Eigenfunctions of the Laplacian and associated Ruelle operator

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Abstract

Let $\Gamma$ be a co-compact Fuchsian group of isometries on the Poincaré disk $D$ and $\Delta$ the corresponding hyperbolic Laplace operator. Any smooth eigenfunction $f$ of $\Delta$, equivariant by $\Gamma$ with real eigenvalue $\lambda = -s(1-s)$, where $s = \frac{1}{2} + it$, admits an integral representation by a distribution $D_{f, s}$ (the Helgason distribution) which is equivariant by $\Gamma$ and supported at infinity $\partial D = S^1$. The geodesic flow on the compact surface $D/\Gamma$ is conjugate to a suspension over a natural extension of a piecewise analytic map $T : S^1 \to S^1$, the so-called Bowen-Series transformation. Let $L_s$ be the complex Ruelle transfer operator associated to the jacobian $-s \ln |T'|$. M. Pollicott showed that $D_{f, s}$ is an eigenfunction of the dual operator $L^*_s$ for the eigenvalue 1. Here we show the existence of a (nonzero) piecewise real analytic eigenfunction $\psi_{f, s}$ of $L_s$ for the eigenvalue 1, given by an integral formula

$$\psi_{f, s}(\xi) = \int \frac{J(\xi, \eta)}{|\xi - \eta|^{2s}} D_{f, s}(d\eta),$$

where $J(\xi, \eta)$ is a $\{0, 1\}$-valued piecewise constant function whose definition depends upon the geometry of the Dirichlet fundamental domain representing the surface $D/\Gamma$.

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1 Introduction

Consider the Laplace operator $\Delta$ defined by

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

on the Lobatchevskii upper half-plane $\mathbb{H} = \{ w = x + iy \in \mathbb{C}; y > 0 \}$, equipped with the hyperbolic metric $ds_\mathbb{H} = \frac{|dw|}{y}$, and the eigenvalue problem

$$\Delta f = -s(1-s)f,$$

where $s$ is of the form $s = \frac{1}{2} + it$, with $t$ is real. We shall also consider the same corresponding Laplace operator

$$\Delta = \frac{1}{4} (1 - |z|^2)^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

and eigenvalue problem

$$\Delta f = -s(1-s)f,$$

defined on the Poincaré disk $\mathbb{D} = \{ z = x + yi \in \mathbb{C}; |z| < 1 \}$, equipped with the metric $ds_\mathbb{D} = 2 \frac{|dz|}{1-|z|^2}$.

Helgason showed in [11] and [12] that any eigenfunction $f$ associated to this eigenvalue problem can be obtained by means of a generalized Poisson representation

$$f(w) = \int_{-\infty}^{\infty} \left( \frac{1+t^2}{x-t^2} + \frac{2}{y^2} \right)^s \mathcal{D}_f(t), \quad \text{for } w \in \mathbb{H},$$

or

$$f(z) = \int_{\partial \mathbb{D}} \left( \frac{1-|z|^2}{|z-\xi|^2} \right)^s \mathcal{D}_f(\xi), \quad \text{for } z \in \mathbb{D},$$

where $\mathcal{D}_f$ or $\mathcal{D}_f^\mathbb{H}$ are analytic distributions called from now on Helgason’s distributions. We have used the canonical isometry between $z \in \mathbb{D}$ and $w \in \mathbb{H}$, namely $w = i \frac{1+z}{1+z}$ or $z = i \frac{w}{1+w}$. The hyperbolic metric is given in $\mathbb{H}$ and in $\mathbb{D}$ by

$$ds_\mathbb{H}^2 = \frac{dx^2 + dy^2}{y^2}, \quad ds_\mathbb{D}^2 = \frac{4(dx^2 + dy^2)}{(1-|z|^2)^2}.$$
We shall be interested in a more restricted problem, where the eigenfunction \( f \) is also automorphic with respect to a co-compact Fuchsian group \( \Gamma \), i.e., a discrete subgroup of the group of Möbius transformations (see [20], [25], [5]) with compact fundamental domain. It is known that the eigenvalues \( \lambda = s(1 - s) = \frac{1}{4} + t^2 \) form a discrete set of positive real numbers with finite multiplicity and accumulating at \( +\infty \) (see [13]).

M. Pollicott showed [21] that the Helgason’s distribution can be seen as a generalized eigenmeasure of the dual complex Ruelle transfer operator associated to a subshift of finite type defined at infinity. Let \( T_L \) be the left Bowen-Series transformation that acts on the boundary \( S^1 = \partial \mathbb{D} \) and is associated to a particular set of generators of \( \Gamma \). The precise definition of \( T_L \) has been given in [8], [22], [23], [24], and more geometrical descriptions have then been given in [1] and [18]. Specific examples of the Bowen-Series transformation have been studied in [17] and [4] for the modular surface and in [3] for a symmetric compact fundamental domain of genus two. The map \( T_L \) is known to be piecewise \( \Gamma \)-Möbius constant, Markovian with respect to a partition \( \{ I^L_k \} \) of intervals of \( S^1 \), on which the restriction of \( T_L \) is constant and equal to an element \( \gamma_k \) of \( \Gamma \), transitive and orbit equivalent to \( \Gamma \). Let \( \mathcal{L}_s^L \) be the complex Ruelle transfer operator associated to the map \( T_L \) and the potential \( A_L = -s \ln |T'_L| \), namely

\[
(\mathcal{L}_s^L \psi)(\xi') = \sum_{T_L(\xi) = \xi'} e^{A_L(\xi)} \psi(\xi) = \sum_{T_L(\xi) = \xi'} \frac{\psi(\xi)}{|T'_L(\xi)|^s},
\]

where the summation is taken over all pre-images \( \xi \) of \( \xi' \) under \( T_L \). Here \( T'_L \) denotes the Jacobian of \( T_L \) with respect to the canonical Lebesgue measure on \( S^1 \). In the case of an automorphic eigenfunction \( f \) of \( \Delta \), Pollicott showed that the corresponding Helgason distribution \( D_{f,s} \) satisfies the dual functional equation

\[
(\mathcal{L}_s^L)^*(D_{f,s}) = D_{f,s}
\]
or, according to Pollicott’s terminology, the parameter \( s \) is a (dual) Perron-Frobenius value, that is, 1 is an eigenvalue for the dual Ruelle transfer operator.

Although suggested in [21], it is not clear whether \( s \) could be a Perron-Frobenius value, that is, whether 1 could also be an eigenvalue for \( \mathcal{L}_s^L \), not only for \( (\mathcal{L}_s^L)^* \). Our goal in this paper is to show that this is actually the case.

The three main ingredients we use are the following:
• Otal’s proof of Helgason’s distribution in [19], giving more precise information on $D_{f,s}$ and enabling us to integrate piecewise $C^1$ test functions, instead of real analytic globally defined test functions;

• a more careful reading of [1], [18], [8], and [24], or a careful study of a particular example in [16], which enables us to construct a piecewise $\Gamma$-Möbius baker transformation (“arithmetically” conjugate to the geodesic billiard);

• the existence of a kernel that we introduced in [3], which enables us to permute past and future coordinates and transfer a dual eigendistribution to a piecewise real analytic eigenfunction. Haydn (in [10]) has introduced a similar kernel in a more abstract setting, without geometric considerations.

More precisely, we prove the following theorem

**Theorem 1.** Let $\Gamma$ be a co-compact Fuchsian group of the hyperbolic disk $\mathbb{D}$ and $\Delta$ the corresponding hyperbolic Laplace operator. Let $\lambda = s(1 - s)$, with $s = \frac{1}{2} + it$, and let $f$ be an eigenfunction of $-\Delta$, automorphic with respect to $\Gamma$, that is, $\Delta f = -\lambda f$ and $f \circ \gamma = f$, for every $\gamma \in \Gamma$. Then there exists a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ on $\mathbb{S}^1$ that is a solution of the functional equation

$$L_s^L(\psi_{f,s}) = \psi_{f,s},$$

where $L_s^L$ is the complex Ruelle transfer operator associated to the left Bowen-Series transformation $T_L : \mathbb{S}^1 \to \mathbb{S}^1$ and the potential $A_L = -s \ln |T_L'|$.

Moreover, $\psi_{f,s}$ admits an integral representation via Helgason’s distribution $D_{f,s}$, representing $f$ at infinity, and a geometric positive kernel $k(\xi, \eta)$ defined on a finite set of disjoint rectangles $\bigcup_k I_k^L \times Q_k^R \subset \mathbb{S}^1 \times \mathbb{S}^1$, namely,

$$\psi_{f,s}(\xi) = \int_{Q_k^R} k^s(\xi, \eta) D_{f,s}^R(\eta) = \int_{Q_k^R} \frac{1}{|\xi - \eta|^{2s}} D_{f,s}^R(\eta),$$

for every $\xi \in I_k^L$, where $I_k^L$ and $Q_k^R$ are intervals of $\mathbb{S}^1$ on which disjoint closure, and $\{I_k^L\}_k$ is a partition of $\mathbb{S}^1$ where $T_L$ is injective, Markovian and piecewise $\Gamma$-Möbius constant.
Lewis [14] and, later, Lewis and Zagier [15], started a different approach to understand Maass wave forms. They were able to identify in a bijective way Maass wave forms of $\text{PSL}(2, \mathbb{Z})$ and solutions of a functional equation with 3 terms closely related to Mayer’s transfer operator. Their setting is strongly dependent of the modular group. Our theorem 1 may be viewed as part of their program for co-compact Fuchsian groups. The Helgason distribution has been used by S. Zelditch in [26] to generalise microlocal analysis on hyperbolic surfaces, by L. Flaminio and G. Forni in [9], to study invariant distributions by the horocycle flow, and by N. Anantharaman and S. Zelditch in [2], to understand the “Quantum Unique Ergodicity Conjecture”.

2 Preliminary results

Let $\Gamma$ be a co-compact Fuchsian group of the Poincaré disk $\mathbb{D}$. We denote by $d(w, z)$ the hyperbolic distance between two points of $\mathbb{D}$, given by the Riemannian metric $ds^2 = 4(dx^2 + dy^2)/(1 - |z|^2)^2$. Let $M = \mathbb{D}/\Gamma$ be the associated compact Riemann surface, $N = T^1 M$ the unit tangent bundle, and $\Delta$ the Laplace operator on $M$. Let $f : M \to \mathbb{R}$ be an eigenfunction of $-\Delta$ or, in other words, a $\Gamma$-automorphic function $f : \mathbb{D} \to \mathbb{R}$ satisfying $\Delta f = -s(1-s)f$ for the eigenvalue $\lambda = s(1-s) > \frac{1}{4}$ and such that $f \circ \gamma = f$, for every $\gamma \in \Gamma$. We know that $f$ is $C^\infty$ and uniformly bounded on $\mathbb{D}$. Thanks to Helgason’s representation theorem, $f$ can be represented as a superposition of horocycle waves, given by the Poisson kernel

$$P(z, \xi) := e^{b_{\xi}(0, z)} = \frac{1 - |z|^2}{|z - \xi|^2},$$

where $b_{\xi}(w, z)$ is the the Busemann cocycle between two points $w$ and $z$ inside the Poincaré disk, observed from a point at infinity $\xi \in \mathbb{S}^1$, defined by

$$b_{\xi}(w, z) := "d(w, \xi) - d(z, \xi)" = \lim_{t \to \xi} d(w, t) - d(z, t),$$

where the limit is uniform in $t \to \xi$ in any hyperbolic cone at $\xi$. Helgason’s theorem states that

$$f(z) = \int_{\mathbb{D}} P^s(z, \xi) D_{f,s}(\xi) = \langle D_{f,s}, P^s(z, \cdot) \rangle$$
for some analytic distribution $D_{f,s}$ acting on real analytic functions on $S^1$. Unfortunately, Helgason's work is too general and is valid for any eigenfunction not necessarily equivariant by a group. For bounded $C^2$ functions $f$, Otal [19] has shown that the distribution $D_{f,s}$ has stronger properties and can be defined in a simpler manner.

We first recall some standard notations in hyperbolic geometry. We call $d(z, z_0)$ the hyperbolic distance between two points: for instance, the distance from the origin is given by $d(O, \tanh(\frac{r}{2})e^{i\theta}) = r$. Let $C(O, r)$ denote the set of points in $\mathbb{D}$ at hyperbolic distance $r$ from the origin,

$$C(O, r) = \{ z \in \mathbb{D}; |z| = \tanh(\frac{r}{2}) \}$$

and, more generally, given for any interval $I$ at infinity and any point $z_0 \in \mathbb{D}$, let $C(z_0, r, I)$ denote the angular arc at the hyperbolic distance $r$ from $z_0$ delimited at infinity by $I$, that is,

$$C(z_0, r, I) = \{ z \in \mathbb{D}; z \in [[z_0, \xi]] \text{ for some } \xi \in I \text{ and } d(z, z_0) = r \},$$

where $[[z_0, \xi]]$ denotes the geodesic ray from $z_0$ to the point $\xi$ at infinity. Let $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ denote the exterior normal derivative to $C(O, r)$ and $|dz|_\mathbb{D} = \sinh(r) \, d\theta$ the hyperbolic arc length on $C(O, r)$.

**Theorem 2.** ([19]) Let $f$ be a bounded $C^2$ eigenfunction satisfying $\Delta f = -s(1 - s)f$. Then:

1. There exists a continuous linear functional $D_{f,s}$ acting on $C^1$ functions of $S^1$, defined by

$$\int \psi(\xi) D_{f,s}(\xi) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_{C(O, r)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|_\mathbb{D},$$

where $c(s)$ is a non zero normalizing constant such that $\langle D_{f,s}, 1 \rangle = f(0)$, and $\psi(z)$ is any $C^1$ extension of $\psi(\xi)$ to a neighborhood of $S^1$.

2. $D_{f,s}$ represents $f$ in the following sense:

$$f(z) = \int [P(z, \xi)]^s D_{f,s}(\xi), \quad \forall \ z \in \mathbb{D}.$$
3. For all $0 \leq \alpha \leq 2\pi$, the following limit exists:

$$
\tilde{D}_{f,s}(\alpha) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_0^\alpha e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) \left( \tanh\left(\frac{r}{2}\right)e^{i\theta} \right) \sinh(r) d\theta.
$$

The convergence is uniform in $\alpha \in [0, 2\pi]$ and $\tilde{D}_{f,s}(0) = 0$.

4. $\tilde{D}_{f,s}$ can be extended to $\mathbb{R}$ as a $\frac{1}{2}$-Hölder continuous function satisfying:

(a) $\tilde{D}_{f,s}(\theta + 2\pi) = \tilde{D}_{f,s}(\theta) + f(0)$, for every $\theta \in \mathbb{R}$,

(b) for any $C^1$ function $\psi : S^1 \to \mathbb{C}$, denoting $\tilde{\psi}(\theta) = \psi(\exp i\theta)$,

$$
\int \psi(\xi) D_{f,s}(\xi) = \tilde{\psi}(0) f(0) - \int_0^{2\pi} \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{D}_{f,s}(\theta) d\theta.
$$

Using similar technical tools as Otal, one can prove the following extension of $D_{f,s}$ on piecewise $C^1$ functions, that is, on functions not necessarily continuous but which admit a $C^1$ extension on each interval $[\xi_k, \xi_{k+1}]$ of some finite and ordered subdivision $\{\xi_0, \xi_1, \ldots, \xi_{r-1}\}$ of $S^1$.

**Proposition 3.** Let $f$ and $D_{f,s}$ be as in Theorem 2.

1. For any interval $I \subset S^1$ and any function $\psi : I \to \mathbb{C}$, which is $C^1$ on the closure of $I$ and null outside $I$, the following limit exists:

$$
\int \psi(\xi) D_{f,s}(\xi) := \frac{1}{c(s)} \lim_{r \to +\infty} \int_{C(O,r,I)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) \left| dz \right|_D
$$

where again $\psi(z)$ is any $C^1$ extension of $\psi(\xi)$ to a neighborhood of $S^1$.

2. For any $0 \leq \alpha < \beta \leq 2\pi$ and any $C^1$ function $\psi$ on the interval $I = [\exp(\imath\alpha), \exp(\imath\beta)]$,

$$
\int \psi(\xi) D_{f,s}(\xi) = \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_\alpha^\beta \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{D}_{f,s}(\theta) d\theta,
$$

where $\tilde{D}_{f,s}$ and $\tilde{\psi}(\theta)$ have been defined in Theorem 2.
Proof. Given $\alpha \in [0, 2\pi]$, let $I = \{\mathrm{e}^{i\theta} \mid 0 \leq \theta \leq \alpha\}$ be an interval in $S^1$, and $\psi$ a $C^1$ function defined on a neighborhood of $S^1$. Denote $\tilde{\psi}(r, \theta) = \psi(\tanh(\frac{r}{2})e^{i\theta})$ and $K(r, \theta) = e^{-sr}(\frac{\partial f}{\partial n} + sf)(\tanh(\frac{r}{2}e^{i\theta}))\sinh(r)$. Then

$$\frac{1}{c(s)} \int_{C(0,r,I)} \psi(z)e^{-sr}\left(\frac{\partial f}{\partial n} + sf\right) |dz|_D$$

$$= \int_0^\alpha \tilde{\psi}(r, \beta)K(r, \beta) d\beta$$

$$= \int_0^\alpha \left[ \tilde{\psi}(r, \alpha) + \int_\beta^\alpha - \frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta) d\theta \right]K(r, \beta) d\beta$$

$$= \tilde{\psi}(r, \alpha) \int_0^\alpha K(r, \beta) d\beta - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta) \left[ \int_0^\theta K(r, \beta) d\beta \right] d\theta.$$  

Since $\int_0^\alpha K(r, \beta) d\beta \to \tilde{D}_{f,s}(\alpha)$ uniformly in $\alpha \in [0, 2\pi]$, the left-hand side of the previous equality converges to

$$\int \psi(\xi) \mathbf{1}_{\{\xi \in I\}} \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(\alpha)\tilde{D}_{f,s}(\alpha) - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(\theta)\tilde{D}_{f,s}(\theta) d\theta.$$  

The second part of the proposition follows subtracting such an expression from another one, such as

$$\int \psi(\xi) \mathbf{1}_{\{\xi = \mathrm{e}^{i\theta}; 0 \leq \theta \leq \beta\}} \tilde{D}_{f,s}(\xi) - \int \psi(\xi) \mathbf{1}_{\{\xi = \mathrm{e}^{i\theta}; 0 \leq \theta \leq \alpha\}} \tilde{D}_{f,s}(\xi).$$

If, in addition, we assume that $f$ is equivariant with respect to a cocompact Fuchsian group $\Gamma$, Pollicott observed in [21] that $\mathcal{D}_{f,s}$, acting on real analytic functions, is equivariant by $\Gamma$, that is, satisfies $\gamma^*(\mathcal{D}_{f,s})(\xi) = |\gamma'(\xi)|^s\mathcal{D}_{f,s}(\xi)$, for all $\gamma \in \Gamma$. Because Otal’s construction is more precise and implies that Helgason’s distribution also acts on piecewise $C^1$ functions, the above equivariance property can be improved in the following way.

**Proposition 4.** Let $f : \mathbb{D} \to \mathbb{R}$ be a $C^2$ function, $I \subset S^1$ an interval and $\psi : I \to \mathbb{C}$ a $C^1$ function on the closure of $I$. If $f$ satisfies $f \circ \gamma = f$, for some $\gamma \in \Gamma$, (f is not necessarily automorphic), then

$$\langle \mathcal{D}_{f,s}, \psi \circ \gamma^{-1} \rangle'_{|\gamma' \circ \gamma^{-1}|^s_1} \mathbf{1}_{\gamma(I)} = \langle \mathcal{D}_{f,s}, \psi \rangle_1.$$
The main difficulty here is to transfer the equivariance property \( f \circ \gamma = f \) to an equivalent property for the extension of \( Df,s \) to piecewise \( C^1 \) functions. If \( I = \mathbb{S}^1 \) and \( \psi \) is real analytic, then, by uniqueness of the representation, Proposition 4 is easily proved. It seems that just knowing the fact that \( Df,s \) is the derivative of some H"older function is not enough to reach a conclusion. The following proof uses Otal’s approach and, essentially, the extension of \( Df,s \) described in Part 1 of Proposition 3.

**Proof of Proposition 4.** First we prove the proposition for \( \psi = 1 \). Let \( g(z) = \exp(-sd(O, z)) \). By definition of \( Df,s \), we obtain

\[
\int 1_I(\xi) Df,s(\xi) = \lim_{r \to +\infty} \frac{1}{c(s)} \int_{C(O, r', I)} \left( g' \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) |dz|_D
\]

where \( r' = r + d(O, O') \), \( O' = \gamma(O) \) and \( g' = g \circ \gamma^{-1} \). Notice that the domain bounded by the circle \( C(O', r') \) contains the circle \( C(O, r) \). Let \( P'Q' \) be the positively oriented arc \( C(O, r, \gamma(I)) \) and \( P''Q'' \) be the arc \( C(O', r', \gamma(I)) \). Then the two geodesic segments \([P, P']\) and \([Q, Q']\) belong to the annulus \( r \leq d(z, O) \leq r + 2d(O, O') \) and their length is uniformly bounded.

We now use Green’s formula to compute the right hand side of the above expression. Let \( \Omega \) denote the domain delimited by \( P, P', Q, Q' \) using the corresponding arcs and geodesic segments, and let \( dv = \sinh(r) \, dr \, d\theta \) be the hyperbolic volume element. We obtain

\[
\int_{P'Q'} \left( g' \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) |dz|_D = \int_{P''Q''} \left( g' \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) |dz|_D
\]

\[
- \int_{[P, P']} \cdots |dz|_D - \int_{[Q', Q']} \cdots |dz|_D
\]

\[
+ \int_{\Omega} (g' \Delta f - f \Delta g') \, dv.
\]

When \( r \) tends to infinity, the last three terms at the right-hand side tend to 0, since along the geodesic segments \([P, P']\) and \([Q, Q']\), the gradient \( \nabla g' \) is uniformly bounded by \( \exp(-\frac{1}{2}r) \) and

\[
g' \Delta f - f \Delta g' = sg'f \sinh(d(z, O'))^{-2} \quad \text{and} \quad \frac{\partial}{\partial n} g' + sg'
\]
are uniformly bounded by a constant times \(\exp(-\frac{5}{2}r)\) in the domain \(\Omega\), for the first expression, and by a constant times \(\exp(-\frac{3}{2}r)\) on \(C(\mathcal{O}, r)\), for the second expression. It follows that

\[
\int 1_I(\xi)D_{f,s}(\xi) = \lim_{r \to +\infty} \frac{1}{c(s)} \int_{C(\mathcal{O}, r, \gamma(I))} g'(\frac{\partial f}{\partial n} + sf) \ |dz|_D
\]

\[
= \lim_{r \to +\infty} \frac{1}{c(s)} \int_{C(\mathcal{O}, r, \gamma(I))} [\psi(z)]^s e^{-sr} \left(\frac{\partial f}{\partial n} + sf\right) \ |dz|_D,
\]

where \(\psi(z) = \exp(d(O, z) - d(O, \gamma^{-1}(z)))\). Now we observe that

\[
\left\{
\begin{array}{ll}
\psi(z) = \exp s \left(d(O, z) - d(\gamma(O), z)\right), & \text{for } z \in \mathbb{D}, \\
\psi(\xi) = \exp b_\xi(O, \gamma(O)) = |\gamma' \circ \gamma^{-1}(\xi)|^{-1}, & \text{for } \xi \in \partial \mathbb{D},
\end{array}
\right.
\]

actually coincides with a real analytic function \(\Psi(z)\) defined in a neighborhood of \(S^1\), given explicitly by

\[
\Psi(z) = \left(\frac{(1 + |z|^2)}{(1 + |\gamma^{-1}(z)|^2 |\gamma' \circ \gamma^{-1}(z)|)}\right)^s.
\]

Thus we proved that

\[
\int 1_I(\xi) D_{f,s}(\xi) = \int \frac{1_{\gamma(I)}(\xi)}{|\gamma' \circ \gamma^{-1}(\xi)|^s} D_{f,s}(\xi).
\]

Now we prove the general case. We use the same notation for the lifting \(\gamma : \mathbb{R} \mapsto \mathbb{R}\) of a Möbius transformation \(\gamma : S^1 \mapsto S^1\). The lifting satisfies \(\gamma(\alpha + 2\pi) = \gamma(\alpha) + 2\pi\), \(\exp(i\gamma(\alpha)) = \gamma(\exp(i\alpha))\) and \(\gamma'(\alpha) = |\gamma'(\alpha)|\), for all \(\alpha \in \mathbb{R}\). Using Proposition 3, we obtain

\[
\tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha) = \frac{\tilde{D}_{f,s} \circ \gamma(\beta)}{\gamma'(\beta)^s} - \frac{\tilde{D}_{f,s} \circ \gamma(\alpha)}{\gamma'(\alpha)^s} - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left(\frac{1}{(\gamma' \circ \gamma^{-1} (\theta))^s}\right) \tilde{D}_{f,s}(\theta) \, d\theta.
\]

For any \(C^1\) function \(\psi(\xi)\) defined on \(I\), we denote \(\tilde{\psi}(\theta) = \psi(\exp i\theta)\), and obtain
\[ LHS := \int \psi(\xi) \mathbf{1}_{I}(\xi) \mathcal{D}_{f,s}(\xi) \]
\[ = \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_{\alpha}^{\beta} \frac{\partial}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta) \, \mathrm{d}\theta \]
\[ = \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_{\alpha}^{\beta} \frac{\partial}{\partial \theta} \left( \tilde{\psi} \circ \gamma^{-1}(\theta) \right) \tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta) \, \mathrm{d}\theta \]
\[ = \tilde{\psi}(\beta) \left( \tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha) \right) - \int_{\alpha}^{\beta} \frac{\partial}{\partial \theta} \left( \tilde{\psi}(\gamma^{-1}\theta) - \tilde{\mathcal{D}}_{f,s}(\alpha) \right) \, \mathrm{d}\theta. \]

We now use the above equivariance and replace both \( \tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha) \) and \( \tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta) - \tilde{\mathcal{D}}_{f,s}(\alpha) \) by the corresponding formula involving \( \tilde{\mathcal{D}}_{f,s} \circ \gamma(\beta) \), \( \tilde{\mathcal{D}}_{f,s} \circ \gamma(\alpha) \), \( \tilde{\mathcal{D}}_{f,s}(\theta) \). Thus
\[ LHS = \int \frac{\psi \circ \gamma^{-1}(\xi)}{|\gamma'(\gamma^{-1}(\xi))|^s} \mathbf{1}_{I}(\xi) \mathcal{D}_{f,s}(\xi). \]

Following [1], [8], [22], [23], [24] and [18] for the general case and [16] for a specific example we recall the definition of the left \( T_L \) and right \( T_R \) Bowen-Series transformation. The hyperbolic surface we are interested in is given by the quotient of the hyperbolic disk \( \mathbb{D} \) by a cocompact Fuchsian group \( \Gamma \). Given a point \( \mathcal{O} \in \mathbb{D} \), let
\[ D_{\Gamma, \mathcal{O}} = \{ z \in \mathbb{D}; d(z, \mathcal{O}) < d(z, \gamma(\mathcal{O})), \quad \forall \gamma \in \Gamma \} \]

Denote the corresponding Dirichlet domain, a convex fundamental domain with compact closure in \( \mathbb{D} \), admitting an even number of geodesic sides and an even number of vertices, some of which may be elliptic. More precisely, the boundary of \( D_{\Gamma, \mathcal{O}} \) is a disjoint union of semi-closed geodesic segments \( S_{-r}^L, \ldots, S_{-1}^L, S_1^L, \ldots, S_r^L \), closed to the left and open to the right, or, equivalently, to a union of semi-closed geodesic segments \( S_{-r}^R, \ldots, S_{-1}^R, S_1^R, \ldots, S_r^R \).
closed to the right and open to the left; for each \( k \), the intervals \( S^L_k \) and \( S^R_k \) have the same endpoints and \( S^L_k \) is associated to \( S^R_k \) by an element \( a_k \in \Gamma \) satisfying \( a_k(S^L_k) = S^R_{-k} \). The elements \( a_k \) generate \( \Gamma \) and satisfy \( a_{-k} = a_k^{-1} \), for \( k = \pm 1, \pm 2, \ldots, \pm r \).

To define the two Bowen-Series transformations \( T_L \) and \( T_R \) geometrically, we need to impose a geometric condition on \( \Gamma \): following [8], [22] and [24], we say that \( \Gamma \) satisfies the even corner property if, for each \( 1 \leq |k| \leq r \), the complete geodesic line through \( S^L_k \) is equal to a disjoint union of \( \Gamma \)-translates of the sides \( S^L_l \), with \( 1 \leq |l| \leq r \). Some \( \Gamma \) do not satisfy this geometric property. Nevertheless, any two co-compact Fuchsian groups \( \Gamma \) and \( \Gamma' \), with identical signature, are geometrically isomorphic, that is, there exists a group isomorphism \( h_\ast : \Gamma \to \Gamma' \) and a quasi-conformal orientation preserving homeomorphism \( h : \mathbb{D} \to \mathbb{D} \) admitting an extension to a conjugating homeomorphism \( h : \partial \mathbb{D} \to \partial \mathbb{D} \), that is,

\[
h(\gamma(z)) = h_\ast(\gamma)(h(z)), \quad \forall \gamma \in \Gamma.
\]

An important observation in [8], [22] and [24] is that any co-compact Fuchsian group is geometrically isomorphic to a Fuchsian group with identical signature and satisfying the even corner property. We are going to recall the Bowen and Series construction in the case that \( \Gamma \) possesses the even corner property and will show that their main conclusions remain valid under geometric isomorphisms.

The complete geodesic line associated to a side \( S^L_k \) cuts the boundary at infinity \( S^1 \) at two points \( s^L_k \) and \( s^R_k \), positively oriented with respect to \( s^L_k \), the oriented geodesic line \( ]s^L_k, s^R_k[ \) seeing the origin \( O \) to the left. Both end points \( s^L_k \) and \( s^R_k \) are neutrally stable with respect to the associated generator \( a_k \), that is, \( |a_k'(s^L_k)| = |a_k'(s^R_k)| = 1 \). The family of open intervals \( ]s^L_k, s^R_k[ \) covers \( S^1 \); since these intervals \( ]s^L_k, s^R_k[ \) overlap each other, there is no canonical partition adapted to this covering. Nevertheless, we may associate two well defined partitions, the left partition \( A_L \) and the right partition \( A_R \).

The former consists of disjoint half-closed intervals,

\[
A_L = \{ A^L_{-r}, \ldots, A^L_{-1}, A^L_1, \ldots, A^L_r \},
\]

given by \( A^L_k = [s^L_k, s^L_{(k)}] \) where \( s^L_{(k)} \) denotes the nearest point \( s^L_k \) after \( s^L_k \), according to a positive orientation. Each \( A^L_k \) belongs to the unstable domain of the hyperbolic element \( a_k \), that is, \( |a_k'(\xi)| \geq 1 \), for each \( \xi \in A^L_k \). By
definition, the left Bowen-Series transformation $T_L : S^1 \rightarrow S^1$ is given by

$$T_L(\xi) = a_k(\xi), \quad \text{if} \quad \xi \in A^L_k.$$ 

Analogously, $S^1$ can be partitioned into half-closed intervals

$$A_R = \{A^R_{-r}, \ldots, A^R_{-1}, A^R_1, \ldots, A^R_r\},$$

where $A^R_k = [s^R_{j(k)}, s^R_k]$, and $s^R_{j(k)}$ denotes the nearest $s^R_j$ before $s^R_k$, according to a positive orientation. The right Bowen-Series transformation is given by

$$T_R(\eta) = a_k(\eta), \quad \text{if} \quad \eta \in A^R_k.$$ 

The two partitions $A^L$ and $A^R$ generate two ways of coding a trajectory. Let $\gamma_L : S^1 \rightarrow \Gamma$ and $\gamma_R : S^1 \rightarrow \Gamma$ be the left and right symbolic coding defined by

$$\gamma_L[\xi] = a_k, \quad \text{if} \quad \xi \in A^L_k, \quad \text{and} \quad \gamma_R[\eta] = a_k, \quad \text{if} \quad \eta \in A^R_k.$$ 

In particular, $T_R(\eta) = \gamma_R[\eta](\eta)$ and $T_L(\xi) = \gamma_L[\xi](\xi)$, for each $\xi \in S^1$. Also, it is known that $T^2_R$ and $T^2_L$ are expanding. Series, in [22], [23] and [24], and later, Adler and Flatto in [1], proved that $T_L$ (respectively $T_R$) is Markov with respect to a partition of $I^L = \{I^L_k\}_{k=1}^q$ (respectively $I^R = \{I^R_l\}_{l=1}^q$) that is finer than $A^L$ (respectively $A^R$). The semi-closed intervals $I^L_k$ and $I^R_l$ are of the same kind as $A^L_k$ and $A^R_l$, and have the same closure.

**Definition 5.** A dynamical system $(S^1, T, \{I_k\})$ is said to be a piecewise $\Gamma$-Möbius Markov transformation if $T : S^1 \rightarrow S^1$ is a surjective map, and $\{I_k\}$ is a finite partition of $S^1$ into intervals such that:

1. for each $k$, $T(I_k)$ is a union of adjacent intervals $I_l$;

2. for each $k$, the restriction of $T$ to $I_k$ coincides with an element $\gamma_k \in \Gamma$;

3. some finite iterate of $T$ is uniformly expanding.

**Theorem 6.** ([8], [24]) For any co-compact Fuchsian group $\Gamma$, there exists a piecewise $\Gamma$-Möbius Markov transformation $(S^1, T, \{I_k\})$ which is transitive and orbit equivalent to $\Gamma$.

The Ruelle transfer operator can be defined for any piecewise $C^2$ Markov transformation $(S^1, T, \{I_k\})$ and any potential function $A$. Actually, we need a particular complex transfer operator given by the potential

$$A = -s \ln |T'|.$$
For any function $\psi : \mathbb{S}^1 \to \mathbb{C}$, define

\[
(L_s(\psi))(\xi') = \sum_{T(\xi) = \xi'} e^{A(\xi)} \psi(\xi) = \sum_{T(\xi) = \xi'} \frac{\psi(\xi)}{|T'(\xi)|^s},
\]

where the summation is taken over all preimages $\xi$ of $\xi'$ under $T$. We modify $L_s$ slightly, so that it acts on the space of piecewise $C^1$ functions. Let $\{I_k\}_{k=1}^q$ be a partition of $\mathbb{S}^1$. Given a piecewise $C^1$ function and $\oplus_{k=1}^q \psi_k \in \oplus_{k=1}^q C^1(I_k)$ set

\[
L_s^L \psi = \oplus_{l=1}^q \phi_l,
\]

where $\phi_l = \sum_{I_l \subset T(I_k)} \psi_k \circ T_{k,l}^{-1} \frac{|T' \circ T_{k,l}^{-1}|^s}{T' \circ T_{k,l}^{-1}|}$,

and $T_{k,l}^{-1}$ denotes the restriction to $I_l$ of the inverse of $T : I_k \to T(I_k) \supset I_l$.

**Proposition 7.** Let $\Gamma$ be a co-compact Fuchsian group. Let $s = \frac{1}{2} + it$ and $f$ be an automorphic eigenfunction of $-\Delta$, that is, $\Delta f = -s(1-s)f$. Let $(\mathbb{S}^1, T, \{I_k\})$ be a piecewise $\Gamma$-Möbius Markov transformation and $L_s$ be the Ruelle transfer operator corresponding to the observable $A = -s \ln |T'|$. Then the Helgason distribution $D_{f,s}$ satisfies

\[
(L_s)^* D_{f,s} = D_{f,s}.
\]

**Proof.** Let $\oplus_{k=1}^q \psi_k$ be a piecewise $C^1$ function in $\oplus_{k=1}^q C^1(I_k)$. Using Proposition 4,

\[
\int (L_s \psi)(\xi') D_{f,s}(\xi) = \int \left( \sum_{l=1}^q \int_{I_l} (L_s \psi)_l(\xi) D_{f,s}(\xi) \right)
\]

\[
= \sum_{T(I_k) \supset I_l} \int_{I_l} \frac{\psi_k \circ T_{k,l}^{-1}(\xi)}{|T' \circ T_{k,l}^{-1}|^s} D_{f,s}(\xi)
\]

\[
= \sum_{T(I_k) \supset I_l} \int_{T^{-1}(I_l) \cap I_k} \psi_k(\xi) D_{f,s}(\xi)
\]

\[
= \sum_{k=1}^q \int_{I_k} \psi_k(\xi) D_{f,s}(\xi) = \int \psi(\xi) D_{f,s}(\xi).
\]

$\blacksquare$
Series in [24], Adler and Flatto in [1], and Morita in [18] noticed that $T_L$ admits a natural extension $\hat{T} : \hat{\Sigma} \mapsto \hat{\Sigma}$ strongly related to $T_R$. We also showed the existence of such a $\hat{T}$ in [16], and it was an important step in the proof of Theorem 3 of [16]. The following definition explains how the two maps $T_L$ and $T_R$ are glued together in an abstract way.

**Definition 8.** Let $\Gamma$ be a co-compact Fuchsian group. A dynamical system $(\hat{\Sigma}, \hat{T}, \{I^L_k\}, \{I^R_l\}, J)$ is said to be a piecewise $\Gamma$-Möbius baker transformation if it admits a description as follows.

1. $\{I^L_k\}$ and $\{I^R_l\}$ are finite partitions of $\mathbb{S}^1$ into disjoint intervals; $J(k,l)$ is a $\{0,1\}$-valued function, and $\hat{\Sigma}$ is the subset of $\mathbb{S}^1 \times \mathbb{S}^1$ defined by
   $$\hat{\Sigma} = \bigsqcup_{J(k,l)=1} I^L_k \times I^R_l.$$

2. For each $k$, $Q^R_k = \bigsqcup \{I^R_l; J(k,l) = 1\}$ is an interval whose closure is disjoint from $\overline{I}^L_k$. For each $l$, $Q^L_l = \bigsqcup \{I^L_k; J(k,l) = 1\}$ is an interval whose closure is disjoint from $\overline{I}^R_l$. Let $I^L_k(\xi) = I^L_k$ and $Q^L_l(\eta) = Q^L_l$, for $\xi \in I^L_k$. Let $I^R_l(\eta) = I^R_l$ and $Q^R_k(\xi) = Q^R_k$, for $\eta \in I^R_l$.

3. $\hat{T} : \hat{\Sigma} \rightarrow \hat{\Sigma}$ is bijective and is given by
   $$\{ \hat{T}(\xi, \eta) = (T_L(\xi), S_R(\xi, \eta)), \hat{T}^{-1}(\xi', \eta') = (S_L(\xi', \eta'), T_R(\eta')) \}$$
   for certain maps $T_L, T_R : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and $S_L, S_R : \hat{\Sigma} \rightarrow \mathbb{S}^1$.

4. $(\mathbb{S}^1, T_L, \{I^L_k\})$ and $(\mathbb{S}^1, T_R, \{I^R_l\})$ are piecewise $\Gamma$-Möbius Markov transformations. There exist two functions $\gamma^L : \mathbb{S}^1 \rightarrow \Gamma$, respectively $\gamma^R : \mathbb{S}^1 \rightarrow \Gamma$, that are piecewise constant on each $I^L_k$, respectively $\{I^R_l\}$, and satisfying
   $$\{ \hat{T}(\xi, \eta) = (\gamma^L(\xi)(\xi), \gamma^L(\eta)(\eta)), \hat{T}^{-1}(\xi', \eta') = (\gamma^R(\eta')(\xi'), \gamma^R(\xi')(\eta')) \}$$

The maps $T_L$ and $T_R$ are called the left and right Bowen-Series transformations, whereas $\gamma^L$ and $\gamma^R$ are the left and right Bowen-Series codings. Finally, we say that $J$ is the incidence matrix, which we extend as a function on $\mathbb{S}^1 \times \mathbb{S}^1$ defining
$$\{ J(\xi, \eta) = 1, \text{ if } (\xi, \eta) \in \hat{\Sigma}, \}$$
$$\{ J(\xi, \eta) = 0, \text{ if } (\xi, \eta) \notin \hat{\Sigma}. \}$$
Notice that this definition is equivariant by geometric isomorphisms. For co-compact Fuchsian groups satisfying the even corner property, Adler and Flatto in [1], Series in [24] (and, for a particular example, in [16]) obtained geometrically the existence of a piecewise $\Gamma$-Möbius baker transformation with left $T_L$ and right $T_R$ maps orbit equivalent to $\Gamma$. By geometric isomorphism considerations, we obtain more generally the following.

**Proposition 9.** ([1], [24], [16]) For any co-compact Fuchsian group $\Gamma$, there exists a piecewise $\Gamma$-Möbius baker transformation with left and right Bowen-Series transformations that are transitive and orbit equivalent to $\Gamma$.

The two maps $T_L$ and $T_R$ are related to the action of the group $\Gamma$ on the boundary $S^1$. The baker transformation $(\hat{\Sigma}, \hat{T})$ encodes this action into a unique dynamical system. For later reference, we state two further properties of this encoding.

**Remark 10.**

1. The two codings $\gamma_L$ and $\gamma_R$ are reciprocal, in the following sense:

   $$\gamma_R[\eta'] = \gamma_L^{-1}[\xi], \quad \text{whenever} \quad (\xi', \eta') = \hat{T}(\xi, \eta).$$

2. For any $\xi'$ and $\eta$ in $S^1$, there is a bijection between the two finite sets

   $$\{\xi; (\xi, \eta) \in \hat{\Sigma} \text{ and } T_L(\xi) = \xi'\}, \quad \{\eta'; (\xi', \eta') \in \hat{\Sigma} \text{ and } T_R(\eta') = \eta\}.$$
and \( q = q(x, y) \in \partial D_{T, O} \) for every oriented geodesic \([y, x]\), \((x, y) \in X\), such that \([q, p] = [[y, x]] \cap \tilde{D}_{T, O} \) has the same orientation as \([[y, x]]\).

For a geodesic passing through a corner, \( p = q \), unless the geodesic is tangent to a side of \( D_{T, O} \). We are now in a position to define a geometric Poincaré section \( B : X \to X \). If \((x, y) \in X\), the geodesic \([[y, x]]\) leaves \( \tilde{D}_{T, O} \) at \( p = p(x, y) \in S_i \), for some side \( S_i \). Since \( S_i^L \) and \( S_i^R \) are permuted by the generator \( a_i \), the new geodesic \( a_i([[y, x]]) = [[y', x']] \) enters again the fundamental domain at a new point \( q' = q(x', y') \) with \( q' = a_i(p) \in S_i^R \). By definition, \( B(x, y) = (x', y') \) and the map \( B : X \to X \) is called a geodesic billiard like the codings as for \( T_L \) and \( T_R \), we introduce two geometric codings \( \gamma_B : X \to \Gamma \) and \( \bar{\gamma}_B : X \to \Gamma \) given by

\[
\begin{cases}
\gamma_B[x, y] = a_i & \text{if } p(x, y) \in S_i^L, \\
\bar{\gamma}_B[x, y] = a_i & \text{if } q(x, y) \in S_i^R. 
\end{cases}
\]

Now the geodesic billiard can be defined by

\[
\begin{align*}
B(x, y) &= (\gamma_B[x, y](x), \gamma_B[x, y](y)), \\
B^{-1}(x', y') &= (\bar{\gamma}_B[x', y'](x'), \bar{\gamma}_B[x', y'](y')).
\end{align*}
\]

Notice that \( \bar{\gamma}_B \circ B = \gamma_B^{-1} \). The map \( B \) is very close to be a baker transformation: \( B \) and \( B^{-1} \) have the same structure as \( \tilde{T} \) and \( \tilde{T}^{-1} \), and \( \gamma_B \) (respectively, \( \bar{\gamma}_B \)) plays the role as \( \gamma_L \) (respectively, \( \gamma_R \)). The main difference is that \( \gamma_B[x, y] \) depends on both \( x \) and \( y \), but \( \gamma_L[\xi] \) depends only on \( \xi \). Nevertheless, we have the following crucial result.

**Theorem 11.** ([1], [24], [16]) There exists a \( \Gamma \)-Möbius baker transformation \((\Sigma, \tilde{T})\) conjugate to \((X, B)\). More precisely, there exists a map \( \rho : X \to \Gamma \) such that \( \pi(x, y) = (\rho[x, y](x), \rho[x, y](y)) \), defines a conjugating map \( \pi : X \to \Sigma \) between \( \tilde{T} \) and \( B \), such that \( \tilde{T} \circ \pi = \pi \circ B \). Equivalently, \( \gamma_L \circ \pi \) and \( \gamma_B \) are cohomologous over \((X, B)\), that is, \( \gamma_L \circ \pi \rho = \rho \circ B \gamma_B \), and \( \gamma_R \circ \pi \) and \( \bar{\gamma}_B \) are cohomologous over \((X, B)\), that is, \( \gamma_R \circ \pi \rho = \rho \circ B^{-1} \bar{\gamma}_B \).

### 3 Proof of Theorem 1

We want to associate to any eigenfunction \( f \) of the Laplace operator a nonzero piecewise real analytic function \( \psi_{f, s} \) that is a solution of the functional equation

\[
\mathcal{L}_s^L(\psi_{f, s}) = \psi_{f, s}, \quad \text{where } \mathcal{L}_s^L(\psi)(\xi') = \sum_{T_L(\xi) = \xi'} \frac{\psi(\xi)}{|T_L'(\xi)|^s}.
\]
The main idea is to use a kernel $k(\xi, \eta)$ introduced in Theorem 7 of [3], as well by in Haydn in [10], and by Bogomolny and Carioli in [6] and [7], in the context of double-sided subshifts of finite type. We begin by extending this definition to include baker transformations.

**Definition 12.** Let $(\hat{\Sigma}, \hat{T})$ be a piecewise $\Gamma$-Möbius baker transformation, with $T_L$ and $T_R$ the left and right Bowen-Series transformations. Let $A_L: S^1 \to \mathbb{C}$ and $A_R: S^1 \to \mathbb{C}$ be two potential functions. We say that $A_L$ and $A_R$ are in involution if there exists a nonzero kernel $k: \hat{\Sigma} \to \mathbb{C}^*$, called an involution kernel, such that

$$k(\xi, \eta)e^{A_L(\xi)} = k(\xi', \eta')e^{A_R(\eta')}, \quad \text{whenever} \quad (\xi', \eta') = \hat{T}(\xi, \eta) \in \hat{\Sigma}.$$ 

The kernel $k$ is extended to $S^1 \times S^1$ by $k(\xi, \eta) = 0$, for $(\xi, \eta) \notin \hat{\Sigma}$.

**Remark 13.**

1. Let $W(\xi, \eta) = \ln k(\xi, \eta)$, for $(\xi, \eta) \in \hat{\Sigma}$. Then $A_L$ and $A_R$ are cohomologous, that is $A_L - A_R \circ \hat{T} = W \circ \hat{T} - W$.

2. If $A_L(\xi)$ is Hölder, then there exists a Hölder function $A_R(\eta)$ (depending only on $\eta$) in involution with $A_L$ with a Hölder involution kernel.

3. If $L_L$ and $L_R$ are the two Ruelle transfer operators associated to $A_L$ and $A_R$, if $A_L$ and $A_R$ are in involution with respect to a kernel $k$, and if $\nu$ is an eigenmeasure of $L_R$, that is, $L_R^\nu(\nu) = \lambda \nu$, then $\psi(\xi) = \int k(\xi, \eta) d\nu(\eta)$ is an eigenfunction of $L_L$, that is, $L_L^\psi = \lambda \psi$.

These remarks appeared first in [10] and were later rediscovered in [3], in the context of a subshift of finite type. The proofs in this general context can be easily reproduced. The third remark suggests a strategy to obtain the eigenfunction $\psi_{f,s}$, by taking $A_L = -s \ln |T'_L|$, $A_R = -s \ln |T'_R|$ and replacing $\nu$ by the distribution $D_{f,s}$. All there is left to prove is that $-\ln |T'_L|$ and $-\ln |T'_R|$ are in involution with respect to a piecewise $C^1$ involution kernel. It so happens that this involution kernel exists and is given by the Gromov distance.

**Definition 14.** The Gromov distance $d(\xi, \eta)$ between two points $\xi$ and $\eta$ at infinity is given by

$$d^2(\xi, \eta) = \exp \left( -b_\xi(O, z) - b_\eta(O, z) \right),$$

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for any point $z$ on the geodesic line $[[\xi, \eta]]$. Notice that this definition depends on the choice of the origin $O$ (but not on $z \in [[\xi, \eta]]$).

In the Poincaré disk model, $(\xi, \eta) \in S^1 \times S^1$, or in the upper half-plane, $(s, t) \in \mathbb{R} \times \mathbb{R}$, the Gromov distance takes the simple form

$$d^2(\xi, \eta) = \frac{1}{4}|\xi - \eta|^2, \quad \text{or} \quad d^2(s, t) = \frac{|s - t|^2}{(1 + s^2)(1 + t^2)}.$$  

**Lemma 15.** Let $T_L : S^1 \to S^1$ and $T_R : S^1 \to S^1$ be the two left and right Bowen-Series transformations of a $\Gamma$-Möbius Markov baker transformation $(\tilde{\Sigma}, \tilde{T})$. Then the two potential functions $A_L(\xi) = -\ln|T_L'(\xi)|$ and $A_R(\eta) = -\ln|T_R'(\eta)|$ are in involution and

$$A_L(\xi) - A_R(\eta') = W(\xi', \eta') - W(\xi, \eta), \quad \text{for} \quad (\xi', \eta') = \tilde{T}(\xi, \eta) \in \tilde{\Sigma},$$

where $W(\xi, \eta) = b_\xi(O, z) + b_\eta(O, z)$ and $z$ is any point of the geodesic line $[[\xi, \eta]]$. In particular, $k(\xi, \eta) = \exp(W(\xi, \eta)) = 4/d^2(\xi, \eta)$ is an involution kernel.

**Proof of Lemma 15.** To simplify the notation, we call $(\xi', \eta') = \tilde{T}(\xi, \eta)$, $\gamma_L = \gamma_L[\xi]$, and $\gamma_R = \gamma_R[\eta']$. We also recall the relation $\gamma_R = \gamma_L^{-1}$. Then, choosing any point $z \in [[\xi, \eta]]$, we get

$$A_L(\xi) - A_R(\eta') = -b_\xi(O, \gamma_L^{-1}O) + b_{\eta'}(O, \gamma_R^{-1}O)
= -b_\xi(O, z) - b_\xi(z, \gamma_L^{-1}O)
+ b_{\eta'}(O, \gamma_L(z)) + b_{\eta'}(\gamma_L(z), \gamma_R^{-1}O)
= W(\xi', \eta') - W(\xi, \eta),$$

where $W(\xi', \eta') = b_{\eta'}(O, \gamma_L(z)) - b_\xi(z, \gamma_L^{-1}O)$ and $W(\xi, \eta) = b_\xi(O, z) - b_{\eta'}(\gamma_L(z), \gamma_R^{-1}O)$. \[\blacksquare\]

Notice that if $A(\xi)$ and $\tilde{A}(\eta)$ are in involution by a positive kernel $k(\xi, \eta)$, then $sA(\xi)$ and $s\tilde{A}(\eta)$ are in involution by $k(\xi, \eta)^s$.

**Lemma 16.** Let $T_L : S^1 \to S^1$ and $T_R : S^1 \to S^1$ be the two left and right Bowen-Series transformations of a $\Gamma$-Möbius Markov baker transformation $(\tilde{\Sigma}, \tilde{T})$. Let $A_L : S^1 \to \mathbb{R}$ and $A_R : S^1 \to \mathbb{R}$ be two potential functions in involution with respect to a kernel $k(\xi, \eta)$. Let $L_L$ and $L_R$ be the two Ruelle transfer operators associated to $A_L$ and $A_R$. Then, for any $\xi' \in S^1$ and $\eta \in S^1$,

$$L_R(k(\xi', \cdot))(\eta) = L_L(k(\cdot, \eta))(\xi').$$
Proof. Given $\xi' \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$, the two finite sets

$$\{\eta' \in \mathbb{S}^1; \; T_R(\eta') = \eta, \; J(\xi', \eta') = 1\}, \quad \{\xi \in \mathbb{S}^1; \; T_L(\xi) = \xi', \; J(\xi, \eta) = 1\}$$

are in bijection. Thus, we obtain

$$L_R(k(\xi', \cdot))(\eta) = \sum_{T_R(\eta') = \eta} k(\xi', \eta')e^{A_R(\eta')}$$

$$= \sum_{T_L(\xi) = \xi'} k(\xi, \eta)e^{A_L(\xi)} = L_L(k(\cdot, \eta))(\xi')$$

Theorem 1 now follows immediately from lemmas 15 and 16.

Proof of Theorem 1. We first prove that $\psi_{f,s}(\xi) = \int k(\xi, \eta)\mathcal{D}_{f,s}(\eta)$, with $k(\xi, \eta) = J(\xi, \eta)/d^2(\xi, \eta)$, is a solution of the equation $L^L_\gamma \psi_f = \psi_f$. In fact, we have

$$\psi_{f,s}(\xi') = \int k^s(\xi', \eta')\mathcal{D}_{f,s}(\eta') = \int L^L_\gamma(k^s(\xi', \cdot))(\eta)\mathcal{D}_{f,s}(\eta)$$

$$= \int L^L_\gamma(k^s(\cdot, \eta')(\xi')\mathcal{D}_{f,s}(\eta) = (L^L_\gamma \psi_{f,s})(\xi').$$

We next prove that $\psi_{f,s}(\xi') \neq 0$. Suppose on the contrary that $\psi_{f,s}(\xi') = 0$ for each $\xi' \in \mathbb{S}^1$. Following Haydn [10], we introduce step functions of the form

$$\tilde{\chi}(\xi', \eta') = \chi \circ pr_1 \circ T^{-1}(\xi', \eta'),$$

where $\chi = \chi(\xi)$ depends only on $\xi$. For instance, for some fixed $\xi'$, let $\chi$ be the characteristic function of the interval $I^L(n, \xi) = \cap_{k=0}^n T_L^{-k}(I^L \circ T_L^{n-1}(\xi))$, for some $\xi$ such that $T_L^n(\xi) = \xi'$. Let $Q^R(\xi) = \{\eta \in \mathbb{S}^1; J(\xi, \eta) = 1\}$ and write

$$\gamma_L[n, \xi] = \gamma_L[T_L^{n-1}(\xi)] \cdots \gamma_L[T_L(\xi)] \gamma_L[\xi], \; Q^R(n, \xi) = \gamma_L[n, \xi]Q^R(\xi).$$

Then $\tilde{\chi}$ equals the characteristic function of the rectangle $I^L(\xi') \times Q^R(n, \xi)$ and $Q^R(\xi')$ is equal to the disjoint union of the intervals $Q^R(n, \xi)$, for all $\xi$ such that $T_L^n(\xi) = \xi'$. We also denote by $\Delta(\xi')$ the set of endpoints of $Q^R(n, \xi)$, for all $T_L^n(\xi) = \xi'$, and observe that $\Delta(\xi')$ is a dense subset of $Q^R(\xi')$. Using the same ideas as in Lemma 16, we obtain

$$\int \tilde{\chi}(\xi', \eta')k^s(\xi', \eta')\mathcal{D}_{f,s}(\eta') = (L^L_\gamma)^n(\chi \psi_{f,s})(\xi') = 0, \quad \forall \xi' \in \mathbb{S}^1.$$
In particular, if \( \hat{\alpha}(\xi') < \hat{\beta}(\xi') < \hat{\alpha}(\xi') + 2\pi \) are chosen such that \( \exp i\hat{\alpha}(\xi') \) and \( \exp i\hat{\beta}(\xi') \) are the two endpoints of the interval \( Q_R(\xi') \), if \( \tilde{k}(\theta) = k(\xi', \exp i\theta) \), then
\[
\tilde{k}(\beta) \tilde{D}_{f,s}(\beta) = \tilde{k}(\hat{\alpha}(\xi')) \tilde{D}_{f,s}(\hat{\alpha}(\xi')) + \int_{\hat{\alpha}(\xi')}^{\hat{\beta}(\xi')} \frac{\partial \tilde{k}}{\partial \theta} \tilde{D}_{f,s}(\theta) d\theta.
\]
for every \( \beta \in [\hat{\alpha}(\xi'), \hat{\beta}(\xi')] \cap \Delta(\xi') \). Since \( \tilde{k}(\theta) \neq 0 \), for each \( \theta \in [\hat{\alpha}(\xi'), \hat{\beta}(\xi')] \), we conclude that the above equality applies to all \( \beta \in [\hat{\alpha}(\xi'), \hat{\beta}(\xi')] \), the two functions \( k(\beta) \tilde{D}_{f,s}(\beta) \) and \( \tilde{D}_{f,s}(\beta) \) are \( C^1 \), and
\[
\int_{\hat{\alpha}(\xi')}^{\hat{\beta}(\xi')} k(\theta) \frac{\partial \tilde{D}_{f,s}}{\partial \theta} d\theta = 0, \quad \forall \beta \in [\hat{\alpha}(\xi'), \hat{\beta}(\xi')].
\]
Therefore, \( \tilde{D}_{f,s}(\theta) \) is a constant function on each \([\hat{\alpha}(\xi'), \hat{\beta}(\xi')]\), thus everywhere on \( S^1 \). It follows that the distribution \( D_{f,s} \) would have to be equal to zero, which is impossible, because it represents a nonzero eigenfunction \( f \).

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References


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