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Equilibrium measures for rational maps

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Abstract. For a polynomial map the measure of maximal entropy is the equilibrium measure for the logarithm potential in the Julia set [**1**], [**4**].

Here we will show that in the case where f is a rational map such that $f(\infty)=\infty$ and the Julia set is bounded, then the two measures mentioned above are equal if and only if f is a polynomial.

Let $\bar{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ be the Riemann sphere and $f\colon \mathbb{C}\rightrightarrows$ be an analytic endomorphism of degree $n\geq 2$. Then f can be written as a rational function $f(z)=P(z)/Q(z)$ where P and Q are relatively prime polynomials such that either P or Q has degree n. The purpose of this paper is to show that, if $f(\infty)=\infty$ and the measure of maximum entropy for the rational map f is equal to the measure of equilibrium for the logarithm potential in the Julia set, then f is a polynomial.

By a previous result of Brolin [**3**], we can conclude, in the case $f(\infty)=\infty$, that the two conditions above are equivalent.

Let a be a fixed point in $\mathbb{C}$, and $z_i^m(a)$ the n^m roots of $f^m(z)=a$ (counted with algebraic multiplicity).

Define a measure $u_m(a)$ $m\in\mathbb{N}$, by

$$u_m(a)=\frac{1}{n^m}\sum_{i=1}^{n^m}\delta_{z_i^m(a)},$$

where δ_x is the Dirac measure on x. Let M be the space of probabilities on the Borel σ-algebra of $\bar{\mathbb{C}}$ endowed with the weak topology.

In [**4**], [**6**] it was shown that, with the possible exception of two values of a, the sequence $u_m(a)$ converges in M to an f-invariant probability u, independent of a, and this measure is the unique measure of maximum entropy for f.

Brolin [**3**] showed that for f a polynomial, the measure u above, is the measure of equilibrium for the logarithm potential in the Julia set (see [**7**], [**4**] for reference). Here we will prove the following theorem:

THEOREM. *Let f be a rational map such that $f(\infty)=\infty$. If the measure of maximum entropy is equal to the measure of equilibrium for the logarithm potential, then f is a polynomial.*

We should like to make some remarks about the measure μ of maximum entropy before the proof of the theorem. Suppose $f(\infty)=\infty$.

Let $f(z)=P(z)/Q(z)$, such that:

$$P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,$$
$$Q(z)=b_kz^k+b_{k-1}z^{k-1}+\cdots+b_1z+b_0,$$

$Q(z)$ of degree $k<n$, and $F(z)$ (see [**1**]) such that

$$\frac{F'(z)}{F(z)}=\int\frac{1}{z-x}\,du(x).$$

The map $F(z)$ satisfies the Bochner functional equation [**1**], $F(f(z))=(F(z))^n$ when f is a polynomial.

The main step in the proof of the theorem is to show that for a rational map $f(z)$ we have $F(f(z))=F(z)^n/Q(z)$.

In the ferromagnetic Ising model proposed by Bessis, Mehta and Moussa [**2**], a renormalization procedure was defined associated to the polynomial $(z-q)^2$ and a functional equation for the associated Diophantine moment problem was obtained:

$$G(z)=\frac{1}{1-qz}G\left(\left(\frac{1}{1-qz}\right)^2\right).$$

In the case $f(\infty)=\infty$, one can define a renormalization procedure in the Julia set in the following way: Let $L\colon M\to M$ be such that for δ_x, the Dirac measure on x,

$$L(\delta_x)=\frac{1}{n}\sum_{i=1}^{n}\delta_{x^i(x)},$$

where $\{x^i(x)\}$ are the n roots of $f(z)=x$. As the finite sums of Dirac measures are dense in M, L is defined uniquely by the previous condition. The fixed measure, for L, is the solution to the renormalization procedure. By [**4**], u is the only solution to the procedure. This renormalization procedure is analogous to the one presented in [**2**] for the parameters $q>2$. This is another characterization of the measure of maximum entropy when f is a polynomial. In terms of the map $F(z)$ this characterization is equivalent to $F(f(z))=F(z)^n$. For a rational map f, the functional equation $F(f(z))=F(z)^n/Q(z)$ has the following electrostatic heuristic interpretation: As

$$\log|F(z)|=n\log|F(z)|-\log|Q(z)|=n\log|F(z)|-\log|b_k|-\sum_{i=1}^{k}\log|z-a_i|,$$

where a_i are the k poles of f, then we conclude that an amount of opposite charge in the poles of f has to be considered when f is rational.

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THEOREM. *Let f be a rational map such that $f(\infty)=\infty$. If the measure of maximum entropy is equal to the measure of equilibrium for the logarithm potential, then f is a polynomial.*

Proof. Let f be such that $f(\infty)=\infty$.

Let us fix a point a in the Julia set. From now on we will write $z_i^m(a)=z_i^m$. As the Julia set is invariant by f and f^{-1}, we have that z_i^m is a point in the Julia set for all $m\in\mathbb{N}$ and $i\in\{1,2,\ldots,n^m\}$. For each $m\in\mathbb{N}$, let the polynomial $f_m(z)=\prod_{i=1}^{n^m}(z-z_i^m)$, and $F_m(z)=(\prod_{i=1}^{n^m}(z-z_i^m))^{1/n^m}$, defined in a neighbourhood of ∞. It is easy to see that there exists a neighbourhood V of ∞, such that for all $m\in\mathbb{N}$, $F_m(z)$ is well defined. Since for $z\in U\subset V$, $m\in\mathbb{N}$, $|F_m(z)|$ is smaller than the distance from U to the Julia set, we have, by the Picard theorem, that there exists a subsequence of $F_m(z)$, converging in the topology of uniform convergence in the compact sets to an analytical map $F(z)$ of the form

$$F(z)=z\left[1+\sum_{k=1}^{\infty}A_kz^{-k}\right].$$

Therefore $\log(F(z)/z)$ is well defined in a neighbourhood of ∞.

Claim A. (I) $\log(F(z)/z)=-\sum_{m=1}^{\infty}(1/m)M_mz^{-m}$ (where $M_m=\int x^m\,du(x)$) in a neighbourhood G of ∞.

(II) For any $y\in G$ there exists a neighbourhood U_y of y in G such that for any $z\in U_y$, $\log F(z)=\int\log(z-x)\,du(x)$. We will suppose U_y (in polar coordinates) of the form $U_y=\{(r,\theta)\,|\,r_1<r,\ \theta_1<\theta<\theta_2\}$ where $0<\theta_1<\theta_2<2\pi$ and $r_1>0$.

(III) $F'(z)/F(z)=\sum_{m=0}^{\infty}M_mz^{-m-1}=\int(1/(z-x))\,du(x)$ for $z\in G$ (where $M_m=\int x^m\,du(x)$).

(IV) $\forall z\in G, \log|F(z)|=\int\log|z-x|\,du(x)$.

Remark. The claim above was stated in Hille [**5**] for z_i^m, the zeros of the Fekete polynomials of a set E, where E is a continuum whose complement G is simply connected and u is the equilibrium measure for the logarithm potential in E. In that case $F(z)$ is the Riemann map for G with derivative 1 in ∞.

Proof of claim A. (I)

$$\begin{aligned}\log\frac{F_m(z)}{z}&=\frac{1}{n^m}\sum_{i=1}^{n^m}\log(1-z_i^mz^{-1})\\&=\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{n^m}\sum_{i=1}^{n^m}(z_i^m)^k\right]z^{-k}\\&=-\sum_{k=1}^{\infty}\frac{1}{k}M_{m,k}z^{-k},\end{aligned}$$

where

$$M_{m,k}=\frac{1}{n^m}\sum_{i=1}^{n^m}(z_i^m)^k.$$

Therefore taking limits

$$\log\frac{F(z)}{z}=-\sum_{k=1}^{\infty}\frac{1}{k}M_kz^{-k}.$$

(II) Consider a branch of F in U_y and apply (I).

(III) For any z in the domain of F, take the derivative of $\log F(z)$ in a small neighbourhood of z.

(IV) Consider the real part of $\log F(z)$.

Now let

$$F_1(z) = n \int \frac{1}{z-x}\, du(x)$$

$$F_2(z) = f'(z) \int \frac{1}{f(z)-x}\, du(x).$$

As we suppose $f(\infty)=\infty$ and $f(z)=P(z)/Q(z)$ has degree n, we have that $P(z)$ has degree n. Suppose $Q(z)$ has degree k. Therefore $k<n$. As

$$\frac{f'(z)}{f(z)} = \frac{P'(z)Q(z)-Q'(z)P(z)}{P(z)Q(z)}$$

and the upper polynomial has degree less than $n+k$, we have that $\lim_{z\to\infty} f'(z)/f(z)=0$. Therefore by (III)

$$\begin{aligned}
\lim_{z\to\infty} F_2(z) &= \lim_{z\to\infty} \frac{f'(z)}{n} \cdot F_1(f(z)) \\
&= \lim_{z\to\infty} \frac{f'(z)}{n} \sum_{m=0}^{\infty} M_m (f(z))^{-m-1} \\
&= \lim_{z\to\infty} \frac{1}{n}\left(\frac{f'(z)}{f(z)} + M_1 \frac{f'(z)}{f(z)} \cdot \frac{1}{f(z)} + M_2 \frac{f'(z)}{f(z)} \frac{1}{(f(z))^2} + \cdots\right) \\
&= 0.
\end{aligned}$$

Therefore $F_1(z)$ and $F_2(z)$ are analytical in a neighbourhood of ∞, and $F_2(z)$ is of the form

$$F_2(z) = \sum_{m=0}^{\infty} B_m z^{-m-1}, \qquad B_m \in \mathbb{C}.$$

Remember that by (IV) we have

$$F_1(z) = n\frac{F'(z)}{F(z)} = n\left(\sum_{m=0}^{\infty} M_m z^{-m-1}\right),$$

where $M_m = \int x^m\, du(x)$.

Now let C be a Jordan Curve in $\mathbb{C}$ which contains the Julia set and all the poles of f. We suppose also that C is close to ∞ and in the domain of $F_1(z)$ and $F_2(z)$. Let $m \geq 1$. Then

$$\begin{aligned}
\int_C F_2(z) z^m\, dz &= \int_C \left(\int \frac{f'(z)}{f(z)-x}\, du(x)\right) z^m\, dz \\
&= \int \left(\int_C \frac{z^m f'(z)}{f(z)-x}\, dz\right) du(x) \\
&= 2\pi i \int \left(\sum_{i=1}^{n} (x_i(x))^m - \sum_{j=1}^{k} (p_j)^m\right) du(x),
\end{aligned}$$

where for $i \in \{1, 2, \ldots, n\}$, $x_i(x)$ are the n roots of $f(z)=x$, and for $j \in \{1, 2, \ldots, k\}$, p_j

are the k poles of f counted with multiplicity. Therefore by the definition of u,

$$\begin{aligned}\int_C F_2(z)z^m\,dz &= \int \sum_{i=1}^{n} (x_i(x))^m\,du(x) - \sum_{j=1}^{k} (p_j)^m \\ &= n\int x^m\,du(x) - \sum_{j=1}^{k} (p_j)^m \\ &= nM_m - \sum_{j=1}^{k} (p_j)^m.\end{aligned}$$

Then we have

$$\begin{aligned}F_2(z) &= \sum_{m=0}^{\infty} B_m z^{-m-1} = B_0 z^{-1} + \sum_{m=1}^{\infty}\left(\int_C F_2(z)z^m\,dz\right)\frac{z^{-m-1}}{2\pi i} \\ &= B_0 z^{-1} + \sum_{m=1}^{\infty}\left(nM_m - \sum_{j=1}^{k}(p_j)^m\right)z^{-m-1} \\ &= B_0 z^{-1} + (F_1(z) - nz^{-1}) - \sum_{m=1}^{\infty}\left(\sum_{j=1}^{k}(p_j)^m\right)z^{-m-1} \\ &= B_0 z^{-1} + (F_1(z) - nz^{-1}) - z^{-1}\left(\sum_{j=1}^{k}\left(\sum_{m=1}^{\infty}(p_j)^m z^{-m}\right)\right) \\ &= B_0 z^{-1} + (F_1(z) - nz^{-1}) - z^{-1}\sum_{j=1}^{k}\left(\frac{1}{1-(p_j/z)} - 1\right) \\ &= B_0 z^{-1} + (F_1(z) - nz^{-1}) - z^{-1}\sum_{j=1}^{k}\left(\frac{z}{z-p_j}\right) + kz^{-1} \\ &= B_0 z^{-1} + F_1(z) - (n-k)z^{-1} - \sum_{j=1}^{k}\left(\frac{1}{z-p_j}\right).\end{aligned}$$

Let us compute B_0 now:

$$\begin{aligned}B_0 &= \lim_{z\to\infty} zF_2(z) = \lim_{z\to\infty}\frac{1}{n}\cdot z\cdot f'(z)F_1(f(z)) \\ &= \lim_{z\to\infty} z\frac{f'(z)}{f(z)}z + B_1\frac{f'(z)}{(f(z))^2}z + B_2\frac{f'(z)}{(f(z))^3}z + \cdots \\ &= \lim_{z\to\infty}\frac{f'(z)}{f(z)}z.\end{aligned}$$

An easy computation shows that

$$\lim_{z\to\infty}\frac{f'(z)}{f(z)}z = n-k, \qquad \text{if } f(z) = \frac{P(z)}{Q(z)},$$

where $P(z) = z^n + a_{n-1}z^{n-1} + \cdots$ and $Q(z) = b_k z^k + b_{k-1}z^{k-1} + \cdots$. Therefore $B_0 = n-k$. Then we have that

$$\begin{aligned}F_2(z) &= (n-k)z^{-1} + F_1(z) - (n-k)z^{-1} - \sum_{j=1}^{k}\left(\frac{1}{z-p_j}\right) \\ &= F_1(z) - \sum_{j=1}^{k}\left(\frac{1}{z-p_j}\right).\end{aligned}$$

Take a determination of $\log(F(f(z))Q(z)/F(z)^n)$ in G, such that $\log(1) = 0$.

Therefore for each $y \in G$ there exists a neighbourhood U_y contained in G such that for $z \in U_y$, (see claim A(II)), we have:

$$\begin{aligned}
&\frac{d}{dz}\left(\int \log(f(z)-x)\,du(x)\right)\\
&\quad = f'(z)\cdot\int \frac{1}{f(z)-x}\,du(x)\\
&\quad = F_2(z) = n\frac{d}{dz}\left(\int \log(z-x)\,du(x)\right) - \sum_{j=1}^{k}\frac{d}{dz}(\log(z-p_j))\Big)\\
&\quad = \frac{d}{dz}\left(n\int \log(z-x)\,du(x) - \log\left(\prod_{j=1}^{k}(z-p_j)\right)b_k\right)\\
&\quad = \frac{d}{dz}\left(n\int \log(z-x)\,du(x) - \log Q(z)\right).
\end{aligned}$$

Therefore there exists a constant $c \in \mathbb{C}$, such that,

$$c = \int \log(f(z)-x)\,du(x) - n\int \log(z-x)\,du(x) + \log Q(z).$$

From the last equality we have

$$c \sim \log F(f(z)) - n\log F(z) + \log Q(z), \qquad \text{for } z \sim \infty.$$

As $\log f(z) - n\log(z) + k\log z + \log b_k \sim 0$ for z in U_y close to ∞, we have

$$c \sim \log\frac{F(f(z))}{(f(z))} - n\log\frac{F(z)}{z} + \log\frac{Q(z)}{b_k z^k}$$

for U_y close to ∞. As $\lim_{z\to\infty}\log(F(z)/z) = 0$ and $\lim_{z\to\infty}\log(Q(z)/b_k z^k) = 0$, we have that $c = 0$. Therefore $F(f(z)) = F(z)^n/Q(z)$, for all $z \in G$.

Now let $h(z)$ be the harmonic function $h(z) = \log|F(z)| = \int \log|z-x|\,du(x)$.

Taking real parts in the expression $\log F(f(z)) = n\log F(z) + \log Q(z)$, we have in G

$$\begin{aligned}
h(f(z)) &= \log|F(f(z))| = n\log|F(z)| - \log|Q(z)|\\
&= nh(z) - \log|Q(z)|.
\end{aligned}$$

The map $h(z)$ is equal to minus the Green function of the set J with pole in the infinity, (it is called the equilibrium potential [7]) by the assumption that the measure u is the equilibrium measure for the logarithm potential. Therefore $h(z)$ has an extension to all the plane (but a set of capacity zero on the Julia set) and it is continuous in the Julia set (but a set of capacity zero). These points are called regular points and this property is a conformal invariant, therefore for these points we have $h(f(z)) = nh(z) - \log|Q(z)|$.

Now observe that in the infinite set of points of the Julia set such that $h(z)$ is defined, and it is equal, let us say, to d, we have $|Q(z)| = e^{n(d-1)}$. This follows from the invariance of f. As the lemniscate $|Q(z)| = e^{n(d-1)}$ is an analytical set, and the map $|Q(f(z))|$ is analytical, and constant in an infinite set of points, we have that a component of the lemniscate is invariant by f. With the same procedure we obtain that the component of the lemniscate is invariant by f^{-1}.

There is no point of the Julia set in the complement of the lemniscate, because every point in the Julia set can be approximated by a regular point and the lemniscate is a closed set.

We claim that one whole component of the lemniscate is in the Julia set, and the Julia set does not intersect any other component. If one component is completely contained in J, then since that component is open in J, J is the orbit of that component and must be a union of components. If there were more than one component in J, then since the lemniscate is an immersion of a union of circles, $\bar{\mathbb{C}} - J$ would have more than two components, hence infinitely many components, giving a contradiction since the lemniscate only has finitely many components. (For any rational map, $\bar{\mathbb{C}} - J$ must have one, two or infinitely many components.)

Therefore all we have to prove is that the Julia set contains completely at least one component of the lemniscate. Suppose not so. Then J must contain no arcs of the lemniscate, and $\bar{\mathbb{C}} - J$ must be connected. Since ∞ is in $\bar{\mathbb{C}} - J$ and ∞ is fixed, ∞ must be either an attractive fixed point or the centre of a Siegel domain. The former must be true, because a Siegel domain would be a component of $\bar{\mathbb{C}} - J$ and would have to have other, distinct components as pre-images. So $\bar{\mathbb{C}} - J$, being connected, must be the attractive basin of ∞. An arc of the lemniscate is in $\bar{\mathbb{C}} - J$ by assumption. The union of the components of the lemniscate which intersect J is invariant by f (because J is invariant by f, and the lemniscate is an analytical set). So the orbit of this arc is in the attractive basin of ∞, hence the orbit is unbounded. This is not possible because the lemniscate is bounded.

Therefore the claim is proved and the Julia set contains an analytical arc. Under these conditions we can use a result of Brolin ([**3**, theorem 9.1]) that asserts that the Julia set is a circle. If $|Q(z)| = e^{n(d-1)}$ is a circle we have that $Q(z) = z^d$. As a lemniscate is invariant, we have that f is a Blaschke product and $Q(z) = z^d$. This is clearly not possible. Therefore we have a contradiction and the two measures in the theorem are different.

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