Eigenfunctions of the Laplacian and associated Ruelle operator

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Abstract

Let Γ be a co-compact Fuchsian group of isometries on the Poincaré disk $\mathbb{D}$ and Δ the corresponding hyperbolic Laplace operator. Any smooth eigenfunction f of Δ, equivariant by Γ with real eigenvalue $\lambda = -s(1-s)$, where $s = \frac{1}{2} + it$, admits an integral representation by a distribution $\mathcal{D}_{f,s}$ (the Helgason distribution) which is equivariant by Γ and supported at infinity $\partial\mathbb{D} = \mathbb{S}^1$. The geodesic flow on the compact surface $\mathbb{D}/\Gamma$ is conjugate to a suspension over a natural extension of a piecewise analytic map $T : \mathbb{S}^1 \to \mathbb{S}^1$, the so-called Bowen-Series transformation. Let $\mathcal{L}_s$ be the complex Ruelle transfer operator associated to the jacobian $-s \ln |T'|$. M. Pollicott showed that $\mathcal{D}_{f,s}$ is an eigenfunction of the dual operator $\mathcal{L}_s^*$ for the eigenvalue 1. Here we show the existence of a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ of $\mathcal{L}_s$ for the eigenvalue 1, given by an integral formula

$$\psi_{f,s}(\xi) = \int \frac{J(\xi,\eta)}{|\xi-\eta|^{2s}} \, \mathcal{D}_{f,s}(d\eta),$$

where $J(\xi,\eta)$ is a $\{0,1\}$-valued piecewise constant function whose definition depends upon the geometry of the Dirichlet fundamental domain representing the surface $\mathbb{D}/\Gamma$.

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1 Introduction

Consider the Laplace operator Δ defined by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

on the Lobatchevskii upper half-plane $\mathbb{H} = \{w = x+iy \in \mathbb{C}; y > 0\}$, equipped with the hyperbolic metric $ds_{\mathbb{H}} = \frac{|dw|}{y}$, and the eigenvalue problem

$$\Delta f = -s(1-s)f,$$

where s is of the form $s = \frac{1}{2} + it$, with t is real. We shall also consider the same corresponding Laplace operator

$$\Delta = \frac{1}{4}(1-|z|^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

and eigenvalue problem

$$\Delta f = -s(1-s)f,$$

defined on the Poincaré disk $\mathbb{D} = \{z = x + yi \in \mathbb{C}; |z| < 1\}$, equipped with the metric $ds_{\mathbb{D}} = 2\frac{|dz|}{1-|z|^2}$.

Helgason showed in [11] and [12] that any eigenfunction f associated to this eigenvalue problem can be obtained by means of a generalized Poisson representation

$$\begin{cases} f(w) = \displaystyle\int_{-\infty}^{\infty} \left(\frac{(1+t^2)y}{(x-t)^2+y^2} \right)^s \mathcal{D}^{\mathbb{H}}_{f,s}(t), & \text{for } w \in \mathbb{H}, \\ \text{or} & \\ f(z) = \displaystyle\int_{\partial\mathbb{D}} \left(\frac{1-|z|^2}{|z-\xi|^2} \right)^s \mathcal{D}^{\mathbb{D}}_{f,s}(\xi), & \text{for } z \in \mathbb{D}, \end{cases}$$

where $\mathcal{D}^{\mathbb{D}}_{f,s}$ or $\mathcal{D}^{\mathbb{H}}_{f,s}$ are analytic distributions called from now on *Helgason's distributions*. We have used the canonical isometry between $z \in \mathbb{D}$ and $w \in \mathbb{H}$, namely $w = i\frac{1-z}{1+z}$ or $z = \frac{i-w}{i+w}$. The hyperbolic metric is given in $\mathbb{H}$ and in $\mathbb{D}$ by

$$ds^2_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}, \quad ds^2_{\mathbb{D}} = \frac{4(dx^2+dy^2)}{(1-|z|^2)^2}.$$

We shall be interested in a more restricted problem, where the eigenfunction f is also automorphic with respect to a co-compact Fuchsian group Γ, i. e., a discrete subgroup of the group of Möbius transformations (see [20], [25], [5]) with compact fundamental domain. It is known that the eigenvalues $\lambda = s(1-s) = \frac{1}{4} + t^2$ form a discrete set of positive real numbers with finite multiplicity and accumulating at $+\infty$ (see [13]).

M. Pollicott showed [21] that the Helgason's distribution can be seen as a generalized eigenmeasure of the dual complex Ruelle transfer operator associated to a subshift of finite type defined at infinity. Let T_L be the left Bowen-Series transformation that acts on the boundary $\mathbb{S}^1 = \partial\mathbb{D}$ and is associated to a particular set of generators of Γ. The precise definition of T_L has been given in [8], [22], [23], [24], and more geometrical descriptions have then been given in [1] and [18]. Specific examples of the Bowen-Series transformation have been studied in [17] and [4] for the modular surface and in [3] for a symmetric compact fundamental domain of genus two. The map T_L is known to be piecewise Γ-Möbius constant, Markovian with respect to a partition $\{I_k^L\}$ of intervals of $\mathbb{S}^1$, on which the restriction of T_L is constant and equal to an element γ_k of Γ, transitive and orbit equivalent to Γ. Let $\mathcal{L}_s^L$ be the *complex Ruelle transfer operator* associated to the map T_L and the potential $A_L = -s \ln |T_L'|$, namely

$$(\mathcal{L}_s^L \psi)(\xi') = \sum_{T_L(\xi)=\xi'} e^{A_L(\xi)} \psi(\xi) = \sum_{T_L(\xi)=\xi'} \frac{\psi(\xi)}{|T_L'(\xi)|^s},$$

where the summation is taken over all pre-images ξ of ξ' under T_L. Here T_L' denotes the Jacobian of T_L with respect to the canonical Lebesgue measure on $\mathbb{S}^1$. In the case of an automorphic eigenfunction f of Δ, Pollicott showed that the corresponding Helgason distribution $\mathcal{D}_{f,s}$ satisfies the dual functional equation

$$(\mathcal{L}_s^L)^*(\mathcal{D}_{f,s}) = \mathcal{D}_{f,s}$$

or, according to Pollicott's terminology, the parameter s is a *(dual) Perron-Frobenius value*, that is, 1 is an eigenvalue for the dual Ruelle transfer operator.

Although suggested in [21], it is not clear whether s could be a *Perron-Frobenius value*, that is, whether 1 could also be an eigenvalue for $\mathcal{L}_s^L$, not only for $(\mathcal{L}_s^L)^*$. Our goal in this paper is to show that this is actually the case.

The three main ingredients we use are the following;

- Otal's proof of Helgason's distribution in [19], giving more precise information on $\mathcal{D}_{f,s}$ and enabling us to integrate piecewise $\mathcal{C}^1$ test functions, instead of real analytic globally defined test functions;
- a more careful reading of [1], [18], [8], and [24], or a careful study of a particular example in [16], which enables us to construct a piecewise Γ-Möbius baker transformation ("arithmetically" conjugate to the geodesic billiard);
- the existence of a kernel that we introduced in [3], which enables us to permute past and future coordinates and transfer a dual eigendistribution to a piecewise real analytic eigenfunction. Haydn (in [10]) has introduced a similar kernel in a more abstract setting, without geometric considerations.

More precisely, we prove the following theorem

Theorem 1. *Let Γ be a co-compact Fuchsian group of the hyperbolic disk $\mathbb{D}$ and Δ the corresponding hyperbolic Laplace operator. Let $\lambda = s(1-s)$, with $s = \frac{1}{2} + it$, and let f be an eigenfunction of $-\Delta$, automorphic with respect to Γ, that is, $\Delta f = -\lambda f$ and $f \circ \gamma = f$, for every $\gamma \in \Gamma$. Then there exists a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ on $\mathbb{S}^1$ that is a solution of the functional equation*

$$\mathcal{L}_s^L(\psi_{f,s}) = \psi_{f,s},$$

where $\mathcal{L}_s^L$ is the complex Ruelle transfer operator associated to the left Bowen-Series transformation $T_L : \mathbb{S}^1 \to \mathbb{S}^1$ and the potential $A_L = -s \ln |T_L'|$.

Moreover, $\psi_{f,s}$ admits an integral representation via Helgason's distribution $\mathcal{D}_{f,s}^{\mathbb{D}}$, representing f at infinity, and a geometric positive kernel $k(\xi, \eta)$ defined on a finite set of disjoint rectangles $\cup_k I_k^L \times Q_k^R \subset \mathbb{S}^1 \times \mathbb{S}^1$, namely,

$$\psi_{f,s}(\xi) = \int_{Q_k^R} k^s(\xi, \eta)\, \mathcal{D}_{f,s}^{\mathbb{D}}(\eta) = \int_{Q_k^R} \frac{1}{|\xi - \eta|^{2s}}\, \mathcal{D}_{f,s}^{\mathbb{D}}(\eta),$$

for every $\xi \in I_k^L$, where I_k^L and Q_k^R are intervals of $\mathbb{S}^1$ on which disjoint closure, and $\{I_k^L\}_k$ is a partition of $\mathbb{S}^1$ where T_L is injective, Markovian and piecewise Γ-Möbius constant.

Lewis [14] and, later, Lewis and Zagier [15], started a different approach to understand Maass wave forms. They where able to identify in a bijective way Maass wave forms of $PSL(2,\mathbb{Z})$ and solutions of a functional equation with 3 terms closely related to Mayer's transfer operator. Their setting is strongly dependent of the modular group. Our theorem 1 may be viewed as part of their program for co-compact Fuchsian groups. The Helgason distribution has been used by S. Zelditch in [26] to generalise microlocal analysis on hyperbolic surfaces, by L. Flaminio and G. Forni in [9], to study invariant distributions by the horocycle flow, and by N. Anantharaman and S. Zelditch in [2], to understand the "Quantum Unique Ergodicity Conjecture".

2 Preliminary results

Let Γ be a co-compact Fuchsian group of the Poincaré disk $\mathbb{D}$. We denote by $d(w,z)$ the hyperbolic distance between two points of $\mathbb{D}$, given by the Riemannian metric $ds^2 = 4(dx^2+dy^2)/(1-|z|^2)^2$. Let $M = \mathbb{D}/\Gamma$ be the associated compact Riemann surface, $N = T^1M$ the unit tangent bundle, and Δ the Laplace operator on M. Let $f : M \to \mathbb{R}$ be an eigenfunction of $-\Delta$ or, in other words, a Γ-automorphic function $f : \mathbb{D} \to \mathbb{R}$ satisfying $\Delta f = -s(1-s)f$ for the eigenvalue $\lambda = s(1-s) > \frac{1}{4}$ and such that $f \circ \gamma = f$, for every $\gamma \in \Gamma$. We know that f is $\mathcal{C}^\infty$ and uniformly bounded on $\mathbb{D}$. Thanks to Helgason's representation theorem, f can be represented as a superposition of horocycle waves, given by the *Poisson kernel*

$$P(z,\xi) := e^{b_\xi(\mathcal{O},z)} = \frac{1-|z|^2}{|z-\xi|^2},$$

where $b_\xi(w,z)$ is the the *Busemann cocycle* between two points w and z inside the Poincaré disk, observed from a point at infinity $\xi \in \mathbb{S}^1$, defined by

$$b_\xi(w,z) := \text{“}d(w,\xi) - d(z,\xi)\text{”} = \lim_{t \to \xi} d(w,t) - d(z,t),$$

where the limit is uniform in $t \to \xi$ in any hyperbolic cone at ξ. Helgason's theorem states that

$$f(z) = \int_{\mathbb{D}} P^s(z,\xi)\, \mathcal{D}_{f,s}(\xi) = \langle \mathcal{D}_{f,s}, P^s(z,.)\rangle$$

for some analytic distribution $\mathcal{D}_{f,s}$ acting on real analytic functions on $\mathbb{S}^1$. Unfortunately, Helgason's work is too general and is valid for any eigenfunction not necessarily equivariant by a group. For bounded $\mathcal{C}^2$ functions f, Otal [19] has shown that the distribution $\mathcal{D}_{f,s}$ has stronger properties and can be defined in a simpler manner.

We first recall some standard notations in hyperbolic geometry. We call $d(z, z_0)$ the hyperbolic distance between two points: for instance, the distance from the origin is given by $d(\mathcal{O}, \tanh(\frac{r}{2})e^{i\theta}) = r$. Let $\mathcal{C}(\mathcal{O}, r)$ denote the set of points in $\mathbb{D}$ at hyperbolic distance r from the origin,

$$\mathcal{C}(\mathcal{O}, r) = \{z \in \mathbb{D};\ |z| = \tanh(\tfrac{r}{2})\}$$

and, more generally, given for any interval I at infinity and any point $z_0 \in \mathbb{D}$, let $\mathcal{C}(z_0, r, I)$ denote the angular arc at the hyperbolic distance r from z_0 delimited at infinity by I, that is,

$$\mathcal{C}(z_0, r, I) = \{z \in \mathbb{D};\ z \in [[z_0, \xi]] \text{ for some } \xi \in I \text{ and } d(z, z_0) = r\},$$

where $[[z_0, \xi]]$ denotes the geodesic ray from z_0 to the point ξ at infinity. Let $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ denote the exterior normal derivative to $\mathcal{C}(\mathcal{O}, r)$ and $|dz|_{\mathbb{D}} = \sinh(r)\, d\theta$the hyperbolic arc length on $\mathcal{C}(\mathcal{O}, r)$.

Theorem 2. ([19]) *Let f be a bounded $\mathcal{C}^2$ eigenfunction satisfying $\Delta f = -s(1-s)f$. Then:*

1. *There exists a continuous linear functional $\mathcal{D}_{f,s}$ acting on $\mathcal{C}^1$ functions of $\mathbb{S}^1$,defined by*

$$\int \psi(\xi)\, \mathcal{D}_{f,s}(\xi) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O},r)} \psi(z) e^{-sr} \left(\frac{\partial f}{\partial n} + sf \right) |dz|_{\mathbb{D}},$$

   where $c(s)$ is a non zero normalizing constant such that $\langle \mathcal{D}_{f,s}, \mathbf{1} \rangle = f(0)$, and $\psi(z)$ is any $\mathcal{C}^1$ extension of $\psi(\xi)$ to a neighborhood of $\mathbb{S}^1$.

2. *$\mathcal{D}_{f,s}$ represents f in the following sense:*

$$f(z) = \int \big[P(z, \xi)\big]^s\, \mathcal{D}_{f,s}(\xi), \quad \forall\ z \in \mathbb{D}.$$

   $\mathcal{D}_{f,s}$ is unique and is called the Helgason distribution of f.

3. *For all $0 \leq \alpha \leq 2\pi$, the following limit exists:*

$$\tilde{\mathcal{D}}_{f,s}(\alpha) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_0^\alpha e^{-sr} \left(\frac{\partial f}{\partial n} + sf \right) \left(\tanh(\frac{r}{2}) e^{i\theta} \right) \sinh(r) d\theta.$$

   The convergence is uniform in $\alpha \in [0, 2\pi]$ and $\tilde{\mathcal{D}}_{f,s}(0) = 0$.

4. *$\tilde{\mathcal{D}}_{f,s}$ can be extended to $\mathbb{R}$ as a $\frac{1}{2}$-Hölder continuous function satisfying:*

   (a) *$\tilde{\mathcal{D}}_{f,s}(\theta + 2\pi) = \tilde{\mathcal{D}}_{f,s}(\theta) + f(0)$, for every $\theta \in \mathbb{R}$,*

   (b) *for any $\mathcal{C}^1$ function $\psi : \mathbb{S}^1 \to \mathbb{C}$, denoting $\tilde{\psi}(\theta) = \psi(\exp i\theta)$,*

$$\int \psi(\xi) \, \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(0) f(0) - \int_0^{2\pi} \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta) \, d\theta.$$

Using similar technical tools as Otal, one can prove the following extension of $\mathcal{D}_{f,s}$ on piecewise $\mathcal{C}^1$ functions, that is, on functions not necessarily continuous but which admit a $\mathcal{C}^1$ extension on each interval $[\xi_k, \xi_{k+1}]$ of some finite and ordered subdivision $\{\xi_0, \xi_1, \ldots, \xi_{r-1}\}$ of $\mathbb{S}^1$.

Proposition 3. *Let f and $\mathcal{D}_{f,s}$ be as in Theorem 2.*

1. *For any interval $I \subset \mathbb{S}^1$ and any function $\psi : I \to \mathbb{C}$, which is $\mathcal{C}^1$ on the closure of I and null outside I, the following limit exists:*

$$\int \psi(\xi) \, \mathcal{D}_{f,s}(\xi) := \frac{1}{c(s)} \lim_{r \to +\infty} \int_{\mathcal{C}(\mathcal{O},r,I)} \psi(z) e^{-sr} \left(\frac{\partial f}{\partial n} + sf \right) |dz|_{\mathbb{D}}$$

   where again $\psi(z)$ is any $\mathcal{C}^1$ extension of $\psi(\xi)$ to a neighborhood of $\mathbb{S}^1$.

2. *For any $0 \leq \alpha < \beta \leq 2\pi$ and any $\mathcal{C}^1$ function ψ on the interval $I = [\exp(i\alpha), \exp(i\beta)]$,*

$$\int \psi(\xi) \, \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(\beta) \tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{\mathcal{D}}_{f,s}(\alpha) - \int_\alpha^\beta \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta) \, d\theta,$$

   where $\tilde{\mathcal{D}}_{f,s}$ and $\tilde{\psi}(\theta)$ have been defined in Theorem 2.

Proof. Given $\alpha \in [0, 2\pi]$, let $I = \{e^{i\theta} \mid 0 \leq \theta \leq \alpha\}$ be an interval in S^1, and ψ a $\mathcal{C}^1$ function defined on a neighborhood of $\mathbb{S}^1$. Denote $\tilde{\psi}(r, \theta) = \psi(\tanh(\frac{r}{2})e^{i\theta})$ and $K(r, \theta) = e^{-sr}\Big(\frac{\partial f}{\partial n} + sf\Big)\Big(\tanh(\frac{r}{2}e^{i\theta}\Big)\sinh(r)$. Then

$$\begin{aligned}
&\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O},r,I)} \psi(z)e^{-sr}\left(\frac{\partial f}{\partial n} + sf\right)|dz|_{\mathbb{D}} \\
&\quad = \int_0^\alpha \tilde{\psi}(r, \beta)K(r, \beta)\,d\beta \\
&\quad = \int_0^\alpha \Big[\tilde{\psi}(r, \alpha) + \int_\beta^\alpha -\frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta)\,d\theta\Big]K(r, \beta)\,d\beta \\
&\quad = \tilde{\psi}(r, \alpha)\int_0^\alpha K(r, \beta)\,d\beta - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta)\Big[\int_0^\theta K(r, \beta)\,d\beta\Big]\,d\theta.
\end{aligned}$$

Since $\int_0^\alpha K(r, \beta)\,d\beta \to \tilde{\mathcal{D}}_{f,s}(\alpha)$ uniformly in $\alpha \in [0, 2\pi]$, the left-hand side of the previous equality converges to

$$\int \psi(\xi)\,\mathbf{1}_{\{\xi \in I\}}\,\mathcal{D}_{f,s}(\xi) = \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(\theta)\tilde{\mathcal{D}}_{f,s}(\theta)\,d\theta.$$

The second part of the proposition follows subtracting such an expression from another one, such as

$$\int \psi(\xi)\,\mathbf{1}_{\{\xi = e^{i\theta};\,0 \leq \theta \leq \beta\}}\,\tilde{\mathcal{D}}_{f,s}(\xi) - \int \psi(\xi)\,\mathbf{1}_{\{\xi = e^{i\theta};\,0 \leq \theta \leq \alpha\}}\,\tilde{\mathcal{D}}_{f,s}(\xi).$$

∎

If, in addition, we assume that f is equivariant with respect to a co-compact Fuchsian group Γ, Pollicott observed in [21] that $\mathcal{D}_{f,s}$, acting on real analytic functions, is equivariant by Γ, that is, satisfies $\gamma^*(\mathcal{D}_{f,s})(\xi) = |\gamma'(\xi)|^s\mathcal{D}_{f,s}(\xi)$, for all $\gamma \in \Gamma$. Because Otal's construction is more precise and implies that Helgason's distribution also acts on piecewise $\mathcal{C}^1$ functions, the above equivariance property can be improved in the following way.

Proposition 4. *Let $f : \mathbb{D} \to \mathbb{R}$ be a $\mathcal{C}^2$ function, $I \subset \mathbb{S}^1$ an interval and $\psi : I \to \mathbb{C}$ a $\mathcal{C}^1$ function on the closure of I. If f satisfies $f \circ \gamma = f$, for some $\gamma \in \Gamma$, (f is not necessarily automorphic), then*

$$\langle \mathcal{D}_{f,s}, \frac{\psi \circ \gamma^{-1}}{|\gamma' \circ \gamma^{-1}|^s}\,\mathbf{1}_{\gamma(I)}\rangle = \langle \mathcal{D}_{f,s}, \psi\,\mathbf{1}_I\rangle.$$

The main difficulty here is to transfer the equivariance property $f \circ \gamma = f$ to an equivalent property for the extension of $\mathcal{D}_{f,s}$ to piecewise $\mathcal{C}^1$ functions. If $I = \mathbb{S}^1$ and ψ is real analytic, then, by uniqueness of the representation, Proposition 4 is easily proved. It seems that just knowing the fact that $\mathcal{D}_{f,s}$ is the derivative of some Hölder function is not enough to reach a conclusion. The following proof uses Otal's approach and, essentially, the extension of $\mathcal{D}_{f,s}$ described in Part 1 of Proposition 3.

Proof of Proposition 4. First we prove the proposition for $\psi = 1$. Let $g(z) = \exp(-sd(\mathcal{O}, z))$. By definition of $\mathcal{D}_{f,s}$, we obtain

$$\begin{aligned}\int \mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi) &= \lim_{r\to+\infty} \frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O},r',I)} \left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right) |dz|_{\mathbb{D}} \\ &= \lim_{r\to+\infty} \frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O}',r',\gamma(I))} \left(g'\frac{\partial f}{\partial n} - f\frac{\partial g'}{\partial n}\right) |dz|_{\mathbb{D}},\end{aligned}$$

where $r' = r + d(\mathcal{O}, \mathcal{O}')$, $\mathcal{O}' = \gamma(\mathcal{O})$ and $g' = g \circ \gamma^{-1}$. Notice that the domain bounded by the circle $\mathcal{C}(\mathcal{O}', r')$ contains the circle $\mathcal{C}(\mathcal{O}, r)$. Let $\overline{PQ}$ be the positively oriented arc $\mathcal{C}(\mathcal{O}, r, \gamma(I))$ and $\overline{P'Q'}$ be the arc $\mathcal{C}(\mathcal{O}', r', \gamma(I))$. Then the two geodesic segments $[[P, P']]$ and $[[Q, Q']]$ belong to the annulus $r \le d(z, \mathcal{O}) \le r + 2d(\mathcal{O}, \mathcal{O}')$ and their length is uniformly bounded.

We now use Green's formula to compute the right hand side of the above expression. Let Ω denote the domain delimited by P, P', Q', Q using the corresponding arcs and geodesic segments, and let $dv = \sinh(r)\,dr\,d\theta$ be the hyperbolic volume element. We obtain

$$\begin{aligned}\int_{\overline{P'Q'}} \left(g'\frac{\partial f}{\partial n} - f\frac{\partial g'}{\partial n}\right) |dz|_{\mathbb{D}} = & \int_{\overline{PQ}} \left(g'\frac{\partial f}{\partial n} - f\frac{\partial g'}{\partial n}\right) |dz|_{\mathbb{D}} \\ & - \int_{[[P,P']]} \cdots |dz|_{\mathbb{D}} - \int_{[[Q',Q]]} \cdots |dz|_{\mathbb{D}} \\ & + \iint_{\Omega} (g'\Delta f - f\Delta g')\, dv.\end{aligned}$$

When r tends to infinity, the last three terms at the right-hand side tend to 0, since along the geodesic segments $[P, P']$ and $[Q, Q']$, the gradient $\nabla g'$ is uniformly bounded by $\exp(-\frac{1}{2}r)$ and

$$g'\Delta f - f\Delta g' = sg'f\sinh(d(z, \mathcal{O}'))^{-2} \quad \text{and} \quad \frac{\partial}{\partial n} g' + sg'$$

are uniformly bounded by a constant times $\exp(-\frac{5}{2}r)$ in the domain Ω, for the first expression, and by a constant times $\exp(-\frac{3}{2}r)$ on $\mathcal{C}(\mathcal{O},r)$, for the second expression. It follows that

$$\begin{aligned}\int \mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi) &= \lim_{r\to+\infty}\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O},r,\gamma(I))} g'\left(\frac{\partial f}{\partial n}+sf\right)|dz|_{\mathbb{D}} \\ &= \lim_{r\to+\infty}\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O},r,\gamma(I))}\big[\psi(z)\big]^s e^{-sr}\left(\frac{\partial f}{\partial n}+sf\right)|dz|_{\mathbb{D}},\end{aligned}$$

where $\psi(z)=\exp\left(d(\mathcal{O},z)-d(\mathcal{O},\gamma^{-1}(z))\right)$. Now we observe that

$$\begin{cases}\psi(z)=\exp s\left(d(\mathcal{O},z)-d(\gamma(\mathcal{O}),z)\right), & \text{for } z\in\mathbb{D},\\ \psi(\xi)=\exp b_\xi(\mathcal{O},\gamma(\mathcal{O}))=|\gamma'\circ\gamma^{-1}(\xi)|^{-1}, & \text{for } \xi\in\partial\mathbb{D},\end{cases}$$

actually coincides with a real analytic function $\Psi(z)$ defined in a neighborhood of $\mathbb{S}^1$, given explicitly by

$$\Psi(z)=\left(\frac{(1+|z|)^2}{(1+|\gamma^{-1}(z)|)^2|\gamma'\circ\gamma^{-1}(z)|}\right)^s.$$

Thus we proved proved that

$$\int \mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi)=\int\frac{\mathbf{1}_{\gamma(I)}(\xi)}{|\gamma'\circ\gamma^{-1}(\xi)|^s}\,\mathcal{D}_{f,s}(\xi).$$

Now we prove the general case. We use the same notation for the lifting $\gamma:\mathbb{R}\mapsto\mathbb{R}$ of a Möbius transformation $\gamma:\mathbb{S}^1\mapsto\mathbb{S}^1$. The lifting satisfies $\gamma(\alpha+2\pi)=\gamma(\alpha)+2\pi$, $\exp(i\gamma(\alpha))=\gamma(\exp(i\alpha))$ and $\gamma'(\alpha)=|\gamma'(\alpha)|$, for all $\alpha\in\mathbb{R}$. Using Proposition 3, we obtain

$$\begin{aligned}&\tilde{\mathcal{D}}_{f,s}(\beta)-\tilde{\mathcal{D}}_{f,s}(\alpha)\\ &\quad=\frac{\tilde{\mathcal{D}}_{f,s}\circ\gamma(\beta)}{\gamma'(\beta)^s}-\frac{\tilde{\mathcal{D}}_{f,s}\circ\gamma(\alpha)}{\gamma'(\alpha)^s}-\int_{\gamma(\alpha)}^{\gamma(\beta)}\frac{\partial}{\partial\theta}\left(\frac{1}{(\gamma'\circ\gamma^{-1}(\theta))^s}\right)\tilde{\mathcal{D}}_{f,s}(\theta)\,d\theta.\end{aligned}$$

For any $\mathcal{C}^1$ function $\psi(\xi)$ defined on I, we denote $\tilde{\psi}(\theta)=\psi(\exp i\theta))$, and obtain

$$
\begin{aligned}
LHS &:= \int \psi(\xi)\, \mathbf{1}_I(\xi)\, \mathcal{D}_{f,s}(\xi) \\
&= \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_\alpha^\beta \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta)\, d\theta \\
&= \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left(\tilde{\psi} \circ \gamma^{-1}(\theta)\right) \tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta)\, d\theta \\
&= \tilde{\psi}(\beta) \left(\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha)\right) - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial \tilde{\psi}(\gamma^{-1}\theta)}{\partial \theta} \left(\tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta) - \tilde{\mathcal{D}}_{f,s}(\alpha)\right) d\theta.
\end{aligned}
$$

We now use the above equivariance and replacee both $\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha)$ and $\tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta) - \tilde{\mathcal{D}}_{f,s}(\alpha)$ by the corresponding formula involving $\tilde{\mathcal{D}}_{f,s} \circ \gamma(\beta)$, $\tilde{\mathcal{D}}_{f,s} \circ \gamma(\alpha)$, $\tilde{\mathcal{D}}_{f,s}(\theta)$. Thus

$$
\begin{aligned}
&LHS \\
&= \frac{\tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s} \circ \gamma(\beta)}{\gamma'(\beta)^s} - \frac{\tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s} \circ \gamma(\alpha)}{\gamma'(\alpha)^s} - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left(\frac{\tilde{\psi}(\gamma^{-1}\theta)}{\gamma'(\gamma^{-1}\theta)^s} \right) \tilde{\mathcal{D}}_{f,s}(\theta)\, d\theta \\
&= \int \frac{\psi \circ \gamma^{-1}(\xi)}{|\gamma' \circ \gamma^{-1}(\xi)|^s}\, \mathbf{1}_{\gamma(I)}\, \mathcal{D}_{f,s}(\xi).
\end{aligned}
$$

■

Following [1], [8], [22], [23], [24] and [18] for the general case and [16] for a specific example we recall the definition of the left T_L and right T_R Bowen-Series transformation. The hyperbolic surface we are interested in is given by the quotient of the hyperbolic disk $\mathbb{D}$ by a co-compact Fuchsian group Γ. Given a point $\mathcal{O} \in \mathbb{D}$, let

$$
D_{\Gamma,\mathcal{O}} = \{z \in \mathbb{D}; d(z, \mathcal{O}) < d(z, \gamma(\mathcal{O})), \quad \forall\, \gamma \in \Gamma\}
$$

Denote the corresponding Dirichlet domain, a convex fundamental domain with compact closure in $\mathbb{D}$, admitting an even number of geodesic sides and an even number of vertices, some of which may be elliptic. More precisely, the boundary of $D_{\Gamma,\mathcal{O}}$ is a disjoint union of semi-closed geodesic segments $S^L_{-r}, \cdots, S^L_{-1}, S^L_1, \cdots, S^L_r$, closed to the left and open to the right, or, equivalently, to a union of semi-closed geodesic segments $S^R_{-r}, \cdots, S^R_{-1}, S^R_1, \cdots, S^R_r$, closed to the right and open to the left; for each k, the intervals S_k^L and S_k^R have the same endpoints and S_k^L is associated to S_{-k}^R by an element $a_k \in \Gamma$ satisfying $a_k(S_k^L) = S_{-k}^R$. The elements a_k generate Γ and satisfy $a_{-k} = a_k^{-1}$, for $k = \pm 1, \cdots, \pm r$.

To define the two Bowen-Series transformations T_L and T_R geometrically, we need to impose a geometric condition on Γ: following [8], [22] and [24], we say that Γ satisfies the *even corner* property if, for each $1 \leq |k| \leq r$, the complete geodesic line through S_k^L is equal to a disjoint union of Γ-translates of the sides S_l^L, with $1 \leq |l| \leq r$. Some Γ do not satisfy this geometric property. Nevertheless, any two co-compact Fuchsian groups Γ and Γ', with identical signature, are geometrically isomorphic, that is, there exists a group isomorphism $h_* : \Gamma \to \Gamma'$ and a quasi-conformal orientation preserving homeomorphism $h : \mathbb{D} \to \mathbb{D}$ admitting an extension to a conjugating homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}$, that is,

$$h(\gamma(z)) = h_*(\gamma)(h(z)), \quad \forall\ \gamma \in \Gamma.$$

An important observation in [8], [22] and [24] is that any co-compact Fuchsian group is geometrically isomorphic to a Fuchsian group with identical signature and satisfying the *even corner* property. We are going to recall the Bowen and Series construction in the case that Γ possesses the *even corner* property and will show that their main conclusions remain valid under geometric isomorphisms.

The complete geodesic line associated to a side S_k^L cuts the boundary at infinity $\mathbb{S}^1$ at two points s_k^L and s_k^R, positively oriented with respect to s_k^L, the oriented geodesic line $]]s_k^L, s_k^R[[$ seeing the origin $\mathcal{O}$ to the left. Both end points s_k^L and s_k^R are neutrally stable with respect to the associated generator a_k, that is, $|a_k'(s_k^L)| = |a_k'(s_k^R)| = 1$. The family of open intervals $]s_k^L, s_k^R[$ covers $\mathbb{S}^1$; since these intervals $]s_k^L, s_k^R[$ overlap each other, there is no canonical partition adapted to this covering. Nevertheless, we may associate two well defined partitions, the left partition $\mathcal{A}_L$ and the right partition $\mathcal{A}_R$. The former consists of disjoint half-closed intervals,

$$\mathcal{A}_L = \{A_{-r}^L, \cdots, A_{-1}^L, A_1^L, \cdots, A_r^L\},$$

given by $A_k^L = [s_k^L, s_{l(k)}^L[$ where $s_{l(k)}^L$ denotes the nearest point s_l^L after s_k^L, according to a positive orientation. Each A_k^L belongs to the unstable domain of the hyperbolic element a_k, that is, $|a_k'(\xi)| \geq 1$, for each $\xi \in A_k^L$. By definition, the left Bowen-Series transformation $T_L : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is given by

$$T_L(\xi) = a_k(\xi), \qquad \text{if} \quad \xi \in A_k^L.$$

Analogously, $\mathbb{S}^1$ can be partitioned into half-closed intervals

$$\mathcal{A}_R = \{A_{-r}^R, \cdots, A_{-1}^R, A_1^R, \cdots, A_r^R\},$$

where $A_k^R =]s_{j(k)}^R, s_k^R]$, and $s_{j(k)}^R$ denotes the nearest s_j^R before s_k^R, according to a positive orientation. The right Bowen-Series transformation is given by

$$T_R(\eta) = a_k(\eta), \qquad \text{if} \quad \eta \in A_k^R.$$

The two partitions A^L and A^R generate two ways of coding a trajectory. Let $\gamma_L : \mathbb{S}^1 \mapsto \Gamma$ and $\gamma_R : \mathbb{S}^1 \mapsto \Gamma$ be the left and right symbolic coding defined by

$$\gamma_L[\xi] = a_k, \quad \text{if } \xi \in A_k^L, \qquad \text{and} \quad \gamma_R[\eta] = a_k, \quad \text{if } \eta \in A_k^R.$$

In particular, $T_R(\eta) = \gamma_R[\eta](\eta)$ and $T_L(\xi) = \gamma_L[\xi](\xi)$, for each $\xi \in \mathbb{S}^1$. Also, it is known that T_R^2 and T_L^2 are expanding. Series, in [22], [23] and [24], and later, Adler and Flatto in [1], proved that T_L (respectively T_R) is Markov with respect to a partition of $\mathcal{I}^L = \{I_k^L\}_{k=1}^q$ (respectively $\mathcal{I}^R = \{I_l^R\}_{l=1}^q$) that is finer than $\mathcal{A}_L$ (respectively $\mathcal{A}_R$). The semi-closed intervals I_k^L and I_l^R are of the same kind as A_k^L and A_l^R, and have the same closure.

Definition 5. *A dynamical system* $(\mathbb{S}^1, T, \{I_k\})$ *is said to be a piecewise* Γ*-Möbius Markov transformation if* $T : \mathbb{S}^1 \to \mathbb{S}^1$ *is a surjective map, and* $\{I_k\}$ *is a finite partition of* $\mathbb{S}^1$ *into intervals such that:*

1. *for each* k*,* $T(I_k)$ *is a union of adjacent intervals* I_l*;*
2. *for each* k*, the restriction of* T *to* I_k *coincides with an element* $\gamma_k \in \Gamma$*;*
3. *some finite iterate of* T *is uniformly expanding.*

Theorem 6. ([8], [24]) *For any co-compact Fuchsian group* Γ*, there exists a piecewise* Γ*-Möbius Markov transformation* $(\mathbb{S}^1, T, \{I_k\})$ *which is transitive and orbit equivalent to* Γ*.*

The Ruelle transfer operator can be defined for any picewise $\mathcal{C}^2$ Markov transformation $(\mathbb{S}^1, T, \{I_k\})$ and any potential function A. Actually, we need a particular complex transfer operator given by the potential

$$A = -s \ln |T'|.$$

For any function $\psi : \mathbb{S}^1 \to \mathbb{C}$, define

$$(\mathcal{L}_s(\psi))(\xi') = \sum_{T(\xi)=\xi'} e^{A(\xi)}\psi(\xi) = \sum_{T(\xi)=\xi'} \frac{\psi(\xi)}{|T'(\xi)|^s},$$

where the summation is taken other all preimages ξ of ξ' under T. We modify $\mathcal{L}_s$ slightly, so that it acts on the space of piecewise $\mathcal{C}^1$ functions. Let $\{I_k\}_{k=1}^q$ be a partition of S^1. Given a piecewise C^1 function and $\oplus_{k=1}^q \psi_k \in \oplus_{k=1}^q \mathcal{C}^1(\bar{I}_k)$ set

$$\mathcal{L}_s^L \psi = \oplus_{l=1}^q \phi_l, \quad \text{where} \quad \phi_l = \sum_{I_l \subset T(I_k)} \frac{\psi_k \circ T_{k,l}^{-1}}{|T' \circ T_{k,l}^{-1}|^s},$$

and $T_{k,l}^{-1}$ denotes the restriction to I_l of the inverse of $T : I_k \to T(I_k) \supset I_l$.

Proposition 7. *Let Γ be a co-compact Fuchsian group. Let $s = \frac{1}{2} + it$ and f be an automorphic eigenfunction of $-\Delta$, that is, $\Delta f = -s(1-s)f$. Let $(\mathbb{S}^1, T, \{I_k\})$ be a piecewise Γ-Möbius Markov transformation and $\mathcal{L}_s$ be the Ruelle transfer operator corresponding to the observable $A = -s \ln |T'|$. Then the Helgason distribution $\mathcal{D}_{f,s}$ satisfies*

$$(\mathcal{L}_s)^* \mathcal{D}_{f,s} = \mathcal{D}_{f,s}.$$

Proof. Let $\oplus_{k=1}^q \psi_k$ be a piecewise $\mathcal{C}^1$ function in $\oplus_{k=1}^q \mathcal{C}^1(\bar{I}_k)$. Using Proposition 4,

$$\begin{aligned}
\int (\mathcal{L}_s\psi)(\xi)\, \mathcal{D}_{f,s}(\xi) &= \sum_{l=1}^q \int_{I_l} (\mathcal{L}_s\psi)_l(\xi)\, \mathcal{D}_{f,s}(\xi) \\
&= \sum_{T(I_k)\supset I_l} \int_{I_l} \frac{\psi_k \circ T_{k,l}^{-1}}{|T' \circ T_{k,l}^{-1}|^s}(\xi)\, \mathcal{D}_{f,s}(\xi) \\
&= \sum_{T(I_k)\supset I_l} \int_{T^{-1}(I_l)\cap I_k} \psi_k(\xi)\, \mathcal{D}_{f,s}(\xi) \\
&= \sum_{k=1}^q \int_{I_k} \psi_k(\xi)\, \mathcal{D}_{f,s}(\xi) = \int \psi(\xi)\, \mathcal{D}_{f,s}(\xi).
\end{aligned}$$

■

Series in [24], Adler and Flatto in [1], and Morita in [18] noticed that T_L admits a natural extension $\hat{T} : \hat{\Sigma} \mapsto \hat{\Sigma}$ strongly related to T_R. We also showed the existence of such a $\hat{T}$ in [16], and it was an important step in the proof of Theorem 3 of [16]. The following definition explains how the two maps T_L and T_R are glued together in an abstract way.

Definition 8. *Let Γ be a co-compact Fuchsian group. A dynamical system $(\hat{\Sigma}, \hat{T}, \{I_k^L\}, \{I_l^R\}, J)$ is said to be a piecewise Γ-Möbius baker transformation if it admits a description as follows.*

1. *$\{I_k^L\}$ and $\{I_l^R\}$ are finite partitions of $\mathbb{S}^1$ into disjoint intervals; $J(k,l)$ is a $\{0,1\}$-valued function, and $\hat{\Sigma}$ is the subset of $\mathbb{S}^1 \times \mathbb{S}^1$ defined by*
$$\hat{\Sigma} = \coprod_{J(k,l)=1} I_k^L \times I_l^R.$$
2. *For each k, $Q_k^R = \coprod\{I_l^R; J(k,l) = 1\}$ is an interval whose closure is disjoint from $\bar{I}_k^L$. For each l, $Q_l^L = \coprod\{I_k^L; J(k,l) = 1\}$ is an interval whose closure is disjoint from $\bar{I}_l^R$. Let $I^L(\xi) = I_k^L$ and $Q^R(\xi) = Q_k^R$, for $\xi \in I_k^L$. Let $I^R(\eta) = I_l^R$ and $Q^L(\eta) = Q_l^L$, for $\eta \in I_l^R$.*
3. *$\hat{T} : \hat{\Sigma} \to \hat{\Sigma}$ is bijective and is given by*
$$\begin{cases} \hat{T}(\xi,\eta) &= (T_L(\xi), S_R(\xi,\eta)), \\ \hat{T}^{-1}(\xi',\eta') &= (S_L(\xi',\eta'), T_R(\eta')), \end{cases}$$
for certain maps $T_L, T_R : \mathbb{S}^1 \to \mathbb{S}^1$ and $S_L, S_R : \hat{\Sigma} \to \mathbb{S}^1$.
4. *$(\mathbb{S}^1, T_L, \{I_k^L\})$ and $(\mathbb{S}^1, T_R, \{I_l^R\})$ are piecewise Γ-Möbius Markov transformations. There exist two functions $\gamma_L : \mathbb{S}^1 \to \Gamma$, respectively $\gamma_R : \mathbb{S}^1 \to \Gamma$, that are piecewise constant on each I_k^L, respectively $\{I_l^R\}$, and satisfying*
$$\begin{cases} \hat{T}(\xi,\eta) &= (\gamma_L[\xi](\xi), \gamma_L[\xi](\eta)) \\ \hat{T}^{-1}(\xi',\eta') &= (\gamma_R[\eta'](\xi'), \gamma_R[\eta'](\eta')) \end{cases}$$

The maps T_L and T_R are called the left and right Bowen-Series transformations., whereas γ_L and γ_R are the left and right Bowen-Series codings. Finally, we say that J is the incidence matrix, which we extend as a function on $\mathbb{S}^1 \times \mathbb{S}^1$ defining
$$\begin{cases} J(\xi,\eta) = 1, & \text{if} \quad (\xi,\eta) \in \hat{\Sigma}, \\ J(\xi,\eta) = 0, & \text{if} \quad (\xi,\eta) \notin \hat{\Sigma}. \end{cases}$$

Notice that this definition is equivariant by geometric isomorphisms. For co-compact Fuchsian groups satisfying the *even corner* property, Adler and Flatto in [1], Series in [24] (and, for a particular example, in [16]) obtained geometrically the existence of a piecewise Γ-Möbius baker transformation with left T_L and right T_R maps orbit equivalent to Γ. By geometric isomorphism considerations, we obtain more generally the following.

Proposition 9. ([1], [24], [16]) *For any co-compact Fuchsian group Γ, there exists a picewise Γ-Möbius baker transformation with left and right Bowen-Series transformations that are transitive and orbit equivalent to Γ.*

The two maps T_L and T_R are related to the action of the group Γ on the boundary $\mathbb{S}^1$. The baker transformation $(\hat{\Sigma}, \hat{T})$ encodes this action into a unique dynamical system. For later reference, we state two further properties of this encoding.

Remark 10.

1. The two codings γ_L and γ_R are reciprocal, in the following sense:

$$\gamma_R[\eta'] = \gamma_L^{-1}[\xi], \quad \text{whenever} \quad (\xi', \eta') = \hat{T}(\xi, \eta).$$

2. For any ξ' and η in $\mathbb{S}^1$, there is a bijection between the two finite sets

$$\{\xi; (\xi, \eta) \in \hat{\Sigma} \text{ and } T_L(\xi) = \xi'\}, \quad \{\eta'; (\xi', \eta') \in \hat{\Sigma} \text{ and } T_R(\eta') = \eta\}.$$

In order to better understand this baker transformation, we briefly explain how $(\hat{\Sigma}, \hat{T})$ is conjugate to a specific Poincaré section of the geodesic flow on the surface $N = T^1M$. We assume for the rest of this section that Γ satisfies the *even corner* property.

Since $D_{\Gamma,\mathcal{O}}$ is a convex fundamental domain, every geodesic (modulo Γ) cuts $\partial D_{\Gamma,\mathcal{O}}$ at two distinct points p and q, unless the geodesic is tangent to one of the sides of $D_{\Gamma,\mathcal{O}}$. These tangent geodesics correspond to a finite union of closed geodesics. We could have parametrized the set of oriented geodesics by all pairs $(p, q) \in \partial D_{\Gamma,\mathcal{O}} \times \partial D_{\Gamma,\mathcal{O}}$, with p and q not belonging to the same side of $D_{\Gamma,\mathcal{O}}$, but we prefer to introduce the space X of all $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$ oriented geodesics $[[y, x]]$, either cutting the interior of $D_{\Gamma,\mathcal{O}}$ or passing through one of the corners of $D_{\Gamma,\mathcal{O}}$ and seeing $\mathcal{O}$ to the left. Using these notations, we define the two intersection points $p = p(x, y) \in \partial D_{\Gamma,\mathcal{O}}$

and $q = q(x,y) \in \partial D_{\Gamma,\mathcal{O}}$ for every oriented geodesic $[[y,x]]$, $(x,y) \in X$, such that $[[q,p]] = [[y,x]] \cap \bar{D}_{\Gamma,\mathcal{O}}$ has the same orientation as $[[y,x]]$.

For a geodesic passing through a corner, $p = q$, unless the geodesic is tangent to a side of $D_{\Gamma,\mathcal{O}}$. We are now in a position to define a geometric Poincaré section $B : X \to X$. If $(x,y) \in X$, the geodesic $[[y,x]]$ leaves $D_{\Gamma,\mathcal{O}}$ at $p = p(x,y) \in S_i$, for some side S_i^L. Since S_i^L and S_{-i}^R are permuted by the generator a_i, the new geodesic $a_i([[y,x]]) = [[y',x']]$ enters again the fundamental domain at a new point $q' = q(x',y')$ with $q' = a_i(p) \in S_{-i}^R$. By definition, $B(x,y) = (x',y')$ and the map $B : X \to X$ is called a geodesic billiard like the codings as for T_L and T_R, we introduce two geometric codings $\gamma_B : X \to \Gamma$ and $\bar{\gamma}_B : X \to \Gamma$ given by

$$\begin{cases} \gamma_B[x,y] = a_i & \text{if} \quad p(x,y) \in S_i^L, \\ \bar{\gamma}_B[x,y] = a_i & \text{if} \quad q(x,y) \in S_i^R. \end{cases}$$

Now the geodesic billiard can be defined by

$$\begin{cases} B(x,y) &= (\gamma_B[x,y](x), \gamma_B[x,y](y)), \\ B^{-1}(x',y') &= (\bar{\gamma}_B[x',y'](x'), \bar{\gamma}_B[x',y'](y')). \end{cases}$$

Notice that $\bar{\gamma}_B \circ B = \gamma_B^{-1}$. The map B is very close to be a baker transformation: B and B^{-1} have the same structure as $\hat{T}$ and $\hat{T}^{-1}$, and γ_B (respectively, $\bar{\gamma}_B$) plays the role as γ_L (respectively, γ_R). The main difference is that $\gamma_B[x,y]$ depends on both x and y, but $\gamma_L[\xi]$ depends only on ξ. Nevertheless, we have the following crucial result.

Theorem 11. ([1], [24], [16]) *There exists a Γ-Möbius baker transformation $(\hat{\Sigma}, \hat{T})$ conjugate to (X,B). More precisely, there exists a map $\rho : X \to \Gamma$ such that $\pi(x,y) = (\rho[x,y](x), \rho[x,y](y))$, defines a conjugating map $\pi : X \to \hat{\Sigma}$ between $\hat{T}$ and B, such that $\hat{T} \circ \pi = \pi \circ B$. Equivalently, $\gamma_L \circ \pi$ and γ_B are cohomologous over (X,B), that is, $\gamma_L \circ \pi \rho = \rho \circ B \gamma_B$, and $\gamma_R \circ \pi$ and $\bar{\gamma}_B$ are cohomologous over (X,B), that is, $\gamma_R \circ \pi \rho = \rho \circ B^{-1} \bar{\gamma}_B$.*

3 Proof of Theorem 1

We want to associate to any eigenfunction f of the Laplace operator a nonzero piecewise real analytic function $\psi_{f,s}$ that is a solution of the functional equation

$$\mathcal{L}_s^L(\psi_{f,s}) = \psi_{f,s}, \quad \text{where} \quad \mathcal{L}_s^L(\psi)(\xi') = \sum_{T_L(\xi)=\xi'} \frac{\psi(\xi)}{|T_L'(\xi)|^s}.$$

The main idea is to use a kernel $k(\xi,\eta)$ introduced in Theorem 7 of [3], as well by in Haydn in [10], and by Bogomolny and Carioli in [6] and [7], in the context of double-sided subshifts of finite type. We begin by extending this definition to include baker transformations.

Definition 12. *Let $(\hat{\Sigma},\hat{T})$ be a piecewise Γ-Möbius baker transformation, with T_L and T_R the left and right Bowen-Series transformations. Let $A_L : \mathbb{S}^1 \to \mathbb{C}$ and $A_R : \mathbb{S}^1 \to \mathbb{C}$ be two potential functions. We say that A_L and A_R are in involution if there exists a nonzero kernel $k : \hat{\Sigma} \to \mathbb{C}^*$, called an involution kernel, such that*

$$k(\xi,\eta)e^{A_L(\xi)} = k(\xi',\eta')e^{A_R(\eta')}, \quad \textit{whenever} \quad (\xi',\eta') = \hat{T}(\xi,\eta) \in \hat{\Sigma}.$$

The kernel k is extended to $\mathbb{S}^1 \times \mathbb{S}^1$ by $k(\xi,\eta) = 0$, for $(\xi,\eta) \notin \hat{\Sigma}$.

Remark 13.

1. Let $W(\xi,\eta) = \ln k(\xi,\eta)$, for $(\xi,\eta) \in \hat{\Sigma}$. Then A_L and A_R are cohomologous, that is $A_L - A_R \circ \hat{T} = W \circ \hat{T} - W$.

2. If $A_L(\xi)$ is Hölder, then there exists a Hölder function $A_R(\eta)$ (depending only on η) in involution with A_L with a Hölder involution kernel.

3. If $\mathcal{L}_L$ and $\mathcal{L}_R$ are the two Ruelle transfer operators associated to A_L and A_R, if A_L and A_R are in involution with respect to a kernel k, and if ν is an eigenmeasure of $\mathcal{L}_R$, that is, $\mathcal{L}_R^*(\nu) = \lambda\nu$, then $\psi(\xi) = \int k(\xi,\eta)\,d\nu(\eta)$ is an eigenfunction of $\mathcal{L}_L$, that is, $\mathcal{L}_L(\psi) = \lambda\psi$.

These remarks appeared first in [10] and were later rediscovered in [3], in the context of a subshift of finite type. The proofs in this general context can be easily reproduced. The third remark suggests a strategy to obtain the eigenfunction $\psi_{f,s}$, by taking $A_L = -s\ln|T_L'|$, $A_R = -s\ln|T_R'|$ and replacing ν by the distribution $\mathcal{D}_{f,s}$. All there is left to prove is that $-\ln|T_L'|$ and $-\ln|T_R'|$ are in involution with respect to a piecewise $\mathcal{C}^1$ involution kernel. It so happens that this involution kernel exists and is given by the Gromov distance.

Definition 14. *The Gromov distance $d(\xi,\eta)$ between two points ξ and η at infinity is given by*

$$d^2(\xi,\eta) = \exp\Big(-b_\xi(\mathcal{O},z) - b_\eta(\mathcal{O},z)\Big),$$

for any point z on the geodesic line $[[\xi,\eta]]$. Notice that this definition depends on the choice of the origin $\mathcal{O}$ (but not on $z \in [[\xi,\eta]]$).

In the Poincaré disk model, $(\xi,\eta) \in \mathbb{S}^1 \times \mathbb{S}^1$, or in the upper half-plane, $(s,t) \in \mathbb{R} \times \mathbb{R}$, the Gromov distance takes the simple form

$$d^2(\xi,\eta) = \frac{1}{4}|\xi-\eta|^2, \quad \text{or} \quad d^2(s,t) = \frac{|s-t|^2}{(1+s^2)(1+t^2)}.$$

Lemma 15. *Let $T_L : \mathbb{S}^1 \to \mathbb{S}^1$ and $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ be the two left and right Bowen-Series transformations of a Γ-Möbius Markov baker transformation $(\hat\Sigma, \hat T)$. Then the two potential functions $A_L(\xi) = -\ln|T_L'(\xi)|$ and $A_R(\eta) = -\ln|T_R'(\eta)|$ are in involution and*

$$A_L(\xi) - A_R(\eta') = W(\xi',\eta') - W(\xi,\eta), \quad \textit{for} \quad (\xi',\eta') = \hat T(\xi,\eta) \in \hat\Sigma,$$

where $W(\xi,\eta) = b_\xi(\mathcal{O},z) + b_\eta(\mathcal{O},z)$ and z is any point of the geodesic line $[[\xi,\eta]]$. In particular, $k(\xi,\eta) = \exp(W(\xi,\eta)) = 4/d^2(\xi,\eta)$ is an involution kernel.

Proof of Lemma 15. To simplify the notation, we call $(\xi',\eta') = \hat T(\xi,\eta)$, $\gamma_L = \gamma_L[\xi]$, and $\gamma_R = \gamma_R[\eta']$. We also recall the relation $\gamma_R = \gamma_L^{-1}$. Then, choosing any point $z \in [[\xi,\eta]]$, we get

$$\begin{aligned} A_L(\xi) - A_R(\eta') &= -b_\xi(\mathcal{O},\gamma_L^{-1}\mathcal{O}) + b_{\eta'}(\mathcal{O},\gamma_R^{-1}\mathcal{O}) \\ &= -b_\xi(\mathcal{O},z) - b_\xi(z,\gamma_L^{-1}\mathcal{O}) \\ &\quad + b_{\eta'}(\mathcal{O},\gamma_L(z)) + b_{\eta'}(\gamma_L(z),\gamma_R^{-1}\mathcal{O}) \\ &= W(\xi',\eta') - W(\xi,\eta), \end{aligned}$$

where $W(\xi',\eta') = b_{\eta'}(\mathcal{O},\gamma_L(z)) - b_\xi(z,\gamma_L^{-1}\mathcal{O})$ and $W(\xi,\eta) = b_\xi(\mathcal{O},z) - b_{\eta'}(\gamma_L(z),\gamma_R^{-1}\mathcal{O})$. ■

Notice that if $A(\xi)$ and $\bar A(\eta)$ are in involution by a positive kernel $k(\xi,\eta)$, then $sA(\xi)$ and $s\bar A(\eta)$ are in involution by $k(\xi,\eta)^s$.

Lemma 16. *Let $T_L : \mathbb{S}^1 \to \mathbb{S}^1$ and $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ be the two left and right Bowen-Series transformations of a Γ-Möbius Markov baker transformation $(\hat\Sigma, \hat T)$. Let $A_L : \mathbb{S}^1 \to \mathbb{R}$ and $A_R : \mathbb{S}^1 \to \mathbb{R}$ be two potential functions in involution with respect to a kernel $k(\xi,\eta)$. Let $\mathcal{L}_L$ and $\mathcal{L}_R$ be the two Ruelle transfer operators associated to A_L and A_R. Then, for any $\xi' \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$,*

$$\mathcal{L}_R(k(\xi',\cdot))(\eta) = \mathcal{L}_L(k(\cdot,\eta))(\xi').$$

Proof. Given $\xi' \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$, the two finite sets

$$\{\eta' \in \mathbb{S}^1;\ T_R(\eta') = \eta,\ J(\xi', \eta') = 1\}, \quad \{\xi \in \mathbb{S}^1;\ T_L(\xi) = \xi',\ J(\xi, \eta) = 1\}$$

are in bijection. Thus, we obtain

$$\begin{aligned} \mathcal{L}_R(k(\xi', \cdot))(\eta) &= \sum_{T_R(\eta')=\eta} k(\xi', \eta') e^{A_R(\eta')} \\ &= \sum_{T_L(\xi)=\xi'} k(\xi, \eta) e^{A_L(\xi)} = \mathcal{L}_L(k(\cdot, \eta))(\xi') \end{aligned}$$

■

Theorem 1 now follows immediately from lemmas 15 and 16.

Proof of Theorem 1. We first prove that $\psi_{f,s}(\xi) = \int k(\xi, \eta)^s \mathcal{D}_{f,s}(\eta)$, with $k(\xi, \eta) = J(\xi, \eta)/d^2(\xi, \eta)$, is a solution of the equation $\mathcal{L}_s^L \psi_f = \psi_f$. In fact, we have

$$\begin{aligned} \psi_{f,s}(\xi') &= \int k^s(\xi', \eta')\, \mathcal{D}_{f,s}(\eta') = \int \mathcal{L}_s^R(k^s(\xi', \cdot))(\eta)\, \mathcal{D}_{f,s}(\eta) \\ &= \int \mathcal{L}_s^L(k^s(\cdot, \eta)(\xi')\, \mathcal{D}_{f,s}(\eta) = (\mathcal{L}_s^L \psi_{f,s})(\xi'). \end{aligned}$$

We next prove that $\psi_{f,s} \neq 0$. Suppose on the contrary that $\psi_{f,s}(\xi') = 0$ for each $\xi' \in \mathbb{S}^1$. Following Haydn [10], we introduce step functions of the form

$$\bar{\chi}(\xi', \eta') = \chi \circ pr_1 \circ \hat{T}^{-1}(\xi', \eta'),$$

where $\chi = \chi(\xi)$ depends only on ξ. For instance, for some fixed ξ', let χ be the characteristic function of the interval $I^L(n, \xi) = \cap_{k=0}^n T_L^{-k}(I^L \circ T_L^k(\xi))$, for some ξ such that $T_L^n(\xi) = \xi'$. Let $Q^R(\xi) = \{\eta \in \mathbb{S}^1; J(\xi, \eta) = 1\}$ and write

$$\gamma_L[n, \xi] = \gamma_L[T_L^{n-1}(\xi)] \cdots \gamma_L[T_L(\xi)]\gamma_L[\xi], \quad Q^R(n, \xi) = \gamma_L[n, \xi] Q^R(\xi).$$

Then $\bar{\chi}$ equals the characteristic function of the rectangle $I^L(\xi') \times Q^R(n, \xi)$ and $Q^R(\xi')$ is equal to the disjoint union of the intervals $Q^R(n, \xi)$, for all ξ such that $T_L^n(\xi) = \xi'$. We also denote by $\Delta(\xi')$ the set of endpoints of $Q^R(n, \xi)$, for all $T_L^n(\xi) = \xi'$, and observe that $\Delta(\xi')$ is a dense subset of $Q^R(\xi')$. Using the same ideas as in Lemma 16, we obtain

$$\int \bar{\chi}(\xi', \eta') k^s(\xi', \eta') \mathcal{D}_{f,s}(\eta') = (\mathcal{L}_s^L)^n (\chi \psi_{f,s})(\xi') = 0, \quad \forall\ \xi' \in \mathbb{S}^1.$$

In particular, if $\tilde{\alpha}(\xi') < \tilde{\beta}(\xi') < \tilde{\alpha}(\xi')+2\pi$ are chosen such that $\exp i\tilde{\alpha}(\xi')$ and $\exp i\tilde{\beta}(\xi')$ are the two endpoints of the interval $Q^R(\xi')$, if $\tilde{k}(\theta) = k(\xi', \exp i\theta)$, then

$$\tilde{k}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) = \tilde{k}(\tilde{\alpha}(\xi'))\tilde{\mathcal{D}}_{f,s}(\tilde{\alpha}(\xi')) + \int_{\tilde{\alpha}(\xi')}^{\beta} \frac{\partial \tilde{k}}{\partial \theta}\tilde{\mathcal{D}}_{f,s}(\theta)\, d\theta.$$

for every $\beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')] \cap \Delta(\xi')$. Since $\tilde{k}(\theta) \neq 0$, for each $\theta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, we conclude that the above equality applies to all $\beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, the two functions $\tilde{k}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta)$ and $\tilde{\mathcal{D}}_{f,s}(\beta)$ are $\mathcal{C}^1$, and

$$\int_{\tilde{\alpha}(\xi'))}^{\beta} k(\theta)\frac{\partial \tilde{\mathcal{D}}_{f,s}}{\partial \theta}\, d\theta = 0, \quad \forall\, \beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')].$$

Therefore, $\tilde{\mathcal{D}}_{f,s}(\theta)$ is a constant function on each $[\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, thus everywhere on $\mathbb{S}^1$. It follows that the distribution $\mathcal{D}_{f,s}$ would have to be equal to zero, which is impossible, because it represents a nonzero eigenfunction f. ■

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References

[1] R. Adler, L, Flatto, Geodesic flows, interval maps and symbolic dynamics, *Bull. Amer. Math. Soc.*, vol. 25 (1991), pp. 229–334.

[2] N. Anantharaman and S. Zelditch, Patterson-Sullivan distributions and quantum ergodicity, *Ann. Henri Poincaré*, vol. 8 (2007), pp. 361–426.

[3] A. Baraviera, A. O. Lopes and Ph. Thieullen, A large deviation principle for equilibrium states of Hölder potentials: the zero temperature case, *Stochastics and Dynamics*, vol. 6:1 (2006).

[4] T. Bedford, M. Keane and C. Series, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Oxford Univ Press (1991).

[5] M. Bekka and M. Mayer, *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces*, Cambridge Univ. Press (2000).

[6] E.B. Bogomolny and M. Carioli, Quantum maps of geodesic flows on surfaces of constant negative curvature, *IV International Conference on "Path Integrals from meV to MeV"*, Tutzing May, (1992), pp. 18–21.

[7] E.B. Bogomolny and M. Carioli, Quantum maps from transfer operator. *Physica D*, vol. 67 (1993), pp. 88–112.

[8] R. Bowen and C. Series, Markov maps associated to Fuchsian groups, *Pub. Math. IHES*, vol. 50 (1979), pp. 153-170.

[9] L. Flaminio and G. Forni, Invariant distributions and the time averages for horocycle flows, *Duke Math. J.*, vol. 119 (2003), pp. 465–526.

[10] N. Haydn. Gibbs' functionals on subshifts. *Commun. Math. Phys.* vol. 134 (1990), pp. 217–236.

[11] S. Helgason, *Analysis on Lie Groups and Homogeneous Spaces*, A.M.S., 1972.

[12] S. Helgason, *Topics in Harmonic Analysis on Homogeneous Spaces*, Birkhauser, 1981.

[13] H. Iwaniec, *Spectral Methods of Automorphic Forms*, AMS (2002).

[14] J. B. Lewis, Spaces of holomorphic functions equivalent to the even Maass cusp forms, *Invent. Math.*, vol. 127 (1997), pp. 271–306.

[15] J. Lewis, D. Zagier, Period functions for Maass wave forms, *Annals of Mathematics*, vol. 153 (2001), pp. 191–258.

[16] A.O. Lopes and P. Thieullen, Mather theory and the Bowen-Series transformation, *Annal. Inst. Henry Poincaré, Analyse non-linéaire*, vol. 23:5 (2006), pp. 663–682.

[17] D. H. Mayer, Thermodynamic formalism approach to Selberg's zeta function for PSL $(2, \mathbb{Z})$, *Bull. Amer. Math. Soc.*, vol. 25:1 (1991), pp. 55–60.

[18] T. Morita, Markov systems and transfer operators associated with cofinite Fuchsian groups. *Erg. Theory and Dyn. Syst.* vol. 17 (1997), pp. 1147–1181.

[19] J-P. Otal, Sur les fonctions propres du laplacien du disque hyperbolique, *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 327:2 (1998), pp. 161–166.

[20] S.J. Patterson, The limit set of a Fuchsian group, *Acta Math.*, vol. 136 (1976), pp. 241–273.

[21] M. Pollicott, Some applications of thermodynamic formalism to manifolds with constant negative curvature, *Advances in Mathematics*, vol. 85 (1991), pp. 161–192.

[22] C. Series, Symbolic dynamics for geodesic flows. *Acta Math.*, vol. 146 (1981), pp. 103–128.

[23] C. Series, The modular surface and continuous fractions, *J. London Math. Soc.*, vol. 31 (1985), pp. 69–80.

[24] C. Series, Geometrical Markov coding on surfaces of constant negative curvature, *Erg. Th. and Dynam. Sys.*, vol. 6 (1986), pp. 601–625.

[25] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, *Publ. Math. IHES*, vol. 50 (1979), pp. 171–202.

[26] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, *Duke Math. J.*, vol. 55 (1987), pp. 919–941.

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