 \int_\Omega g d\mu_f \frac{d}{dt} w_{f+tg}(x) \Big|_{t=0} + \frac{d^2}{dt^2} w_{f+tg}(x) \Big|_{t=0} \\
&= \frac{d^2}{dt^2} e^{p(t)} \Big|_{t=0} + \frac{d^2}{dt^2} w_{f+tg}(x) \Big|_{t=0} .
\end{aligned}$$

Then,

$$\begin{aligned}
& \int_\Omega \frac{d^2}{dt^2} (e^{p(t)} w_{f+tg}) \Big|_{t=0} d\mu_f \\
&= \int_\Omega \frac{d^2}{dt^2} e^{p(t)} \Big|_{t=0} d\mu_f + \int_\Omega \frac{d^2}{dt^2} w_{f+tg} \Big|_{t=0} d\mu_f \quad (24) \\
&= \frac{d^2}{dt^2} e^{p(t)} \Big|_{t=0} + \int_\Omega \frac{d^2}{dt^2} w_{f+tg} \Big|_{t=0} d\mu_f .
\end{aligned}$$

Besides that, since $f \in \mathcal{N}(\Omega)$, we have

$$\frac{d^2}{dt^2} e^{p(t)}|_{t=0} = p''(0) + (p'(0))^2 = p''(0) .$$

Therefore, from (23), (24), and the assumption $\beta'(0) = p'(0) = \int_{\Omega} g d\mu_f = 0$, we get that

$$p''(0) = \frac{d^2}{dt^2} e^{p(t)}|_{t=0} = \int_{\Omega} g^2 d\mu_f + 2 \int_{\Omega} g \cdot \left(\frac{d}{dt} w_{f+tg} \right) |_{t=0} d\mu_f . \quad (25)$$

Furthermore, given $n \in \mathbb{N}$, following the same reasoning as above, but now for the expression

$$\mathcal{L}_{f+tg}^n(w_{f+tg}) = e^{n p(t)} w_{t+fg} ,$$

we get that

$$\begin{aligned} n p''(0) &= n \frac{d^2}{dt^2} e^{p(t)}|_{t=0} \\ &= \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right)^2 d\mu_f + 2 \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right) \cdot \left(\frac{d}{dt} w_{f+tg} \right) |_{t=0} d\mu_f . \end{aligned} \quad (26)$$

From (26) it follows that for all $n \in \mathbb{N}$ the following holds

$$\begin{aligned} p''(0) &= \frac{d^2}{dt^2} e^{p(t)}|_{t=0} \\ &= \frac{1}{n} \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right)^2 d\mu_f + \frac{2}{n} \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right) \cdot \left(\frac{d}{dt} w_{f+tg} \right) |_{t=0} d\mu_f . \end{aligned} \quad (27)$$

As $\int_{\Omega} g d\mu_f = 0$ and $\frac{d}{dt} w_{f+tg}(\cdot)|_{t=0}$ is a bounded function on Ω , it follows from the Ergodic Theorem (and Dominated Convergence Theorem) that

$$\beta''(0) = p''(0) = \frac{d^2}{dt^2} e^{p(t)}|_{t=0} = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right)^2 d\mu_f . \quad (28)$$

Now, taking $r := t - s$, and calculating $\frac{d^2}{dr^2} e^{p(r)}|_{r=0}$, we obtain that

$$\beta''(s) = p''(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \left(\sum_{j=0}^{n-1} (g \circ \sigma^j - \beta'(s)) \right)^2 d\mu_{f+sg}$$

□

The former Theorem implies the following:

Corollary 10. Consider $\beta(t) := P(f + tg) - P(f)$, $t \in \mathbb{R}$, where $f, g \in \text{Lip}(\Omega)$ and $\int_{\Omega} g d\mu_f = 0$. Then,

$$\begin{aligned} \sigma_f^2(g) &:= \beta''(0) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \left(\sum_{j=0}^{n-1} g \circ \sigma^j \right)^2 d\mu_f \\ &= \int_{\Omega} g^2 d\mu_f + 2 \int_{\Omega} g \cdot \left(\frac{d}{dt} w_{f+tg} \right) \Big|_{t=0} d\mu_f . \end{aligned} \quad (29)$$

The value $\beta''(s)$ stated at Lemma 9 is called the *asymptotic variance* for the Lipschitz-Gibbs probability μ_{f+sg} . This value is the one to be used as a variance of the Gaussian distribution obtained in the central limit theorem for the Lipschitz-Gibbs probability μ_f (see [31], [10] or [9]). Our proof adapts the reasoning used in [19] for our setting.

Remark 11. Note that in the case when g is co-boundary to a constant $c \in \mathbb{R}$, i.e., there exists $v \in C(\Omega)$ such that $g = v - v \circ \sigma + c$. Then, for any $s \in \mathbb{R}$, we have $p'(s) = c$ and, therefore, for any $s \in \mathbb{R}$, it follows that $p''(s) = 0$. In Proposition 4.2 in [31] it was shown the converse, i.e., when, $p''(s) = 0$ for all $s \in \mathbb{R}$, then, g is co-boundary to a constant.

We will assume from now on that $g \in \text{Lip}(\Omega)$ and not co-boundary to a constant. In this case, it follows from Remark 11 that the map $t \mapsto P(f + tg)$ is strictly convex.

Before presenting the main theorem of this paper, we will introduce the following well-known technical result which allows us to prove properties of convergence of a family of probability distributions using the point-wise convergence of their corresponding moment generating functions (for more details see Theorem 2 in section XIII in [16]).

Theorem 12. (Continuity Thm [16, 30]) Consider a sequence of Borel probability densities $(\rho_n)_{n=1}^{\infty}$ on $\mathbb{R}$ and its corresponding moment generating functions given by $\varphi_n(z) := \int_{\mathbb{R}} e^{zt} d\rho_n(t)$. If for each value $z \in \mathbb{R}$ we have

$$\lim_{n \rightarrow \infty} \varphi_n(z) = \varphi(z) := \int_{\mathbb{R}} e^{zt} d\phi_{\alpha, \sigma^2}(t) = e^{z\alpha + z^2 \frac{\sigma^2}{2}} .$$

That is, φ is the moment generating function of the Gaussian density ϕ_{α, σ^2} on $\mathbb{R}$, then, the sequence of probability densities $(\rho_n)_{n=1}^{\infty}$ converges weakly to the Gaussian probability ϕ_{α, σ^2} when $n \rightarrow \infty$.

Remark 13. The former theorem implies that for, either a simple function of the form $u := \sum_{j=1}^k a_j 1_{A_j}$ or a function $u \in C(\mathbb{R})$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} u(t) d\rho_n(t) = \int_{\Omega} u(t) d\phi_{\alpha, \sigma^2}(t) .$$

Now we are able to state the main result of this section, which is a very generalized version of the CLT.

Theorem 14. Denote by ϕ_{α, σ^2} the Gaussian density on $\mathbb{R}$ with mean α and variance σ^2 , consider $\beta(t) := P(f + tg) - P(f)$, so, $\beta'(0) = \int g d\mu_f$ and $\sigma^2 = \beta''(0)$. Let u be, either a simple function $u := \sum_{j=1}^k a_j 1_{A_j}$ or a function $u \in C(\mathbb{R})$. Then, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} u \left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (g \circ \sigma^j - \beta'(0)) \right) d\mu_f = \int_{\mathbb{R}} u(t) d\phi_{0, \beta''(0)}(t). \quad (30)$$

Proof. It is well-known that the moment generation function for the Gaussian distribution $\phi_{0, \beta''(0)}$ with mean 0 and variance $\beta''(0)$ is given by the expression

$$\varphi(z) := \int_{\mathbb{R}} e^{zt} d\phi_{0, \beta''(0)}(t) = e^{z^2 \frac{\beta''(0)}{2}}.$$

For each $n \in \mathbb{N}$, denote by ρ_n the Borel probability measure on $\mathbb{R}$, such that, the interval (a, b) has ρ_n -measure equal to

$$\rho_n(a, b) := \mu_f \left(\left\{ x \in \Omega \mid \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (g(\sigma^j(x)) - \beta'(0)) \in (a, b) \right\} \right).$$

Let φ_n be the corresponding moment generation function associated with the probability distribution ρ_n , i.e., the one satisfying for each $z \in \mathbb{R}$

$$\varphi_n(z) = \int_{\mathbb{R}} e^{zt} d\rho_n(t).$$

We will show that for z fixed, we get that

$$\lim_{n \rightarrow \infty} \varphi_n(z) = \varphi(z) := e^{z^2 \frac{\beta''(0)}{2}},$$

that is, the map φ is the moment generation function of $\phi_{0, \beta''(0)}$.

At the first, consider the Taylor's polynomial of order 2 at zero for the map $\beta(t)$, which is given by

$$\beta(t) := \beta'(0)t + \frac{1}{2}\beta''(0)t^2 + o(t^3), \quad (31)$$

where $\beta'(0) = \int_{\Omega} g d\mu_f$ and $\beta''(0)$ is given by (29).

Besides that, assume that $\int_{\Omega} w_{f+tg} d\rho_{f+tg} = 1$, for all $t \in \mathbb{R}$, where w_{f+tg} and ρ_{f+tg} satisfy the expressions appearing in (3).

Observe that $\rho_n = (\gamma_n)_* \mu_f$, with $\gamma_n := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (g \circ \sigma^j - \beta'(0))$. Then, the moment generating function for the Borel probability measure ρ_n is given by the expression

$$\begin{aligned}
\varphi_n(z) &= \int_{\mathbb{R}} e^{zt} d\rho_n(t) \\
&= \int_{\Omega} e^{z \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (g \circ \sigma^j - \beta'(0))} d\mu_f \\
&= e^{-z \frac{n}{\sqrt{n}} \beta'(0)} \int_{\Omega} e^{z \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ \sigma^j} w_f \frac{1}{\lambda_f^n} d((\mathcal{L}_f^*)^n(\rho_f)) \\
&= e^{-z \frac{n}{\sqrt{n}} \beta'(0)} \frac{1}{\lambda_f^n} \int_{\Omega} \mathcal{L}_{f+z \frac{1}{\sqrt{n}} g}^n(w_f) d\rho_f \\
&= e^{-z \frac{n}{\sqrt{n}} \beta'(0)} \frac{1}{\lambda_f^n} \frac{\lambda_{f+z \frac{1}{\sqrt{n}} g}^n}{\lambda_{f+z \frac{1}{\sqrt{n}} g}^n} \int_{\Omega} \mathcal{L}_{f+z \frac{1}{\sqrt{n}} g}^n(w_f) d\rho_f \\
&= e^{-z \frac{n}{\sqrt{n}} \beta'(0) + n \beta(z \frac{1}{\sqrt{n}})} \frac{1}{\lambda_{f+z \frac{1}{\sqrt{n}} g}^n} \int_{\Omega} \mathcal{L}_{f+z \frac{1}{\sqrt{n}} g}^n(w_f) d\rho_f \\
&= e^{n(-z \beta'(0) \frac{1}{\sqrt{n}} + \beta(z \frac{1}{\sqrt{n}}))} \int_{\Omega} \frac{\mathcal{L}_{f+z \frac{1}{\sqrt{n}} g}^n(w_f)}{\lambda_{f+z \frac{1}{\sqrt{n}} g}^n} d\rho_f.
\end{aligned} \tag{32}$$

On the other hand, by the continuity of the maps $f \mapsto w_f$ and $f \mapsto \rho_f$, it follows immediately that

$$\lim_{n \rightarrow \infty} w_{f+z \frac{1}{\sqrt{n}} g} = w_f \quad \text{and} \quad \lim_{n \rightarrow \infty} \rho_{f+z \frac{1}{\sqrt{n}} g} = \rho_f. \tag{33}$$

In order to estimate (32), observe that taking the potential $f+z \frac{1}{\sqrt{n}} g$ instead of f in (4) and choosing w_f instead of w in (4) as well, by the assumption $\int_{\Omega} w_f d\rho_f = 1$, it follows that any $x \in \Omega$ satisfies the limit

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{\mathcal{L}_{f+z \frac{1}{\sqrt{n}} g}^n(w_f)(x)}{\lambda_{f+z \frac{1}{\sqrt{n}} g}^n} &= \lim_{n \rightarrow \infty} w_{f+z \frac{1}{\sqrt{n}} g}(x) \int_{\Omega} w_f d\rho_{f+z \frac{1}{\sqrt{n}} g} \\
&= w_f(x) \int_{\Omega} w_f d\rho_f = w_f(x).
\end{aligned} \tag{34}$$

Besides that, since $\|\mathcal{L}_{f+z \frac{1}{\sqrt{n}} g}^n(w_f)\|_{\infty} < \infty$ for any $n \in \mathbb{N}$, we can take into account the reasoning in (34) for estimating the limit in (32).

Indeed, first note that $\lim_{n \rightarrow \infty} \beta(z \frac{1}{\sqrt{n}}) = \beta(0) = 0$. Therefore, up to a multiplicative constant, from (31), (32), (33), (34) and the assumption $\int w_f d\rho_f = 1$,

it will follow that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \varphi_n(z) &= \lim_{n \rightarrow \infty} \int_{\Omega} e^{-zt} d\rho_n(t) \\
&= \lim_{n \rightarrow \infty} e^{n(-z\beta'(0)\frac{1}{\sqrt{n}} + \beta(z\frac{1}{\sqrt{n}}))} \int_{\Omega} \frac{\mathcal{L}_{f+z\frac{1}{\sqrt{n}}g}^n(w_f)}{\lambda_{f+z\frac{1}{\sqrt{n}}g}^n} d\rho_f \\
&= e^{z^2 \frac{1}{2} \beta''(0)} \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\mathcal{L}_{f+z\frac{1}{\sqrt{n}}g}^n(w_f)}{\lambda_{f+z\frac{1}{\sqrt{n}}g}^n} d\rho_f \\
&= e^{z^2 \frac{1}{2} \beta''(0)} \int_{\Omega} w_f d\rho_f = e^{z^2 \frac{1}{2} \beta''(0)},
\end{aligned} \tag{35}$$

where $e^{z^2 \frac{1}{2} \beta''(0)}$ is the moment generation function for the Gaussian probability distribution $\phi_{0, \beta''(0)}$.

Finally, from the above, we get that the following limit holds

$$\lim_{n \rightarrow \infty} \varphi_n(z) = e^{z^2 \frac{1}{2} \beta''(0)} = \int_{\mathbb{R}} e^{zt} d\phi_{0, \beta''(0)}(t).$$

So, our claim holds by Theorem 12. $\square$

Remark 15. *Theorem 14 implies the following interesting cases: If we take $u(x) := 1_{(-\infty, c)}$ we get that (30) is the same as the expression in (16); when we take $u(x) := x$ we obtain that the limit in (30) is equal to $\beta'(0) = \int_{\Omega} g d\mu_f$; moreover, choosing $u(x) = x^2$, it follows that the limit in (30) is equal to the value $\beta''(0) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} (\sum_{j=0}^{n-1} g \circ \sigma^j)^2 d\mu_f$.*

5 First and Second Derivatives

Given an *a priori* measure $\nu \in \mathcal{M}_1(M)$ and a potential $f \in \text{Lip}(\Omega)$, consider as in (2) the Ruelle operator $\mathcal{L}_f$ given by

$$\mathcal{L}_f(w)(x) := \int_M e^{f(ax)} w(ax) d\nu(a).$$

Assume that $f \in \mathcal{N}(\Omega)$ and denote by μ_f the probability such that $\mathcal{L}_f^*(\mu_f) = \mu_f$. Remember that the set of all μ_f 's, obtained when $f \in \mathcal{N}(\Omega)$, is also denoted by $\mathcal{N}(\Omega)$ and any element in this set is called a Lipschitz-Gibbs probability.

The entropy of $\mu_f \in \mathcal{N}(\Omega)$, where $f = \log J$ (see [22]), is given by the expression

$$h(\mu_f) = - \int_{\Omega} \log J d\mu_f = - \int_{\Omega} f d\mu_f. \tag{36}$$

Denote by $N : \text{Lip}(\Omega) \rightarrow \mathcal{N}(\Omega)$ the function sending a $f \in \text{Lip}(\Omega)$ into its associated normalized potential $N(f) =: \log J_f \in \mathcal{N}(\Omega)$, it is well known that the function $f \mapsto N(f)$ is analytic (see Theorem 3.5 in [17]). Denote by $\mathcal{C}$ the

set of Lipschitz continuous functions $\psi : \Omega \rightarrow \mathbb{R}$ of the form $\psi = w - (w \circ \sigma) + c$, where $c \in \mathbb{R}$ and $w \in \text{Lip}(\Omega)$.

Theorem 3.3 in [17] claims that for $f \in \mathcal{N}(\Omega)$, the vector spaces $\text{Ker}(\mathcal{L}_f)$ and $\mathcal{C}$ intersect trivially and, even more

$$\text{Lip}(\Omega) = \text{Ker}(\mathcal{L}_f) \oplus \mathcal{C} .$$

Therefore, any potential $\varphi \in \text{Lip}(\Omega)$, can be written in the form

$$\varphi = \xi + \left(w - (w \circ \sigma) + c \right), \quad (37)$$

with $\xi \in \text{Ker}(\mathcal{L}_f)$ and $w := (I - \mathcal{L}_f)^{-1}(c - \mathcal{L}_f(\varphi)) \in \text{Lip}(\Omega)$.

Moreover, if $\varphi \in \mathcal{N}(\Omega)$ is co-boundary to $\psi \in \mathcal{N}(\Omega)$, then, the associated ξ appearing in (37) is the same for both φ and ψ .

For a function $\eta \in \text{Ker}(\mathcal{L}_f)$ which satisfies $\int_{\Omega} \eta d\mu_f = 0$, a natural normalization is to assume that $\int_{\Omega} \eta^2 d\mu_f = 1$.

Remark 16. Given $f, g \in \text{Lip}(\Omega)$ and a function $\varphi \in \text{Lip}(\Omega)$ in the form (37) note that

$$\int_{\Omega} \varphi d\mu_{f+g} - \int_{\Omega} \varphi d\mu_f = \int_{\Omega} \xi d\mu_{f+g} - \int_{\Omega} \xi d\mu_f .$$

Therefore, for computations like the calculus of the derivatives

$$\frac{d}{dt} \int_{\Omega} \varphi d\mu_{f+tg}$$

we can assume that $\varphi \in \text{Ker}(\mathcal{L}_f)$, i.e., to take $\varphi = \xi$.

Remark 17. Note that when $\rho \in \text{Ker}(\mathcal{L}_f)$, we get that

$$(I - \mathcal{L}_f)^{-1} \rho = \rho. \quad (38)$$

Theorem 18. [Thm 3.5 in [17]] The derivative of the map N at the point f , denoted by DN_f , is the linear projection of the space $\text{Lip}(\Omega)$ on $\text{Ker}(\mathcal{L}_{N(f)})$ in the direction of $\mathcal{C}$.

Given a normalized Hölder potential f , a function g and small t , it will be necessary later to consider $t \rightarrow N(f + tg)$, and the associated derivative on t .

Theorem 19. [17] Consider the map $\eta \mapsto \mu_{f+\eta}$, where $f, \eta \in \text{Lip}(\Omega)$. Assume that $f \in \mathcal{N}(\Omega)$, $\varphi \in \text{Lip}(\Omega)$ satisfies $\int_{\Omega} \varphi d\mu_f = 0$, and finally that $\eta \in \text{Ker}(\mathcal{L}_f)$. Then,

$$\int_{\Omega} \varphi d\mu_{f+\eta} = \int_{\Omega} (I - \mathcal{L}_f)^{-1}(\varphi) \eta d\mu_f + O(\|\varphi\| \|\eta\|^2). \quad (39)$$

Now, assuming that $f, g \in \text{Lip}(\Omega)$ are general (not necessarily in the kernel of $\mathcal{L}_f$), Theorem C in [17] claims

$$\frac{d}{dt} \int_{\Omega} \varphi d\mu_{f+tg}|_{t=0} = \int_{\Omega} (I - \mathcal{L}_f)^{-1}(\varphi_f) \cdot DN_f(g) d\mu_f \quad (40)$$

where $\varphi_f = \varphi - \int_{\Omega} \varphi d\mu_f$.

Finally, if we consider $f \in \mathcal{N}(\Omega)$, $\eta \in \text{Ker}(\mathcal{L}_f)$, and do not assume that $\int_{\Omega} \varphi d\mu_f = 0$, we get the more general result

$$\int_{\Omega} \varphi d\mu_{f+\eta} = \sum_{j=0}^{\infty} \int_{\Omega} \varphi (\eta \circ \sigma^j) d\mu_f + O(\|\varphi\| \|\eta\|^2). \quad (41)$$

Proof. The first claim (39) and the last claim (41) are stated in Theorem 4.1 in Section 4 in [17]. Expression (40) is the claim of Theorem C in Section 1.2 in [17]. $\square$

A useful result follows from the above:

Theorem 20. [17] Assume that $f, \eta, \varphi \in \text{Lip}(\Omega)$, $f \in \mathcal{N}(\Omega)$, and consider the law $t \mapsto \mu_{f+t\eta}$, where $\eta \in \text{Ker}(\mathcal{L}_f)$. Then,

$$\frac{d}{dt} \int_{\Omega} \varphi d\mu_{f+t\eta}|_{t=0} = \int_{\Omega} \varphi \eta d\mu_f. \quad (42)$$

Proof. For a proof see (2) in Theorem 5.1 in [17]. $\square$

In the next results we will use in a fundamental way the property that any function can be decompose in the form (37).

Corollary 21. Assume $f, g, \varphi \in \text{Lip}(\Omega)$, with $f \in \mathcal{N}(\Omega)$. Consider the law $t \mapsto \mu_{f+tg}$, where g does not necessarily belong to the kernel of $\mathcal{L}_f$. Assume that there are $v, w \in \mathcal{C}(\Omega)$ such that $g = \eta + (v - v \circ \sigma)$ and $\varphi = \xi + (w - w \circ \sigma) + c$, where $\eta, \xi \in \text{Ker}(\mathcal{L}_f)$, and $c \in \mathbb{R}$. Then, we have,

$$\frac{d}{dt} \int_{\Omega} \varphi d\mu_{f+tg}|_{t=0} = \int_{\Omega} \xi \eta d\mu_f. \quad (43)$$

Proof. From (37) and Remark 17 we can take

$$\frac{d}{dt} \int_{\Omega} \varphi d\mu_{f+tg}|_{t=0} = \frac{d}{dt} \int_{\Omega} \xi d\mu_{f+tg}|_{t=0}.$$

From (40)

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \xi d\mu_{f+tg}|_{t=0} &= \int_{\Omega} (I - \mathcal{L}_f)^{-1}(\xi) \cdot DN_f(g) d\mu_f \\ &= \int_{\Omega} \xi \cdot \eta d\mu_f. \end{aligned} \quad (44)$$

$\square$

A different proof of (45) was presented in [17].

In the following proposition, we study the behavior of the entropy $h(\mu_f)$ for a Lipschitz-Gibbs probability $\mu_f \in \mathcal{N}(\Omega)$.

Proposition 22. *Consider the path $t \mapsto \mu_{f+t\eta}$, where f and η belong to $\text{Lip}(\Omega)$. Assume that $f = \log J \in \mathcal{N}(\Omega)$ and $\eta \in \text{Ker}(\mathcal{L}_f)$. Denoting $h(t) := h(\mu_{f+t\eta})$ the entropy of the Lipschitz-Gibbs probability $\mu_{f+t\eta}$, we get*

$$\frac{d}{dt}h(t)|_{t=0} = - \int_{\Omega} f \eta d\mu_f, \quad (45)$$

and

$$\frac{d^2}{dt^2}h(t)|_{t=0} = - \left(\int_{\Omega} \eta^2 d\mu_f + \int_{\Omega} f \eta^2 d\mu_f \right) = - \int_{\Omega} (\log J + 1) \eta^2 d\mu_f. \quad (46)$$

Proof. First note that it follows from the chain rule and Theorem 20:

$$\begin{aligned} \int_{\Omega} \eta d\mu_{f+t\eta} &= \frac{d}{dt}P(f+t\eta) = \frac{d}{dt} \left(h(t) + \int_{\Omega} (f+t\eta) d\mu_{f+t\eta} \right) \\ &= \frac{d}{dt}h(t) + \int_{\Omega} \eta d\mu_{f+t\eta} + \int_{\Omega} (f+t\eta) \eta d\mu_{f+t\eta}. \end{aligned} \quad (47)$$

Since $\eta \in \text{Ker}(\mathcal{L}_f)$, we get $\int_{\Omega} \eta d\mu_f = 0$. This implies that

$$\frac{d}{dt}h(t)|_{t=0} = - \int_{\Omega} f \eta d\mu_f.$$

Furthermore, from (47), it follows that

$$\begin{aligned} \frac{d^2}{dt^2}h(t)|_{t=0} &= - \frac{d}{dt} \left(\int_{\Omega} (f+t\eta) \eta d\mu_{f+t\eta} \right) |_{t=0} \\ &= - \left(\int_{\Omega} f \eta^2 d\mu_f + \int_{\Omega} \eta^2 d\mu_f \right) \\ &= - \int_{\Omega} (\log J + 1) \eta^2 d\mu_f \\ &= - \int_{\Omega} (\log J + \log(e)) \eta^2 d\mu_f = - \int_{\Omega} \log(Je) \eta^2 d\mu_f. \end{aligned}$$

It is natural to assume a normalization condition $\int_{\Omega} \eta^2 d\mu_f = 1$. In this case

$$- \int_{\Omega} \log(Je) \eta^2 d\mu_f \geq - \int_{\Omega} \log(e) \eta^2 d\mu_f = -1.$$

The function η may enhance points x where the value $(J(x)e)$ is close to zero (producing large values for $\frac{d^2}{dt^2}h(t)|_{t=0}$) or points x where the value $(J(x)e)$ is close to e (producing values $\frac{d^2}{dt^2}h(t)|_{t=0}$ close to -1). $\square$

Proposition 23. Assume, $f, g \in \text{Lip}(\Omega)$, with g not necessarily on $\text{Ker}(\mathcal{L}_f)$. Then,

$$\frac{d}{dt}h(\mu_{f+tg}) = - \int_{\Omega} \zeta \eta d\mu_f, \quad (48)$$

where $f = \zeta + (u - u \circ \sigma) + c$ and $g = \eta + (v - v \circ \sigma)$, with $\zeta, \eta \in \text{Ker}(\mathcal{L}_f)$ and $c \in \mathbb{R}$.

Proof. If g is not necessarily on the kernel of $\mathcal{L}_f$, we get from (40) that

$$\begin{aligned} \int_{\Omega} g d\mu_f &= \frac{d}{dt}P(f+tg)|_{t=0} \\ &= \frac{d}{dt} \left(h(\mu_{f+tg}) + \int_{\Omega} (f+tg) d\mu_{f+tg} \right) |_{t=0} \\ &= \frac{d}{dt} h(\mu_{f+tg})|_{t=0} + \int_{\Omega} g d\mu_f \\ &\quad + \int_{\Omega} (I - \mathcal{L}_f)^{-1} \left(f - \int_{\Omega} f d\mu_f \right) \cdot DN_f(g) d\mu_f. \end{aligned} \quad (49)$$

where we used the derivative of the product in the last line.

Using expression (37) for $f = \zeta + (u - u \circ \sigma) + c$, with $\zeta \in \text{Ker}(\mathcal{L}_f)$, we get for the derivative the equivalent expression

$$\frac{d}{dt} \int_{\Omega} f d\mu_{f+tg}|_{t=0} = \frac{d}{dt} \int_{\Omega} \zeta d\mu_{f+tg}|_{t=0}.$$

Now in (40), taking $\varphi = \zeta$, from remark 17, we get

$$\begin{aligned} \frac{d}{dt} h(\mu_{f+tg})|_{t=0} &= - \int_{\Omega} (I - \mathcal{L}_f)^{-1}(\zeta) DN_f(g) d\mu_f \\ &= - \int_{\Omega} \zeta DN_f(g) d\mu_f. \end{aligned} \quad (50)$$

We assumed that $g = \eta + (v - v \circ \sigma)$, where $\eta \in \text{Ker}(\mathcal{L}_f)$, then,

$$\frac{d}{dt} h(\mu_{f+tg}) = - \int_{\Omega} \zeta DN_f(g) d\mu_f = - \int_{\Omega} \zeta \eta d\mu_f. \quad (51)$$

□

Corollary 24. Given $f, g \in \text{Lip}(\Omega)$, assume that $f = \zeta + (u - u \circ \sigma) + c$ and $g = \eta + (v - v \circ \sigma)$, with $\zeta, \eta \in \text{Ker}(\mathcal{L}_f)$ and $\int_{\Omega} \eta^2 d\mu_f = 1$. Then, the maximal value of

$$\frac{d}{dt}h(\mu_{f+tg}) = - \int_{\Omega} \zeta \eta d\mu_f, \quad (52)$$

among all the η 's on the kernel of the operator $\mathcal{L}_f$, indicates the direction of the larger increasing of entropy for $h(\mu_{f+tg})$, when t is close to zero.

Given a Hölder function φ , the pressure of φ is

$$P(\varphi) = \sup_{\mu \in \mathcal{M}_T(\Omega)} \left\{ h(\mu) + \int_{\Omega} \varphi d\mu \right\}. \quad (53)$$

Given $\varphi \in \text{Lip}(\Omega)$, the *Pressure problem* about find the value $P(\varphi)$ (see the expression in (53)) for σ -invariant probabilities μ is one of the main issues in Thermodynamic Formalism (see [31] and [22]). For analyzing this problem a concept of entropy is required and this topic was addressed via (36). It is known the existence of a unique maximizing solution $\mu_B \in \mathcal{N}(\Omega)$, for $P(\varphi)$, where $B \in \mathcal{N}(\Omega)$, and μ_B is called the equilibrium probability for φ (and also for B). Even more, the functions B and φ are co-boundary each other (see [31]).

It is natural to consider variations of the law $t \mapsto f + tg$ and the directional derivative

$$\frac{d}{dt} \left(h(\mu_{f+tg}) + \int_{\Omega} \varphi d\mu_{f+tg} \right) |_{t=0}. \quad (54)$$

For fixed φ , we can ask about the maximal value of (15) for η in the kernel of $\mathcal{L}_f$, when η satisfies $\int_{\Omega} \eta^2 = 1$. In this direction we can take advantage of expressions (43) and (45).

In the next theorem, we are interested in the directional derivatives associated with such a problem, but when g is not necessarily on the Kernel of $\mathcal{L}_f$.

Theorem 25. *Given $\varphi \in \text{Lip}(\Omega)$, denote by $\mu_B \in \mathcal{N}(\Omega)$, the unique maximizing solution for $P(\varphi)$, where $B \in \mathcal{N}(\Omega)$ is normalized. Consider another normalized potential f and the associated equilibrium probability $\mu_f \in \mathcal{N}(\Omega)$, where $\mu_f \neq \mu_B$. Assume that $f = \zeta + (u - u \circ \sigma) + c$, $g = \eta + (v - v \circ \sigma)$ and $\varphi = \xi + (w - w \circ \sigma) + k$, where $\zeta, \eta, \xi \in \text{Ker}(\mathcal{L}_f)$ and $c, k \in \mathbb{R}$. In this case, we do not assume that g is in the Kernel of $\mathcal{L}_f$, but we assume for simplification that it has integral $\int g d\mu_f = 0$. An equivalent expression for (15), for a fixed g (and therefore for the associated $\eta \in \text{Ker}(\mathcal{L}_f)$) is:*

$$\int_{\Omega} (\xi - \zeta) \eta d\mu_f. \quad (55)$$

Proof. From Corollary 21 and Proposition 23, we have

$$\begin{aligned} \frac{d}{dt} h(\mu_{f+tg}) |_{t=0} + \frac{d}{dt} \int_{\Omega} \varphi d\mu_{f+tg} |_{t=0} &= - \int_{\Omega} \zeta \eta d\mu_f + \int_{\Omega} \xi \eta d\mu_f \\ &= \int_{\Omega} (\xi - \zeta) \eta d\mu_f. \end{aligned} \quad (56)$$

□

Note that in the case (54) is critical for all variations η on the kernel (that is, (54) = 0), it will also be critical for all variations g not in the kernel (but with μ_f mean zero).

Note also that when μ_f , which is Lipschitz-Gibbs probability for the normalized potential f , maximizes (53), the functions f and φ are co-boundaries. In this case $\xi - \zeta = 0$ and (15) is equal to zero in any direction η . Therefore, the probability μ_B is a critical point for the directional derivatives of the law $t \mapsto h(\mu_{f+tg}) + \int_{\Omega} \varphi d\mu_{f+tg}$.

In Subsection 5.1 we show for some particular examples of μ_f a more explicit expression for (55) (see for instance expression (66)).

Another expression for the second derivative of pressure can be obtained in the following way.

Theorem 26. *Given $f, g \in \text{Lip}(\Omega)$ and $t \in \mathbb{R}$, assume $f \in \mathcal{N}(\Omega)$ and denote by $p(t) := P(f + tg)$. Then,*

$$p''(0) = \frac{d^2}{dt^2} P(f + tg)|_{t=0} = \int_{\Omega} \left(g - \int_{\Omega} g d\mu_f + \frac{d}{dt} (\varphi_t - \varphi_t \circ \sigma)|_{t=0} \right)^2 d\mu_f, \quad (57)$$

where $w_{f+tg} = e^{\varphi_t}$ is the eigenfunction for $f + tg$.

Proof. We assume that for all t (small and close to 0) and any x

$$\int_M J_t(ax) d\nu(a) = \int_M e^{(f(ax)+tg(ax))+[\varphi_t(ax)-\varphi_t(x)]-p(t)} d\nu(a) = 1.$$

where $J_t := e^{f+tg+\varphi_t-\varphi_t \circ \sigma - p(t)}$. We know that

$$p'(0) = \frac{d}{dt} P(f + tg)|_{t=0} = \int_{\Omega} g d\mu_f, \quad (58)$$

$$\frac{d}{dt} \int_{\Omega} (\varphi_t - \varphi_t \circ \sigma) d\mu_f|_{t=0} = 0, \quad (59)$$

and

$$\frac{d^2}{dt^2} \int_{\Omega} (\varphi_t - \varphi_t \circ \sigma) d\mu_f|_{t=0} = 0.$$

Moreover,

$$\begin{aligned} 0 &= \frac{d^2}{dt^2} \int_{\Omega} \int_M e^{(f(ax)+tg(ax))+[\varphi_t(ax)-\varphi_t(x)]-p(t)} d\nu(a) d\mu_f(x) \\ &= \frac{d}{dt} \int_{\Omega} \int_M J_t(ax) \left(g(ax) + \frac{d}{dt} (\varphi_t(ax) - \varphi_t(x)) - \frac{d}{dt} p(t) \right) d\nu(a) d\mu_f(x) \\ &= \int_{\Omega} \int_M J_t(ax) \left(g(ax) + \frac{d}{dt} (\varphi_t(ax) - \varphi_t(x)) - \frac{d}{dt} p(t) \right)^2 \\ &\quad + J_t(ax) \left(\frac{d^2}{dt^2} (\varphi_t(ax) - \varphi_t(x)) - \frac{d^2}{dt^2} p(t) \right) d\nu(a) d\mu_f(x). \end{aligned}$$

Then, from the above and (59), it follows that

$$\begin{aligned}
0 &= \frac{d^2}{dt^2} \int_{\Omega} \int_M e^{(f(ax)+tg(ax))+[\varphi_t(ax)-\varphi_t(x)]-p(t)} d\nu(a) d\mu_f(x)|_{t=0} \\
&= \int_{\Omega} \int_M \left(g(ax) + \frac{d}{dt}(\varphi_t(ax) - \varphi_t(x))|_{t=0} - \frac{d}{dt}p(t)|_{t=0} \right)^2 \\
&\quad + \left(\frac{d^2}{dt^2}(\varphi_t(ax) - \varphi_t(x))|_{t=0} - \frac{d^2}{dt^2}p(t)|_{t=0} \right) d\nu(a) d\mu_f(x) \\
&= \int_{\Omega} \int_M \left(g(ax) + \frac{d}{dt}(\varphi_t(ax) - \varphi_t(x))|_{t=0} - \frac{d}{dt}p(t)|_{t=0} \right)^2 - \frac{d^2}{dt^2}p(t)|_{t=0} d\nu(a) d\mu_f(x) .
\end{aligned}$$

Finally, since we have $\mu_f \in \mathcal{M}_T(\Omega)$, it follows that

$$\frac{d^2}{dt^2}p(t)|_{t=0} = \int_{\Omega} \left(g - \int_{\Omega} g d\mu_f + \frac{d}{dt}(\varphi_t - \varphi_t \circ \sigma)|_{t=0} \right)^2 d\mu_f .$$

□

Theorem 27. Suppose $f \in \mathcal{N}(\Omega)$, we are going to take derivative on the direction $g \in \text{Lip}(\Omega)$. Denote by $w(t, x) := w_{f+tg}(x)$ the eigenfunction for $\mathcal{L}_{f+tg}$ associated to the eigenvalue λ_{f+tg} . Then, the function $x \mapsto \frac{\partial}{\partial t}w(t, x)|_{t=0}$ satisfies the following expression for all $x \in \Omega$

$$(\mathcal{L}_f - I)\left(\frac{\partial}{\partial t}w(t, \cdot)\right)|_{t=0} = \int_{\Omega} g d\mu_f - \mathcal{L}_f(g) . \quad (60)$$

Proof. We will estimate for each $t \in \mathbb{R}$ the partial derivative of the eigenfunction $\frac{\partial}{\partial t}w(t, x)|_{t=0}$. Indeed, taking derivative on t , we obtain

$$\frac{\partial}{\partial t}\mathcal{L}_{f+tg}(w(t, \cdot))(x) = \mathcal{L}_{f+tg}(g w(t, \cdot))(x) + \mathcal{L}_{f+tg}\left(\frac{\partial}{\partial t}w(t, \cdot)\right)(x).$$

On the other hand, for all $x \in \Omega$ we have

$$\frac{\partial}{\partial t}(\lambda_{f+tg} w(t, x)) = w(t, x) \frac{d}{dt}\lambda_{f+tg} + \lambda_{f+tg} \frac{\partial}{\partial t}w(t, x) .$$

Then,

$$\mathcal{L}_{f+tg}(g w(t, \cdot))(x) + \mathcal{L}_{f+tg}\left(\frac{\partial}{\partial t}w(t, \cdot)\right)(x) = w(t, x) \frac{d}{dt}\lambda_{f+tg} + \lambda_{f+tg} \frac{\partial}{\partial t}w(t, x) . \quad (61)$$

Therefore, when $t = 0$ we get

$$\mathcal{L}_f(g)(x) + \mathcal{L}_f\left(\frac{\partial}{\partial t}w(t, \cdot)|_{t=0}\right)(x) = \int_{\Omega} g d\mu_f + \frac{\partial}{\partial t}w(t, x)|_{t=0} .$$

Finally,

$$(\mathcal{L}_f - I)\left(\frac{\partial}{\partial t}w(t, \cdot)|_{t=0}\right) = \int_{\Omega} g d\mu_f - \mathcal{L}_f(g) . \quad (62)$$

□

For fixed f, g , the eigenfunction w_t , t close to 0, of the Ruelle operator $\mathcal{L}_{f+tg}$, satisfies

$$f + tg = B_t - \log w_t + \log(w_t \circ \sigma) + \log \lambda_t,$$

where λ_t is the eigenvalue and B_t is the associated normalized potential. In order to get control of all elements in this coboundary equation it is worthwhile to get some information in $\frac{d}{dt}w_t$. In the case of a general g , we get the implicit expression (60). The operator $\mathcal{L}_f - I$ is not injective due to the fact that $(\mathcal{L}_f - I)(1) = 0$; however some functions can be eventually inverted.

The last result is new. The next one, was stated and proved in [23]. We can get it now as a corollary of Theorem 27.

Corollary 28. *If $g \in \text{Ker}(\mathcal{L}_f)$ and $w(t, x) := w_{f+tg}(x)$ is the main eigenfunction of $\mathcal{L}_{f+tg}$, then, $\frac{\partial}{\partial t}w(t, x)|_{t=0}$ is a constant function independent of x .*

Proof. Since we assume $g \in \text{Ker}(\mathcal{L}_f)$, it follows that $\int_{\Omega} g d\mu_f = 0$ as well. Therefore, from (62) we get that

$$(\mathcal{L}_f - I)\left(\frac{\partial}{\partial t}w(t, \cdot)|_{t=0}\right) = 0.$$

The above, in particular implies that $(\frac{\partial}{\partial t}w(t, x)|_{t=0})$ is constant. □

5.1 Examples with explicit computation

We consider in this section the symbolic space $\Omega := \{0, 1\}^{\mathbb{N}}$ and the shift σ acting on it; this corresponds to take the space of symbols $M = \{0, 1\}$.

In this subsection, we want to show that for some particular examples of μ one can get more explicit expressions which are helpful in applications of Theorem 25 (see expression (66)). In order to do that we take advantage of the existence of an orthonormal basis in the Kernel of a certain Ruelle operator (see [23] and [11]).

We denote by P the line stochastic matrix (with all entries positive)

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{pmatrix}. \quad (63)$$

Consider the invariant Markov probability μ obtained from the stochastic matrix $(P_{i,j})_{i,j=0,1}$ and the unique initial **left invariant vector of probability** $\pi = (\pi_0, \pi_1) \in [0, 1]^2$.

The associated normalized potential $B = \log J$ satisfies: $J : \{0, 1\}^{\mathbb{N}} \rightarrow (0, 1)$ is constant equal

$$J_{i,j} = \frac{\pi_i P_{i,j}}{\pi_j}$$

on the **cylinder** $[ij] := \{x \in \Omega | x_1 = i; x_2 = j\}$, $i, j = 0, 1$.

Let $\omega := (\omega_1, \omega_2, \dots, \omega_n) \in \{0, 1\}^n$ be a finite word on the symbols $\{0, 1\}$. Denote by $[\omega]$ the cylinder set $[\omega] := \{x \in \Omega | x_1 = \omega_1, \dots, x_n = \omega_n\}$. The empty word, denoted by $\emptyset$, is also considered a finite word.

Given μ as above, consider the Hilbert space $L^2(\mu)$ equipped with the inner product $\langle f, g \rangle = \int_{\Omega} f g d\mu$.

The family of Hölder functions

$$e_{\omega} := \frac{1}{\sqrt{\mu([\omega])}} \sqrt{\frac{P_{\omega_n,1}}{P_{\omega_n,0}}} \mathbf{1}_{[\omega 0]} - \frac{1}{\sqrt{\mu([\omega])}} \sqrt{\frac{P_{\omega_n,0}}{P_{\omega_n,1}}} \mathbf{1}_{[\omega 1]}, \quad (64)$$

$\omega := (\omega_1, \omega_2, \dots, \omega_n) \in \{0, 1\}^n$, is an orthonormal set for $L^2(\mu)$ (see [18] for a general expression and [23] for (64); see also [11]). In order to get a (Haar) basis we have to add $e_{\emptyset} := \frac{1}{\sqrt{\mu([0])}} \mathbf{1}_{[0]} - \frac{1}{\sqrt{\mu([1])}} \mathbf{1}_{[1]}$ to this family.

Given a finite word $\omega := (\omega_1, \omega_2, \dots, \omega_n) \in \{0, 1\}^n$, we denote

$$a_{\omega} := \frac{\sqrt{\pi_{\omega_1}}}{\sqrt{\pi_0} \sqrt{P_{0,\omega_1}}} e_{0\omega} - \frac{\sqrt{\pi_{\omega_1}}}{\sqrt{\pi_1} \sqrt{P_{1,\omega_1}}} e_{1\omega},$$

Finally, we set

$$\hat{a}_{\omega} := \frac{1}{\|a_{\omega}\|} a_{\omega}. \quad (65)$$

the normalization of a_{ω} .

In order to get a complete orthonormal set for the kernel of the Ruelle operator $\mathcal{L}_{\log J}$ acting on $L^2(\mu)$, we will have to add to the functions of the form (65) two more functions: $\hat{a}_{\emptyset}^0$ and $\hat{a}_{\emptyset}^1$, which are described in Definition 6.8 in [23]. Denote

$$V_1 = \pi_1 P_{10} I_{[00]} - \pi_0 P_{00} I_{[10]}$$

and

$$V_2 = \pi_0 P_{01} I_{[11]} - \pi_1 P_{11} I_{[01]},$$

and, finally set

$$\hat{a}_{\emptyset}^0 = \frac{1}{|V_1|} V_1,$$

and

$$\hat{a}_{\emptyset}^1 = \frac{1}{|V_2|} V_2.$$

Denote by $\{0, 1\}^*$ the set of all the finite words on $\{0, 1\}$, i.e., $\{0, 1\}^* := \emptyset \cup \bigcup_{n=1}^{\infty} \{0, 1\}^n$. One of the main results in [23] is

Theorem 29. *The family $(\hat{a}_{\omega})_{\omega \in \{0, 1\}^*}$, plus the two functions $\hat{a}_{\emptyset}^0$ and $\hat{a}_{\emptyset}^1$, determine an orthonormal set on the kernel of the Ruelle operator $\mathcal{L}_{\log J}$ acting on $L^2(\mu)$.*

The set of elements in this orthonormal family will be generically described by $(\hat{a}_{\omega})_{\omega}$ (we allow ω to be the empty word $\emptyset$.)

An item of major interest here is to express the Lipschitz function $\eta \in \mathcal{L}_{\log J}$ on the form

$$\eta = \sum_{\omega \in \{0, 1\}^*} \eta_{\omega} \hat{a}_{\omega},$$

where $\eta_\omega := \langle \eta, \hat{a}_\omega \rangle$.

An important issue in expression (56) in Theorem 25 was to express $f = \zeta + (u - u \circ \sigma) + c$, $g = \eta + (v - v \circ \sigma)$ and $\varphi = \xi + (w - w \circ \sigma) + k$, where $\zeta, \eta, \xi \in \text{Ker}(\mathcal{L}_f)$ and $c, k \in \mathbb{R}$. Here we take $g = \eta$, and $f = \log J$, where J was defined above.

Therefore, given φ we set

$$\zeta = \sum_{\omega \in \{0,1\}^*} f_\omega \hat{a}_\omega,$$

and

$$\xi = \sum_{\omega \in \{0,1\}^*} \varphi_\omega \hat{a}_\omega.$$

Since the family $(\hat{a}_\omega)_\omega$ is orthonormal on $\text{Ker}(\mathcal{L}_{\log J})$, it follows that

$$\begin{aligned} \int_{\Omega} (\xi - \zeta) \eta d\mu &= \int_{\Omega} \left(\sum_{\omega \in \{0,1\}^*} (\varphi_\omega - f_\omega) \hat{a}_\omega \sum_{\omega \in \{0,1\}^*} \eta_\omega \hat{a}_\omega \right) d\mu \\ &= \sum_{\omega \in \{0,1\}^*} (\varphi_\omega - f_\omega) \eta_\omega. \end{aligned} \tag{66}$$

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