Eigenfunctions of the Laplacian and eigenfunctions of the associated Ruelle operator

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Abstract

Let Γ be a co-compact Fuchsian group of isometries on the Poincaré disk $\mathbb{D}$ and $\Delta$ be the corresponding hyperbolic Laplacian. Any smooth eigenfunction $f$ of $\Delta$ equivariant by $\Gamma$ with real eigenvalue $\lambda = s(1 - s)$, $s = \frac{1}{2} + it$, admits an integral representation by a distribution $\mathcal{D}_{f,s}$, called Helgason’s distribution, equivariant by $\Gamma$ and supported at infinity $\partial \mathbb{D} = S^1$. The geodesic flow on the compact surface $\mathbb{D}/\Gamma$ is conjugate to a suspension over a natural extension of a piecewise analytic map $T : S^1 \to S^1$, called Bowen-Series transformation. Let $\mathcal{L}_s$ be the complex Ruelle transfer operator associated to the jacobian $-s \ln |T'|$. M. Pollicott showed that $\mathcal{D}_{f,s}$ is an eigenfunction of the dual operator $\mathcal{L}_s^*$ for the eigenvalue 1. We show the existence of a (non zero) piecewise real analytic function $\psi_{f,s}$, eigenfunction of $\mathcal{L}_s$ for the eigenvalue 1, given by an integral formula

$$\psi_{f,s}(\xi) = \int \frac{J(\xi, \eta)}{|\xi - \eta|^{2s}} \mathcal{D}_{f,s}(d\eta)$$

where $J(\xi, \eta)$ is a $\{0, 1\}$-value piecewise constant function whose definition depends on the geometry of the a Dirichlet fundamental domain representing the surface $\mathbb{D}/\Gamma$.

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1 Introduction

We consider the Laplacian operator $\Delta$ defined on the Lobachevskii upper half-plane $\mathbb{H} = \{ w = x + iy \in \mathbb{C}, y > 0 \}$ equipped with the hyperbolic metric $ds_{\mathbb{H}} = \frac{|dw|}{y}$ and the eigenvalue problem

$$\Delta = y^2 \left( \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right), \quad \Delta f = -s(1-s)f,$$

for $s$ of the form $s = \frac{1}{2} + it$ where $t$ is real. In a similar way we shall also consider the corresponding Laplacian and eigenvalue problem defined on the Poincaré disk $\mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \}$ equipped with $ds_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}$ by

$$\Delta = \frac{1}{4}(1-|z|^2)^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad \Delta f = -s(1-s)f.$$

Helgason showed in [11] and [12] that any eigenfunction $f$ associated to the equation above, can be obtained by means of a generalized Poisson representation

$$\left\{ \begin{array}{ll}
  f(w) = \int_{-\infty}^{\infty} \left( \frac{(1+t^2)y}{(x-t)^2 + y^2} \right)^s D_{f,s}^\mathbb{H}(t), & \text{for } w \in \mathbb{H} \\
  f(z) = \int_{\partial \mathbb{D}} \left( \frac{1-|z|^2}{|z-\xi|^2} \right)^s D_{f,s}^\mathbb{D}(\xi), & \text{for } z \in \mathbb{D}
\end{array} \right.$$ 

where $D_{f,s}^\mathbb{D}$ or $D_{f,s}^\mathbb{H}$ are analytic distributions that we call from now on the Helgason’s distribution. We have used the canonical isometry between $z \in \mathbb{D}$ and $w \in \mathbb{H}$, namely $w = \frac{1+z}{1+z}$ or $z = \frac{i-w}{1+w}$. The hyperbolic metric is given in $\mathbb{H}$ and in $\mathbb{D}$ by

$$ds_{\mathbb{D}}^2 = \frac{dx^2 + dy^2}{y^2}, \quad ds_{\mathbb{H}}^2 = \frac{4(dx^2 + dy^2)}{(1-|z|^2)^2}.$$

We shall be interested in a more restricted problem where the eigenfunction $f$ is in addition automorphic with respect to a co-compact Fuchsian group $\Gamma$, a discrete subgroup of the group of Möbius transformations (see [20], [25], [5]) with compact fundamental domain. It is known that the eigenvalues $\lambda = s(1-s) = \frac{1}{4} + t^2$ form a discrete set of positive real numbers with finite multiplicity and accumulating at $+\infty$ see [13].
M. Pollicott realized in [21] that such a distribution can be seen as a generalized eigenmeasure of the dual complex Ruelle transfer operator associated to a subshift of finite type defined at infinity. Let $T_L$ be the left Bowen-Series transformation acting on the boundary $S^1 = \partial \mathbb{D}$ associated to a particular set of generators of $\Gamma$. The precise definition of $T_L$ has been given first in [8], [22], [23], [24]. More geometrical descriptions have then been given in [1] and [18]. Specific examples have been studied in [17], [4] for the modular surface and in [3] for a symmetric compact fundamental domain of genus two. The map $T_L$ is known to be piecewise $\Gamma$-Möbius constant, Markovian with respect to a partition $\{I_k^L\}$ of intervals of $S^1$ on which the restriction of $T_L$ is constant equal to an element $\gamma_k$ of $\Gamma$, transitive and orbit equivalent to $\Gamma$. Let $L_s^L$ be the complex Ruelle transfer operator associated to the map $T_L$ and the potential $A_L = -s \ln |T'_L|$, namely

$$(L_s^L \psi)(\xi') = \sum_{T_L(\xi) = \xi'} e^{A_L(\xi)} \psi(\xi) = \sum_{T_L(\xi) = \xi'} \frac{\psi(\xi)}{|T'_L(\xi)|^s}$$

where the summation is taken over all $\xi$ pre-images of $\xi'$ by $T_L$ and $T'_L$ denotes the jacobian of $T_L$ with respect to the canonical Lebesgue measure on $S^1$. In the case of an automorphic eigenfunction $f$ of $\Delta$, Pollicott showed that the corresponding Helgason's distribution $D_{f,s}$ satisfies the dual functional equation

$$(L_s^L)^* (D_{f,s}) = D_{f,s}$$

or according to Pollicott’s terminology, the parameter $s$ is called a (dual) Perron-Frobenius value, that is 1 is an eigenvalue for the dual Ruelle transfer operator.

Although suggested in [21], it is not clear whether $s$ could be a Perron-Frobenius value, that is whether 1 could be an eigenvalue for $L_s^L$ and not for $(L_s^L)^*$. Our goal is to show that it is actually the case. We would like to thank I. Efrat for showing us the reference [6] and F. Ledrappier for the references [14] and [15].

The three main ingredients we use are summarized here:

- Otal’s proof of Helgason’s distribution in [19] giving more precise information on $D_{f,s}$ and enabling us to integrate piecewise $C^1$ test functions instead of real analytic globally defined test functions,

- a more careful reading of [1], [18], [8], [24] or a careful study of a particular example in [16], which enables us to construct a piecewise
Γ-Möbius baker transformation ("arithmetically" conjugate to the geodesic billiard),

- the existence of a kernel that we introduced in [3] which enable us to permute past and future coordinates and transfer a dual eigendistribution to a piecewise real analytic eigenfunction. Haydn has introduced in [10] a similar kernel in a more abstract setting without geometric consideration.

More precisely we show

**Theorem 1.** Let $\Gamma$ be a co-compact Fuchsian group of the hyperbolic disk $\mathbb{D}$ and $\Delta$ be the corresponding hyperbolic Laplacian. Let $\lambda = s(1-s)$, $s = \frac{1}{2} + it$ and $f$ be an eigenfunction of $-\Delta$, automorphic with respect to $\Gamma$: $\Delta f = -\lambda f$ and $f \circ \gamma = f$, for every $\gamma \in \Gamma$. Then there exists a (non-zero) piecewise real analytic eigenfunction $\psi_{f,s}$ on $\mathbb{S}^1$ solution of the functional equation

$$\mathcal{L}_s^L(\psi_{f,s}) = \psi_{f,s}$$

where $\mathcal{L}_s^L$ is the complex Ruelle transfer operator associated to the left Bowen-Series transformation $T_L : \mathbb{S}^1 \to \mathbb{S}^1$ and the potential $A_L = -s \ln |T_L'|$.

Moreover $\psi_{f,s}$ admits an integral representation using Helgason’s distribution $\mathcal{D}_{f,s}^D$ representing $f$ at infinity and a geometric positive kernel $k(\xi,\eta)$ defined on a finite set of disjoint rectangles $\bigcup_k I^L_k \times Q^R_k \subset \mathbb{S}^1 \times \mathbb{S}^1$

$$\psi_{f,s}(\xi) = \int_{Q^R_k} k^s(\xi,\eta) \mathcal{D}_{f,s}^D(\eta) = \int_{Q^R_k} \frac{1}{|\xi - \eta|^{2s}} \mathcal{D}_{f,s}^D(\eta), \quad \forall \xi \in I^L_k$$

where $I^L_k$ and $Q^R_k$ are intervals of $\mathbb{S}^1$ with disjoint closure and $\{I^L_k\}_k$ is a partition of $\mathbb{S}^1$ where $T_L$ is injective, Markovian and piecewise $\Gamma$-Möbius constant.

Lewis [14] and later Lewis and Zagier [15] initiated a different approach to understand Maass wave forms. They where able to identify in a bijective way Maass wave forms of $PSL(2, \mathbb{Z})$ and solutions of a functional equation with 3 terms closely related to Mayer’s transfer operator. Their setting is strongly dependent of the modular group. Theorem 1 may be viewed as part of their program for co-compact Fuchsian groups. The Helgason’s distribution has been used by S. Zelditch in [26] to generalise microlocal analysis on hyperbolic surfaces, by L. Flaminio and G. Forni in [9] study invariant distributions by the horocycle flow and by N. Anantharaman and S. Zelditch in [2] to understand the “Quantum Unique Ergodicity Conjecture”.

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2 Preliminary results

Let $\Gamma$ be a co-compact Fuchsian group of the Poincaré disk $\mathbb{D}$. We denote by $d(w, z)$ the hyperbolic distance between two points of $\mathbb{D}$ given by the Riemannian metric $ds^2 = 4(dx^2 + dy^2)/(1 - |z|^2)^2$. Let $M = \mathbb{D}/\Gamma$ be the associated compact Riemann surface and $N = T^1 M$ be the unit tangent bundle. Let $\Delta$ be the Laplacian operator on $M$ and $f : M \to \mathbb{R}$ be an eigenfunction of $-\Delta$ or in other words, a $\Gamma$-automorphic function $f : \mathbb{D} \to \mathbb{R}$ satisfying $\Delta f = -s(1-s)f$ for the eigenvalue $\lambda = s(1-s) > \frac{1}{4}$ and $f \circ \gamma = f$ for every $\gamma \in \Gamma$. We know that $f$ is a $C^\infty$ function uniformly bounded on $\mathbb{D}$. Thanks to Helgason's representation theorem, $f$ can be represented as a superposition of horocycle waves given by the Poisson kernel

$$P(z, \xi) := e^{b_\xi(O,z)} = \frac{1 - |z|^2}{|z - \xi|^2},$$

where $b_\xi(w, z)$ is the the Busemann cocycle between two points $w$ and $z$ inside the Poincaré disk observed from a point at infinity $\xi \in S^1$

$$b_\xi(w, z) := "d(w, \xi) - d(z, \xi)" = \lim_{t \to \xi} d(w, t) - d(z, t)$$

uniformly in $t \to \xi$ in any hyperbolic cone at $\xi$. Helgason’s theorem says that

$$f(z) = \int_{\mathbb{D}} P^s(z, \xi) D_{f,s}(\xi) = \langle D_{f,s}, P^s(z, \cdot) \rangle$$

for some analytic distribution $D_{f,s}$ acting on real analytic functions $\psi(\xi)$ on $S^1$. Unfortunately, Helgason’s work is too general and is valid for any eigenfunction not necessarily equivariant by a group. For bounded $C^2$ function $f$, Otal has shown that the distribution $D_{f,s}$ has stronger properties and can be defined in a simpler manner [19].

We first recall some standard notations in hyperbolic geometry. We call $d(z, z_0)$ the hyperbolic distance between two points: for instance the distance from the origin is given by $d(O, \tanh(\frac{r}{2})e^{i\theta}) = r$. Let $C(O, r)$ be the set of points in $\mathbb{D}$ at the hyperbolic distance $r$ from the origin,

$$C(O, r) = \{z \in \mathbb{D} \ s.t. \ |z| = \tanh(\frac{r}{2})\}$$

and more generally for any interval $I$ for any point $z_0 \in \mathbb{D}$, let $C(z_0, r, I)$ be the angular arc at the hyperbolic distance $r$ from $z_0$ delimited at infinity by
\( \mathcal{C}(z_0, r, I) = \{ z \in \mathbb{D} \text{ s.t. } d(z, z_0) = r, \ z \in [[z_0, \xi]] \text{ for some } \xi \in I \} \).

where \([[z_0, \xi]]\) denotes the geodesic ray from \(z_0\) to \(\xi\) at infinity. Let \(\frac{\partial}{\partial n} = \frac{\partial}{\partial r}\) be the exterior normal derivative to \(\mathcal{C}(O, r)\) and \(|dz|_D = \sinh(r) d\theta\) be the hyperbolic arclength on \(\mathcal{C}(O, r)\).

**Theorem 2.** (Otal [19]) Let \(f\) be a bounded \(C^2\) eigenfunction satisfying \(\Delta f = -s(1-s)f\). Then

1. There exists a continuous linear functionnal \(\mathcal{D}_{f,s}\) acting on \(C^1\) functions defined on \(S^1\) by
   \[
   \int \psi(\xi) \mathcal{D}_{f,s}(\xi) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_{\mathcal{C}(O, r)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|_D
   \]
   where \(c(s)\) is a non-zero normalizing constant so that \(\langle \mathcal{D}_{f,s}, 1 \rangle = f(0)\) and \(\psi(z)\) is any \(C^1\) extension of \(\psi(\xi)\) on a neighborhood of \(S^1\).

2. \(\mathcal{D}_{f,s}\) represents \(f\) in the following sense
   \[
   f(z) = \int [P(z, \xi)]^s \mathcal{D}_{f,s}(\xi), \quad \forall \ z \in \mathbb{D}.
   \]
   \(\mathcal{D}_{f,s}\) is unique and is called the Helgason distribution of \(f\).

3. For all \(0 \leq \alpha \leq 2\pi\), the following limit exists
   \[
   \tilde{\mathcal{D}}_{f,s}(\alpha) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_0^\alpha e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) \left( \tanh \left( \frac{r}{2} \right) e^{i\theta} \right) \sinh(r) d\theta.
   \]
   The convergence is uniform in \(\alpha \in [0, 2\pi]\) and \(\tilde{\mathcal{D}}_{f,s}(0) = 0\).

4. \(\tilde{\mathcal{D}}_{f,s}\) can be extended on \(\mathbb{R}\) as a \(\frac{1}{2}\)-Hölder continuous function satisfying
   (a) for all \(\theta \in \mathbb{R}\), \(\tilde{\mathcal{D}}_{f,s}(\theta + 2\pi) = \tilde{\mathcal{D}}_{f,s}(\theta) + f(0)\),
   (b) for any \(C^1\) function \(\psi : S^1 \to \mathbb{C}\) and \(\tilde{\psi}(\theta) = \psi(\exp i\theta)\)
   \[
   \int \psi(\xi) \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(0)f(0) - \int_0^{2\pi} \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta) d\theta.
   \]
Using similar technical tools as in Otal, one can prove the following extension of $D_{f,s}$ on piecewise $C^1$ functions, that is on functions not necessarily continuous which admit a $C^1$ extension on each interval $[\xi_k, \xi_{k+1}]$ for some finite and ordered subdivision \{\xi_0, \xi_1, \ldots, \xi_{r-1}\} of $S^1$.

**Proposition 3.** Let $f$ and $D_{f,s}$ be as in Theorem 2

1. For any interval $I \subset S^1$ and any function $\psi : I \to \mathbb{C}$, $C^1$ on the closure of $I$ and null outside $I$, the following limit exists

$$
\int \psi(\xi) D_{f,s}(\xi) := \frac{1}{c(s)} \lim_{r \to +\infty} \int_{\mathcal{C}(\mathbb{O}, r, I)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|_{\mathbb{D}}
$$

where again $\psi(z)$ is any $C^1$ extension of $\psi(\xi)$ on a neighborhood of $S^1$.

2. For any $0 \leq \alpha < \beta \leq 2\pi$, for any $C^1$ function $\psi$ on the interval $I = [\exp(i\alpha), \exp(i\beta)]$ and $\tilde{\psi}(\theta) = \psi(\exp i\theta)$,

$$
\int \psi(\xi) D_{f,s}(\xi) = \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_\alpha^\beta \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{D}_{f,s}(\theta) d\theta
$$

where $\tilde{D}_{f,s}$ has been defined in Theorem 2.

**Proof of proposition 3.** Let be $\alpha \in [0, 2\pi]$, $I = \{e^{i\theta} | 0 \leq \theta \leq \alpha\}$ an interval, $\psi$ a $C^1$ function defined on a neighborhood of $S^1$, $\tilde{\psi}(r, \theta) = \psi(\tanh(\frac{r}{2}) e^{i\theta})$ and $K(r, \theta) = e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) \left( \tanh(\frac{r}{2}) e^{i\theta} \right) \sinh(r)$. Then

$$
\frac{1}{c(s)} \int_{\mathcal{C}(\mathbb{O}, r, I)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|_{\mathbb{D}}
$$

$$
= \int_0^\alpha \tilde{\psi}(r, \beta) K(r, \beta) d\beta
$$

$$
= \int_0^\alpha \left[ \tilde{\psi}(r, \alpha) + \int_\beta^\alpha -\frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta) d\theta \right] K(r, \beta) d\beta
$$

$$
= \tilde{\psi}(r, \alpha) \int_0^\alpha K(r, \beta) d\beta - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta) \left[ \int_0^\theta K(r, \beta) d\beta \right] d\theta.
$$

Since $\int_0^\alpha K(r, \beta) d\beta \to \tilde{D}_{f,s}(\alpha)$ uniformly in $\alpha \in [0, 2\pi]$, the left hand side of the previous equality converges to

$$
\int \psi(\xi) 1_{\xi \in I_1} D_{f,s}(\xi) = \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(\theta) \tilde{D}_{f,s}(\theta) d\theta.
$$
The second part of the proposition follows by subtracting two such expressions

\[
\int \psi(\xi) \mathbf{1}_{\{\xi = e^{i\theta} | 0 \leq \theta \leq \beta\}} \tilde{D}_{f,s}(\xi) - \int \psi(\xi) \mathbf{1}_{\{\xi = e^{i\theta} | 0 \leq \theta \leq \alpha\}} \tilde{D}_{f,s}(\xi).
\]

\[\blacksquare\]

If in addition we assume that \( f \) is equivariant with respect to a co-compact Fuchsian group \( \Gamma \), Pollicott observed in \([21]\) that \( D_{f,s} \), acting on real analytic functions, equivariant by \( \Gamma \), that is satisfies \( \gamma^*(D_{f,s})(\xi) = |\gamma'(\xi)|^s D_{f,s}(\xi) \) for all \( \gamma \in \Gamma \). Because Otal’s construction is more precise and implies that Helgason’s distribution also acts on piecewise \( C^1 \) functions, the above equivariance property can be improved in the following way.

**Proposition 4.** Let \( f : \mathbb{D} \to \mathbb{R} \) be a \( C^2 \) function, \( \gamma \in \Gamma \), \( I \subset \mathbb{S}^1 \) be an interval and \( \psi : I \to \mathbb{C} \) be a \( C^1 \) function on the closure of \( I \). \( f \) satisfies \( f \circ \gamma = f \) (not necessarily automorphic), then

\[
\langle D_{f,s}, \frac{\psi \circ \gamma^{-1}}{|\gamma' \circ \gamma^{-1}|^s} \mathbf{1}_{\gamma(I)} \rangle = \langle D_{f,s}, \psi \mathbf{1}_I \rangle.
\]

The main difficulty is to transfer the equivariance property \( f \circ \gamma = f \) to an equivalent property for the extension of \( D_{f,s} \) on piecewise \( C^1 \) functions. If \( I = \mathbb{S}^1 \) and \( \psi \) is real analytic, then by uniqueness of the representation, proposition 4 is easily proved. It seems that just knowing the fact that \( D_{f,s} \) is the derivative of some Hölder function is not enough to conclude. The following proof uses Otal’s approach and essentially the extension of \( D_{f,s} \) as described in part 1 of proposition 3.

**Proof of proposition 4.** Part one. We first prove proposition 4 for \( \psi = 1 \). Let \( g(z) = \exp(-sd(O, z)) \). By definition of \( D_{f,s} \) we obtain

\[
\int 1_I(\xi) D_{f,s}(\xi) = \lim_{r \to +\infty} \frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O}, r', I)} \left( g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) |dz|_{\mathbb{D}}
\]

\[
= \lim_{r \to +\infty} \frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O}', r', \gamma(I))} \left( g' \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) |dz|_{\mathbb{D}}
\]

where \( r' = r + d(O, \mathcal{O}') \), \( \mathcal{O}' = \gamma(\mathcal{O}) \) and \( g' = g \circ \gamma^{-1} \). Notice that the domain bounded by the circle \( \mathcal{C}(\mathcal{O}', r') \) contains the circle \( \mathcal{C}(\mathcal{O}, r) \). Let \( PQ \)
be the positively oriented arc $C(O, r, \gamma(I))$ and $\overrightarrow{PQ'}$ be the arc $C(O', r', \gamma(I))$. Then the two geodesic segments $[[P, P']]$ and $[[Q, Q']]$ belong to the annulus $r \leq d(z, O) \leq r + 2d(O, O')$ and their length is uniformly bounded. We use now Green’s formula to compute the right hand side: we denote by $\Omega$ the domain delimited by $P, P', Q, Q'$ using the corresponding arcs and geodesic segments and $dv = \sinh(r) \, dr \, d\theta$ the hyperbolic volume element

\[
\int_{P'Q'} \left( g' \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) |dz|_D = \int_{PQ} \left( g' \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) |dz|_D
- \int_{[[P, P']]} \cdots |dz|_D - \int_{[[Q', Q]]} \cdots |dz|_D
+ \int_{\Omega} (g' \Delta f - f \Delta g') \, dv
\]

When $r$ tends to infinity the last three terms in the right hand side go to 0: along the geodesic segments $[P, P']$ and $[Q, Q']$, the gradient $\nabla g'$ is uniformly bounded by $\exp(-\frac{1}{2}r)$ and

\[
g' \Delta f - f \Delta g' = sg' f \sinh(d(z, O'))^{-2} \quad \text{and} \quad \frac{\partial}{\partial n} g' + sg'
\]

are bounded uniformly by a constant times $\exp(-\frac{5}{2}r)$ in the domain $\Omega$ for the first expression and by a constant times $\exp(-\frac{3}{2}r)$ on $C(O, r)$ for the second expression. We get

\[
\int 1_I(\xi) D_{f,s}(\xi) = \lim_{r \to +\infty} \frac{1}{c(s)} \int_{C(O, r, \gamma(I))} g' \left( \frac{\partial f}{\partial n} + sf \right) |dz|_D
= \lim_{r \to +\infty} \frac{1}{c(s)} \int_{C(O, r, \gamma(I))} \left[ \psi(z) \right]^s e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|_D
\]

where $\psi(z) = \exp (d(O, z) - d(O, \gamma^{-1}(z)))$. The new observation is that

\[
\left\{ \begin{array}{ll}
\psi(z) = \exp s (d(O, z) - d(\gamma(O), z)), & \text{for } z \in \mathbb{D}, \\
\psi(\xi) = \exp b_{\xi}(O, \gamma(O)) = |\gamma' \circ \gamma^{-1}(\xi)|^{-1}, & \text{for } \xi \in \partial \mathbb{D},
\end{array} \right.
\]

coincides actually with a real analytic function $\Psi(z)$ defined in a neighborhood of $S^1$ given explicitly by

\[
\Psi(z) = \left( \frac{(1 + |z|)^2}{(1 + |\gamma^{-1}(z)|)^2 |\gamma' \circ \gamma^{-1}(z)|} \right)^s.
\]
We hence have proved
\[
\int 1_I(\xi) \mathcal{D}_{f,s}(\xi) = \int \frac{1_{\gamma(I)}(\xi)}{|\gamma' \circ \gamma^{-1}(\xi)|^s} \mathcal{D}_{f,s}(\xi).
\]

Part two. We now prove the general case. We use the same notation for the lift \(\gamma : \mathbb{R} \mapsto \mathbb{R}\) of a Möbius transformation \(\gamma : S^1 \mapsto S^1\). The lift satisfies \(\gamma(\alpha + 2\pi) = \gamma(\alpha) + 2\pi\), \(\exp(i\gamma(\alpha)) = \gamma(\exp(i\alpha))\) and \(\gamma'(\alpha) = |\gamma'(\alpha)|\) for all \(\alpha \in \mathbb{R}\). Using Proposition 3, we have proved
\[
\tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha) = \frac{\tilde{D}_{f,s} \circ \gamma(\beta)}{\gamma'(\beta)^s} - \frac{\tilde{D}_{f,s} \circ \gamma(\alpha)}{\gamma'(\alpha)^s} - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left( \frac{1}{(\gamma' \circ \gamma^{-1}(\theta))^s} \right) \tilde{D}_{f,s}(\theta) d\theta.
\]

For any \(C^1\) functions \(\psi(\xi)\) defined on \(I\), \(\tilde{\psi}(\theta) = \psi(\exp i\theta)\), we have
\[
LHS := \int \psi(\xi) 1_I(\xi) \mathcal{D}_{f,s}(\xi)
\]
\[
= \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_{\alpha}^{\beta} \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{D}_{f,s}(\theta) d\theta
\]
\[
= \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left( \tilde{\psi} \circ \gamma^{-1}(\theta) \right) \tilde{D}_{f,s}(\gamma^{-1}\theta) d\theta
\]
\[
= \tilde{\psi}(\beta) \left( \tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha) \right) - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial \tilde{\psi}(\gamma^{-1}\theta)}{\partial \theta} \left( \tilde{D}_{f,s}(\gamma^{-1}\theta) - \tilde{D}_{f,s}(\alpha) \right) d\theta.
\]

We now use the equivariance we have proved in part one and substitute both \(\tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha)\) and \(\tilde{D}_{f,s}(\gamma^{-1}\theta) - \tilde{D}_{f,s}(\alpha)\) by the corresponding formula involving \(\tilde{D}_{f,s} \circ \gamma(\beta)\), \(\tilde{D}_{f,s} \circ \gamma(\alpha)\), \(\tilde{D}_{f,s}(\theta)\) and finally obtain
\[
LHS
\]
\[
= \frac{\tilde{\psi}(\beta) \tilde{D}_{f,s} \circ \gamma(\beta)}{\gamma'(\beta)^s} - \frac{\tilde{\psi}(\alpha) \tilde{D}_{f,s} \circ \gamma(\alpha)}{\gamma'(\alpha)^s} - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left( \frac{\tilde{\psi}(\gamma^{-1}\theta)}{\gamma'(\gamma^{-1}\theta)^s} \right) \tilde{D}_{f,s}(\theta) d\theta
\]
\[
= \int \frac{\psi \circ \gamma^{-1}(\xi)}{|\gamma' \circ \gamma^{-1}(\xi)|^s} 1_{\gamma(I)} \mathcal{D}_{f,s}(\xi).
\]

\[\square\]
Following [1], [8], [22], [23], [24], [18] for the general case and [16] for a specific example we recall the definition of the left $T_L$ and right $T_R$ Bowen-Series transformation. The hyperbolic surface we are interested in is given by the quotient of the hyperbolic disk $\mathbb{D}$ by a co-compact Fuchsian group $\Gamma$.

Let $O \in \mathbb{D}$ be a given point and $D_{\Gamma, O} = \{ z \in \mathbb{D} | d(z, O) < d(z, \gamma(O)) \} \forall \gamma \in \Gamma$

be the corresponding Dirichlet domain. One can prove that $D_{\Gamma, O}$ is a convex fundamental domain with compact closure in $\mathbb{D}$, admitting an even number of geodesic sides and an even number of vertices some of those may be elliptic. More precisely the boundary of $D_{\Gamma, O}$ is equal to a disjoint union of semi-closed geodesic segments $S_{-r}^L, \ldots, S_{-1}^L, S_1^L, \ldots, S_r^L$, closed to the left and open to the right, or equivalently to a union of semi-closed geodesic segments $S_{-r}^R, \ldots, S_{-1}^R, S_1^R, \ldots, S_r^R$, closed to the right and open to the left so that each $S_k^L$ and $S_k^R$ have the same endpoints. Each side $S_k^L$ is associated to $S_{-k}^R$ by an element $a_k \in \Gamma$ satisfying $a_k(S_k^L) = S_{-k}^R$. The set $\{a_k, 1 \leq |k| \leq r\}$ generates $\Gamma$ and satisfies $a_{-k} = a_k^{-1}$ for all $k = \pm 1, \ldots, \pm r$.

In order to define geometrically the two Bowen-Series transformations, $T_L$ and $T_R$, one needs to impose a geometric condition on $\Gamma$: we say, following [8], [22], [24], that $\Gamma$ satisfies the “even corner” property if, for each side $S_k^L$, $1 \leq |k| \leq r$, of $D_{\Gamma, O}$, the complete geodesic line through $S_k^L$ is equal to a disjoint union of $\Gamma$-translates of the sides $\{S_l^L, 1 \leq |l| \leq r\}$. Some $\Gamma$ do not satisfy this geometric property. Nevertheless any two co-compact Fuchsian groups, $\Gamma$ and $\Gamma'$, with identical signature are geometrically isomorphic, that is, there exist a group isomorphism $h_* : \Gamma \rightarrow \Gamma'$ and a quasi-conformal orientation preserving homeomorphism $h : \mathbb{D} \rightarrow \mathbb{D}$ admitting an extension to a homeomorphism $h : \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ and conjugating the two actions,

$$h(\gamma(z)) = h_*(\gamma)(h(z)), \forall \gamma \in \Gamma.$$

An important observation in [8], [22] and [24] is that, any co-compact Fuchsian group is geometrically isomorphic to a Fuchsian group with identical signature and satisfying the “even corner” property. We are going to recall the Bowen and Series construction in the case $\Gamma$ possesses the “even corner” property and show that their main conclusions remain valid under geometric isomorphism.

The complete geodesic line associated to a side $S_k^L$ cuts the boundary at infinity $\mathbb{S}^1$ at two points $s_k^L$ and $s_k^R$ (positively oriented with respect to $s_k^L$),
the oriented geodesic line \( ]s_k^L, s_k^R[ \) sees the origin \( O \) on the left. The two end points \( s_k^L \) and \( s_k^R \) are both neutrally stable with respect to the associated generator \( a_k \), that is \( |a_k'(s_k^L)| = |a_k'(s_k^R)| = 1 \). The family of open intervals \( ]s_k^L, s_k^R[ \) covers \( S^1 \). Since the intervals \( ]s_k^L, s_k^R[ \) overlap one to each other, there is no canonical partition adapted to this covering. We can nevertheless associate two well defined partitions, the left and the right partition, \( \mathcal{A}_L \) and \( \mathcal{A}_R \). The left partition is made of disjoint semi-closed intervals

\[
\mathcal{A}_L = \{ A^L_{-\infty}, \ldots, A^L_{-1}, A^L_1, \ldots, A^L_{+\infty} \}.
\]

where \( A^L_k = [s_k^L, s_{l(k)}^L] \) and \( s_{l(k)}^L \) denotes the nearest point \( s_j^L \) after \( s_k^L \) according to a positive orientation. Each \( A^L_k \) belongs to the unstable domain of the hyperbolic element \( a_k \), that is \( |a_k'(\xi)| \geq 1 \) for all \( \xi \in A^L_k \). By definition the left Bowen-Series transformation \( T_L : S^1 \mapsto S^1 \) is given by

\[
T_L(\xi) = a_k(\xi), \quad \text{whenever} \quad \xi \in A^L_k.
\]

In the same manner \( S^1 \) can be partitioned into semi-closed intervals

\[
\mathcal{A}_R = \{ A^R_{-\infty}, \ldots, A^R_{-1}, A^R_1, \ldots, A^R_{+\infty} \}.
\]

where \( A^R_k = ]s_j^R, s_{l(k)}^R[ \) and \( s_{l(k)}^R \) denotes the nearest \( s_j^R \) before \( s_k^R \) according to a positive orientation. The right Bowen-Series transformation is defined as

\[
T_R(\eta) = a_k(\eta), \quad \text{whenever} \quad \eta \in A^R_k.
\]

These two partitions generate two ways of coding a trajectory. Let \( \gamma_L : S^1 \mapsto \Gamma \) and \( \gamma_R : S^1 \mapsto \Gamma \) be the left and right symbolic coding defined by

\[
\gamma_L[\xi] = a_k, \quad \text{if} \quad \xi \in A^L_k, \quad \gamma_R[\eta] = a_k, \quad \text{if} \quad \eta \in A^R_k.
\]

In particular \( T_R(\eta) = \gamma_R[\eta](\eta) \) and \( T_L(\xi) = \gamma_L[\xi](\xi) \) for all \( \xi \in S^1 \). One shows that \( T^2_R \) and \( T^2_L \) are expanding. Series in [22], [23], [24] and later Adler and Flatto [1] proved that \( T_L \) resp. \( T_R \) are Markov with respect to a partition in \( \mathcal{T}^L = \{ I^L_k \}_{k=1}^\infty \) resp. \( \mathcal{T}^R = \{ I^R_i \}_{i=1}^\infty \) finer than \( \mathcal{A}_L \) resp. \( \mathcal{A}_R \). \( I^L_k \) and \( I^R_i \) are semi-closed intervals of the same kind as \( A^L_k \) and \( A^R_i \) and have the same closure.

**Definition 5.** A dynamical system \( (S^1, T, \{ I_k \}) \) is said to be a piecewise \( \Gamma \)-Möbius Markov transformation if \( T : S^1 \mapsto S^1 \) is a surjective map, \( \{ I_k \} \) is a finite partition of \( S^1 \) into intervals such that
1. for each $k$, $T(I_k)$ is a union of adjacent intervals $I_l$.

2. for each $k$, the restriction of $T$ on $I_k$ coincides with an element $\gamma_k \in \Gamma$.

3. some finite iterate of $T$ is uniformly expanding.

**Theorem 6.** ([8], [24]) For any co-compact Fuchsian group $\Gamma$, there exists a piecewise $\Gamma$-Möbius Markov transformation $(S^1, T, \{I_k\})$ which is transitive and orbit equivalent to $\Gamma$.

The Ruelle transfer operator can be defined for any piecewise $C^2$ Markov transformation $(S^1, T, \{I_k\})_{k=1}^q$ and any potential function $A$. We actually need a particular complex transfer operator given by the potential

$$A = -s \ln |T'|.$$

For any function $\psi : S^1 \to \mathbb{C}$, define

$$(L_s(\psi))(\xi') = \sum_{T(\xi) = \xi'} e^{A(\xi)} \psi(\xi) = \sum_{T(\xi) = \xi'} \frac{\psi(\xi)}{|T'(\xi)|^s},$$

where the summation is taken over all preimages $\xi$ of $\xi'$ under $T$. We modify slightly $L_s$ so that it moreover acts on the space of piecewise $C^1$ functions $\bigoplus_{k=1}^q \psi_k \in \bigoplus_{k=1}^q C^1(\bar{I}_k)$ in the following way

$$L_s^L \psi = \bigoplus_{l=1}^q \phi_l$$

where

$$\phi_l = \sum_{I_l \subset T(I_k)} \psi_k \circ T_{k,l}^{-1} \left| T' \circ T_{k,l}^{-1} \right|^s,$$

where $T_{k,l}^{-1}$ is the restriction to $I_l$ of the inverse of $T : I_k \to T(I_k) \supset I_l$.

**Proposition 7.** Let $\Gamma$ be a co-compact Fuchsian group. Let $s = \frac{1}{2} + it$ and $f$ be an automorphic eigenfunction of $-\Delta$, $\Delta f = -(1-s)f$. Let $(S^1, T, \{I_k\})$ be a piecewise $\Gamma$-Möbius Markov transformation, $L_s$ be the Ruelle transfer operator corresponding to the observable $A = -s \ln |T'|$. Then the Helgason’s distribution $D_{f,s}$ satisfies

$$(L_s)^* D_{f,s} = D_{f,s}.$$

**Proof.** Let $\bigoplus_{k=1}^q \psi_k$ be a piecewise $C^1$ function in $\bigoplus_{k=1}^q C^1(\bar{I}_k)$, then using Proposition 4

13
\[ \int (L_s \psi)(\xi) D_{f,s}(\xi) = \sum_{l=1}^q \int_{I_l} (L_s \psi)_l(\xi) D_{f,s}(\xi) \]
\[ = \sum_{T(I_k) \supset I_l} \int_{I_l} \psi_k \circ T_{k,l}^{-1}(\xi) D_{f,s}(\xi) \]
\[ = \sum_{T(I_k) \supset I_l} \int_{T^{-1}(I_l) \cap I_k} \psi_k(\xi) D_{f,s}(\xi) \]
\[ = \sum_{k=1}^q \int_{I_k} \psi_k(\xi) D_{f,s}(\xi) = \int \psi(\xi) D_{f,s}(\xi). \]

Series in [24], Adler and Flatto in [1], Morita in [18] noticed that \( T_L \) admits a natural extension \( \hat{T} : \Sigma \mapsto \hat{\Sigma} \) strongly related to \( T_R \). We also showed the existence of such a \( \hat{T} \) in [16] which was an important step in the proof of Theorem 3 of [16]. The following definition explains how the two maps \( T_L \) and \( T_R \) are glued together in an abstract way.

**Definition 8.** Let \( \Gamma \) be a co-compact Fuchsian group. A dynamical system \((\Sigma, \hat{T}, \{I^L_k\}, \{I^R_l\}, J)\) is said to be a piecewise \( \Gamma \)-Möbius baker transformation if it admits the following description

1. \( \{I^L_k\} \) and \( \{I^R_l\} \) are finite partitions of \( S^1 \) into disjoint intervals; \( J(k,l) \) is a \( \{0,1\} \)-value function, \( \hat{\Sigma} \) is a subset of \( S^1 \times S^1 \) defined by

\[ \hat{\Sigma} = \bigsqcup_{J(k,l)=1} I^L_k \times I^R_l, \]

2. For each \( k \), \( Q^R_k = \bigsqcup \{I^R_l \mid J(k,l) = 1\} \) is an interval whose closure is disjoint from \( \bar{I}^L_k \). For each \( l \), \( Q^L_l = \bigsqcup \{I^L_k \mid J(k,l) = 1\} \) is an interval whose closure is disjoint from \( \bar{I}^R_l \). Let \( I^L(\xi) = I^L_k \) and \( Q^R(\xi) = Q^R_k \) if \( \xi \in I^L_k \). Let \( I^R(\eta) = I^R_l \) and \( Q^L(\eta) = Q^L_l \) if \( \eta \in I^R_l \).

3. \( \hat{T} : \Sigma \to \hat{\Sigma} \) is bijective and has the form

\[ \begin{cases} \hat{T}(\xi,\eta) = (T_L(\xi), S_R(\xi, \eta)), \\ \hat{T}^{-1}(\xi',\eta') = (S_L(\xi', \eta'), T_R(\eta')) \end{cases} \]

for some maps, \( T_L, T_R : S^1 \to S^1 \) and \( S_L, S_R : \hat{\Sigma} \to S^1 \).
4. \((S^1, T_L, \{I^L_k\})\) and \((S^1, T_R, \{I^R_k\})\) are piecewise \(\Gamma\)-Möbius Markov transformations. There exist two functions \(\gamma_L : S^1 \rightarrow \Gamma\) resp. \(\gamma_R : S^1 \rightarrow \Gamma\), piecewise constant on each \(\{I^L_k\}\) resp. \(\{I^R_k\}\), such that

\[
\begin{align*}
\hat{T}(\xi, \eta) &= (\gamma_L[\xi](\xi), \gamma_L[\xi](\eta)) \\
\hat{T}^{-1}(\xi', \eta') &= (\gamma_R[\eta'](\xi'), \gamma_R[\eta'](\eta'))
\end{align*}
\]

\(T_L\) and \(T_R\) are called the left and right Bowen-Series transformations. \(\gamma_L\) and \(\gamma_R\) are called the left and right Bowen-Series codings. \(J\) is called the incidence matrix and is extended as a function on \(S^1 \times S^1\) by

\[
\begin{align*}
J(\xi, \eta) &= 1 \quad \text{if} \quad (\xi, \eta) \in \hat{\Sigma} \\
J(\xi, \eta) &= 0 \quad \text{if} \quad (\xi, \eta) \notin \hat{\Sigma}
\end{align*}
\]

Notice that this definition is equivariant by geometric isomorphisms. For co-compact Fuchsian groups satisfying the “even corner” property, Adler and Flatto in [1], Series in [24] and for a particular example [16] obtained geometrically the existence of a piecewise \(\Gamma\)-Möbius baker transformation with left \(T_L\) and right \(T_R\) maps orbit equivalent to \(\Gamma\). By geometric isomorphism, we obtain more generally

**Proposition 9.** ([1], [24], [16]) For any co-compact Fuchsian group \(\Gamma\), there exists a piecewise \(\Gamma\)-Möbius baker transformation with left and right Bowen-Series transformations transitive and orbit equivalent to \(\Gamma\).

The two maps \(T_L\) and \(T_R\) are related to the action of the group \(\Gamma\) on the boundary \(S^1\). The baker transformation \((\hat{\Sigma}, \hat{T})\) encodes this action into a unique dynamical system. Notice for later

**Remark 10.**

1. The two codings \(\gamma_L\) and \(\gamma_R\) are reciprocal in the following sense

\[
\gamma_R[\eta] = \gamma_L^{-1}[\xi] \quad \text{whenever} \quad (\xi', \eta') = \hat{T}(\xi, \eta).
\]

2. For any \(\xi'\) and \(\eta\) in \(S^1\), there is a bijection between the two finite sets

\[
\{\xi : (\xi, \eta) \in \hat{\Sigma} \text{ and } T_L(\xi) = \xi'\}, \quad \{\eta' : (\xi', \eta') \in \hat{\Sigma} \text{ and } T_R(\eta') = \eta\}.
\]
In order to better understand this baker transformation, we briefly explain how \((\hat{\Sigma}, \hat{T})\) is conjugate to a specific Poincaré section of the geodesic flow on the surface \(N = T^1M\). We assume for the rest of this section that \(\Gamma\) satisfies the “even corner” property.

As \(D_{\Gamma,0}\) is a convex fundamental domain every geodesic (modulo \(\Gamma\)) cuts \(\partial D_{\Gamma,0}\) at two distinct points \(p\) and \(q\), unless the geodesic is tangent to one of the sides of \(D_{\Gamma,0}\). These tangent geodesics correspond to a finite union of closed geodesics. We could have parametrized the set of oriented geodesics by all \((p,q)\in\partial D_{\Gamma,0}\times\partial D_{\Gamma,0}\). We could have parametrized the set of oriented geodesics by all \((p,q)\in\partial D_{\Gamma,0}\times\partial D_{\Gamma,0}\), then cutting the interior of \(D_{\Gamma,0}\) or passing through one of the corners of \(D_{\Gamma,0}\) and seeing \(\mathcal{O}\) to the left. Using these notations, we define for every oriented geodesic \([[y, x]]\), \((x, y)\in X\), the two intersections points \(p = p(x, y)\in\partial D_{\Gamma,0}\) and \(q = q(x, y)\in\partial D_{\Gamma,0}\) so that \([[q, p]] = [[y, x]]\cap D_{\Gamma,0}\) and have the same orientation as \([[y, x]]\).

For a geodesic passing through a corner, \(p = q\) unless the geodesic is tangent to a side of \(D_{\Gamma,0}\). We are now in a position to define a geometric Poincaré section \(B : X \to X\). If \((x, y)\in X\), the geodesic \([[y, x]]\) leaves \(D_{\Gamma,0}\) at \(p = p(x, y)\in S_i\) for some side \(S_i^L\). Since \(S_i^L\) and \(S_i^R\) are permuted by the generator \(a_i\), the new geodesic \(a_i([[y, x]]) = [[y', x']]\) enters again the fundamental domain at a new point \(q' = q(x', y')\) with \(q' = a_i(p)\in S_i^R\).

By definition \(B(x, y) = (x', y')\). The map \(B : X \to X\) is called geodesic billiard. As for \(T_L\) and \(T_R\) we introduce two geometric codings \(\gamma_B : X \to \Gamma\) and \(\bar{\gamma}_B : X \to \Gamma\) defined by

\[
\begin{align*}
\gamma_B[x, y] &= a_i \quad \text{whenever} \quad p(x, y) \in S_i^L \\
\bar{\gamma}_B[x, y] &= a_i \quad \text{whenever} \quad q(x, y) \in S_i^R
\end{align*}
\]

The geodesic billiard can then be defined by

\[
\begin{align*}
B(x, y) &= (\gamma_B[x, y](x), \gamma_B[x, y](y)) \\
B^{-1}(x', y') &= (\bar{\gamma}_B[x', y'](x'), \bar{\gamma}_B[x', y'](y'))
\end{align*}
\]

Notice that \(\bar{\gamma}_B \circ B = \gamma_B^{-1}\). The map \(B\) is very close to be a baker transformation: \(B\) and \(B^{-1}\) have the same structure as \(\hat{T}\) and \(\hat{T}^{-1}\), \(\gamma_B\) (resp. \(\bar{\gamma}_B\)) plays the role of \(\gamma_L\) (resp. \(\gamma_R\)). The main difference is that \(\gamma_B[x, y]\) depends on both \(x\) and \(y\) although \(\gamma_L[\xi]\) depends only on \(\xi\). Nevertheless we have the following crucial result.
Theorem 11. ([1], [24], [16]) There exists a $\Gamma$-Möbius baker transformation $(\hat{\Sigma}, \hat{T})$ conjugate to $(X, B)$. More precisely there exists a map $\rho : X \to \Gamma$ such that, $\pi : X \to \hat{\Sigma}$ defined by $\pi(x, y) = (\rho(x, y)(x), \rho(x, y)(y))$, is a conjugating map between $\hat{T}$ and $B$, $\hat{T} \circ \pi = \pi \circ B$. In an equivalent way, $\gamma_L \circ \pi$ and $\gamma_B$ are cohomologuous over $(X, B)$, $\gamma_L \circ \pi = \rho \circ B \gamma_B$, $\gamma_R \circ \pi$ and $\bar{\gamma}_B$ are cohomologuous over $(X, B)$, $\gamma_R \circ \pi = \rho \circ B^{-1} \bar{\gamma}_B$.

3 Proof of the main theorem 1

We want to associate to any eigenfunction $f$ of the Laplacian, a non zero piecewise real analytic function $\psi_{f,s}$ solution of the functional equation

$$L_s^L(\psi_{f,s}) = \psi_{f,s}, \quad \text{where} \quad L_s^L(\psi)(\xi') = \sum_{T_L(\xi) = \xi'} \frac{\psi(\xi)}{|T_L^s(\xi)|^s}.$$

The main idea is to use a kernel $k(\xi, \eta)$ introduced in Theorem 7 in [3], in Haydn [10], or in Bogomolny and Carioli [6], [7], in the context of double-sided subshifts of finite type. We begin by extending this definition for baker transformations.

Definition 12. Let $(\hat{\Sigma}, \hat{T})$ be a piecewise $\Gamma$-Möbius baker transformation, $T_L$ and $T_R$ be the left and right Bowen-Series transformations. Let $A_L : S^1 \to \mathbb{C}$ and $A_R : S^1 \to \mathbb{C}$ be two potential functions. We say that $A_L$ and $A_R$ are in involution if there exists a non zero kernel $k : \hat{\Sigma} \to \mathbb{C}^*$, called an involution kernel, such that

$$k(\xi, \eta)e^{A_L(\xi)} = k(\xi', \eta')e^{A_R(\eta')} \quad \text{whenever} \quad (\xi', \eta') = \hat{T}(\xi, \eta) \in \hat{\Sigma}.$$

The kernel $k$ is extended on $S^1 \times S^1$ by putting $k(\xi, \eta) = 0$ whenever $(\xi, \eta) \notin \hat{\Sigma}$.

Remark 13.

1. Let $W(\xi, \eta) = \ln k(\xi, \eta)$ for all $(\xi, \eta) \in \hat{\Sigma}$. Then $A_L$ and $A_R$ are cohomologuous, that is $A_L - A_R \circ \hat{T} = W \circ \hat{T} - W$.

2. If $A_L(\xi)$ is H"older, then there exists an H"older function $A_R(\eta)$ (depending only on $\eta$) in involution with $A_L$ with an H"older involution kernel.
3. If $\mathcal{L}_L$ and $\mathcal{L}_R$ are the two Ruelle transfer operators associated to $A_L$ and $A_R$, if $A_L$ and $A_R$ are in involution with respect to a kernel $k$, if $\nu$ is an eigenmeasure of $\mathcal{L}_R$, $\mathcal{L}_R^*(\nu) = \lambda \nu$, then $\psi(\xi) = \int k(\xi, \eta) d\nu(\eta)$ is an eigenfunction of $\mathcal{L}_L$, $\mathcal{L}_L(\psi) = \lambda \psi$.

These remarks have been noticed first in [10] and rediscovered later in [3] in the context of a subshift of finite type. The proofs in this general context can be reproduced easily. The third remark suggests a strategy to obtain the eigenfunction $\psi_{f,s}$ by taking $A_L = -s \ln |T'_L|$, $A_R = -s \ln |T'_R|$ and replacing $\nu$ by the distribution $D_{f,s}$. One is left to prove that $- \ln |T'_L|$ and $- \ln |T'_R|$ are in involution with respect to a piecewise $C^1$ involution kernel. It happens that this involution kernel exists and is given by the Gromov distance.

**Definition 14.** We call Gromov distance between two points at infinity $\xi$ and $\eta$, the distance $d(\xi, \eta)$ given by

$$d^2(\xi, \eta) = \exp \left( - b_\xi(O, z) - b_\eta(O, z) \right)$$

for any point $z$ of the geodesic line $[[\xi, \eta]]$. Notice that this definition depends on the choice of the origine $O$ (but not on $z \in [[\xi, \eta]]$).

In the Poincaré disk model, $(\xi, \eta) \in S^1 \times S^1$, or in the Poincaré upper half-plane, $(s, t) \in \mathbb{R} \times \mathbb{R}$, the Gromov distance takes the simple form

$$d^2(\xi, \eta) = \frac{1}{4} |\xi - \eta|^2, \quad \text{or} \quad d^2(s, t) = \frac{|s - t|^2}{(1 + s^2)(1 + t^2)}.$$

**Lemma 15.** Let $T_L : S^1 \to S^1$ and $T_R : S^1 \to S^1$ be the two left and right Bowen-Series transformations of a $\Gamma$-Möbius Markov baker transformation $(\Sigma, \hat{T})$. Then the two potential functions $A_L(\xi) = -\ln |T'_L(\xi)|$ and $A_R(\eta) = -\ln |T'_R(\eta)|$ are in involution

$$A_L(\xi) - A_R(\eta') = W(\xi', \eta') - W(\xi, \eta), \quad \text{whenever} \quad (\xi', \eta') = \hat{T}(\xi, \eta) \in \hat{\Sigma}$$

where $W(\xi, \eta) = b_\xi(O, z) + b_\eta(O, z)$ and $z$ is any point of the geodesic line $[[\xi, \eta]]$. In particular $k(\xi, \eta) = \exp(W(\xi, \eta)) = 4/d^2(\xi, \eta)$ is an involution kernel.
Proof of lemma 15. To simplify the notations, we call \((\xi', \eta') = \hat{T}(\xi, \eta), \gamma_L = \gamma_L[\xi], \gamma_R = \gamma_R[\eta']\). We also recall the relation \(\gamma_R = \gamma_L^{-1}\). Then, choosing any point \(z \in [[\xi, \eta]]\), we get

\[
A_L(\xi) - A_R(\eta') = -b_\xi(O, \gamma_L^{-1}O) + b_{\eta'}(O, \gamma_R^{-1}O) \\
= -b_\xi(O, z) - b_\xi(z, \gamma_L^{-1}O) \\
+ b_{\eta'}(O, \gamma_L(z)) + b_{\eta'}(\gamma_L(z), \gamma_R^{-1}O) \\
= W(\xi', \eta') - W(\xi, \eta),
\]

where \(W(\xi', \eta') = b_{\eta'}(O, \gamma_L(z)) - b_\xi(z, \gamma_L^{-1}O)\) and \(W(\xi, \eta) = b_\xi(O, z) - b_{\eta'}(\gamma_L(z), \gamma_R^{-1}O)\).

\[\square\]

Notice that if \(A(\xi)\) and \(\bar{A}(\eta)\) are in involution by a positive kernel \(k(\xi, \eta)\) then \(sA(\xi)\) and \(s\bar{A}(\eta)\) are in involution by \(k(\xi, \eta)^s\).

Lemma 16. Let \(T_L : S^1 \to S^1\) and \(T_R : S^1 \to S^1\) be the two left and right Bowen-Series transformations of a \(\Gamma\)-Möbius Markov baker transformation \((\hat{\Sigma}, \hat{T})\). Let \(A_L : S^1 \to \mathbb{R}\) and \(A_R : S^1 \to \mathbb{R}\) be two potential functions in involution with respect to a kernel \(k(\xi, \eta)\). Let \(\mathcal{L}_L\) and \(\mathcal{L}_R\) be the two Ruelle transfer operators associated to \(A_L\) and \(A_R\). Then for any \(\xi' \in S^1\) and \(\eta \in S^1\)

\[
\mathcal{L}_R(k(\xi', \cdot))(\eta) = \mathcal{L}_L(k(\cdot, \eta))(\xi').
\]

Proof. Let \(\xi' \in S^1\) and \(\eta \in S^1\) then the two finite sets

\[
\{\eta' \in S^1 : T_R(\eta') = \eta, J(\xi', \eta') = 1\}, \quad \{\xi \in S^1 : T_L(\xi) = \xi', J(\xi, \eta) = 1\}
\]

are in bijection. We thus obtain

\[
\mathcal{L}_R(k(\xi', \cdot))(\eta) = \sum_{T_R(\eta') = \eta} k(\xi', \eta')e^{A_R(\eta')} \\
= \sum_{T_L(\xi) = \xi'} k(\xi, \eta)e^{A_L(\xi)} = \mathcal{L}_L(k(\cdot, \eta))(\xi')
\]

\[\square\]

Theorem 1 now follows immediately from lemmas 15 and 16.
Proof of Theorem 1. We first prove that \( \psi_{f,s}(\xi) = \int k(\xi,\eta)^{n} D_{f,s}(\eta) \) is solution of the equation \( L_{s}^{L} \psi_{f} = \psi_{f} \) with \( k(\xi,\eta) = J(\xi,\eta)/d^{2}(\xi,\eta) \).

\[
\psi_{f,s}(\xi') = \int k^{s}(\xi',\eta') D_{f,s}(\eta') = \int L_{s}^{R}(k^{s}(\xi',\cdot))(\eta) D_{f,s}(\eta) = \int L_{s}^{L}(k^{s}(\cdot,\eta')) D_{f,s}(\eta) = (L_{s}^{L} \psi_{f,s})(\xi').
\]

We next prove that \( \psi_{f,s} \neq 0 \). By contradiction, suppose that \( \psi_{f,s}(\xi') = 0 \) for all \( \xi' \in S^{1} \). Following Haydn [10], we introduce step functions of the form

\[
\tilde{\chi}(\xi',\eta') = \chi \circ pr_{1} \circ \hat{T}^{-1}(\xi',\eta'),
\]

where \( \chi = \chi(\xi) \) depends only on \( \xi \). For instance, for some fixed \( \xi' \), let \( \chi \) be the characteristic function of the interval \( I^{L}(n,\xi) = \cap_{k=0}^{n} T_{L}^{-k}(I^{L} \circ T_{L}^{k}(\xi)) \) for some \( \xi \) such that \( T_{L}^{n}(\xi) = \xi' \). Let \( Q^{R}(\xi') = \{ \eta \in S^{1} : J(\xi',\eta) = 1 \} \) and

\[
\gamma_{L}[n,\xi] = \gamma_{L}[T_{L}^{-1}(\xi)] \cdots \gamma_{L}[T_{L}(\xi)] \gamma_{L}[\xi], \quad Q^{R}(n,\xi) = \gamma_{L}[n,\xi] Q^{R}(\xi).
\]

Then \( \tilde{\chi} \) is equal to the indicatrix function of the rectangle \( I^{L}(\xi') \times Q^{R}(n,\xi) \) and \( Q^{R}(\xi') \) is equal to the disjoint union of the intervals \( Q^{R}(n,\xi) \) for all \( \xi \) such that \( T_{L}^{n}(\xi) = \xi' \). We also call \( \Delta(\xi') \) the set of endpoints of \( Q^{R}(n,\xi) \) for all \( T_{L}^{n}(\xi) = \xi' \) and notice that \( \Delta(\xi') \) is a dense set of \( Q^{R}(\xi') \). Using the same ideas as in lemma 16, we obtain

\[
\int \tilde{\chi}(\xi',\eta') k^{s}(\xi',\eta') D_{f,s}(\eta') = (L_{s}^{L})^{n}(\chi \psi_{f,s})(\xi') = 0, \quad \forall \xi' \in S^{1}.
\]

In particular, if \( \bar{\alpha}(\xi') < \bar{\beta}(\xi') < \bar{\alpha}(\xi') + 2\pi \) are chosen such that \( \exp i\bar{\alpha}(\xi') \) and \( \exp i\bar{\beta}(\xi') \) are the two endpoints of the interval \( Q^{R}(\xi') \), if \( \bar{k}(\theta) = k(\xi', \exp i\theta) \), then for every \( \beta \in [\bar{\alpha}(\xi'), \bar{\beta}(\xi')] \cap \Delta(\xi') \),

\[
\tilde{k}(\beta) D_{f,s}(\beta) = \bar{k}(\bar{\alpha}(\xi')) \bar{k}(\bar{\beta}(\xi')) + \int_{\bar{\alpha}(\xi')}^{\beta} \frac{\partial \bar{k}}{\partial \theta} D_{f,s}(\theta) d\theta.
\]

Since \( \bar{k}(\theta) \neq 0 \) for all \( \theta \in [\bar{\alpha}(\xi'), \bar{\beta}(\xi')] \), we conclude successively that the above equality applies for all \( \beta \in [\bar{\alpha}(\xi'), \bar{\beta}(\xi')] \), the two functions \( \bar{k}(\beta) D_{f,s}(\beta) \) and \( D_{f,s}(\beta) \) are \( C^{1} \),

\[
\int_{\bar{\alpha}(\xi')}^{\beta} k(\theta) \frac{\partial D_{f,s}}{\partial \theta} d\theta = 0, \quad \forall \beta \in [\bar{\alpha}(\xi'), \bar{\beta}(\xi')]
\]
$\mathcal{D}_{f,s}(\theta)$ is a constant function on each $[\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$ and therefore everywhere on $S^1$. The distribution $\mathcal{D}_{f,s}$ would be equal to zero which is impossible since it represents a non zero eigenfunction $f(z)$.

References


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