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Dimension Spectra and a Mathematical Model for Phase Transition

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Here we will present a mathematical model for phase transition. In general terms, a phase transition occurs when we consider an evolution of a system depending on a continuous external parameter and a sudden appearance of a discontinuity of the behaviour of the system happens. A simple example of this fact happens with the water, that turns into ice at zero degrees. Other interesting examples appear in several different problems in physics, among them the ferromagnetic Ising model.

We do not claim the model can explain any of the Physical Phenomena mentioned above. We just want to present a rigorous mathematical formulation of the dimension spectrum theory for a certain class of systems and show that they present, in some cases, the phenomena of phase transition. This model is based on the concept of generalized dimension that was introduced previously in the literature. In order to establish the concept in a rigorous base, we first introduce some probability measures that will play the role of probability laws to choose the center of balls at random. In this way, we will have generalized dimension and the spectra of dimensions in a totally rigorous way for hyperbolic rational maps. These systems do not present phase transition. As far as we know, these probability laws were never used before in the definition of generalized dimension. Using the fact that techniques of hyperbolic rational maps can be applied for some special cases of non-hyperbolic rational maps, it is also shown that the existence of phase transition in the setting of generalized dimension. There exists a relation of the setting of generalized dimension with the setting of the pressure. The setting of generalized dimension uses capacity (or box counting) dimension in some way, and we believe this is closer to the procedure that physicists use to understand phase transition. It will appear in the mathematical model of phase transition presented here a continuous evolution of a measure (equilibrium state) and then a sudden discontinuity, in fact, a jump for a Dirac measure in a fixed point. This phenomena can be understood in the correct context as a spontaneous magnetization. This will be more carefully explained in the paper. The more important point in this paper, is the relation of the setting of pressure and generalized dimension. When phase transition occurs in one setting, it also occurs in the other. In principle, the setting of generalized dimension is simpler to work with. The setting of pressure is, nevertheless, closer to the concept of Gibbs states.

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1. Introduction

Phase transition and the spectrum of dimension have been extensively studied in the physics literature in the last years (see [13, 14, 17, 44]). In the work of P. Collet, J. Lebowitz, and A. Porzio [5], D. Rand [38], and T. Bohr and D. Rand [2], several different cases of dimension of spectrum are considered, and rigorous results were obtained. In [19], we consider the dimension spectrum of the maximal measure of a hyperbolic rational map on the Riemann sphere, and we showed that phase transitions do not occur.

Here we will use several ideas introduced in the above-mentioned work to formulate, in a rigorous way, concepts like generalized dimension and others, that have been previously considered in physics literature.

The spectrum of dimension refers to an initially fixed measure that one wants to analyze. In our case we will consider, as we already did in [19], this measure, the maximal entropy measure u [11, 24]. This measure, here, will play the role of the Boltzman factor in the physical model.

This maximal entropy measure, also called maximal measure, can be seen as a particular case of the equilibrium state (also called Gibbs state), in the case where there is no external thermal source (see [3, 41–43, 47]). If one considers thermal external sources, then one has to consider the pressure [42]. The equilibrium states we will consider here are maximal pressure measures. This kind of statistical mechanics approach was devised by D. Ruelle and R. Bowen [42, 43]. In general the main references in thermodynamic formalism were not concerned with phase transition.

Now we consider some of the problems in physics in which the dimension spectrum and phase transition are related. A variety of complicated fractal sets appears in nonlinear physics. In diffusion limited aggregation, the probability of a random walker landing next to a given site of the aggregate is of interest. In percolation the distribution of voltages across different elements in a random resistor network may be of interest [13, 14, 17]. All these examples can be better analyzed by dividing an object in boxes (or pieces) labeled by indexes, and in both cases one will have to work with fractal sets and the notion of dimension.

In order to obtain better formalization for these problems, several mathematical physicists began to consider a one-dimensional map or a diffeomorphism as a model for the creation of boxes and labels (see [12–14, 17, 34, 44]).

The generalized dimension appears in the above context in a natural way, as a similar procedure used in physics to understand the applied examples mentioned before [13, 14, 17].

Here we relate this concept with the pressure and, from that, equilibrium states for the pressure will appear in a natural way that will be followed with the variation of an external parameter. The pressure of thermodynamic formalism considered here is equal to -1 times the pressure that people in statistical physics are used to.

These equilibrium states should not be confused with the maximal measure u (analogous to the Boltzman factor) that will remain fixed during the process of phase transition (generalized dimension setting).

In phase transition, the important and physically relevant place to see the evolution of these equilibrium measures is, nevertheless, in the setting of generalized dimension and not in the setting of pressure. This will be more carefully explained in Section 6. In fact, for each value of the parameter, the associated equilibrium state is the probability law that one has to consider in order to choose at random the center of the above-mentioned boxes (see Remark 8).

In the setting of pressure, the concept of Hausdorff dimension appears in a natural way, and in the setting of generalized dimension, capacity (or box counting) dimension is more natural. It is well know that such concepts do not always give the same result [22].

An essential point in Section 5 will be the fact that Hausdorff dimension of measures and capacity dimension of measures are the same for very general systems. This relation is presented in a theorem of L. S. Young in [48], where it is used with some ideas of F. Ledrapier about the capacity dimension of a measure (see also [22]).

The q-generalized dimension $D(q)$ is usually defined as

$$D(q) = \frac{1}{q-1} \lim_{\xi \to 0} \frac{\log \sum (p_i(\xi))^q}{\log \xi} \qquad \text{for } q \geq 0,$$

where the summation is in a subcovering with the small number of atoms (more than one could exist), among the possible partitions of balls of radius ξ with center in points of the set. In our notation $p_i(\xi)$ is the u-mass of a ball of the minimal partition.

For general systems, as far as we know, there is no rigorous justification for the existence of such a limit. One of the main reasons why we have to consider thermodynamic formalism, pressure, Hausdorff dimension of measures, and capacity of measures is to formalize in a rigorous way the concept of generalized dimension. This will be done for hyperbolic rational maps in Section 5.

In the case that the rational map is hyperbolic, there is no phase transition [19]. We will show here that in the case that the rational map is subhyperbolic (see Section 2 for definitions), then we will have, in several cases, phase transition at the level of equilibrium states (measures). This result is stated in Theorem 3 in Section 6.

We are not sure about the existence of phase transition in the case of the general nonhyperbolic rational map, nevertheless, we believe that a set of positive Lebesgue measures can exist in the set of parameters, such that phase transition occurs. We wonder if the techniques used by M. Rees in [39, 40] can be applied in this situation. The techniques used in [40] are inspired by the Jacobson theorem about real quadratic polynomials in the real line [16].

Phase transition in general is associated with the lack of differentiability of a scalar function that gives some important information about our system. If a discontinuity of the derivative of this function exists, then by definition we have a first-order phase transition. In the case that this first derivative is continuous, but the second derivative is discontinuous, we say that we have a second-order phase transition. Here we will just consider first-order transitions. Here we will show the existence of a first-order transition in the above sense and, also, a transition in terms of measures of equilibrium, that is, a sudden discontinuity of the equilibrium measure when we change continuously an external parameter (temperature). We will say that these transitions happen, respectively, at Level-1 and Level-2.

The main tool to formulate the problem that we consider here is thermodynamic formalism. The problem about the extension of the formulation obtained here to the general rational map is related to the fact that, in this case, there is not a complete understanding of equilibrium states (in terms of existence and uniqueness) for pressure.

For C^k-maps it is known that in some cases there is no measure of maximal entropy (see [42, 47] for detailed references). For the general rational map, there is always a measure of maximal entropy [11, 24].

The work presented here, in fact, gives a brief idea of some of the difficulties that can happen with the question of uniqueness for the maximal pressure measure problem in the general case of rational maps. In this work we follow a unique equilibrium state until a certain transition value of the parameter, where another different equilibrium measure suddenly appears (therefore, we do not have uniqueness in this transition point), and then the old solution disappears and we begin to follow the new equilibrium measure that suddenly appeared and (after the bifurcation point) which is also unique.

In one of the situations covered by our model, we can change an external parameter and observe a transition value in which the charge distribution located in a segment, suddenly jumps for the charge distribution located in one of the extremal points of this segment (see Remark 11). We would like to explain the physical meaning of this sudden appearance of a jump from a measure to a Dirac delta in a fixed point when we change an external parameter.

Consider the one-sided one-dimensional lattice $\mathbb{N}$ and, at each point of the lattice, the possibility of the occurrence of spin $+$ or $-$. This model appears when one considers an Ising model with a wall effect. A measure in $\{+, -\}^{\mathbb{N}}$ can be seen as a Gibbs state [42]. In a model well known in statistical mechanics, a certain external parameter t is decreased until a certain transition value is attained, then there exists a spontaneous magnetization. As is very well known, this clearly corresponds to the appearance of a Dirac delta in the point $\{+, +, +, +, \dots\}$ (or $\{-, -, -, -, \dots\}$) of $\{+, -\}^{\mathbb{N}}$ (see also [22, 18]).

Note that $\{+, +, +, +, \dots\}$ is a fixed point for the shift. It is also well known that by means of Markov partitions, we can make a change of coordinates and obtain that the shift on $\{+, -\}^{\mathbb{N}}$ is dynamically equivalent with a one-dimensional map of degree 2 in the interval. This consideration shows that the sudden appearance of a Dirac delta in an extremal of a segment (as we had mentioned before, it will appear in our model for phase transition) can be seen as a spontaneous magnetization [22].

In [18] some of the above ideas are applied in an example where appears three equilibrium states in the transition value. The example presented in the present paper is analogous to the Ising Model case, and the example in [18] is analogous to the Potts Model.

Another situation in which we have a discontinuity of the Hausdorff dimension appears in [6]. In this work a one-parameter family from an Anosov system to a DA-system is presented and there is a discontinuity of the Hausdorff dimension of the nonwandering set. This example has a completely different nature from the considerations presented here.

This paper is divided in the following way: in Section 2 we describe results for rational maps in a topological point of view. In Section 3 the ergodic theory of rational maps is considered. After that, in Sections 4, 5, 6, and 7 we consider, respectively, the dimension spectrum, generalized dimension, phase transition, and generalized entropies. In the Appendix we prove a technical proposition.

We thank A. Politi and I. Kan for a conversation about the subject of one-dimensional dynamics.

The present paper was written before references [22, 18, 21]. In fact the results presented there were inspired by the results presented here. We refer the reader to [22, 21] for models of concrete physical problems using an appropriate modification of the ideas presented here.

2. Dynamics of Rational Maps

Consider $f(z)$ a rational map (that is, the quotient of two complex polynomials) in the Riemann sphere (the compactification of the complex plane with the point in ∞) of degree d. A complex polynomial is also a rational map.

DEFINITION 1. The Julia set of f is the closure of the set of expanding periodic points, that is, the closure of the set

$$\{z \in \mathbb{C} | \exists n \in \mathbb{N} \text{ such that } f^n(z) = z \text{ and } |(f^n)'(z)| > 1\}.$$

We will denote the Julia set by J.

We refer the reader to [1, 7, 9, 25, 29] for general results about the topological point of view of the theory of rational maps. The Julia set in most of the cases is a fractal [7].

DEFINITION 2. We say that a rational map f is hyperbolic if there exist $c > 0$ and $\lambda > 1$ such that for all z in the Julia set, and all $n \in \mathbb{N}$ $|(f^n)'(z)| \geq c\lambda^n$.

DEFINITION 3. We will say that f is subhyperbolic, if there exist a metric γ (perhaps with singularities) and $B > 1$ such that for any $x, y \in J$,

$$\gamma(f(x), f(y)) \geq B\gamma(x, y).$$

We will not explain more intricate details about such metric γ because it is not essential for us here, how such metric is obtained for maps satisfying conditions of Examples 2 and 4. We just use some very well known theorems that allow one to use techniques of expanding systems (rational hyperbolic maps) to subhyperbolic rational maps. These theorems are related to the existence of equilibrium states. We refer the reader to [15; 9, Expose No. V, Section 4, Proposition 3]. See also [25, p. 87] for mathematical details about such subhyperbolic metrics. It is easy to see that any hyperbolic rational map is subhyperbolic.

EXAMPLE 1. For $f(z) = z^d$, the Julia set is the unit circle. This map is hyperbolic with $c = 1$ and $\lambda = d$ in Definition 2. In this case the Julia set has dimension 1.

This example is quite special, in general the Julia set is not a smooth curve, as happen for instance for $f(z) = z^2 + a$, where a is a small complex number and $a \neq 0$, $a \neq -4$.

EXAMPLE 2. Suppose $f(z)$ is such that all critical points are not periodic, but are eventually periodic, that means, any critical point, after a finite number of iterations hits a periodic expanding point. In this case is known (Sullivan [45]) that the Julia set is the all Riemann sphere. It is also known in this case that f is not hyperbolic, but there exist a metric γ with singularities, such that f is subhyperbolic ([8, 9], also [25, p. 87]).

EXAMPLE 3. An specific example of the above situation is $f(z) = ((z - 2)/z)^2$ in the complex variable z. In this case, the critical points are 2 and ∞, and $f(2) = 0$, $f(0) = \infty$, $f(\infty) = 1$, and finally $f(1) = 1$. Note also that $f'(1) = -4$.

This example is known as Lattes example. It is obtained as a quotient map of a Weierstrass elliptic function.

In the case $f(z) = ((z-2)/z)^2$ we are considering the "complex multiplication" endomorphism $A\colon z \to \sqrt{2}\, iz$ of T_i^2 (see [25, pp. 86–89]).

Remark 1. Other possible rational maps satisfying the hypothesis of Example 2 can be obtained with this procedure above mentioned, using different "complex multiplications" (see [25, p. 89]).

DEFINITION 4. Given a periodic point p, of period n, we call $(f^n)'(p)$ the exponent of p.

All periodic points with exponent of modulus larger than one are in the Julia set. In the other way, all periodic points with exponent of modulus smaller than one, are not in the Julia set and there is just a finite number of such periodic points.

For more detailed information about the other possibilities, we refer the reader to [1, 25].

DEFINITION 5. Given a point $z \in \mathbb{C}$, the upper Liapunov number of z, is the number

$$\limsup_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|.$$

DEFINITION 6. In a similar way, given a point $z \in \mathbb{C}$, the lower Liapunov number of z is the number

$$\liminf_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|.$$

DEFINITION 7. Given a point $z \in \mathbb{C}$, if the upper and lower Liapunov numbers of z are equal, we will say that z has the Liapunov number

$$\lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|.$$

Remark 2. It follows easily from the considerations about $f(z) = ((z-2)/2)^2$ and its relation with the endomorphism $A\colon z \to \sqrt{2}\, iz$, that every point in $\mathbb{C}$, different from 1, has upper Liapunov number $\log\sqrt{2}$ (this fact will be proved in the Appendix). The Liapunov number of 1 is $\log 4$. Note that this last value is strictly larger than $\log\sqrt{2}$. This situation will be explored later in Section 6. The same phenomena also happens for other rational maps mentioned in Remark 1. Therefore, the possible "complex multiplication" and periods associated to the Weierstrass elliptic function will be important later on in this paper. Note the important point that f preserves a measure (the maximal measure) with a density with respect to

2-dimensional Lebesgue measure. If u denotes the maximal measure, then up to a finite number of points we have for $z \in \mathbb{C}$, $u(B(z, \xi)) \approx \xi^2$.

DEFINITION 8. The orbit of z, is the set $\{f^n(z) | n \in \mathbb{N}\}$.

In this way we can say the maps satisfying the hypothesis of Example 2 are the ones that have orbits of the critical point finite.

DEFINITION 9. A periodic point p is called an attractor if the exponent of p has modulus smaller than one.

Any attracting periodic orbit has a neighbourhood V such that all points z in V satisfy the fact that $\lim_{n \to \infty} f^n(z)$ converges to the orbit of p. This orbit has the cardinality of the period of p.

EXAMPLE 4. Suppose f is such that some trajectories of the critical points are attracted to an attracting periodic orbit and the other critical points are eventually periodic. In this case, the Julia set is an arbor (tree), and the rational map is subhyperbolic, but not hyperbolic (see Douady and Hubbard [9]). In this case the Julia set has two-dimensional Lebesgue measure zero.

EXAMPLE 5. Consider $f(z) = 1 - 2z^2$ in the complex plane. This is a specific example of a map satisfying the hypothesis of Example 4. In this case 0 and ∞ are the critical points, $f(\infty) = \infty$, $f(0) = 1$, $f(1) = -1$, and finally $f(-1) = -1$. It is well known that the Julia set of f is an arbor without branches, in fact, the segment $[-1, 1]$. If we consider the map $f(z)$ restricted to the interval $[-1, 1]$, then the change of coordinates $z = \sin \pi y/2$, conjugates $f(z)$ with $l(y) = 1 - 2|y|$ for $y \in [-1, 1]$ (see [34]). Observe the analogy of this situation with Example 3. In this case $l(y) = 1 - 2|y|$ plays the role of A: $z \to \sqrt{2}iz$ of Example 3.

Remark 3. In the same way as in Example 3, all Liapunov numbers of $z \in (-1, 1)$ are equal to $\log 2$, but the point -1 has Liapunov number $\log 4$ (see Appendix). Observe that we have the same property as Remark 1, namely, the periodic orbit that is hit by the image of the orbit of the critical point has Liapunov number strictly larger than any other possible Liapunov number for z in $(-1, 1)$.

EXAMPLE 6. The Tchebycheff polynomials seen as a complex map also satisfy the hypothesis of Example 4. In fact, in this case the Julia set is also $[-1, 1]$.

Remark 4. The maps of Examples 1, 2, 6 are the ones with parabolic orbifold [49, 8].

In the case that f is hyperbolic, the Julia set always has two-dimensional Lebesgue measure zero [1, 25, 29]. An important conjecture exists that claims that the set of hyperbolic rational maps is dense in the set of all possible rational maps (it is known that it is open). See [29] for important results related to the conjecture.

In Herman [15] one can find other kinds of rational maps, such that the Julia set is the all-Riemann sphere, but they are not included in the situation of Example 2.

Related to the above-mentioned conjecture, we mention the result of M. Rees, which claims that a set of positive Lebesgue measure. In the set of possible parameters of a rational map of degree d exists, such that the Julia set of f belonging to this set is the all sphere S^2. In this case, of course, the rational map is not hyperbolic. Most of these cases are not of the kind mentioned in Example 2, but the set of positive Lebesgue measure is obtained as the closure of a set of maps satisfying the hypothesis of Example 2 (see [39, 40]).

All the Examples 2 to 6 are degenerated cases, where not all the strong results that are known for hyperbolic rational maps can be proved.

3. Ergodic Theory of Rational Maps

Let $M(f)$ be the set of invariant probabilities for f, that is, the set of measures v such that $v(f^{-1}(A)) = v(A)$, for any Borel set A on $\mathbb{R}^2$ and also $v(\mathbb{R}^2) = 1$. The support of all these invariant measures is the Julia set J. We will not consider the measures concentrated in the finite number of attracting periodic orbits of f (they are not in J).

Definition 10. For a continuous function $g = J \to \mathbb{R}$ and $v \in M(f)$, we will define the pressure of v with respect to g by

$$h(v) + \int g(z)\,dv(z),$$

where $h(v)$ denotes the entropy of v (see [27, 47] for the definition of entropy and general results in ergodic theory). We will denote the pressure of v with respect to g by $P(v, g)$.

Definition 11. We will call $P(g) = \sup\{P(v, g) | v \in M(f)\}$ the topological pressure of the function g.

In the case that f is hyperbolic and g is Hölder-continuous, there always exists always a unique measure that attains such supremum. This measure is ergodic. These measures are sometimes called equilibrium state for g or a Gibbs measure in the context of thermodynamic formalism [3, 42, 43].

We refer the reader to [41] for results related to the Hausdorff dimension of the Julia set of hyperbolic rational maps and its relation with the pressure.

DEFINITION 12. In the case that there exists a probability in $M(f)$ that we will denote by $v(g)$, such that $P(g) = h(v(g)) + \int g(z)\, d(v(g))(z)$, we will call this measure a maximal pressure measure for $g\colon J \to \mathbb{R}$.

In the case that f is subhyperbolic and g is Hölder continuous, the same proof of existence of an equilibrium state for hyperbolic rational maps works, using the metric γ of Definition 3.

Here we will need to consider $g(z) = -t \log|f'(z)|$, where t in $\mathbb{R}$ is an external parameter that is considered fixed in the moment. In this case $g(z)$ is not Hölder-continuous if there exists a singularity ($f'(x) = 0$) in the Julia set, as happens in Examples 2 to 5.

The difficulties of this situation can be handle in the following way. First consider smooth approximations of $-t \log|g'(z)|$ around the critical point, then obtain the solution for the Hölder-continuous case, and finally make a limit of the measures obtained in each approximation. The weak limit will be a maximal pressure measure for $-t \log|f'(z)| = g(z)$. The metric γ is degenerated around the critical points, and in this way we obtain the existence of an equilibrium state, but we cannot assure uniqueness anymore.

Another different way to show the existence of equilibrium states for $-t\log|f'(z)|$ is to use the techniques of L. Mendoza in [33, Section 2]. The same proof presented there then works for the general subhyperbolic case. In the general case, the existence of equilibrium states for $-t \log|f'(z)| = g(z)$ with $t \in \mathbb{R}$ fixed are not known.

For $g(z)$ constant equal to zero, nevertheless, the situation is well understood in the case of the general rational map, as we now explain.

DEFINITION 13. For g constant equal to zero, a maximal pressure measure for g is called a maximal measure.

Let z_0 be a point in the Riemann sphere, and for each $n \in \mathbb{N}$, let us denote $z(n, i, z_0)$, $i \in \{1, 2, \ldots, d^n\}$ the d^n-solutions (with multiplicity) of the equation $f^n(z) = z_0$. Denote the Dirac delta measure on z by $\delta(z)$. Let $u(n, z_0)$ be the probability

$$d^{-n} \sum_{i=1}^{d^n} \delta(z(n, i, z_0)).$$

In [11, 24] it was shown that for any z_0 (but at most two exceptional points), and independent of z_0, the weak limit $\lim_{n\to\infty} u(n, z_0) = u$ exists and the measure u is the maximal measure of the rational map f.

Hyperbolicity is not assumed to obtain this result. We have also that u is ergodic and has entropy $\log d$. This u is the unique measure of maximal entropy.

DEFINITION 14. For any real $t \in \mathbb{R}$ we will denote $P(t) = P(g)$, when $g(z) = -t \log|f'(z)|$.

From [41, 42] it is known that $P(t)$ is convex and real analytic in the variable t, when f is hyperbolic. When f is subhyperbolic, $P(t)$ can have points where it is not differentiable as we will see later.

Following L. S. Young [48] we have the following definition:

DEFINITION 15. For a given probability v we will call the Hausdorff dimension of the measure v, denoted by $\mathrm{HD}(v)$, the value $\inf\{\mathrm{HD}(A) | v(A) = 1,\ A$ Borelean set in $J\}$. Here $\mathrm{HD}(A)$ denotes the Hausdorff dimension of the set A.

DEFINITION 16. For any real $t \in \mathbb{R}$ we will denote $u(t)$ the maximal pressure measure for $g(z) = -t \log|f'(z)|$, in the case $u(t)$ is unique.

It is proved in [28, 30–32] that if f is hyperbolic, then

$$P'(t) = -\int \log|f'(z)| d(u(t))(z) = -h(u(t)) \cdot (\mathrm{HD}(u(t)))^{-1}.$$

Using the metric γ (see Definition 3), one can obtain for subhyperbolic maps the same result as above in the case $P(t)$ is differentiable at t. The proof is absolutely equal to [31, 32].

In the same way as in [19, 23] one can obtain $P(t)$ as

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \int |(f^n)'(z)|^{-t}\, du(z) - \log d,$$

in the subhyperbolic case. The above integrals are finite from [28].

It is also known that for the general rational map and v an ergodic probability, there exist a Borelean set A such that $v(A) = 1$ and $\forall z \in A$,

$$\lim_{\xi \to 0} \frac{\log v(B(z, \xi))}{\log \xi} = \mathrm{HD}(v) = h(v)\left(\int \log|f'(z)|\, dv(z)\right)^{-1},$$

where $B(z, \xi)$ denotes the disk of center z and radius ξ [28].

Another way to obtain $\mathrm{HD}(v)$ is $\mathrm{HD}(v) = \lim_{\delta \to 1} \inf\{$capacity dimension of A, for all Borel sets A such that $v(A) \geq \delta\}$ (see [48]).

For $v \in M(f)$ we will call the value $\int \log|f'(z)|\, dv(z)$, the Liapunov number of the measure v, and we will denote such value by $\mathrm{LE}(v)$. In the case that $v = u(t)$, we will denote $\mathrm{LE}(u(t)) = \mathrm{LE}(t)$, $t \in \mathbb{R}$.

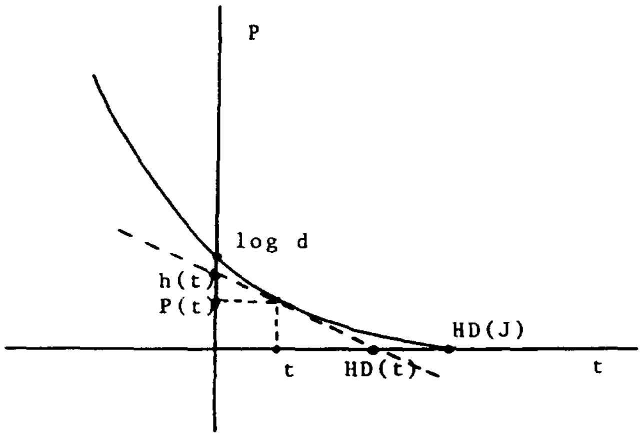

FIGURE 1

We will also denote HD(t) and $h(t)$, respectively, the Hausdorff dimension and the entropy of $u(t)$, $t \in \mathbb{R}$ (see McCluskey and Manning picture on Fig. 1).

Other results about the maximal measure can be obtained in [20, 25, 26, 30, 37, 44, 49]; among them, we point out the result of H. Brolin that shows that the maximal measure for a polynomial map on $\mathbb{C}$ is the charge distribution in the Julia set [4]. This happens, for example, for the polynomial $f(z) = 1 - 2z^2$ (this will be mentioned in Remark 11).

For rational maps that are not polynomials, the charge distribution is always different from the maximal measure [20].

4. THE DIMENSION SPECTRUM FOR THE MAXIMAL MEASURE

In Sections 4 and 5 we consider the dimension spectrum and generalized dimension of the maximal measure u.

DEFINITION 17. Let $J(\alpha)$ be the set of points $z \in J$, such that the limit exists

$$\lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|^{-\alpha} = -\log d.$$

Using the strong results presented in [28] for the general rational map, we can also conclude as in [19] that, if $z \in J(\alpha)$ then

$$\lim_{\xi\to 0} \frac{\log u(B(z,\xi))}{\log \xi} = \alpha.$$

DEFINITION 18. Let $\not{f}(\alpha)$ be the Hausdorff dimension of the set $J(\alpha)$.

The following result was obtained in [19], and later we will consider a slight generalization of it.

THEOREM 1. *Suppose f is a hyperbolic rational map and u is the maximal measure. Then for a given $\alpha \in \mathbb{R}$, there exists a unique $t \in \mathbb{R}$ such that $P'(t) = -\log d/\alpha$ and $\not{f}(\alpha) = \mathrm{HD}(u(t))$, where $u(t)$ is the maximal pressure measure for $-t \log|f'(z)|$. The function $\not{f}$ is real analytic on the variable α.*

Now we will present a generalization of the above result:

THEOREM 2. *Suppose f is a subhyperbolic rational map and u the maximal measure. Suppose for a given α, there exists a $t \in \mathbb{R}$, such that P is differentiable at t and*

$$P'(t) = -\frac{\log d}{\alpha},$$

then $f(\alpha) = HD(u(t))$, where $u(t)$ is a maximal pressure measure for $-t \log|f'(z)|$.

Proof. Under the above conditions the existence of $u(t)$ follows from the same proof of [19], considering the metric γ (mentioned in Definition 3), instead of the usual metric in S^2.

Remark 5. Differentiability of pressure, uniqueness, and existence of equilibrium states are closely related as can be seen in [23, 22].

Remark 6. The important consequence of Theorems 2 and 3 is that you can consider the support of $u(t)$ instead of the set $J(\alpha)$, if you are interested just in questions related with the Hausdorff dimension. More than this fact, the $u(t)$ will be important because they will be the equilibrium states that will change continuously with t until a transition value of the parameter is attained, where there will be a sudden jump for a new equilibrium state. This situation does not happen for hyperbolic systems but can happen for subhyperbolic systems, as we will see later. The transition point will be the value of parameter t where we do not have differentiability of P.

After these two remarks we will return to analyze the $P(t)$ function of a subhyperbolic rational map satisfying the hypothesis of Examples 2, 3, 5, and 6. In this case it was recently proved by A. Zdunik [49] that for such rational maps the pressure $P(t)$ is linear in the interval $[0, \mathrm{HD}(J)]$. For the maps of Examples 2 and 3 the pressure is given by $P(t) = \log d -$

$t(\log d/2)$ for $t \in [0, \mathrm{HD}(J)] = [0, 2]$. Therefore, $P(0) = \log d$ and $P(2) = 0$. For the map of Example 5, $P(0) = \log 2$ and $P(1) = 0$, and therefore $P(t) = \log 2 - t \log 2$ for $t \in [0, 1]$. This result of A. Zdunik follows a previous work of Przyticki, Urbanski, and Zdunik [37], where the asymptotic variance of the maximal measure is considered. The asymptotic variance, by the way, is the second derivative of the pressure in the origin. The linearity of $P(t)$ in $[0, \mathrm{HD}(J)]$ is related to the fact that in the above-mentioned cases this second derivative is zero.

We will show here that due to the fact that the Liapunov number of the expanding fixed point of Examples 3 and 5 is strictly larger than the other possible upper-Liapunov number of points of z in the Julia set, there exists a phase transition at the level of the equilibrium states. Furthermore, $P(t)$ will not be linear, but it is linear by parts in the interval $(-\infty, \mathrm{HD}(J)]$.

For hyperbolic rational maps, $f(z) = z^d$ is the only one such that $u(t) = u$ for all t in R (see [48]).

5. Generalized Dimensions for the Maximal Measure

For a given $t \in \mathbb{R}$ consider q such that $q = P(t)/\log d$. We are using the notation of the end of Section 3. Consider also α such that $P'(t) = -\log d/\alpha$. These three variables are uniquely related in the case P is differentiable on t.

For each q consider $\tilde{J}(\alpha)$ contained in $J(\alpha)$ (see Definition 17), a support of $u(t)$ such that $\mathrm{HD}(\tilde{J}(\alpha)) = f(\alpha) = \mathrm{HD}(t) = \mathrm{HD}(u(t))$. Fix a certain $q \in \mathbb{R}$ (and therefore t and α).

For each $\xi > 0$, $\delta > 0$, and $A \subset \tilde{J}(\alpha)$, such that $u(t)\ (A) \geq 1 - \delta$, consider the minimal number of balls of radius ξ necessary to cover A.

Denote $p_i(\xi)$ the u-mass of a generic element of this minimal cover associated with A and ξ. Now $\Sigma(p_i(\xi))^q$ will denote the sum of all $p_i(\xi)^q$, where i cover all the indices of the above-mentioned partition.

Now in away analogous to that in [48, p. 119], we define

$$\underline{C}(q, A, \delta) = \liminf_{\xi \to 0} \frac{\log \sum p_i(\xi)^q}{\log \xi},$$

$$\overline{C}(q, A, \delta) = \limsup_{\xi \to 0} \frac{\log \sum p_i(\xi)^q}{\log \xi},$$

$$\underline{C}(q, u) = \sup_{\delta \to 0} \inf_{\substack{A \subset J(\alpha) \\ u(t)(A) > 1 - \delta}} \underline{C}(q, A, \delta),$$

and

$$\overline{C}(q,u) = \sup_{\delta\to 0} \inf_{\substack{A\subset J(\alpha) \\ u(t)(A)>a-\delta}} \overline{C}(q,A,\delta).$$

From [28] the hypotheses of Theorem 4.4 in [48] are satisfied; therefore, using the conclusion of this theorem we have that $\underline{C}(q,u) = \overline{C}(q,u)$. We will denote such expressions as $\lim_{\xi\to 0}(\log \Sigma(p_i(\xi))^q/\log \xi)$ to use the standard notation in the dimension spectrum theory.

PROPOSITION 1. *There exists the limit*

$$\lim_{\xi\to 0} \frac{\log \sum (p_i(\xi))^q}{\log \xi} = \alpha q - \mathrm{HD}(t).$$

Proof. Using Theorem 4.4 of [48] we know that the cardinality of a partition of a minimal ξ-cover of a set of large $u(t)$ measure is like $\xi^{-\mathrm{HD}(t)}$.

This could be also obtained using techniques presented in [35, 36]. Now using Theorems 2 and 3, we know that $u(B(z,\xi))^q$ is of order $\xi^{\alpha q}$, because we are assuming $z \in J(\alpha)$. From this we conclude that

$$\lim_{\xi\to 0} \frac{\log \sum (p_i(\xi))^q}{\log \xi} = -\mathrm{HD}(t) + \alpha q.$$

DEFINITION 19. For $q \in \mathbb{R}$, define

$$\tau(q) = \lim_{\xi\to 0} \frac{\log \sum (p_i(\xi))^q}{\log \xi} = \alpha q - \mathrm{HD}(t),$$

where t satisfies the equation $q = P(t)/\log d$ and $p'(t) = -\log d/\alpha$.

We point out that in general (for hyperbolic rational maps), the above definition create several difficulties to allow one to estimate the real value of $\tau(q)$. Nevertheless, we believe this is the only formalization of the concept of generalized dimension that makes sense in the general situation, and corresponds to what one should expect to happen $(\Sigma p_i(\xi)^q \approx \xi^{\tau(q)})$.

The remarkable fact, however, is that the definition is computable in some non-hyperbolic cases. This will be explained later and is related to the fact that in the computable examples the Liapunov numbers are basically constant for most of the points in the Julia set. For hyperbolic rational maps in general there are fluctuations of the Liapunov numbers, in fact z^d is the only counterexample.

Remark 7. Note that if we define $V(t) = P(-t)/\log d$, then we have $V(\tau(q)) = q$. Therefore the lack of C^1-differentiability for $P(t)$ will always associated with the same property for $\tau(q)$.

Remark 8. The measure $u(t)$ is also ergodic for $t \in \mathbb{R}$, therefore $u(t)$ and u are equal or have disjoint support. Anyway, note that if one consider a point in the support of $u(t)$, the balls in $\mathbb{R}^2$ with center on this point and positive radius have positive u-measure. Some authors in the physics literature consider the points where you take the center of balls in $J(\alpha)$ chosen at random. We point out that if one wants to consider this approach, then one have to consider these centers of balls chosen at random with respect to the probability $u(t)$ and not u.

Note that for $q = 1$, we have $P(t)/\log d = q = 1$, and this equation is satisfied for $t = 0$. In this case $\mathrm{HD}(t) = \mathrm{HD}(0) = \mathrm{HD}(u)$ and also $\tau(1) = \mathrm{HD}(u) - \mathrm{HD}(0)$ because $-\mathrm{LE}(t) = P'(0) = -\log d/\alpha = -h(u)/\alpha$ [11, 24], and therefore $\alpha = h(u)/\mathrm{LE}(0) = \mathrm{HD}(u)$. Therefore $\tau(1) = 0$.

Let us compute the derivative of τ in 1, $\lim_{q \to 1} \tau(q)/(q-1) = \tau'(1)$. As $P(-\tau(q)) = q \log d$ by Remark 7, we have

$$\tau'(1) = -\frac{\log d}{P'(\tau(1))} = -\frac{\log d}{P'(0)} = \frac{\log d}{\mathrm{LE}(0)} = \mathrm{HD}(u).$$

This result is in accordance with [34, p. 689]. In our case the information dimension is the Hausdorff dimension of the maximal measure u.

Given $m(x)$ and $n(y)$, we will say that $m(x)$ is the anti-Legendre transform of $n(y)$, if for any $x \in \mathbb{R}$,

$$m(x) = -\inf_{y \in \mathbb{R}} \{n(y) - xy\}.$$

PROPOSITION 2. *The anti-Legendre transform of* $\tau(q)$ *is* $f(\alpha)$, $q \in \mathbb{R}$, $\alpha \in \mathbb{R}$.

Proof. From Theorems 2 and 3, $f(\alpha) = \mathrm{HD}(t)$, where $P'(t) = -\log d \cdot \alpha^{-1}$. Consider $B = \alpha_0 \in \mathbb{R}$ fixed, then $\inf_{q \in \mathbb{R}}\{\tau(q) - qB\}$ is attained when $\tau'(\tilde{q}_0) = B$. As $(d/dq)P(-\tau(q)) = \log d$ (Remark 7), then $-P'(-\tau(\tilde{q}_0))\tau'(\tilde{q}_0) = \log d$. Therefore,

$$P'(-\tau(\tilde{q}_0)) = -\frac{\log d}{B} = -\frac{\log d}{\alpha_0}.$$

How can we discover which $\tilde{q}_0$ satisfies this equation? The answer is $\tilde{q}_0 = q_0$, the one associated with α_0. The reason is that $P(-\tau(q_0)) = q_0 \log d$ and $P(t_0) = q_0 \log d$, therefore $\tau(q_0) = -t_0$ (remember the relation satisfied by q_0, t_0, and α_0). In fact, in this case, $P'(t_0) = -\log d/B$ and therefore we also have $\tau(q_0) = \tau(\tilde{q}_0)$.

Therefore, we have for

$$q_0\alpha_0 - \tau(q_0) = q_0\alpha_0 - q_0\alpha_0 + \mathrm{HD}(t_0) = \mathrm{HD}(t_0) = f(\alpha_0).$$

Therefore we conclude that the anti-Legendre transform of $\tau(q)$ is $\not{f}(\alpha)$.

For an instantaneous geometrical proof of Proposition 2 rotate Fig. 1 an angle of $\pi/2$.

Now we will show that $\tau(0) = -\mathrm{HD}(J)$. From the above Proposition 2, and general arguments of the type: if $\not{f}$ is the Legendre transform of τ, then τ is the Legendre transform of $\not{f}$, we conclude

$$\tau(0) = \inf_{\alpha \in \mathbb{R}} \{\alpha \cdot 0 - \not{f}(\alpha)\} = \inf_{\alpha \in \mathbb{R}} \{-\not{f}(\alpha)\}.$$

The point where $\not{f}(\alpha)$ is maximum is when $t = \mathrm{HD}(J)$, because in this case $\mathrm{HD}(u(t)) = \mathrm{HD}(J)$ (see [41]). Therefore, $\not{f}(\alpha) = \mathrm{HD}(J)$. Therefore, we conclude that $\tau(0) = -\mathrm{HD}(J)$.

Now following the usual terminology, let us define:

DEFINITION 20. The generalized dimension $\mathrm{D}(q)$ for $q \in \mathbb{R} - \{1\}$ is by definition $D(q) = \tau(q)/(q-1)$.

In the case $q = 1$, we define $\mathrm{D}(1) = \mathrm{HD}(u)$, and from $\tau'(1) = \mathrm{HD}(u)$ we conclude $\mathrm{D}(q)$ is continuous. Note also that $\mathrm{D}(0) = \mathrm{HD}(J)$.

6. Phase Transition for the Maximal Measure in Some Non-Hyperbolic Cases

The below result will not be essentially used here, but for sake of completeness we will prove this result in the appendix.

PROPOSITION 3. *Suppose f is a rational map such that* $\log|f'(z)|$ *is bounded above in the Julia set, then there exist an ergodic measure $v \in M(f)$ such that the Liapunov number of this measure v is larger than the upper-Liapunov number of any point z in J.*

The proof of Proposition 3 will be done in the Appendix. Note that we allowed the existence of singularities in the Julia set.

DEFINITION 21. We will call a measure $v \in M(f)$ a measure of maximal Liapunov number, if its Liapunov number is larger or equal to the Liapunov number of any other invariant measure. We will denote such measure by p.

EXAMPLE 7. For the map $f(z) = ((z - 2)/z)^2$, the measure of maximal Liapunov number is the measure concentrated in the point 1 (see Appendix).

EXAMPLE 8. For the map $f(z) = 1 - 2z^2$, the measure of maximal Liapunov number is the measure concentrated in the point -1 (see Appendix).

DEFINITION 22. We will say a rational map has a gap if the Liapunov number of the maximal Liapunov number measure is strictly larger than the Liapunov number of any other invariant measure, that is,

$$\mathrm{LE}(p) > \sup_{v \in M(f)-p} \{\mathrm{LE}(v)\}.$$

Therefore, we can say that $f(z) = ((z - 2)/z)^2$ and $f(z) = 1 - 2z^2$ have a gap (see Appendix). Several different examples satisfying the hypothesis of Remark 1 can have a gap.

PROPOSITION 4. *Suppose f is a rational map that has a gap, then there exist a value $t_0 < 0$ such that for $t < t_0$. $P(t)$ is linear. In fact, $P(t) = h(p) - t\,\mathrm{LE}(p)$.*

Proof. As f has a gap, there exist $A > 0$ such that $\mathrm{LE}(p) > \mathrm{LE}(v) + A$, for any $v \in M(f)$, different from p.

Consider t_0 negative, small enough such that $t_0 A + \log 2 < 0$. Therefore, $h(p) - t\,\mathrm{LE}(p) \geq h(p) - t(\mathrm{LE}(v) + A) \geq h(v) - \log 2 - t\,\mathrm{LE}(v) - tA \geq h(v) - t\,\mathrm{LE}(v) - (tA + \log 2) \geq h(v) - t\,\mathrm{LE}(v)$ for any $v \in M(f)$ and $t < t_0$. From this we conclude that p is the equilibrium state for $P(t)$ when $t < t_0$. The value of $P(t)$ is $P(t) = h(p) - t\,\mathrm{LE}(p)$.

Remark 9. In the case $f(z) = ((z - 2)/z)^2$, we have $h(p) = 0$, $\mathrm{LE}(p) = \log 4$, and therefore $P(t) = -t \log 4$ for t negative enough.

Remark 10. In the case $f(z) = 1 - 2z^2$, we also have $h(p) = 0$, $\mathrm{LE}(p) = \log 4$ and therefore $P(t) = -t \log 4$.

DEFINITION 22. We will say that a rational map f presents phase transition of first-order (or at Level-1) if $P(t)$ is not differentiable.

Note that $P(t)$ can be in some cases differentiable but not infinitely differentiable (see [21]). In this case we should also say we have first-order transition.

DEFINITION 23. We will say that a rational map f presents phase transition at Level-2 if, when we decrease t we continuously follow a unique equilibrium state, then for some transition parameter t_0 appears another equilibrium state. Therefore, we have two different equilibrium states for t_0. After this value, for $t < t_0$, we follow continuously the new equilibrium state that just appear in the transition parameter t_0.

Note from Remark 7 that phase transition of first order is associated also to nondifferentiability of $\tau(q)$ and the generalized dimension $D(q)$.

In [34], it was shown for the first time (as far as we know), the nondifferentiability of $\tau(q)$ for $f(z) = 1 - 2z^2$. Following the definitions given here, and using Remark 7, we say in this case there phase transition exists at Level-1.

From Theorem 1 we have the nonexistence of phase transition at Level-1 and Level-2, for hyperbolic rational maps.

THEOREM 3. *There exist a countable family of nonhyperbolic rational maps with phase transition at Level-*1 *and Level-*2.

Proof. Consider first $f(z) = ((z - 2)/z)^2$, the map of Example 3, then it follows from results of A. Zdunik [49], that the pressure function $P(t)$ is linear and of the form $P(t) = \log 2 - t \log 2/2$, for $t \in [0, \mathrm{HD}(J)] = [0, 2]$.

It is also proved in [49] that the maximal measure is equivalent to the two-dimensional Lebesgue measure. In fact, the maximal measure has a density with singularities in the points 0, ∞, and 1.

As the maximal Liapunov number measure p is concentrated in 1 (see Appendix) we have seen in Remark 9 and Proposition 4, for t negative small enough, we have $P(t) = -t \log 4$. Therefore, we have a phase transition at first level. In the point $t_0 = -\frac{2}{3}$, we have the only point where $P(t)$ is not differentiable.

For $t > t_0$, $P(t)$ has the maximal measure as unique equilibrium state ($u(t)$ has Hausdorff dimension 2 by the McCluskey–Manning picture, and the maximal measure, is the only one with this Hausdorff dimension).

For $t = t_0 = -\frac{2}{3}$, the transition parameter, we have a bifurcation and we have two equilibrium states, the maximal measure and the measure p concentrated in 1. For $t < t_0 = -\frac{2}{3}$, we have p as the unique equilibrium state for $P(t)$ (see Proposition 4).

Therefore, we have just shown the existence of a transition at second level for the rational map $f(z) = ((z - 2)/z)^2$.

Several different examples can be consider with different "complex multiplication" as it was discussed in Example 3 and Remark 1. The result of A. Zdunik about the linearity of $P(t)$ in $t \in [0, \mathrm{HD}(J)]$ is for all the all class of maps satisfying the hypothesis of Examples 2, 3 and 5. In this way, we have a countable family of examples with the same behaviour as $f(z) = ((z - 2)/z)^2$.

Another example where the same situation occurs is with the map $f(z) = 1 - 2z^2$. For the parameter $t = -1$ (in the setting of the pressure $P(t)$) the maximal measure jumps to the measure concentrated in the point -1, when we decrease the parameter t.

There is another setting where we can also show the same result about the phase transition for $f(z) = ((z - 2)/z)^2$ at first level. We just have to consider the function $\tau(q)$ instead of $P(t)$ and use Remark 7. From the considerations of Example 3 (see also [49]), it is known that $u(B(2, \xi)) \approx \xi^2$. Now as $u(f(B(2, \xi))) \approx u(B(f(2), \xi^2))$, we have that, in the point $0 = f(2)$,

$$u(B(0, \xi)) \approx \xi.$$

This follows from the invariance of u and the fact that 2 is a critical point.

As $f(0) = \infty$, we have in the same way $u(B(\infty, \xi)) \approx \xi$, and as $f(\infty) = 1$ and ∞ is critical, we also have $u(B(1, \xi)) \approx \xi^{1/2}$.

In order to compute $\tau(q)$ we have to consider $u(t)$, where t is related in the usual way to q. As it happens that $u(t) = u$, for $t > -\frac{2}{3}$ and $u(t) = \delta(1)$ for $t < -\frac{2}{3}$, then (see end of Remark 2)

$$\sum (p_i(\xi))^q \approx \xi^{-2+2q} \text{ or } \xi^{q/2}.$$

Now considering the lower envelope of the linear maps

$$-2 + 2q \qquad \text{and} \qquad \frac{1}{2}q,$$

we observe linear by part maps with a lack of differentiability in just the point $\frac{4}{3}$. In this point $\tau(q) = \frac{2}{3}$. This corresponds by Remark 7 to the same result obtained before, for $P(t)$, in which the critical value of the parameter was $-\frac{2}{3}$.

This way to look at the problem by means of generalized dimension arguments is analogous to the one used in [34] for the map $f(z) = 1 - 2z^2$.

This is the end of the proof of Theorem 3.

It would be interesting to investigate if there exist examples of a pressure function on $P(t)$, linear for t negative enough, and $P(t)$ nonlinear for $t > 0$. We wonder if it is possible to obtain such examples with Level-2 phase-transition. Denote by p and $u(t_0)$ the two possible equilibrium states in the critical parameter t_0; then we imagine that the situation on this case would be described by Fig. 2.

Note that indeed an example very close to the abovementioned situation can occur (see [21, 22]).

In Fig. 3 we show the graph of $P(t)$ for $f(z) = ((z - 2)/z)^2$, as was mentioned before in Theorem 3.

Remark 11. In the case $f(z) = 1 - 2z^2$, the maximal measure is the charge distribution on $[-1, 1]$, and for the critical parameter $t_0 = -1$, there is a phase-transition, because there is a jump from this charge distribution to the unitary charge in the point -1. Here we are using the results mentioned in the end of Section 3. As we said before in Section 1, this fact can be seen as a model for a sudden magnetization of a ferromagnetic system (see also [18, 21, 22]).

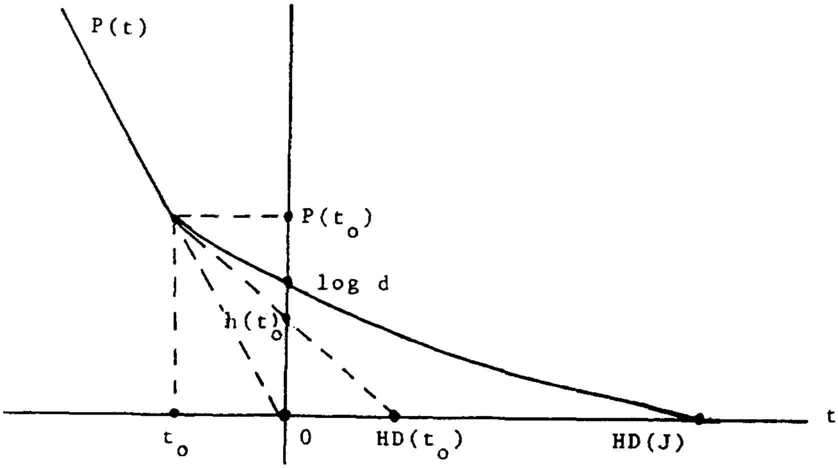

FIGURE 2

7. THE GENERALIZED ENTROPY

In order to complete the description of the main concepts used in dimension spectrum and phase transition, we introduce a formal definition of generalized entropy that should represent the usual one.

Given an invariant measure $v \in M(f)$, consider for each $z \in J$,

$$\lim_{\xi \to 0} \frac{v(B(z,\xi))}{v(f(B(z,\xi)))}.$$

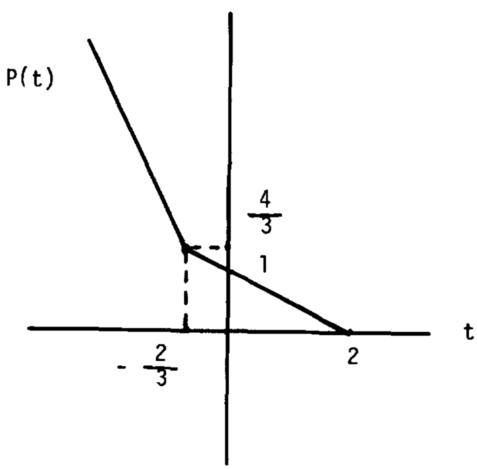

FIGURE 3

The above limit exists υ-almost everywhere by the Radon–Nikodyn theorem (see [28, 18, 23]).

DEFINITION 24. Denote

$$J(z) = \lim_{\xi \to 0} \frac{\upsilon(B(z,\xi))}{\upsilon(f(B(z,\xi)))},$$

for z υ-almost everywhere, the Jacobian of the measure υ in the point z.

It is well known (see [28]) that $h(\upsilon) = -\int \log J(z)\, d\upsilon(z)$. Now we will use some concepts of large deviation theory [10, 23]. For each $x \in \mathbb{R}$ consider the free energy for the random variable $-\log J(z)$ and the measure $\upsilon \in M(f)$: for $u \in \mathbb{R}$,

$$\begin{aligned} c(x) &= \lim_{n\to\infty} \frac{1}{n} \log \int e^{x(-\Sigma_{j=0}^{n-1} \log J(f^j(z)))} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \int \left(\prod_{j=0}^{n-1} J(f^j(z)) \right)^x d\upsilon(z). \end{aligned}$$

From [23, 10], if c is differentiable on 0, we have $c'(0) = -\int \log J(z)\, d\upsilon(z) = h(\upsilon)$.

It is possible to show that if $J(z)$ is Hölder-continuous, then c is differentiable on 0 [26, 18].

DEFINITION 25. Define for $q \in \mathbb{R} - \{0\}$, $E(q)$ the generalized entropy as $E(q) = c(q-1)/q-1$.

If we consider $x = q - 1$, then $x \to 0$, if and only if $q \to 1$. Therefore, we can extend the above definition for $q = 1$ as $E(1) = h(\upsilon)$.

In [21, 22] related results concerning the spectrum of dimension will appear.

In the case υ is the maximal measure, it is easy to see that $J(z) = d^{-1}$ for any $z \in J$, and therefore $E(q)$ is a constant equal to $\log d$ for all possible $q \in \mathbb{R}$.

APPENDIX

PROPOSITION 3. *Suppose f is a rational map such that $\log|f'(z)|$ is bounded above in the Julia set, then there exist an ergodic measure $\upsilon \in M(f)$ such that the Liapunov number of this υ is larger than the upper-Liapunov number of any point z in J.*

This proof is based in the following general lemma that appears in [46, Lemma II.3.5]. We will sketch the proof.

Lemma. Assume that $\{f_n\}$, $n \geq 0$ is a subadditive sequence of upper-semicontinuous real valued bounded maps, defined on a compact metric space A. Then there exist an ergodic f-invariant measure $\upsilon \in M(f)$ such that, for υ-almost every point x,

$$\lim_{n\to\infty} (1/n) f_n(x) = \lim_{n\to\infty} (1/n) \sup_{z\in A} f_n(z).$$

Proof of the lemma. We choose for each $n \in N$, x_n such that $f_n(x_n) = \sup_{z\in A} f_n(z)$, and define a probability measure

$$u_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta\big(f^i(x_n)\big).$$

Consider υ a weak limit of (u_n). Since (f_n) is subadditive, $f_n \leq f_1 + f_1 \circ f + \cdots + f_1 \circ f^{n-1}$, then $(1/n) f_n(x_n) \leq u_n(f_1)$.

If υ is a weak-limit of u_n, then as n goes to ∞, $\upsilon(f_1) \geq \lim_{n\to\infty}(1/n)\sup_{z\in A} f_n(z)$. Now

$$f_{2n} \leq f_2 + f_2 \circ f^2 + \cdots + f_2 \circ f^{2n-2},$$
$$f_{2n} \leq f_1 + f_{2n-1} \circ f,$$
$$f_{2n} \leq f_1 + f_2 \circ f + f_2 \circ f^3 + \cdots + f_2 \circ f^{2n-3} + f_1 \circ f^{2n-1}.$$

If we add the above inequalities, we have

$$2 f_{2n} \leq f_1 + f_2 + f_2 \circ f + f_2 \circ f^2 + \cdots + f_2 \circ f^{2n-3} + f_1 \circ f^{2n-1};$$

then as n goes to ∞, we have

$$\frac{1}{2}\upsilon(f_2) \geq \lim_{n\to\infty} \frac{1}{n} \sup_{z\in A} f_n(z).$$

By induction we can also have

$$\frac{1}{n}\upsilon(f_n) \geq \lim_{n\to\infty} \frac{1}{n} \sup_{z\in A} f_n(z).$$

This is the end of the proof of the lemma.

Proof of Proposition 3. Consider $f_n(z) = \log|(f^n)'(z)|$ and apply the lemma. Now we assume the proposition is proved.

Now we will show the following proposition.

PROPOSITION 5. *The rational maps* $f(z) = ((z-2)/z)^2$ *and* $f(z) = 1 - 2z^2$ *have gaps.*

Proof. Let us first consider $f(z) = 1 - 2z^2$. Remember that $f(0) = -1$, $f(-1) = -1$, $f(-1) = -1$, $|f'(1)| = 4$, and 0 is the critical point. The points z such that $\exists n \in \mathbb{N}$, $f^n(z) = 0$ (the critical point) have Liapunov numbers equal to $-\infty$ and therefore are smaller than $\log 4$.

Suppose z is such that for all $n \in \mathbb{N}$, $f^n(z) \neq 0$.

Consider the change of coordinates $z = h(y) = \sin(y\pi/2)$. The map f in the new coordinates y is given by $l(y) = 1 - 2|y|$. We have also $f = h \circ l \circ h^{-1}$.

Note that the Liapunov number of any such possible y is $\log 2$ for the map l (here we are using $f^n(z) \neq 0$, $\forall n \in \mathbb{N}$ and $f = h \circ l \circ h^{-1}$).

The derivative of h is $h'(z) = (\pi/2)\cos(y\pi/2)$ and is a bounded function with singularities $h'(-1) = h'(1) = 0$.

The equation for the nth iterate of f is $f^n = h \circ l^n \circ h^{-1}$. Therefore,

$$f^{n\prime}(z) = \big(h'((l^n \circ h^{-1})(z)) \cdot \big(l^{n\prime}(h^{-1}(z)) \cdot ((h^{-1})'(z))\big)\big).$$

As $h'(y) \leq \pi/2$, $y \in [-1,1]$, $|l^{n\prime}(y)| = 2^n$, $y \in [-1,1]$, and $(h^{-1})'(z)$ is fixed, then we conclude that for any periodic orbit z,

$$\lim_{n\to\infty} \frac{1}{n} \log|f^{n\prime}(z)| \leq \log 2 < \log 4 = \log|f'(1)|.$$

It is possible to show that any invariant probability can be approximated by probabilities that are convex combinations of probabilities that are sums of Dirac deltas (with equal mass) in the orbit of periodic points [27]. From this fact Proposition 5 easily follows. Therefore, $f(z) = 1 - 2z^2$ has a gap.

Consider now $f(z) = ((z-2)/z)^2$.

Again consider $z \in J$ such that there is no $n \in \mathbb{N}$ such that $f^n(z)$ is a critical point. Remember that 2 and ∞ are the critical points of f and $f(2) = 0$, $f(0) = \infty$, $f(\infty) = 1$, and $f(1) = 1$. We also have $|f'(1)| = 4$.

There is an identification function h given when we consider the quotient of the complex multiplication

$$A: z \to \sqrt{2}\, zi.$$

We will denote such identification by h and it will play the same role as the change of coordinates in the case $f(z) = 1 - 2z^2$ above.

The problem here is that $h'(z)$ can become close to ∞ in the points $h(2)$, $h(0)$, $h(\infty)$, and $h(r)$. Therefore, when we consider

$$f^{n\prime}(z) = \big(h'(l \circ h^{-1})(z)\big(l^{n\prime}(h^{-1})\big)(h^{-1\prime})(z)\big), \tag{$*$}$$

we could have, in principle, problems in controlling the modulus $|f^{n\prime}(z)|$.

We could give a proof in the same lines of the case $1 - 2z^2$, but we will give a proof with a different reasoning.

In order to avoid problems with the point ∞, consider a linear change of coordinates such that f in the new coordinates has the finite points a_2, a_0, a_∞, and a_1, corresponding to 2, 0, ∞, and 1. We will also denote the new function by f.

Consider V a neighbourhood that is the union of balls of the center points a_2, a_0, a_∞, and a_1 and radius ξ.

As h' is bounded outside V, then from (*) we have that, in order to have $(1/n)\log|f^{n\prime}(z)|$ larger than $\log\sqrt{2}$, we have to spend a long time of the orbit of z in the neighbourhood V. As $f(a_2) = a_0$, $f(a_0) = a_\infty$, $f(a_\infty) = a_1$ and $f(a_1) = a_1$, this means to spend a large part of the orbit of z in $B(a_1, \xi)$.

Let us analyze the string that corresponds to a large period of time in $B(a_1, \xi)$. We will show that this is not enough to increase the Liapunov number, because in this situation the orbit has to be very close to the critical point and this has a high order contribution in decreasing the Liapunov number of the orbit. Now we will formalize the above considerations.

Consider the string given by $k, l \in \mathbb{N}$, $0 < k < l < n$, such that.

$$f^{k-1}(z) \notin B(a_1, \xi)$$
$$f^j(z) \in B(a_1, \xi), \qquad j \in \{k, k+1, \ldots, l\}$$

and, finally,

$$f^{l+1}(z) \notin B(a_1, \xi).$$

Denote by $m = l - k$ and $\delta = |f^k(z) - a_1|$. As the derivative $|f'(a_1)| = 4$, then $\delta 4^m < \xi$. As $f'(a_\infty) = 0$, we have then $|f'(f^{k-1}(z))| \approx 2|f^{k-1}(z) - a_\infty|$. Now $|f^{k-1}(z) - a_\infty| \approx |f^k(z) - a_1|^{1/2} = \delta^{1/2}$, and finally

$$|f'(f^{k-1}(z))| \sim 2\delta^{1/2} \sim 2\xi 4^{-1/2m}.$$

Therefore, the value of the string

$|(f^{m+1})'(f^{k-1}(z))| = |f'(f^{k-1}(z))| 1 f^{m\prime}(f^k(z))|$ is of the order $4^{-1/2m} \circ 4^m$.

In this way

$$\frac{1}{m}\log|f^{m+1\prime}(f^{k-1}(z))| \approx \log 4 - \frac{1}{2}\log 4.$$

Therefore, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log|f^{n-1}(z)| \leq \log 4^{1/2} = \log 2.$$

The conclusion is that f has a gap.

An analogous proof for the case $1 - 2z^2$ can be also obtained with simple modifications of the above argument.

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