Interactions, Specifications, DLR probabilities and the Ruelle Operator in the One-Dimensional Lattice

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Abstract

In this paper we consider general continuous potentials on the symbolic space (the one-dimensional lattice $\mathbb{N}$ with a finite number of spins). We show a natural way to associate potentials of Thermodynamic Formalism with Interactions. Using properties of the Ruelle operator we show the relation of the Gibbs Measures of Hölder and Walters potentials considered in Thermodynamic Formalism with the Dobrushin-Lanford-Ruelle Gibbs measures and also with Gibbs measures obtained via the Thermodynamic Limit with boundary conditions. We prove some uniform convergence theorems for finite volume Gibbs measures, we analyze the Long-Range Ising Model and we show that it belongs to the Walters class for certain parameters. This paper also provides a kind of "dictionary" so one of our purposes is to clarify for both the Dynamical Systems and Mathematical Statistical Mechanics communities how these (seemingly distinct) concepts of Gibbs measures are related.

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1 Introduction

The basic idea of the Ruelle Operator remounts to the transfer matrix method introduced by Kramers and Wannier and (independently) by Montroll, on an effort to compute the partition function of the Ising model. In a very famous work [26] published by Lars Onsager in 1944, the transfer matrix method was generalized to the two-dimensional lattice and was employed to successfully compute the partition function for the first neighbors Ising model. As a byproduct, he obtained the critical point at which the model passes through a phase transition. These two historical and remarkable chapters of the theory of transfer operators are related to the study of their actions on finite-dimensional vector spaces.

In a seminal paper in 1968, David Ruelle [30] introduced the transfer operator for an one-dimensional statistical mechanics model with infinite range interactions. This paved the way to the study of transfer operators in infinite-dimensional vector spaces. In this paper, Ruelle proved the existence and uniqueness of the Gibbs measure for a lattice gas system with a potential depending on infinitely many coordinates.

Nowadays, the transfer operators are called Ruelle operators (mainly in Thermodynamic Formalism) and play an important role in Dynamical Systems and Mathematical Statistical Mechanics. They are actually useful tools in several other branches of mathematics.

Roughly speaking, the famous Ruelle-Perron-Frobenius Theorem states that the Ruelle operator for a potential with a certain regularity, in a suitable Banach space, has a unique simple positive eigenvalue (equal to the spectral radius) and a positive eigenfunction. For Hölder continuous potentials the proof of this theorem can be found in [30, 2]. In 1978, Walters obtained the Ruelle-Perron-Frobenius Theorem for a more general setting [38], allowing expansive and mixing dynamical systems together with potentials with summable variation. After the 80's, the literature on Ruelle's theorem became abundant ([2, 12, 20, 27, 3, 33]).

The Ruelle operator was successfully used to study equilibrium measures for a very general class of potentials f. For a normalized potential f on such class the unique probability measure which is the fixed point for the dual of Ruelle operator associated to f it is sometimes called Gibbs probability in Thermodynamic Formalism. Some people on this area can use a different terminology. By the other hand the equilibrium probability for f is the solution of the variational problem (1) of pressure of the potential f. Under some regularity conditions one can show that the unique probability measure which is the solution of the variational problem is also the Gibbs measure associated to the potential f. Some important properties of the equilibrium probability can be derived from the formalism of the Ruelle operator approach. This study turns out to be a very important on topological dynamics and differentiable dynamical systems, with applications on the study of invariant measures for an Anosov diffeomorphism [32] and the meromorphy of Zelberg's zeta function [31].

The so called DLR Gibbs measures were introduced in 1968 and 1969 independently by Dobrushin [8] and Lanford and Ruelle [22]. The abstract formulation in terms of specifications was developed five years latter in [9, 13, 28]. An important stage in the development of the theory was established by the works of Preston, Gruber, Hintermann and Merlini around 1977, Ruelle (1978) and Israel (1979). Preston's work was more focused on the abstract measure theory, while Gruber et. al. concentrated on specific methods for Ising type models, Israel dealt with the variational principle and Ruelle worked towards Gibbsian formalism in Ergodic Theory.

Dobrushin began the study of non-uniqueness of the DLR Gibbs measure and proposed its interpretation as a phase transition. He proved the famous Dobrushin Uniqueness Theorem in 1968, ensuring the uniqueness of the Gibbs measures for a very general class of interactions at very high temperatures ($\beta \ll 1$). This result, together with the rigorous proof of non-uniqueness of the Gibbs measures for the two-dimensional Ising model at low temperatures, is a great triumph of the DLR approach in the study of phase transition in Statistical Mechanics. Some accounts of the general results on the Gibbs Measure theory (from the Statistical Mechanics' viewpoint) can be found in [17, 11, 34, 5, 16, 1].

This work aims to explain the relationship of DLR-Gibbs measures with the usual Gibbs measures considered in the Thermodynamic Formalism. We also present the details of the construction of specifications for continuous functions. In particular, we show how to construct an absolutely uniformly summable specification for any Hölder potential. En route to the proof of this work's main theorem, we show that the DLR approach extends the construction usually considered in Thermodynamics Formalism and then introduce the concept of dual Gibbs measure (which is natural for Ruelle formalism) and show that these measures are equivalent to the DLR-Gibbs measures for the lattice $\mathbb{N}$. We also analyze the relation with the thermodynamic limit with boundary conditions.

2 The Ruelle Operator

Consider the symbolic space $\Omega \equiv \{1, 2, .., d\}^{\mathbb{N}}$ and let $\sigma : \Omega \to \Omega$ be the left shift. As usual we consider Ω as metric space endowed with a metric d defined by

$$d(x, y) = 2^{-N}, \quad \text{where } N = \inf\{i \in \mathbb{N} : x_i \neq y_i\}.$$

The Ruelle operator on the space of continuous function is defined as follows.

Definition 1. *Fix a continuous function $f : \Omega \to \mathbb{R}$. The Ruelle operator $\mathcal{L}_f : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})$ is defined on the function ψ as follows*

$$\mathcal{L}_f(\psi)(x) = \sum_{\substack{y \in \Omega; \\ \sigma(y) = x}} e^{f(y)}\, \psi(y).$$

Normally we call f a potential and $\mathcal{L}_f$ the transfer operator associated to the potential f.

Definition 2. *For a continuous potential f the Pressure of f is defined by*

$$P(f) = \sup_{\nu \in \mathcal{M}(\sigma)} \left\{ h(\nu) + \int_\Omega f \, d\nu \right\},$$

where $\mathcal{M}(\sigma)$ is the set of σ-invariant probabilities measures and $h(\nu)$ the Shannon-Kolmogorov entropy of ν.

A probability $\mu \in \mathcal{M}(\sigma)$ is called an equilibrium state for f if it solves the above variational problem, i.e.,

$$h(\mu) + \int_\Omega f \, d\mu = P(f). \tag{1}$$

Several properties of the equilibrium probability can be obtained via the Ruelle operator approach under some regularity assumptions on f.

From a compactness argument one can show that if f is a continuous potential then there exists at least one equilibrium state, see [35]. In such generality it is possible to exhibit examples where the equilibrium state is not unique, see [19].

A simple and important class where uniqueness holds is the class of Hölder continuous potentials. In this paper, we use this class to exemplify how to deal with problems in Thermodynamic Formalism using the DLR approach. A Hölder function is defined as follows. Fix $0 < \gamma \leq 1$. A function $f : \Omega \to \mathbb{R}$ is γ-Hölder continuous if

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\gamma(x, y)} < +\infty.$$

The space of all the γ-Hölder continuous functions is denote by Hol_γ. When we say that $f : \Omega \to \mathbb{R}$ is Hölder continuous function we mean $f \in \mathrm{Hol}_\gamma$ for some $0 < \gamma \leq 1$.

A classical theorem in Thermodynamic Formalism assures that for any Hölder potential f the equilibrium state is unique.

Definition 3. *Given $f : \Omega \to \mathbb{R}$ the n-variation of f is*

$$\mathrm{var}_n(f) = \sup \{ \, |f(x) - f(y)| : x, y \in \Omega \text{ and } x_i = y_i \ \text{ for all } 0 \leq i \leq n-1 \}.$$

Definition 4. *We say that $f : \Omega \to \mathbb{R}$ is in the Walters class if*

$$\lim_{p \to \infty} [\sup_{n \geq 1} \{ \mathrm{var}_{n+p}(\ f(x) + f(\sigma(x)) + \ldots + f(\sigma^{n-1}(x)) \ ) \ \} \ ] = \ 0.$$

Remark. Any Hölder potential is in the Walters class.

We shown in the Section 4 that the potential $f : \{-1,1\}^{\mathbb{N}} \to \mathbb{R}$ associated to the Long-Range Ising model with interactions of type $1/r^{\alpha}$ is in the Walters class if $\alpha > 2$. We remark that if f is in the Walters class the equilibrium state is unique and all nice properties of the Ruelle operator still valid as for Hölder potentials [37]. When f is just a continuous potential there are examples where more than one equilibrium state exists. We present one of such examples later.

In what follows we present the first definition of Gibbs measure which is common on the dynamical system community. For this one needs introduce before the dual Ruelle operator.

Definition 5. *Given a continuous potential $f : \Omega \to \mathbb{R}$ let $\mathcal{L}_f^*$ be the operator on the set of Borel Measures on Ω defined so that $\mathcal{L}_f^*(\nu)$, for each Borel measure ν, satisfies:*

$$\int_{\Omega} \psi \; d\mathcal{L}_f^*(\nu) = \int_{\Omega} \mathcal{L}_f(\psi)\, d\nu,$$

for all continuous functions ψ.

Remark. The existence of the above operator is guaranteed by the Riesz-Markov Theorem. In fact, for any fixed Borel measure ν we have that $\psi \mapsto \int_{\Omega} \mathcal{L}_f(\psi)\, d\nu$ is a positive linear functional.

Theorem 6 (Ruelle-Perron-Frobenius)**.** *If $f : \Omega \to \mathbb{R}$ is in the Walters class, then there exist a strictly positive function ψ_f (called eigenfunction) and a strictly positive eigenvalue λ_f such that $\mathcal{L}_f(\psi_f) = \lambda_f \psi_f$. The eigenvalue λ_f is simple and it is equal to the spectral radius of the operator. Moreover, there exists a unique probability ν_f on Ω such that $\mathcal{L}_f^*(\nu_f) = \lambda_f\, \nu_f$. After proper normalization the probability $\mu_f = \psi_f\, \nu_f$ is invariant and it is also the equilibrium probability for f.*

Suppose f is in the Walters class. If φ is positive and $\beta > 0$ satisfies $\mathcal{L}_f(\varphi) = \beta\, \varphi$, then $\beta = \lambda_f$ (the spectral radius) and φ is multiple of ψ_f. We also assume that ν_f is a probability measure and ψ_f is chosen in such way that μ_f is also a probability. The main point is: if for some potential f (in the Walters class or not) it is possible to find a positive eigenfunction for the Ruelle operator, then several interesting properties of the system can be obtained, see section 9.

If a potential f satisfies for all $x \in \Omega$ the equation $\mathcal{L}_f(1)(x) = 1$, we say that f is a normalized potential. If f is in Walters class using the Ruelle-Perron-Frobenius Theorem we can construct a normalized potential $\bar{f}$ given by

$$\bar{f} \equiv f + \log \psi_f - \log \psi_f \circ \sigma - \log \lambda_f. \tag{2}$$

We remark that if f is Hölder then $\bar{f}$ is Hölder. For some particular cases of non Hölder potentials one can show the existence of the main eigenvalue and the main positive eigenfunction, see [19] and [36], for instance. As far as we know there is no general result about existence of the main eigenfunction for a continuous potential.

Theorem 7. *Let $f : \Omega \to \mathbb{R}$ be a potential in the Walters class and $\bar{f}$ given by* (2)*. Then there is a unique Borel probability measure $\mu_{\bar{f}}$ such that $\mathcal{L}^*_{\bar{f}}(\mu_{\bar{f}}) = \mu_{\bar{f}}$. The probability measure $\mu_{\bar{f}}$ is σ invariant and for all Hölder functions ψ we have that, in the uniform convergence topology,*

$$\mathcal{L}^n_{\bar{f}}\psi \to \int_\Omega \psi \, d\mu_{\bar{f}}.$$

Definition 8 (Gibbs Measures)**.** *Let $f : \Omega \to \mathbb{R}$ be in the Walters class and $\bar{f}$ its normalization as in* (2)*. The unique fixed point $\mu_{\bar{f}}$ of the dual operator $\mathcal{L}^*_{\bar{f}}$ is called the Gibbs Measure for the potential f.*

If we denote $g = \log \bar{f}$, any fixed point probability for $\mathcal{L}^*_{\bar{f}}$ is also called a g-measure (see [7, 29]).

One of the main accomplishments of Thermodynamic Formalism was to show that the equilibrium probability and the Gibbs probability are the same [27]. Several important properties of the equilibrium probability measure such as decay of correlation, central limit theorem, large deviations and others can be infered from the analysis of the Ruelle operator.

We remark that nowadays there are several definitions of Gibbs measure. In [33], the author presents some of them and some ideas which are relevant to the general problem we treat.

The definition of Gibbs measure given above is based on the Ruelle-Perron-Frobenius Theorem, which is well known when the potential is Hölder or Walters. The aim of this work is to present a generalization of the concept of Gibbs measure that extends this definition.

A natural way to do this is to prove a better version of the Ruelle-Perron-Frobenius Theorem, but our efforts are not in this direction. In the next section, we shall introduce the concept of interaction and then Gibbs Measures will be constructed using the DLR formalism. We proceed to compare both definitions, showing that the DLR approach is more general and constructive, which means that, given a potential, we can directly define the Gibbs measures without studying the spectral properties of the Ruelle Operator.

Note that for a given continuous potential f one can define the associated Ruelle operator $\mathcal{L}_f$, which acts on continuous functions. Using a standard argument (Thychonov-Schauder Theorem), one can show that in this case there exists at least one probability measure ν and $\lambda > 0$ such that $\mathcal{L}^*_f(\nu) = \lambda \, \nu$. If the potential f is continuous, we consider the transformation $\mathcal{T} : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ such that

$\mathcal{T}(\mu) = \rho$, where for any continuous function g we have

$$\int_\Omega g\, d\mathcal{T}(\mu) = \int_\Omega g d\rho = \frac{\int_\Omega \mathcal{L}_f(g)\, d\mu}{\int_\Omega \mathcal{L}_f(1) d\mu}.$$

By the Thychonov-Schauder Theorem there exists a fixed point ν for $\mathcal{T}$. In this case $\lambda = \int_\Omega \mathcal{L}_f(1) d\nu$.

For a general continuous potential f, it seems natural to enlarge the set of the so called Gibbs probabilities in the context of Ruelle operators. As we will see, this generalization agrees with the definition of Gibbs measures used in Statistical Mechanics.

Definition 9 (Dual-Gibbs Measures)**.** *Let $f : \Omega \to \mathbb{R}$ be a continuous function. We call a probability measure ν a Dual-Gibbs probability for f if there exists a positive $\lambda > 0$ such that $\mathcal{L}_f^*(\nu) = \lambda\, \nu$. We denote the set of such probabilities by $\mathcal{G}^*(f)$.*

Notice that, in general, $\mathcal{G}^*(f) \neq \emptyset$, although a probability measure on $\mathcal{G}^*(f)$ needs not be shift-invariant, even if f is Hölder. If f is a Walters potential, it is only possible to get associated eigenprobabilities when λ is the maximal eigenvalue of the Ruelle operator $\mathcal{L}_f$. In this case the eigenprobability ν associated to the maximal eigenvalue is unique.

A natural question is: for a Holder continuous potential f is it possible to have different eigenprobabilities $\nu_1, \nu_2 \in \mathcal{G}^*(f)$ for $\mathcal{L}_f^*$, associated to different eigenvalues λ_1 and λ_2? This is not possible due to properties of the involution kernel (see [25]). These two eigenprobabilities would determine via the involution kernel two positive eigenfunctions for another Ruelle operator $\mathcal{L}_{f^*}$, with the eigenvalues λ_1 and λ_2 (see [18]), where f^* is the dual potential for f. This is not possible (see for instance [27] or Proposition 1 in [25]). Anyway, eigendistributions for $\mathcal{L}_f^*$ may exist (see [18]).

Now we introduce some more notations. The elements in the space $\{1, 2, ..., d\}^{\mathbb{N}}$ shall be denoted by $(x_0, x_1, x_2, ..)$, while the elements of $\hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}}$ will be denoted by $(...x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...)$. So the image of the left shift on $\hat{\sigma}$ on $\hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}}$ will be written as

$$\sigma(..., x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...) = (..., x_{-2}, x_{-1}, x_0 \,|\, x_1, x_2, ...).$$

Suppose we initially consider a potential $\hat{f} : \hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}} \to \mathbb{R}$ and we want to find the equilibrium state $\hat{\mu}$ for the potential described by $\hat{f}$, that is,

$$P(\hat{f}) = \sup_{\nu \in \mathcal{M}(\hat{\sigma})} \left\{ h(\nu) + \int_{\hat{\Omega}} \hat{f}\, d\nu \right\} = h(\hat{\mu}) + \int_{\hat{\Omega}} \hat{f}\, d\hat{\mu}.$$

The Bowen-Ruelle-Sinai approach (see [2] and [27]) to the problem is the following: since there is no reason for the site $0 \in \mathbb{Z}$ to have privileged status in the lattice, we shall consider shift-invariant probability measures. As we are interested in such probabilities, we point out that the equilibrium probabilities for $\hat{f}$ and $g : \{1, 2, .., d\}^{\mathbb{Z}} \to \mathbb{R}$, satisfying $\hat{f} = g + h - h \circ \hat{\sigma}$ for some continuous $h : \{1, 2, .., d\}^{\mathbb{Z}} \to \mathbb{R}$ are the same. The main point is that under some regularity assumptions on the potential f, one can get a special g which depends just on future coordinates, that is, for any pair $x = (..., x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...)$ and $y = (..., y_{-2}, y_{-1} \,|\, x_0, x_1, x_2, ...) \in \hat{\Omega}$, we have $g(x) = g(y)$. We abuse notation by simply writing $g(x) = g(x_0, x_1, x_2, ...)$., so we can think of g as a function on $\{1, 2, .., d\}^{\mathbb{N}}$. Therefore, we can apply the formalism of the Ruelle operator $\mathcal{L}_g$ to study the properties of μ (a probability measure on $\{1, 2, .., d\}^{\mathbb{N}}$), which is the equilibrium state for $g : \{1, 2, .., d\}^{\mathbb{N}} \to \mathbb{R}$. One can show that the equilibrium state for $\hat{f}$ is the probability measure $\hat{\mu}$ (a probability on $\{1, 2, .., d\}^{\mathbb{Z}}$), which is given by the natural extension of μ (see [2] and [27]). Let us elaborate on that: what we call the natural extension of the shift-invariant probability measure μ on $\{1, 2, .., d\}^{\mathbb{N}}$ is the probability measure $\hat{\mu}$ on $\{1, 2, .., d\}^{\mathbb{Z}}$ defined in the following way: for any given cylinder set on the space $\hat{\Omega}$ of the form $[a_k, a_{k+1}, ..., a_{-2}, a_{-1} \,|\, a_0, a_1, a_2, ..a_{k+n}]$, where $a_j \in \{1, 2, .., d\}$ and $j \in \{k, k+1, .., k+n\} \subset \mathbb{Z}$, we define

$$\hat{\mu}\Big([a_k, a_{k+1}, ..., a_{-1} \,|\, a_0, a_1, ..., a_{k+n}]\Big) = \mu\Big([a_k, a_{k+1}, ..., a_{-1}, a_0, a_1, ..a_{k+n}]\Big),$$

where $[a_k, a_{k+1}, ..., a_{-1}, a_0, a_1, ..., a_{k+n}]$ is now a cylinder on Ω. Notice that if μ is shift-invariant, then $\hat{\mu}$ is shift-invariant.

If we initially consider a potential $\hat{f} : \hat{\Omega} \to \mathbb{R}$, then it is natural to denote $\mathcal{G}^*(\hat{f})$ as the set of natural extensions $\hat{\nu}$ of the probabilities ν which are eigenprobabilities of the Ruelle operator $\mathcal{L}_g$ (where g was the coboundary associated by the procedure we just described).

The strategy described above works well if $\hat{f}$ is Hölder (and the g we get is also Hölder). In some cases where $\hat{f}$ (or g) is not Hölder, part of the above formalism also works (see Example 15).

We shall remind that in order to use the Ruelle operator formalism we have to work with potentials f which are defined on the symbolic space Ω, i.e., $f : \Omega = \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$.

We point out that the setting we consider here for DLR, Thermodynamic Limit and such is for problems which are naturally defined on the lattice $\mathbb{N}$, and not on the lattice $\mathbb{Z}$. The theory in $\mathbb{Z}$ could, in principle, differ and we do not address this issue here.

3 Interactions on the Lattice $\mathbb{N}$

From now on the notation $A \Subset \mathbb{N}$ is used meaning that A is an empty or finite subset of $\mathbb{N}$. If to each $A \Subset \mathbb{N}$ is associated a function $\Phi_A : \Omega \to \mathbb{R}$ then we have a family of functions defined on Ω. We denote such family simply by $\Phi = \{\Phi_A\}_{A \Subset \mathbb{N}}$ and Φ will be called an interaction. We shall remark that it is very common for an interaction Φ having several finite subsets A's for which the associated function Φ_A is identically zero and another important observation is that in this work we only treat interactions defined in the lattice $\mathbb{N}$.

The space of interactions has natural structure of a vector space where the sum of two interactions Φ and Ψ, is given by the interaction $(\Phi+\Psi) \equiv \{\Phi_A+\Psi_A\}_{A \Subset \mathbb{N}}$. If $\lambda \in \mathbb{R}$ then $\lambda\Phi = \{\lambda\Phi_A\}_{A \Subset \mathbb{N}}$. This vector space is too big to our our purposes so we concentrated in a proper subspace which is defined below.

We can also consider interactions defined over a general countable set V. If $V = \mathbb{Z}$, for example, then the family Φ is now indexed over the collection of all $A \Subset \mathbb{Z}$. In this case we say that the interaction is defined on the lattice $\mathbb{Z}$. We focus here on interactions Φ defined on the lattice $\mathbb{N}$, in order to relate the DLR-Gibbs measures and the Thermodynamic Formalism.

Definition 10 (Uniformly Absolutely Summable Interaction). *An interaction $\Phi = \{\Phi_A\}_{A \Subset \mathbb{N}}$ is uniformly absolutely summable if satisfies:*

1. *for each $A \Subset \mathbb{N}$ the function $\Phi_A : \Omega \to \mathbb{R}$ depends only on the coordinates that belongs to the set A, for example, if $A = \{1, \ldots, n\}$ then $\Phi_A(x) = \Phi_A(x_1, \ldots, x_n)$.*
2. *Φ satisfies the following regularity condition*

$$\|\Phi\| \equiv \sup_{n \in \mathbb{N}} \sum_{A \Subset \mathbb{N}; A \ni n} \sup_{x \in \Omega} |\Phi_A(x)| < \infty.$$

We remark that $\|\Phi\|$ is a norm on the space of all uniformly absolutely summable interactions and this space endowed with this norm is a Banach Space.

We now present two simple examples of uniformly absolutely summable Interactions.

Example 11. *Let $\Omega = \{0,1\}^{\mathbb{N}}$ and Φ an interaction given by the following expression: for any $n \in \mathbb{N}$*

$$\Phi_A(x) = \begin{cases} x_n x_{n+1} - 1, & \text{if } \ A = \{n, n+1\}; \\ 0, & \text{otherwise.} \end{cases}$$

is an uniformly absolutely summable interaction. In fact, for any $A \Subset \mathbb{N}$ we have that $\Phi_A \equiv 0$ if A is not of the form $\{n, n+1\}$. If $A = \{n, n+1\}$ is clear from the *definition of Φ_A that this function depends only on the coordinates x_n and x_{n+1}. The regularity condition in this example is satisfied once*

$$\|\Phi\| \equiv \sup_{n\in\mathbb{N}} \sum_{A\Subset\mathbb{N};A\ni n} \sup_{x\in\Omega} |\Phi_A(x)| = \sup_{n\in\mathbb{N}} \sup_{x\in\Omega} |\Phi_{n,n+1}(x)| = 1.$$

Example 12. *If $\Omega = \{0,1\}^{\mathbb{N}}$, and $\alpha > 1$ is a fixed parameter, then the interaction Φ given by*

$$\Phi_A(x) = \begin{cases} \dfrac{1}{|n-m|^{\alpha}}(x_n x_m - 1), & \text{if } A = \{n,m\} \text{ and } m \neq n; \\ 0, & \text{otherwise,} \end{cases}$$

is an uniformly absolutely summable interaction. In fact, for any $A \Subset \mathbb{N}$ we have that $\Phi_A \equiv 0$ if $\#A \neq 2$. On the other hand if $A = \{m,n\}$ with $m \neq n$ we have, similarly to the previous example, that Φ_A depends only on the coordinates x_n and x_m. The regularity condition is verified as follows

$$\begin{aligned} \|\Phi\| \equiv \sup_{n\in\mathbb{N}} \sum_{A\Subset\mathbb{N};A\ni n} \sup_{x\in\Omega} |\Phi_A(x)| &= \sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}\setminus\{n\}} \sup_{x\in\Omega} \left| \frac{x_n x_m - 1}{|m-n|^{\alpha}} \right| \\ &\leq \sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}\setminus\{n\}} \frac{1}{|m-n|^{\alpha}} \\ &\leq 2\zeta(\alpha). \end{aligned}$$

Remark. The interactions in the Examples 11 and 12 are the interactions of the short (first neighbors) and long range Ising model on the lattice $\mathbb{N}$, respectively.

Now are ready to state the main theorem of this section. The interaction obtained there will be used in the next section to construct the Gibbs measure for a Hölder potential f using the Specification theory.

Theorem 13. *Let $f : \Omega \to \mathbb{R}$ be a* **continuous potential**. *For each integers $k \geq 1$ and $n \geq 0$ consider the arithmetic progression $A(k,n) \equiv \{k, \ldots, 2k+n\}$ and the function $g_{A(k,n)} : \Omega \to \mathbb{R}$ defined by*

$$g_{A(k,n)}(x) = f(x_k, \ldots, x_{2k+n}, 0, 0, \ldots) - f(x_k, \ldots, x_{2k+n-1}, 0, 0, \ldots),$$

if $n \geq 1$ and in the case $n = 0$ this function is defined by

$$g_{A(k,0)}(x) = f(x_k, \ldots, x_{2k}, 0, 0, \ldots) - f(0, 0, 0, \ldots).$$

Let $\Phi^f = \{\Phi^f_A\}_{A\Subset\mathbb{N}}$ be the interaction on the lattice $\mathbb{N}$ given by

$$\Phi^f_A(x) = \begin{cases} g_{A(k,n)}(x), & \text{if } A = A(k,n); \\ 0, & \text{otherwise,} \end{cases}$$

then we have

$$f(x) - f(0,0,\ldots) = \sum_{A \Subset \mathbb{N}; A \ni 1} \Phi^f_A(x).$$

Moreover if f is Hölder then Φ^f is uniformly absolutely summable interaction.

Proof. We first observe that from the definition of $\Phi_{A(1,n)}(x)$ we have

$$f(0,0,\ldots) + \sum_{A \Subset \mathbb{N}; A \ni 1} \Phi^f_A(x) = f(0,0,\ldots) + \sum_{n=0}^{\infty} \Phi^f_{A(1,n)}(x). \tag{3}$$

The partial sums of the r.h.s. above are given by

$$\begin{aligned}
f(0,0,\ldots) + \sum_{j=0}^{n} \Phi^f_{A(1,j)}(x) = \\
& f(0,0,\ldots) + \Big(f(x_1,x_2,0,\ldots) - f(0,0,\ldots) \Big) \\
& + \Big(f(x_1,x_2,x_3,0,\ldots) - f(x_1,x_2,0,\ldots) \Big) \\
& + \ldots + \Big(f(x_1,x_2,\ldots,x_{2+n},0,0,\ldots) - f(x_1,x_2,\ldots,x_{1+n},0,0,\ldots) \Big) \\
& = f(x_1,x_2,\ldots,x_{2+n},0,0,\ldots).
\end{aligned}$$

Therefore, the partial sums are given by

$$f(0,0,\ldots) + \sum_{k=0}^{n} \Phi^f_{A(1,k)}(x) = f(0,0,\ldots) + f(x_1,x_2,\ldots,x_{2+n},0,0,\ldots)$$

Since f is continuous, we get that

$$f(x_1,x_2,\ldots,x_{2+n},0,0,,\ldots) - f(x_1,x_2,\ldots) \to 0$$

and the first statement follows. Now if we assume that f is Hölder we have

$$|f(x_1,x_2,\ldots,x_{2+n},0,0,,\ldots) - f(x_1,x_2,\ldots)| \leq K(f) 2^{-\gamma(n+2)},$$

where $K(f)$ is the Hölder constant of f and $\gamma \in (0,1]$. This proves that

$$\lim_{n\to\infty} f(x_1,x_2,\ldots,x_n,0,0,,\ldots) = f(x). \tag{4}$$

The next step is to prove that Φ^f is uniformly absolutely summable. So we need to upper bound the sum

$$\|\Phi^f\| \equiv \sup_{n\in\mathbb{N}} \sum_{A \Subset \mathbb{N}; A \ni n} \sup_{x\in\Omega} |\Phi^f_A(x)|.$$

In order to bound this sum we observe that the r.h.s above is equal to

$$\sup_{n\in\mathbb{N}}\left(\sum_{k=1}^{\infty}\sup_{x\in\Omega}\left|\Phi^f_{A(n,k)}(x)\right| + \sum_{k=2}^{\infty}\sup_{x\in\Omega}\left|\Phi^f_{A(n-1,k)}(x)\right| + \ldots + \sum_{k=n}^{\infty}\sup_{x\in\Omega}\left|\Phi^f_{A(1,k)}(x)\right|\right)$$

The general term in the first sum is

$$\sup_{x\in\Omega}\left|\Phi^f_{A(n,k)}(x)\right| = \sup_{x\in\Omega}|g_{A(k,n)}(x)|$$

which is, by definition of $g_{A(k,n)}$ and because of f is Hölder, bounded by

$$\sup_{x\in\Omega}|f(x_k,...,x_{2k+n},0,..)-f(x_k,...,x_{2k+n-1},0,...)| \le K(f)2^{-\gamma(k+n-1)}.$$

So for any $n\in\mathbb{N}$ the first sum in the above supremun is bounded by

$$\sum_{k=1}^{\infty}\sup_{x\in\Omega}\left|\Phi^f_{A(n,k)}(x)\right| \le K(f)\sum_{k=1}^{\infty}2^{-\gamma(k+n-1)} = 2K(f)2^{-\gamma n}.$$

This upper bound allows us to conclude that the above supremum is bounded by $2K(f)\sum_{j=1}^{n}2^{-\gamma j}$ from where we get that

$$\|\Phi^f\| \equiv \sup_{n\in\mathbb{N}}\sum_{A\Subset\mathbb{N};A\ni n}\sup_{x\in\Omega}|\Phi^f_A(x)| \le \sup_{n\in\mathbb{N}} 2K(f)\sum_{j=1}^{n}2^{-\gamma j} \le 4K(f)\zeta(2\gamma).$$

By a direct inspection of the expression below

$$\Phi^f_A(x) = \begin{cases} g_{A(k,n)}(x), & \text{if } \ A = A(k,n); \\ 0, & \text{otherwise} \end{cases}$$

and from definition of $g_{A(k,n)}(x)$, we can verify that $\Phi^f_A(x)$ depends only on the coordinates in A thus we finally complete the proof.

□

Remark 1. The above proof works the same for a continuous potential $f:\hat{\Omega}\to\mathbb{R}$, if we consider interactions Φ on the lattice $\mathbb{Z}$. In this case we would consider arithmetic progressions of the form $A(k,n)$, $k\in\mathbb{Z}$, $n\in\mathbb{N}$.

Remark 2. The Hölder hypothesis considered above can be weakened and the theorem still valid as long as the potential f satisfies

$$\sum_{n\in\mathbb{N}}\sup\{|f(x)-f(y)| : x_i = y_i, \forall i\le n\} < \infty$$

We can also modify the definition of $g_{A(k,n)}(x)$ by using any fixed $y \in \Omega$, i.e., the above proof works if consider $g_{A(k,n)}(x)$ being

$$f(x_k,\ldots,x_{2k+n},y_{2k+n+1},y_{2kn+2},\ldots) - f(x_k,\ldots,x_{2k+n-1},y_{2k+n},y_{2k+n+1},\ldots).$$

For sake of future use, we state below the theorem on its stronger form. Remember that $\mathrm{var}_n(f) = \sup\{|f(x)-f(y)| : x_i = y_i, \forall i \le n\}$.

Theorem 14. *Let $f : \Omega \to \mathbb{R}$ be a continuous potential and a fixed $y \in \Omega$. For each integers $k \ge 1$ and $n \ge 0$ consider the arithmetic progression $A(k,n) \equiv \{k,\ldots,2k+n\}$ and the function $g_{A(k,n)} : \Omega \to \mathbb{R}$ defined by*

$$g_{A(k,n)}(x) = f(x_k,..,x_{2k+n},y_{2k+n+1},y_{2kn+2},..) - f(x_k,..,x_{2k+n-1},y_{2k+n},y_{2k+n+1},..).$$

if $n \ge 1$ and in the case $n = 0$, this function is defined by

$$g_{A(k,0)}(x) = f(x_k,\ldots,x_{2k},y_{2k+1},y_{2k+2},\ldots) - f(y).$$

Let $\Phi^f = \{\Phi^f_A\}_{A\Subset\mathbb{N}}$ be the interaction given by

$$\Phi^f_A(x) = \begin{cases} g_{A(k,n)}(x), & \textit{if } \; A = A(k,n); \\ 0, & \textit{otherwise.} \end{cases}$$

then,

$$f(x) - f(y) = \sum_{A\Subset\mathbb{N};A\ni 1} \Phi^f_A(x).$$

If in addition we have that

$$\sum_{n=2}^{\infty} \mathrm{var}_n(f) < \infty.$$

then, Φ^f is a uniformly absolutely summable interaction.

Remark. Note that given a Hölder or summable variation potential f the interaction we constructed above depends on the choice of y. On the other hand the Gibbs Measure we will construct on the Section 9 using this interaction will be independent of this choice.

Example 15. *An interesting class of potentials $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is presented by P. Walters in [36]. Each potential f on this class depends just on the first strings of zeroes and ones. This class is a generalization of a model initially considered by F. Hofbauer (see [19]). Some of the examples are of Hölder class and some of them are not. In some particular cases of f one can get phase transitions (see [19], [23], [24] and [15]). We observe that the specific examples of [24] [15] [6] are not in the Walters class. In [36] is presented the implicit expression of the main* *eigenfunction λ of the Ruelle operator (page 1342) and the explicit expression of the eigenfunction (page 1341).*

We will consider here a particular class of such family. Denote by L_n, $n \geq 1$, the cylinder set $\overline{\underbrace{000...0}_{n}1}$ and by R_n, $n \geq 1$, the cylinder set $\overline{\underbrace{111...1}_{n}0}$. Consider the sequences a_n, c_n such that $a_n \to a$ and $c_n \to c$, when $n \to \infty$. Consider also the parameters c and d.

Denote by f the following continuous potential (which is not necessarily of Hölder class):

$$f(x) = \begin{cases} a_n, & \text{if } x \in L_n \text{ and } n \geq 2; \\ c_n, & \text{if } x \in R_n \text{ and } n \geq 2; \\ a, & \text{if } x \in L_1 \text{ or } x = 0^\infty; \\ b, & \text{if } x \in R_1; \\ c, & \text{if } x = 1^\infty; \end{cases}$$

Applying the Theorem 14 with $y = 01010101....$ we have that Φ^f is given by

$$g_{A(k,n)}(x) = f(x_k, .., x_{2k+n}, 0, 1, 0, 1, ...) - f(x_k, .., x_{2k+n-1}, 1, 0, 1, 0, ..).$$

Since $x_k, .., x_{2k+n-1}, ...$ is in one of the sets L_j or R_j, $j = 1, 2, 3, ...$ and the string $0, 1, 0, 1, \ldots$ creates (or, not) an interruption of the first string of zeroes and ones of $x_k, .., x_{2k+n-1}..$ we get easily the expression for the interaction Φ^f. Each value $g_{A(k,n)}(x)$ will be a a_j or a c_j.

4 The Hamiltonian and Interactions

Throughout this section $\Phi = \{\Phi_A\}_{A \Subset \mathbb{N}}$ will denote an interaction. For each $n \in \mathbb{N}$ we consider the set $\Lambda_n \equiv \{1, \ldots, n\}$ and the Hamiltonian H_n defined by the following expression

$$H_n(x) = \sum_{\substack{A \Subset \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Phi_A(x). \tag{5}$$

We claim that if ϕ is absolutely summable then the series above is absolutely convergent. In fact, we have the following upper bounds

$$\sum_{\substack{A \Subset \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} |\Phi_A(x)| \leq \sum_{i=1}^{n} \sum_{\substack{A \Subset \mathbb{N} \\ A \ni i}} |\Phi_A(x)| \leq \sum_{i=1}^{n} \sum_{\substack{A \Subset \mathbb{N} \\ A \ni i}} \sup_{x \in \Omega} |\Phi_A(x)| \leq n \|\Phi\|. \tag{6}$$

Proposition 16. *Let $f : \Omega \to \mathbb{R}$ be a potential satisfying $\sum_{n=2}^{\infty} \mathrm{var}_n(f) < \infty$ (for example, Hölder potential) and $\Phi^f = \{\Phi^f_A\}_{A \Subset \mathbb{N}}$ the uniform absolutely summable* *interaction provided by the Theorems 13 and 14. Consider the Hamiltonian defined as in* (5) *with the interaction* $\Phi \equiv \Phi^f$. *Then for any* $n \in \mathbb{N}$ *we have the following equality*

$$H_n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x) + nf(0, 0, \ldots).$$

Proof. The idea is to explore the second conclusion of the Theorem (13). Before, we observe that the following identity holds

$$H_n(x) = \sum_{\substack{A \Subset \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Phi^f_A(x) = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \Phi_{A(j,k)}(x). \tag{7}$$

To make the argument more clear let us expand the sum on r.h.s above

$$\sum_{j=1}^{n} \sum_{k=0}^{\infty} \Phi^f_{A(j,k)}(x) = \sum_{k=0}^{\infty} \Phi^f_{A(1,k)}(x) + \sum_{k=0}^{\infty} \Phi^f_{A(2,k)}(x) + \ldots \sum_{k=0}^{\infty} \Phi^f_{A(n,k)}(x) \tag{8}$$

From the Theorem (13) we know that the first term on r.h.s above satisfies

$$\sum_{k=0}^{\infty} \Phi^f_{A(1,k)}(x) = f(x) - f(0, 0, \ldots).$$

We observe that

$$\sum_{k=0}^{\infty} \Phi^f_{A(2,k)}(x) = f(\sigma x) - f(0, 0, \ldots)$$

The proof is similar to one we give in Theorem (13). We repeat the main step for the reader's convenience. Notice that $f(0, 0, 0, \ldots)$ plus the partial sums of the l.h.s. above is given by

$$\begin{aligned} f(0, 0, \ldots) + \Big(f(x_2, x_3, x_4, \ldots) - f(0, 0, \ldots)\Big) \\ +\Big(f(x_1, ..., x_5, 0, \ldots) - f(x_1, .., x_4, 0, \ldots)\Big) \\ + \ldots + \Big(f(x_1, \ldots, x_{2k+n}, 0, 0, \ldots) - f(x_1, \ldots, x_{2k+n-1}, 0, 0, \ldots)\Big) \\ = f(0, 0, \ldots) + f(x_1, x_2, \ldots, x_{2k+n}, 0, 0, \ldots). \end{aligned}$$

By performing a formal induction we have that for any $j \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \Phi^f_{A(n,j)}(x) = f(\sigma^{j-1}x) - f(0, 0, \ldots). \tag{9}$$

By using (7), (8) and (9) follows that

$$H_n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x) - nf(0, 0, \ldots).$$

$\square$

5 Long-Range Ising Model and Walters Condition

Throughout this section we work on the symbolic space $\hat{\Omega} \equiv \{-1,1\}^{\mathbb{Z}}$ and $\hat{\sigma}$ will denote the left shift on this space. As usual the distance between $x, y \in \hat{\Omega}$ is given by

$$d(x,y) = \frac{1}{2^N}, \quad \text{where } N = \inf\{|i| : x_i \neq y_i\}.$$

For a fixed $\alpha > 1$ we consider the potential $f : \hat{\Omega} \to \mathbb{R}$ defined by

$$f(x) = -\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{x_0 x_n}{n^\alpha}$$

The relation of functions f and interactions Φ^f was described in previous sections. In the present case the Φ^f associated to such f will be specified soon.

Fact 1. The potential f is not γ-Hölder continuous for any $0 < \gamma \leq 1$.
We claim for any fixed $0 < \gamma \leq 1$ that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\gamma(x,y)} = +\infty.$$

It is enough to show that for any $M > 0$ there are x and y such that $d(x,y) = 2^{-N}$ and $2^{N\gamma}|f(x) - f(y)| \geq M$. From the definition of f it follows that

$$2^{N\gamma}|f(x) - f(y)| = 2^{N\gamma} \left| \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{x_0 x_k}{k^\alpha} - \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{y_0 y_k}{k^\alpha} \right|.$$

By taking x and y so that $d(x,y) = 2^N$ (N will be chosen latter), $1 = x_{N+1} = -y_{N+1}$, $1 = x_{-N-1} = -y_{-N-1}$ and $x_i = y_i$ for all $i \in \mathbb{Z} \setminus \{N+1\}$ we can see that the r.h.s above is equals to

$$2^{N\gamma} \left| \frac{4}{(N+1)^\alpha} \right|.$$

Since N is arbitrary and the above expression goes to infinity when N goes to infinity the claim follows.

Fact 2. If $\alpha > 2$ the potential f is in the Walters class.
Indeed, it is easy to see that for $n, p > 0$

$$\mathrm{var}_{n+p}(\, f(x) + f(\sigma(x)) + \ldots + f(\sigma^{n-1}(x))\,) = (n+p)^{-\alpha+1} + (n+p-1)^{-\alpha+1} + \ldots + p^{-\alpha+1}.$$

Therefore, for each fixed $p > 0$ we have that

$$\sup_{n\in\mathbb{N}} \left[\mathrm{var}_{n+p}(\, f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x))\,)\right] \sim \sum_{j=p}^{\infty} j^{-\alpha+1} \sim p^{-\alpha+2},$$

which proves that the Walters condition is satisfied as long as $\alpha > 2$.

The potential f is associated to the following absolutely uniformly summable interaction

$$\Phi_A(x) = \begin{cases} \dfrac{x_n x_m}{|n-m|^\alpha}, & \text{if } \; A = \{n, m\} \text{ and } m \neq n; \\ 0, & \text{otherwise,} \end{cases}$$

meaning that

$$H_n(x) = \sum_{\substack{A\Subset\mathbb{Z} \\ A\cap\Lambda_n\neq\emptyset}} \Phi_A(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x).$$

Fact 3. The interaction $\Phi = \{\Phi_A\}_{A\Subset\mathbb{Z}}$ is absolutely uniformly summable interaction for any $\alpha > 1$.

This fact can be proved working in the same way we have done in the Example 12.

We can ask if the above potential f is cohomologous with a potential g which depends just on future coordinates? Note that the Sinai's Theorem can not be directly applied here to answer this question, because the fact we shown above (f is not Hölder). For the other hand in this case by keep tracking the cancellations we actually can construct the *transfer function* (Definition 1.11 in [33]) using Sinai's idea, see Proposition 1.2 in [27]. This means: there exists $g : \{-1, 1\}^{\mathbb{Z}} \to \mathbb{R}$ and $h : \{-1, 1\}^{\mathbb{Z}} \to \mathbb{R}$, such that:

1. for any $x = (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...)$ we have that $g(x) = g(x_0, x_1, x_2, ...)$,
2. $f = g + h - h \circ \hat{\sigma}$.

We first define $\varphi : \{-1, 1\}^{\mathbb{Z}} \to \{-1, 1\}^{\mathbb{Z}}$ by the expression

$$\varphi(\ldots, x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, \ldots) = \begin{cases} (\ldots, -1, -1 \,|\, x_0, x_1, x_2, \ldots), & \text{if } x_0 = -1; \\ (\ldots, 1, 1 \,|\, x_0, x_1, x_2, \ldots), & \text{if } x_0 = 1. \end{cases}$$

Now we define the transfer function h by

$$h(x) = \sum_{j=0}^{\infty} f(\hat{\sigma}^n(x)) - f(\hat{\sigma}^n(\varphi(x))).$$

Suppose, for instance, that x satisfies $x_0 = -1$, then,

$$\begin{aligned} h(x) = \ & [\, f\,(\ldots, x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, \ldots) \ - \ f\,(\ldots, x_0, x_0 \,|\, x_0, x_1, x_2, \ldots)\,] \\ & + [\, f\,(\ldots, x_{-2}, x_{-1}, x_0 \,|\, x_1, x_2, \ldots) \ - \ f\,(\ldots, x_1, x_1, x_1 \,|\, x_1, x_2, \ldots)\,] \\ & + [\, f\,(\ldots, x_{-2}, x_{-1}, x_0, x_1 \,|\, x_2, \ldots) \ - \ f\,(\ldots, x_2, x_2, x_2, x_2 \,|\, x_2, \ldots)\,] + ... \end{aligned}$$

The case $x_0 = 1$ is similar. Note that if we show that this series is absolutely convergent then it is easy to see that 1) and 2) above holds by taking $g = f - h + h \circ \hat{\sigma}$. By the definition of φ and f we have that

$$\begin{aligned} \sum_{j=0}^{\infty} |f(\hat{\sigma}^n(x)) - f(\hat{\sigma}^n(\varphi(x))| = & \sum_{m=0}^{\infty} \left| x_m \left(\sum_{n=1}^{\infty} \frac{x_{-n}}{(m+n)^\alpha} - \sum_{n=1}^{\infty} \frac{1}{(m+n)^\alpha} \right) \right| \\ \leq & 2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^\alpha} \end{aligned}$$

which is finite as long as $\alpha > 2$. In this case we can also obtain g in explicit form as:

$$g(x_0, x_1, x_2, ...) = -x_0 \sum_{j=1}^{\infty} \frac{x_j}{j^\alpha} - \zeta(\alpha).$$

Remark. The potential $f : \{-1, 1\}^{\mathbb{Z}} \to \mathbb{R}$ is very well known on the Mathematical Statistical Mechanics community. Two classical theorems state that if $1 < \alpha \leq 2$, then there exists a finite positive $\beta_c(\alpha)$ such that if $\beta > \beta_c(\alpha)$ then for the potential βf the set DLR Gibbs measures has at least two elements and if $\beta < \beta_c(\alpha)$ there is exactly one DLR Gibbs Measure. This is a very non-trivial theorem. The case $1 < \alpha < 2$ it was proved by Freeman Dyson in [10] and the famous borderline case $\alpha = 2$ was proved by Frölich and Spencer in [14].

In a future work we will analyze properties which can be derived from the above $g : \{-1, 1\}^{\mathbb{N}} \to \mathbb{R}$ via the Ruelle operator formalism.

6 Boundary conditions

Now we introduce the concept of Gibbsian specification for the lattice $\mathbb{N}$. The presentation will not be focused on general aspects of the theory and we restrict ourselves to the aspect needed to present our main theorem.

Let Ω be the symbolic space, $\mathscr{F}$ the product σ-algebra on Ω and $\Phi = \{\Phi_A\}_{A \Subset \mathbb{N}}$ be an interaction defined on Ω and obtained from a continuous $f : \Omega \to \mathbb{R}$. For some results we will need some more regularity like uniformly absolutely summability.

In what follows we introduce the so called finite volume Gibbs measures with a boundary condition $y \in \Omega$ (see [33]). Fixed $\beta > 0$, $y \in \Omega$ and $n \in \mathbb{N}$. Consider the probability measure in $(\Omega, \mathscr{F})$ so that for any $F \in \mathscr{F}$, we have

$$\mu_n^y(F) = \frac{1}{Z_n^y(\beta)} \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)),$$

where $Z_n^y(\beta)$ is a normalizing factor called partition function given by

$$Z_n^y(\beta) = \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} \exp(-\beta H_n(x)).$$

It follows from (6) that the above expression is finite. So straightforward computations shows that $\mu_n^y(\cdot)$ is a probability measure.

The above expression can be written in the Ruelle operator formalism in the following way. Given a potential f and $-H_n(x) = f(x)+f(\sigma(x)+..+f(\sigma^{n-1}(x))$, then, for $n \in \mathbb{N}$

$$\mu_n^y(F) = \frac{1}{Z_n^y(\beta)} \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)) \;=\; \frac{\mathcal{L}_{\beta f}^n\,(1_F)\,(\sigma^n(y))}{\mathcal{L}_{\beta\, f}^n\,(1)\,(\sigma^n(y))}. \tag{10}$$

In another way

$$\mu_n^y = \frac{1}{\mathcal{L}_{\beta\, f}^n(1)(\sigma^n(y))}[\,(\mathcal{L}_{\beta\, f})^*\,]^n\,(\delta_{\sigma^n(y)}).$$

Note that if f is in the Walters class then

$$\mathcal{L}_f^n\,(1)\,(\sigma^n(y)) = \lambda^n\,\varphi(\sigma^n(y))\,\mathcal{L}_{\bar f}^n\left(\frac{1}{\varphi}\right)(\sigma^n(y)), \tag{11}$$

where λ is the eigenvalue, φ the eigenfunction of the Ruelle operator $\mathcal{L}_f$ and $\bar f = f + \log(\varphi) - \log(\varphi\circ\sigma) - \log\lambda$ is the normalized potential associated to f. So for any $n \in \mathbb{N}$ we have when $\beta = 1$

$$\mu_n^y(F) = \frac{\mathcal{L}_f^n\,(1_F)\,(\sigma^n(y))}{\lambda^n\,\varphi(\sigma^n(y))\,\mathcal{L}_{\bar f}^n\,(\frac{1}{\varphi})\,(\sigma^n(y))} \;=\; \frac{\mathcal{L}_{\bar f}^n\,(\frac{1_F}{\varphi})\,(\sigma^n(y))}{\mathcal{L}_{\bar f}^n\,(\frac{1}{\varphi})\,(\sigma^n(y))}. \tag{12}$$

Which implies that for any given cylinder set F and any $y \in \Omega$, we have that there exists the limit

$$\lim_{n\to\infty} \mu_n^y(F) = \nu(F), \tag{13}$$

where ν is the eigenprobability for $\mathcal{L}_f^*$.

Gibbs Specifications

Proposition 17. *For any fixed $n \in \mathbb{N}$ and $F \in \mathscr{F}$, the mapping $y \mapsto \mu_n^y(F)$ is measurable with respect the σ-algebra $\sigma^n \mathscr{F}$.*

Proof. It is enough to note that a function $y \mapsto z(y)$ is $\sigma^n \mathscr{F}$ measurable iff it is of the form $z(y) = v(\sigma^n(y))$ for some $\mathscr{F}$ measurable function v.

□

Definition 18 (Gibbs Specification Determined by Φ)**.** *Given a general interaction Φ and the associated Hamiltonian, for each $n \in \mathbb{N}$ and $y \in \Omega$, consider the function $K_n : (\mathscr{F}, \Omega) \to \mathbb{R}$ defined by*

$$K_n(F, y) = \frac{1}{Z_n^y(\beta)} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)),$$

where $Z_n^y(\beta)$ is

$$Z_n^y(\beta) = \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} \exp(-\beta H_n(x)).$$

The collection $\{K_n\}_{n \in \mathbb{N}}$ is known as the Gibbsian specification determined by the interaction Φ.

Theorem 19. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a Gibbsian specification determined by an interaction Φ, then: for $n \in \mathbb{N}$*

a) $y \to K_n(F, y)$ is $\sigma^n \mathscr{F}$-measurable

b) $F \mapsto K_n(F, y)$ is a probability measure;

Proof. The proof of a) follows from the Proposition 17 and b) is straightforward.

□

Proposition 20. *Let $\Phi = \{\Phi_A\}_{A \Subset \mathbb{N}}$ and $\Psi = \{\Psi_A\}_{A \Subset \mathbb{N}}$ two uniformly absolutely summable interactions. Suppose that there exist a sequence of real numbers (a_n) such that*

$$H_n(x) = \sum_{\substack{A \Subset \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Phi_A(x) \qquad \text{and} \qquad H_n(x) + a_n = \sum_{\substack{A \Subset \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Psi_A(x).$$

Then both interactions determines the same Gibbsian specification.

Proof. Let $K_n, \tilde{K}_n$ be the Gibbsian specifications determined by Φ and Ψ through H_n and $H_n + a_n$, respectively. If $\tilde{Z}_n^y(\beta)$ is the partition function associated to $\tilde{K}_n$, i.e.,

$$\tilde{K}_n(F, y) = \frac{1}{\tilde{Z}_n^y(\beta)} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x) + a_n),$$

then $\tilde{Z}_n^y(\beta) = \exp(a_n) Z_n^y(\beta)$, recall that $\tilde{Z}_n^y(\beta)$ is obtained taking the numerator of the above expression with $F = \Omega$. Therefore we have

$$\begin{aligned}
\tilde{K}_n(F, y) &= \frac{1}{\tilde{Z}_n^y(\beta)} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x) + a_n) \\
&= \frac{1}{\exp(a_n) Z_n^y(\beta)} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)) \exp(a_n) \\
&= K_n(F, y).
\end{aligned}$$

□

7 Gibbs Specifications for Continuous Functions

Given a continuous potential f we consider the following Hamiltonian

$$H_n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1} x).$$

Note that we are not assuming in this section that the potential f is associated to any interaction. Even though we can define a Gibbsian specification as in the previous section, i.e., for $n \in \mathbb{N}$

$$K_n(F, y) = \frac{1}{Z_n^y(\beta)} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)), \tag{14}$$

since we are assuming that f is continuous the two conclusions of the Theorem 19 holds and in this case we will say that $\{K_n\}_{n \in \mathbb{N}}$ is the Gibbs specification associated to the continuous function f.

8 Thermodynamic Limit

We consider $\{K_n\}_{n \in \mathbb{N}}$ a Gibbsian Specification determined by an uniformly absolutely summable interaction Φ or by a continuous potential f. By definition for any measurable set $F \subset \Omega$ we have

$$K_n(F, y) = \frac{1}{Z_n^y(\beta)} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)).$$

Let (y_n) be a sequence in Ω and $(K_n(\cdot, y_n))$ a sequence of Borel probabilities measures on Ω. From the compactness of $\mathcal{M}(\Omega)$ follows that there is at least one subsequence (n_k) so that $y_{n_k} \to y \in \Omega$ and $K_{n_k}(\cdot, y_{n_k}) \rightharpoonup \mu^y$, i.e., for all continuous function g we have

$$\lim_{k \to \infty} \int_\Omega g(x)\, dK_{n_k}(x, y_{n_k}) = \int_\Omega g\, d\mu^y.$$

So the probability measure μ^y is a cluster point in the weak topology of the set

$$\mathscr{C} \equiv \{K_n(\cdot, y_n) : n \in \mathbb{N} \text{ and } y_n \in \Omega\} \subset \mathcal{M}(\Omega).$$

Definition 21. *(Thermodynamic Limit Gibbs probability) The set $\mathcal{G}^{TL}(\Phi)$ or $\mathcal{G}^{TL}(f)$ is defined as being the closure of the convex hull of the set of the cluster points of $\mathscr{C}$ in $\mathcal{M}(\Omega)$, where*

$$\mathscr{C} \equiv \{K_n(\cdot, y_n) : n \in \mathbb{N} \textit{ and } y_n \in \Omega\}.$$

Any probability in $\mathcal{G}^{TL}(\Phi)$ will be called Thermodynamic Limit Gibbs probability.

Proposition 22. *If f is in the Walters class then $\mathcal{G}^{TL} = \mathcal{G}^*$.*

Proof. This follows from expression (13). □

9 DLR - Gibbs Measures

In this section we show how to extend the concept of Gibbs measures beyond the Hölder and Walters classes given in the Definition 8. This is done by following the Dobrushin-Lanford-Ruelle approach. We will present this concept of **Gibbs measures** for an interaction Φ and for a general continuous potential f.

Finite Volume DLR-equations

Let $\{K_n\}_{n \in \mathbb{N}}$ be a specification determined by and interaction Φ or a continuous potential f. Before present the DLR-equations let us introduce the following notation

$$\int_\Omega g\, dK_n(\cdot, y) \equiv \int_\Omega g(x)\, dK_n(x, y),$$

where g is any bounded measurable function $y \in \Omega$ and $x = (x_1, x_2, ...)$ is just an integration variable which runs over the space Ω.

Theorem 23 (Finite Volume DLR-equations)**.** *Let $\{K_n\}_{n\in\mathbb{N}}$ be a specification determined by a continuous potential f. Then for any continuous function $g : \Omega \to \mathbb{R}$, and any $z \in \Omega$ fixed, we have*

$$\int_\Omega \left[\int_\Omega g(x)\, dK_n(x,y)\right] dK_{n+r}(y,z) = \int_\Omega g(y)\, dK_{n+r}(y,z).$$

Before prove the theorem we prove some facts. The first one is the following lemma.

Lemma 24. *For any $n, r \in \mathbb{N}$, $x = (x_1, x_2, .., x_n)$, $y = (y_1, .., y_n, y_{n+1}, .., y_{n+r})$ and $z = (z_1, z_2, z_n, ..) \in \Omega$ we have*

$$H_{n+r}(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..) - H_n(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..)$$
$$=$$
$$H_{n+r}(x_1, .., x_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..) - H_n(x_1, .., x_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..).$$

In other words, the above difference does not depends on the first n variables.

Proof. By the definition of H_n we have

$$H_{n+r}(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..) - H_n(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..)$$
$$=$$
$$\sum_{j=n}^{n+r-1} f(\sigma^j(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..)).$$

By a simple inspection on the above expression the lemma follows. □

Corollary 25. *For any $x, y, z \in \Omega$ and $n, k \in \mathbb{N}$ we have*

$$H_{n+r}(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..) + H_n(x_1, .., x_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..)$$
$$=$$
$$H_{n+r}(x_1, .., x_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..) + H_n(y_1, .., y_n, y_{n+1}, .., y_{n+r}, z_{n+r+1}, ..).$$

Proof of Theorem 23. Observe that

$$\int_\Omega g(x)\, dK_n(x,\, (y_1, \ldots, y_{n+1}, \ldots, y_{n+r}, z_{n+r+1}, z_{n+r+2}, \ldots)\,)$$
$$=$$
$$\frac{1}{Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)} \quad\times \sum_{\substack{x\in\Omega \\ \sigma^n(x)=(y_{n+1},..,y_{n+r},z_{n+r+1},..)}} g(x)\exp(-\beta H_n(x))$$
$$\equiv$$
$$h(y_1, \ldots, y_{n+r}, z_{n+r+1}, \ldots).$$

Note that the statement of the theorem is equivalent to

$$\frac{1}{Z^z_{n+r}(\beta)} \sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} h(y)\exp(-\beta H_{n+r}(y)) = \frac{1}{Z^z_{n+r}(\beta)} \sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} g(y)\exp(-\beta H_{n+r}(y)).$$

Since $Z^z_{n+r}(\beta) > 0$, the above equation is equivalent to

$$\sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} h(y)\exp(-\beta H_{n+r}(y)) = \sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} g(y)\exp(-\beta H_{n+r}(y)). \tag{15}$$

From the definition of h follows that the l.h.s above is given by

$$\sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} \frac{1}{Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)} \sum_{\substack{x\in\Omega \\ \sigma^{n}(x)=(y_{n+1},..,y_{n+r},z_{n+r+1},..)}} g(x)\exp(-\beta H_n(x))\exp(-\beta H_{n+r}(y))$$

Using the Corollary 25 we can interchange the n first coordinates of the variables on the above exponentials showing that the above expression is equals to

$$\sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} \frac{\exp(-\beta H_n(y_1,..,y_{n+k},z_{n+r+1},..))}{Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)} \times$$
$$\sum_{\substack{x\in\Omega \\ \sigma^{n}(x)=(y_{n+1},..,y_{n+r},z_{n+r+1},..)}} g(x)\exp(-\beta H_{n+r}(x_1,..,x_n,y_{n+1},..,y_{n+r},z_{n+r+1},..)).$$

Note that the above expression is equals to

$$\sum_{y_1,\dots,y_{n+r}} \frac{\exp(-\beta H_n(y_1,..,y_{n+r},z_{n+r+1},..))}{Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)} \times$$
$$\sum_{x_1,\dots,x_n} g(x)\exp(-\beta H_{n+r}(x_1,..,x_n,y_{n+1},..,y_{n+r},z_{n+r+1},..)).$$

By changing the order of the sums we get

$$\sum_{y_{n+1},\dots,y_{n+r}} \sum_{y_1,\dots,y_n} \frac{\exp(-\beta H_n(y_1,..,y_{n+r},z_{n+r+1},..))}{Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)} \times$$
$$\sum_{x_1,\dots,x_n} g(x)\exp(-\beta H_{n+r}(x_1,..,x_n,y_{n+1},..,y_{n+r},z_{n+r+1},..)).$$

Observe that $Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)$ does not depends on $y_1,\dots,y_n$, so from its definition we have

$$\sum_{y_1,\dots,y_n} \frac{\exp(-\beta H_n(y_1,..,y_{n+r},z_{n+r+1},..))}{Z_n^{(y_1,..,y_{n+r},z_{n+r+1},..)}(\beta)} = 1,$$

implying that the previous expression is equals to

$$\sum_{y_{n+1},\ldots,y_{n+r}} \sum_{x_1,\ldots,x_n} g(x)\exp(-\beta H_{n+r}(x_1,..,x_n,y_{n+1},..,y_{n+k},z_{n+r+1},..))$$
$$=$$
$$\sum_{\substack{y\in\Omega \\ \sigma^{n+r}(y)=\sigma^{n+r}(z)}} g(y)\exp(-\beta H_{n+r}(y)).$$

Therefore the Equation (15) holds and the theorem is proved. □

In terms of the Ruelle operator the above theorem claims that for any z, and $n, r \geq 0$,

$$\mathcal{L}_f^{n+r}(h)(\sigma^{n+r}(z)) = \mathcal{L}_f^{n+r}\left(\frac{\mathcal{L}_f^n\,(h)(\,\sigma^n(\cdot)\,)}{\mathcal{L}_f^n(1)(\sigma^n(\cdot)\,)}\right)(\sigma^{n+r}(z)).$$

DLR-equations

Theorem 26 (DLR-equations)**.** *Let $\{K_n\}_{n\in\mathbb{N}}$ be a specification determined by an absolutely summable interaction or by a continuous potential f. If the sequence $K_{n_r}(\cdot, z) \rightharpoonup \mu^z$, when $r\to\infty$, then for any continuous function $g:\Omega\to\mathbb{R}$, we have*

$$\int_\Omega \left[\int_\Omega g(x)\,dK_n(x,y)\right]\,d\mu^z(y) = \int_\Omega g\,d\mu^z$$

Proof. In both cases, where K_n is determined by an absolutely uniformly summable interaction or continuous potential f, the mapping

$$y \mapsto \int_\Omega g(x)\,dK_n(x,y)$$

is continuous for any continuous function g. Taking r large enough the proof follows from the Theorem 23 and the definition of weak topology. □

Definition 27 (DLR-Gibbs Measures)**.** *Let $\{K_n\}_{n\in\mathbb{N}}$ be the Gibbsian Specification on the lattice $\mathbb{N}$ determined by an interaction Φ obtained from a continuous potential f constructed as in the two previous sections. The set of the* **DLR Gibbs measures** *for the interaction Φ (or for potential f) is denoted by $\mathcal{G}^{DLR}(\Phi)$ ($\mathcal{G}^{DLR}(f)$) and given by*

$$\{\mu\in\mathcal{M}(\Omega) : \mu(F|\sigma^n\mathscr{F})(y) = K_n(F,y) \text{ for any } y,\ \mu\text{-a.e. } \forall F\in\mathscr{F} \text{ and } \forall n\in\mathbb{N}\}.$$

Remark. We observe that the set $\mathcal{G}^{DLR}(\Phi)$ or $\mathcal{G}^{DLR}(f)$ for a general uniformly absolutely summable interaction Φ or continuous potential, respectively, is not necessarily an unitary set. But in any case it is a convex and compact subset of $\mathcal{M}(\Omega)$ in the weak topology. We also observe that **for the lattice** $\mathbb{Z}$ even in the case where $\#\mathcal{G}^{DLR}(\Phi) = 1$ or $\#\mathcal{G}^{DLR}(f) = 1$ the unique probability measure on such sets are not necessarily shift invariant. **For the lattice** $\mathbb{Z}$, assuming the hypothesis of uniformly absolutely summability and the shift invariant property in the lattice $\mathbb{Z}$ for the interaction Φ, then, we get that the unique element in $\mathcal{G}^{DLR}(\Phi)$ is invariant for the action of the shift in $\{1, 2..., d\}^{\mathbb{Z}}$ (see Corollary 3.48 in [16]). **For the lattice** $\mathbb{N}$ and a continuous potential the corresponding results are not necessarily the same: the difference between the action by automorphisms and endomorphisms has to be properly considered. Theorems 30 and 34 are examples where results on $\mathbb{N}$ and $\mathbb{Z}$ match.

Theorem 28. *Let $\{K_n\}_{n\in\mathbb{N}}$ be the Gibbsian Specification determined by an interaction Φ obtained from a continuous potential f. A Borel probability measure μ belongs to $\mathcal{G}^{DLR}(\Phi)$ or $\mathcal{G}^{DLR}(f)$ if and only if for all $n \in \mathbb{N}$ and any continuous function $g : \Omega \to \mathbb{R}$, we have*

$$\int_\Omega \left[\int_\Omega g(x)\, dK_n(x, y) \right] d\mu(y) = \int_\Omega g\, d\mu.$$

In other words, $\mu \in \mathcal{G}^{DLR}(\Phi)$ or $\mathcal{G}^{DLR}(f)$ iff μ satisfies the DLR-equations.

Proof. The proof for $\mathcal{G}^{DLR}(\Phi)$ is very classical on the Statistical Mechanics literature. We mimic this proof for $\mathcal{G}^{DLR}(f)$. Suppose that $\mu \in \mathcal{G}^{DLR}(f)$ then it follows from the definition of $\mathcal{G}^{DLR}(f)$ and the basic properties of the conditional expectation that for all $n \in \mathbb{N}$ we have

$$\int_\Omega g\, d\mu = \int_\Omega \mu(g|\sigma^n \mathscr{F})(y)\, d\mu(y) = \int_\Omega \left[\int_\Omega g(x)\, dK_n(x, y) \right] d\mu(y).$$

Conversely, we assume that the DLR-equations are valid for all $n \in \mathbb{N}$ and for any continuous function g. By taking $g = 1_E h$, where $E \in \sigma^n \mathscr{F}$ is a cylinder set and h is an arbitrary continuous function, we have g is continuous and

$$\int_E \left[\int_\Omega h(x)\, dK_n(x, y) \right] d\mu(y) = \int_E h\, d\mu.$$

It follows from the Dominate Convergence Theorem and Monotone Class Theorem that above identity holds for any measurable set $E \in \sigma^n \mathscr{F}$. Since the mapping

$$y \mapsto \int_\Omega h(x)\, dK_n(x, y)$$

is $\sigma^n(\mathscr{F})$-measurable and $E \in \sigma^n(\mathscr{F})$ is an arbitrary measurable set, we have from the definition of conditional expectation that

$$\int_\Omega h(x)\, dK_n(x, y) = \mu(h|\sigma^n \mathscr{F})(y) \quad \mu - \text{a.e.}.$$

Using again the dominate convergence theorem for conditional expectation and monotone class theorem we can show that the above equality holds for $h = 1_F$ where F is a measurable set on Ω, so the result follows. $\square$

Theorem 29. *For any absolutely uniformly summable interaction Φ or a continuous potential f we have that*

$$\mathcal{G}^{TL}(\Phi) \subset \mathcal{G}^{DLR}(\Phi) \qquad \textit{and} \qquad \mathcal{G}^{TL}(f) \subset \mathcal{G}^{DLR}(f).$$

Proof. Suppose that $K_{n_k}(\cdot, z_{n_k}) \rightharpoonup \mu$, i.e., $\mu \in \mathcal{G}^{TL}(f)$. Since f is continuous potential we have that for any continuous function $g : \Omega \to \mathbb{R}$ the mapping

$$y \mapsto \int_\Omega g(x)\, dK_n(x, y)$$

is continuous. Using the definition of weak convergence, the finite volume DLR-equations and continuity of g, respectively, we have

$$\begin{aligned}
\int_\Omega \left[\int_\Omega g(x)\, dK_n(x,y)\right] d\mu(y) &= \lim_{k\to\infty} \int_\Omega \left[\int_\Omega g(x)\, dK_n(x,y)\right] dK_{n_k}(y, z_{n_k}) \\
&= \lim_{r\to\infty} \int_\Omega g(y)\, dK_{n_k}(y, z_{n_k}) \\
&= \int_\Omega g(y)\, d\mu(y).
\end{aligned}$$

The above equation prove that the probability measure μ satisfies the DLR-equation for any continuous function g. By applying the Theorem 28 we conclude that $\mu \in \mathcal{G}^{DLR}(f)$. $\square$

Theorem 30. *Let Φ be an interaction obtained from a continuous potential f We consider the Gibbs specification $\{K_n\}_{n\in\mathbb{N}}$, associated to this interaction Φ or to the function f. Then, in the general case $\mathcal{G}^{DLR}(\Phi) = \mathcal{G}^{TL}(\Phi)$. In particular, $\mathcal{G}^{DLR}(f) = \mathcal{G}^{TL}(f)$.*

Proof. We borrow the proof from [5]. We first remark that $\mathcal{G}^{TL}(\Phi) \subset \mathcal{G}^{DLR}(\Phi)$ is the content of the Theorem 29. Suppose by contradiction that there exists $\mu \in \mathcal{G}^{DLR}(\Phi)$ which is not in the compact set $\mathcal{G}^{TL}(\Phi)$.

Any open neighborhood $V \subset \mathcal{M}(\Omega)$ of μ in the weak topology contains an basis element $B(g, \varepsilon)$, where g is some continuous function and $\epsilon > 0$, of the form

$$B(g, \epsilon) \;=\; \left\{\nu \in \mathcal{M}(\Omega) : \left|\int_\Omega g\, d\,\nu - \int_\Omega g\, d\mu\right| < \epsilon\right\}.$$

Assume for some $B(g, \epsilon)$, that we have $B(g, \epsilon) \cap \mathcal{G}^{TL}(\Phi) = \emptyset$. Note that the mapping $\nu \mapsto \int_\Omega g\, d\nu$ is continuous and convex as a function defined on the convex set $\mathcal{G}^{TL}(\Phi)$.

Without loss of generality we can assume for any $\nu \in \mathcal{G}^{TL}(\Phi)$ that $\int_\Omega g\, d\nu > \int_\Omega g d\mu + \epsilon$. By Theorem 28 we have for any $n \in \mathbb{N}$ that

$$\int_\Omega \left[\int_\Omega g(x)\, dK_n(x,y) \right] \, d\mu(y) = \int_\Omega g \, d\mu.$$

For each n, there exist at least one y_n such that $\int_\Omega g(x)\, dK_n(x,y_n) < \int_\Omega g d\nu + \epsilon$. Now, by considering a convergent subsequence $\lim_{k\to\infty} K_{n_k}(.,y_{n_k}) = \mu \in \mathcal{G}^{TL}(\Phi)$ we reach a contradiction. Therefore, $\mathcal{G}^{DLR}(\Phi) = \mathcal{G}^{TL}(\Phi)$.

$\square$

Note that when f is a Walters potential, for any $z \in \Omega$ we have from (13) that $K_n(.,z) \to \nu$, where $\nu \in \mathcal{G}^*(f)$. Therefore, ν is a DLR-Gibbs Measure for the potential f.

There is another direct way to prove this result for a general continuous potential f which needs not to be on the Bowen or Walters class. To prove this result in such generality is useful to deal with the potentials appearing on [6] and [36].

Theorem 31. *Suppose f is continuous and there exists a positive eigenfunction ψ for the Ruelle operator $\mathcal{L}_f$. Then, $\mathcal{G}^*(f) \subset \mathcal{G}^{DLR}$.*

Proof. Suppose ν is an eigenprobability in $\mathcal{G}^*(f)$. Then $\mu = \psi\,\nu$ is a shift-invariant probability measure. Given a function $g : \Omega \to \mathbb{R}$, consider the expected value of g with respect to the finite volume Gibbs measure with boundary condition y, given by

$$\int_\Omega g(x) K_n(x,y)\, dx = \frac{\mathcal{L}^n_{\beta f}\,(g)\,(\sigma^n(y))}{\mathcal{L}^n_{\beta\, f}\,(1)\,(\sigma^n(y))}.$$

Note that $\frac{\mathcal{L}^n_{\beta f}\,(g)\,(\sigma^n(y))}{\mathcal{L}^n_{\beta\, f}\,(1)\,(\sigma^n(y))}$ is $\sigma^n \mathscr{F}$ measurable. From the Theorem 23 we have for any $n, r \geq 0$ and z that

$$\mathcal{L}_f^{n+r}(h)(\sigma^{n+r}(z)) = \mathcal{L}_f^{n+r} \left(\frac{\mathcal{L}^n_f\,(h)(\,\sigma^n(\cdot)\,)}{\mathcal{L}^n_f(1)(\sigma^n(\cdot)\,)} \right) (\sigma^{n+r}(z)). \tag{16}$$

Given $n \in \mathbb{N}$ and measurable functions g and v we have that

$$\int_\Omega (g\circ\sigma^n)(z)\,\frac{\mathcal{L}_f^n\,(v)\,(\sigma^n(z))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(z))}\,d\,\nu(z) = \int_\Omega \frac{\mathcal{L}_f^n\,(v\,(g\circ\sigma^n))\,(\sigma^n(z))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(z))}\,d\,\nu(z)$$

$$= \int_\Omega \frac{1}{\lambda^n}\,\mathcal{L}_f^n\left[\frac{\mathcal{L}_f^n\,(v\,(g\circ\sigma^n))\,(\sigma^n(\cdot))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(\cdot))}\right](z)\,d\,\nu(z)$$

$$= \int_\Omega \frac{1}{\lambda^n}\,\mathcal{L}_f^n\left[\frac{\mathcal{L}_f^n\,(v\,(g\circ\sigma^n))\,(\sigma^n(\cdot))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(\cdot))}\right](z)\,\frac{1}{\psi(z)}\,d\,\mu(z)$$

since the measure μ is translation invariant the r.h.s above is equals to

$$\int_\Omega \frac{1}{\lambda^n}\,\mathcal{L}_f^n\left[\frac{\mathcal{L}_f^n\,(v\,(g\circ\sigma^n))\,(\sigma^n(\cdot))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(\cdot))}\right](\sigma^n)(z)\ \frac{1}{\psi(\sigma^n(z))}\,d\,\mu(z).$$

Using the equation (16) we can see that the above expression is equals to

$$\int_\Omega \frac{1}{\lambda^n}\,\mathcal{L}_f^n\,[(v\,(g\circ\sigma^n))(\cdot)]\,(\sigma^n)(z)\ \frac{1}{\psi(\sigma^n(z))}\,d\,\mu(z).$$

Using again the translation invariance of μ we get that the above expression is equals to

$$\int_\Omega \frac{1}{\lambda^n}\,\mathcal{L}_f^n\,[(v\,(g\circ\sigma^n))(\cdot)]\,(z)\ \frac{1}{\psi(z)}\,d\,\mu(z) = \int_\Omega \frac{1}{\lambda^n}\,\mathcal{L}_f^n\,[(v\,(g\circ\sigma^n))(\cdot)]\,(z)\ \ d\,\nu(z)$$

$$= \int_\Omega\ (g\circ\sigma^n)\,(z)\,v(z)\ d\,\nu(z)$$

so for each $n\in\mathbb{N}$ follows from the identities we obtained above that

$$\nu(F|\sigma^n\mathscr{F})(y) = \frac{\mathcal{L}_f^n\,(I_F)\,(\sigma^n(y))}{\mathcal{L}_{\beta\,f}^n\,(1)\,(\sigma^n(y))},\qquad \nu-\text{almost surely.}$$

Which immediately implies that ν is DLR-Gibbs measure for f. $\square$

In the next section we address the question of uniqueness of the Gibbs Measures. We provide sufficient conditions for a specification defined by an interaction Φ or by a continuous potential f to have a unique DLR-Gibbs measure. In the sequel we use this theorem to prove that the specification Φ^f for a Hölder potential f as defined in Theorem 13 or the specification for f in Walter class, has a unique Gibbs Measure.

We shown that the unique DLR-Gibbs measure associated to $\Phi^{\bar{f}}$ is the unique fixed point of $\mathcal{L}^*_{\bar{f}}$ the same for f in the Walter class. Since we have a unique Gibbs measure for any Hölder potential or for any f in Walter class follows that the DLR point of view give us a direct and convenient way to define Gibbs Measures for a very large class of potentials and extends the concept of Gibbs measure usually considered in Thermodynamic Formalism. We also explain the relation between the DLR Gibbs measures for $\Phi^{\bar{f}}$ and Φ^f in the Hölder class and the analogous statement for the Walter class.

Uniqueness and Walters Condition

We say that a specification presents phase transition in the DLR, TL or dual sense if, respectively, the sets $\mathcal{G}^{DLR}(\Phi)$, $\mathcal{G}^{TL}(\Phi)$, or $\mathcal{G}^*(\Phi)$ has more than one element. In [6] several examples of such phenomena are described for one-dimensional systems.

We begin this section recalling a classical theorem of the Probability theory. Consider the following decreasing sequence of σ-algebras

$$\mathscr{F} \supset \sigma\mathscr{F} \supset \sigma^2\mathscr{F} \supset \cdots \supset \sigma^n\mathscr{F} \supset \cdots \supset \bigcap_{j=1}^{\infty} \sigma^j\mathscr{F}$$

the Backward Martingale Convergence Theorem states that for all $f : \Omega \to \mathbb{R}$ bounded measurable function we have

$$\mu(f|\sigma^n\mathscr{F}) \to \mu(f| \cap_{j=1}^{\infty} \sigma^j\mathscr{F}), \quad \text{a.s. and in } \ L^1.$$

Theorem 32. *Let $\{K_n\}_{n\in\mathbb{N}}$ be an specification determined by an absolutely uniformly summable interaction Φ or a continuous function f. Let $\mu \in \mathcal{G}^{DLR}(\Phi)$ or $\mu \in \mathcal{G}^{DLR}(f)$. In any of both cases the following conditions are equivalent characterizations of extremality.*

1. *μ is extremal;*
2. *μ is trivial on $\bigcap_{j=1}^{\infty} \sigma^j\mathscr{F}$;*
3. *Any $\bigcap_{j=1}^{\infty} \sigma^j\mathscr{F}$-measurable functions g is constant μ-almost surely.*

Proof. The proof can be found on [17] and [16, p. 151]. □

Theorem 33. *Let $\{K_n\}_{n\in\mathbb{N}}$ be an specification determined by an interaction Φ obtained from a continuous function f. Let μ be an extremal element of the convex space $\mathcal{G}^{DLR}(\Phi)$ or $\mathcal{G}^{DLR}(f)$. Then for μ-almost all y we have*

$$K_n(\cdot, y) \rightharpoonup \mu.$$

Proof. We will prove convergence and not just convergence of subsequence. We will borrow the proof from [16].

Note that it is enough to prove that for μ-almost all y we have

$$K_n(C,y) \to \mu(C) \qquad \forall C \text{ cylinder set.}$$

The crucial fact is that the of all cylinder set is countable. Fix a cylinder C and $n \in \mathbb{N}$ then by the definition of $\mathcal{G}^{DLR}(\Phi)$ or $\mathcal{G}^{DLR}(f)$ there is a set $\Omega_{n,C}$ with $\mu(\Omega_{n,C}) = 1$ and such that $K_n(C,y) = \mu(C|\sigma^n\mathscr{F})(y)$ for all $y \in \Omega_{n,C}$. By the item *3* of the Theorem 32 and definition of conditional expectation it is possible to find a measurable set Ω_C so that $\mu(C) = \mu(C|\cap_{j=1}^{\infty}\sigma^j\mathscr{F})(y)$ for all $y \in \Omega_C$ and $\mu(\Omega_C) = 1$. By the Backward Martingale Convergence Theorem with $f = 1_C$, we have that there is a measurable set Ω'_C with $\mu(\Omega'_C) = 1$ so that

$$\mu(C|\sigma^n\mathscr{F})(y) \to \mu(C|\cap_{j=1}^{\infty}\sigma^j\mathscr{F})(y) \qquad \forall y \in \Omega'_C.$$

Therefore if

$$y \in \bigcap_{\substack{n\in\mathbb{N} \\ C \text{ cylinder set}}} (\Omega_C \cap \Omega'_C \cap \Omega_{n,c})$$

which is a set of μ measure one, we have the desired convergence. □

Theorem 34. *Let Φ be an absolutely uniform summable potential and f a continuous potential. Consider the Hamiltonian*

$$H_n(x) = \sum_{\substack{A\in\mathbb{N} \\ A\cap\Lambda_n\neq\emptyset}} \Phi_A(x) \qquad \text{or} \qquad H_n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x)$$

If

$$D \equiv \sup_{n\in\mathbb{N}} \sup_{\substack{x,y\in\Omega; \\ d(x,y)\leq 2^{-n}}} |H_n(x) - H_n(y)| < \infty$$

then $\#\mathcal{G}^{DLR}(\Phi) = 1$ or $\#\mathcal{G}^{DLR}(f) = 1$, respectively.

Before presenting the proof of this theorem we need one more lemma.

Lemma 35. *Let D be the constant defined on the above theorem. Then, for all $x, y \in \Omega$, all cylinders C and for n large enough, we have*

$$e^{-2D\beta}K_n(C,z) \leq K_n(C,y) \leq e^{2D\beta}K_n(C,z).$$

Proof. The proof presented here follow closely [16]. By the definition of D, uniformly in $n \in \mathbb{N}$, $x, y, z \in \Omega$, we have

$$-D \leq H_n(x_1,\ldots,x_n,z_{n+1},\ldots) - H_n(x_1,\ldots,x_n,y_{n+1},\ldots) \leq D$$

which implies the following two inequalities:

$$\exp(-D\beta)\exp(-\beta H_n(x_1,...,x_n,z_{n+1},...)) \leq \exp(-H_n(x_1,...,x_n,y_{n+1},...))$$

and

$$\exp(-H_n(x_1,...,x_n,y_{n+1},...)) \leq \exp(D\beta)\exp(-\beta H_n(x_1,...,x_n,z_{n+1},...)).$$

Using theses inequalities we get that

$$e^{-\beta D} Z_n^z(\beta) \leq Z_n^y(\beta) \leq e^{\beta D} Z_n^z(\beta).$$

Let C be a cylinder set and suppose that its basis is contained in the set $\{1,\ldots,p\}$. For every $n \geq p$ we have $1_C(x_1,\ldots,x_n,z_{n+1},\ldots) = 1_C(x_1,\ldots,x_n,y_{n+1},\ldots)$. Therefore

$$\begin{aligned}
K_n(C,y) =& \frac{1}{Z_n^y(\beta)} \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} 1_C(x)\exp(-\beta H_n(x)) \\
\leq& \frac{1}{e^{-\beta D} Z_n^z(\beta)} \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(z)}} 1_C(x)\exp(D\beta)\exp(-\beta H_n(x)) \\
=& e^{2D\beta} K_n(C,z).
\end{aligned}$$

The proof of the inequality $e^{-2D\beta}K_n(C,z) \leq K_n(C,y)$ is similar. □

Proof of the Theorem 34. The strategy is to prove that $\mathcal{G}(\Phi)$ or $\mathcal{G}(f)$ has a unique extremal measure and use the Choquet Theorem to conclude the uniqueness of the Gibbs measures.

We give the argument for $\mathcal{G}(f)$, the proof is the same for $\mathcal{G}(\Phi)$. Let μ and ν be extremal measures on $\mathcal{G}(f)$. By the Theorem 33 there exists $y, z \in \Omega$ such that both measures μ and ν are thermodynamic limits of $K_n(\cdot, y)$ and $K_n(\cdot, z)$, respectively, when $n \to \infty$. Using the Lemma 35 we have for any cylinder set C that

$$\mu(C) = \lim_n K_n(C,y) \leq e^{2D\beta} \lim_n K_n(C,z) = e^{2D\beta}\nu(C).$$

Clearly the collection $\mathscr{D} = \{E \in \mathscr{F} : \mu(E) \leq e^{2D\beta}\nu(E)\}$ is a monotone class. Since it contains the cylinder set, which is stable under intersections, we have that $\mathscr{D} = \mathscr{F}$. Therefore $\mu \leq e^{2D\beta}\nu$, in particular $\mu \ll \nu$. This contradict the fact that two distinct extremal gibbs measures are mutually singular, see [17, Theo. 7.7, pag 118]. So have proved that $\mu = \nu$. □

Remark. Note that the condition imposed over D is more general than the Walters condition. In fact, we just have proved uniqueness in the Bowen class (see definition in [36]). Note that for a continuous f we have that $\mathcal{G}^*(f)$ and $\mathcal{G}^{TL}(f)$ are not empty. We just show that the cardinality of $\mathcal{G}^{TL}(f)$ is equal to one.

Theorem 36. *Let $f : \Omega \to \mathbb{R}$ be a Hölder potential and Φ^f the interaction constructed in the Theorem 13. If*

$$H_n(x) = \sum_{\substack{A \Subset \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Phi_A^f(x)$$

then the hypothesis of Theorem 34 holds and therefore Φ^f has a unique DLR Gibbs measure measures for the interaction Φ^f.

Proof. By Proposition 16 and the triangular inequality we have that

$$\begin{aligned} |H_n(x) - H_n(y)| &= |f(x) + \ldots + f(\sigma^n x) - f(y) - \ldots - f(\sigma^n y)| \\ &\leq |f(x) - f(y)| + \ldots + |f(\sigma^n x) - f(\sigma^n y)|. \end{aligned}$$

Given $n \in \mathbb{N}$ if $d(x,y) \leq 2^{-n}$ then the r.h.s above is bounded by $K(f)2^{-n} + K(f)2^{-n+1} + \ldots + K(f)$. So

$$D \equiv \sup_{n \in \mathbb{N}} \sup_{\substack{x,y \in \Omega; \\ d(x,y) \leq 2^{-n}}} |H_n(x) - H_n(y)| < K(f).$$

□

Remark. We point out the following a criteria of [21] for the uniqueness of the eigenprobability of the normalized potential $\bar{f}$ (which is stronger than [4]): For any $\epsilon > 0$ we have that

$$\sum_{n=1}^{\infty} \exp\left[-(\frac{1}{2} + \epsilon)\left(\mathrm{var}_1(\bar{f}) + \mathrm{var}_2(\bar{f}) + ... + \mathrm{var}_n(\bar{f})\right)\right] = \infty.$$

10 Equivalence on the Walters Class

Now we consider a potential f in the Walters class and the specification $\{K_n\}_{n \in \mathbb{N}}$ determined by f. We will show, following the steps in [3], that the unique Gibbs measure $\mu \in \mathcal{G}^{DLR}(f)$ satisfies $\mathcal{L}^*_{\bar{f}}(\mu) = \mu$. Next, we show that if $\bar{f}$ belongs to the Walters class, then

$$\mathcal{G}^*(\bar{f}) = \mathcal{G}^{DLR}(\bar{f}) = \mathcal{G}^{TL}(\bar{f}).$$

We will express al the concepts and results described in previous sections in the language of Ruelle operators. When f is in the Walters class there exist the main eigenfunction and this sometimes simplifies some proofs.

Theorem 37. *Let $f : \Omega \to \mathbb{R}$ be a potential in the Walters class and $\bar{f}$ its normalization given by (2). Let μ be the unique probability belonging to the set $\mathcal{G}^{DLR}(\Phi^{\bar{f}})$ then $\mathcal{L}^*_{\bar{f}}\mu = \mu$. In other words $\mathcal{G}^{DLR}(\bar{f}) \subset \mathcal{G}^*(\bar{f})$.*

Proof. We first prove that for any continuous function $g : \Omega \to \mathbb{R}$ and $y \in \Omega$ fixed, we have

$$\sum_{\substack{x\in\Omega\\ \sigma^n x = y}} g(x)\, K_n(x,y) = \mathcal{L}^n_{\bar{f}}(f)(\sigma^n(y)).$$

It follows directly from the definition of the Ruelle operator that

$$\mathcal{L}^n_{\bar{f}}(1)(\sigma^n(y)) = \sum_{\substack{x\in\Omega\\ \sigma^n x = \sigma^n y}} \exp(\bar{f}(x) + \ldots + \bar{f}(\sigma^{n-1}x)).$$

Since the potential is normalized we have that above sum is equal to one. If $\{K_n\}_{n\in\mathbb{N}}$ is the specification associated to the interaction $\Phi^{\bar{f}}$ then for any continuous potential g we have

$$\sum_{\substack{x\in\Omega\\ \sigma^n x = y}} g(x) K_n(x,y) = \frac{1}{Z^y_n} \sum_{\substack{x\in\Omega\\ \sigma^n x = \sigma^n y}} g(x) \exp(\bar{f}(x) + \ldots + \bar{f}(\sigma^{n-1}x))$$

Note we have canceled the term $nf(0,0,\ldots)$ in the numerator and denominator in the above expression. It is immediate to check that $Z^y_n = \mathcal{L}^n_{\bar{f}}(1)(\sigma^n(y)) = 1$, therefore

$$\sum_{\substack{x\in\Omega\\ \sigma^n x = y}} g(x) K_n(x,y) = \mathcal{L}^n_{\bar{f}}(g)(\sigma^n(y))$$

By the Theorem 30 and a classical result from Thermodynamic Formalism we have that (up to subsequence)

$$\int_\Omega g\, d\mu = \lim_{n\to\infty} \sum_{\substack{x\in\Omega\\ \sigma^n x = y}} g(x) K_n(x,y) = \lim_{n\to\infty} \mathcal{L}^n_{\bar{f}}(g)(\sigma^n(y)) = \int_\Omega g dm,$$

where m is the fixed point of the Ruelle operator. By observing that the g is an arbitrary continuous function it follows from the Riesz-Markov Theorem that $\mu = m$. □

Given a potential f and $-H_n(x) = f(x) + f(\sigma(x) + .. + f(\sigma^{n-1}(x))$, then

$$K_n(F,y) = \frac{1}{Z^y_n(\beta)} \sum_{\substack{x\in\Omega;\\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-\beta H_n(x)) \;=\; \frac{\mathcal{L}^n_{\beta f}\,(1_F)\,(\sigma^n(y))}{\mathcal{L}^n_{\beta\, f}\,(1)\,(\sigma^n(y))}.$$

As long as f is in the Walters class we have

$$\mathcal{L}_f^n\,(1)\,(\sigma^n(y)) = \lambda^n\,\varphi(\sigma^n(y))\,\mathcal{L}_{\bar{f}}^n\,\left(\frac{1}{\varphi}\right)(\sigma^n(y))$$

and we can also rewrite $K_n(F,y)$ as follows:

$$K_n(\cdot,y) = \frac{1}{\mathcal{L}_{\beta\,f}^n(1)(\sigma^n(y))}[\,(\mathcal{L}_{\beta\,f})^*\,]^n\,(\delta_{\sigma^n(y)}). \tag{17}$$

Note that if the potential $\bar{f}$ is normalized, that is $\mathcal{L}_{\bar{f}}1 = 1$, and $H_n = \bar{f}(x) + \bar{f}(\sigma(x)+\ldots+\bar{f}(\sigma^n(x))$, then for all n and all y we have $Z_n^{y,\bar{f}}(1) = \mathcal{L}_{\bar{f}}^n\,(1)\,(\sigma^n(y)) = 1$. Moreover,

$$K_n(F,y) = \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} 1_F(x)\exp(-H_n(x)) \;=\; \mathcal{L}_{\bar{f}}^n\,(1_F)\,(\sigma^n(y))$$

or alternatively $K_n(\cdot,y) = [\,(\mathcal{L}_{\bar{f}})^*\,]^n\,(\delta_{\sigma^n(y)})$. Assuming that f is in the Walters class we have for any continuous function $g:\Omega\to\mathbb{R}$, and any fixed $y\in\Omega$ that

$$\lim_{n\to\infty}\mathcal{L}_{\bar{f}}^{n+1}(g)\,(\sigma^n(y)) = \lim_{n\to\infty}\mathcal{L}_{\bar{f}}^n(g)\,(\sigma^n(y)) = \int_\Omega g\,d\mu,$$

where μ is the fix point for $\mathcal{L}_{\bar{f}}^*$.

Remark: From the above expression follows that if $\bar{f}$ is just continuous but normalized ($\mathcal{L}_{\bar{f}}(1)=1$) and such that for a certain $y\in\Omega$ there exists the limit $m = \lim_{n\to\infty}\mu_n^y$, then,

$$\begin{aligned}
\mathcal{L}_{\bar{f}}^*(m) = \mathcal{L}_{\bar{f}}^*(\lim_{n\to\infty}\mu_n^{y,\bar{f}}) &= \mathcal{L}_{\bar{f}}^*(\lim_{n\to\infty}[\,(\mathcal{L}_{\bar{f}})^*\,]^n\,(\delta_{\sigma^n(y)})\,)\\
&= \lim_{n\to\infty}\;\mathcal{L}_{\bar{f}}^*[\,(\mathcal{L}_{\bar{f}})^*\,]^n\,(\delta_{\sigma^n(y)}\,)\\
&= m.
\end{aligned}$$

In other words: $\mathcal{G}^*(\bar{f}) = \mathcal{G}^{TL}(\bar{f})$.

Theorem 38. *Let $f:\Omega\to\mathbb{R}$ be a potential in the Walters class and $\bar{f}$ its normalization given by* (2)*. Then*

1. *For any $y\in\Omega$ we have that $\mu_n^{y,\bar{f}} = [\,(\mathcal{L}_{\bar{f}})^*\,]^n\,(\delta_{\sigma^n(y)})$ converges when $n\to\infty$ in the weak topology to the unique equilibrium state for f (or $\bar{f}$).*

2. Moreover, for any $y \in \Omega$, we have that

$$\mu_n^{y,f} = \frac{1}{\mathcal{L}_f^n(1)(\sigma^n(y))} \left[(\mathcal{L}_f)^* \right]^n (\delta_{\sigma^n(y)})$$

converges when $n \to \infty$ to the unique eigenprobability ν for $\mathcal{L}_f^*$, associated to the principal eigenvalue.

3. As the set $\mathcal{G}^{TL}(\Phi^f)$ is the closure of the convex hull of the set of weak limits of subsequences $\mu_{n_k}^{y,f}$, $k \to \infty$, we get that $\mathcal{G}^{DLR}(\Phi^f)$ has cardinality 1.

Proof. 1. If $\bar{f}$ is normalized and in the Walters class there is a unique fixed probability μ for $\mathcal{L}_{\bar{f}}^*$. Given any $y \in \Omega$ we have, for any continuous function g, that

$$(\mathcal{L}_{\bar{f}})^n(g)(\sigma^n(y)) \to \int_\Omega g d\mu,$$

as $n \to \infty$. Since $\int_\Omega g\, d\mu_n^{y,\bar{f}} = (\mathcal{L}_{\bar{f}})^n(g)(\sigma^n(y))$ the first claim follows.

2. Since the equilibrium state $\mu = \varphi\, \nu$, where φ is the eigenfunction and ν is the eigenprobability. Given a non-normalized f in the Walters class and φ the main eigenfunction of $\mathcal{L}_f$ considering the normalized potential $\bar{f} = f + \log\varphi - \log(\varphi \circ \sigma) - \log\lambda$, we have

$$(\mathcal{L}_{\bar{f}})^n\left(\frac{1}{\varphi}\right)(\sigma^n(y)) \to \int_\Omega \frac{1}{\varphi} d\mu = \int_\Omega 1\, d\nu.$$

From this convergence and the coboundary property we get that

$$\frac{\varphi(\sigma^n(y))^{-1}\ \mathcal{L}_f^n(\,1\,)(y)}{\lambda^n} \to 1.$$

Analogously for any given continuous function $g : \Omega \to \mathbb{R}$, we have that

$$(\mathcal{L}_{f+\log\varphi-\log(\varphi\circ\sigma)-\log\lambda})^n\left(\frac{g}{\varphi}\right)(\sigma^n(y)) = (\mathcal{L}_{\bar{f}})^n\left(\frac{g}{\varphi}\right)(\sigma^n(y)) \to \int \frac{g}{\varphi} d\mu = \int g\, d\nu.$$

and

$$\frac{\varphi(\sigma^n(y))^{-1}\ \mathcal{L}_f^n(\,g\,)(\sigma^n(y))}{\lambda^n} \to \int g d\nu,$$

when $n \to \infty$. So taking the limit when $n \to \infty$ we obtain

$$\int_\Omega g\, \mu_n^{y,f} = \frac{1}{\mathcal{L}_f^n(1)(\sigma^n(y))} (\mathcal{L}_f)^n (g)(\sigma^n(y)) \ \sim (\mathcal{L}_f)^n (g)(\sigma^n(y)) \frac{1}{\frac{\lambda^n}{\varphi(\sigma^n(y))^{-1}}}$$

$$\to \int g d\nu,$$

$\square$

The above proof also shows that $\mathcal{G}^{DLR}(f) \subset \mathcal{G}^*(f)$. That is, if f is in the Walters class we have that $\{\nu\} = \mathcal{G}^{DLR}(f)$ where ν is the unique eigenprobability for $\mathcal{L}_f^*$. Moreover, if $\bar{f}$ is normalized and in the Walters class we have that $\{\mu\} = \mathcal{G}^{DLR}(\bar{f})$ where μ is the unique fixed point for $\mathcal{L}_{\bar{f}}^*$.

If the potential $\bar{f}$ does not satisfies the hypothesis of the Theorem 34 the limit $\mu_n^{y,\bar{f}}$, when $n \to \infty$, can depends on y. We Remark that there is a certain optimality in Theorem 34 because of the long-range Ising model. When the parameter $\alpha > 2$ in this model, we have shown that the Hypothesis of the Theorem 34 are satisfied and we have only one Gibbs measure for any y. For $1 < \alpha \leq 2$ (where the hypothesis is broken) Dyson, Frölich and Spencer (see [10, 14]) shows that there is more than one Gibbs measure for sufficiently low temperatures. More examples of this phenomenon can also be found in [6, 17].

Theorem 39. *Let $f : \Omega \to \mathbb{R}$ be a potential in the Walters class and $\bar{f}$ its normalization given by* (2)*. If $\nu \in \mathcal{G}^{DLR}(f)$ and $\mu \in \mathcal{G}^{DLR}(\bar{f})$ then for any continuous function g we have that*

$$\int_\Omega g \, d\nu = \int_\Omega \frac{g}{\varphi} \, d\mu.$$

Proof. Let φ be the main eigenfunction associated to the main eigenvalue λ of the operator $\mathcal{L}_f$. Let $\overline{H}_n$ and H_n the Hamiltonians associated to f and $\bar{f}$, respectively

$$-\overline{H}_n^y(x) = \sum_{j=0}^{n-1} \bar{f}(\sigma^j x) \quad \text{and} \quad -H_n^y(x) = \sum_{j=0}^{n-1} f(\sigma^j x)$$

By a simple computation follows that

$$\sum_{j=0}^{n-1} \bar{f}(\sigma^j x) = \sum_{j=0}^{n-1} f(\sigma^j x) + \log \varphi(x) - \log \varphi(\sigma^n x) - n \log \lambda.$$

Therefore, we have

$$\overline{H}_n(x) - H_n(x) = \log \varphi(\sigma^n x) - \log \varphi(x) + n \log \lambda. \tag{18}$$

Note that for any $g : \Omega \to \mathbb{R}$ we have

$$\frac{\displaystyle\sum_{x \in \Omega; \sigma^n x = \sigma^n y} g(x) e^{-\beta H_n(x)}}{\displaystyle\sum_{x \in \Omega; \sigma^n x = \sigma^n y} e^{-\beta H_n(x)}} = \frac{\displaystyle\sum_{x \in \Omega; \sigma^n x = \sigma^n y} g(x) e^{-\beta \overline{H}_n(x) + \beta \log \varphi(\sigma^n x) - \beta \log \varphi(x)}}{\displaystyle\sum_{x \in \Omega; \sigma^n x = \sigma^n y} e^{-\beta \overline{H}_n(x) + \beta \log \varphi(\sigma^n x) - \beta \log \varphi(x)}}$$

In the sums on the r.h.s. above the terms $\exp(\beta \log \varphi(\sigma^n x))$ are constant equal to $\exp(\beta \log \varphi(y))$. So they cancel each other and r.h.s of the above expression is equal to

$$\frac{\displaystyle\sum_{x\in\Omega;\sigma^n x=\sigma^n y} g(x)e^{-\beta\overline{H}_n(x)-\beta\log\varphi(x)}}{\displaystyle\sum_{x\in\Omega;\sigma^n x=\sigma^n y} e^{-\beta\overline{H}_n(x)-\beta\log\varphi(x)}} = \frac{\displaystyle\sum_{x\in\Omega;\sigma^n x=\sigma^n y} \frac{g(x)}{\varphi(x)}e^{-\beta\overline{H}_n(x)}}{\displaystyle\sum_{x\in\Omega;\sigma^n x=\sigma^n y} \frac{1}{\varphi(x)}e^{-\beta\overline{H}_n(x)}}.$$

By dividing and multiplying by the partition function of the Hamiltonian $\overline{H_n}$ and taking the limit when $n \to \infty$ we get from the two previous equations that

$$\int_\Omega g\, d\nu = \int_\Omega \frac{g}{\varphi}\, d\mu$$

□

Remark. From (18) for any Hölder or Walters potential f we have

$$\sup_{n\in\mathbb{N}} \sup_{x\in\Omega} |\overline{H}_n(x) - H_n(x)| = \sup_{n\in\mathbb{N}} \sup_{x\in\Omega} |\log\varphi(x) - \log\varphi(\sigma^n x)| < \infty,$$

where in the last inequality we use that φ is positive everywhere, continuous and Ω is compact. So the specifications K_n and $\bar{K}_n$ are "physically" equivalent, see [17, p.136]. As we already said μ is always σ-invariant for any Hölder potential f. Although μ and ν are associated to equivalent specifications there are cases where ν is not σ invariant. For the other hand in any case $\mu = \nu$ on the tail σ-algebra, i.e., $\cap_{n\in\mathbb{N}}\sigma^n\mathscr{F}$, see Theorem 7.33 in [17].

11 Concluding Remarks

In this paper we have compared the definitions of Gibbs measures defined in terms of the Ruelle operator and DLR specifications. We show how to obtain for potentials in the Walters and Hölder class the Gibbs measures usually considered in the Thermodynamic Formalism via the DLR formalism and prove that the measures obtained from both approaches are the same.

Both approaches have their advantages. For example, using the Ruelle operator we were able to prove some uniform convergence theorems about the finite volume Gibbs measures.

The literature about absolutely uniformly summable interactions is vast and this approach allow us to consider non translation invariant potentials and also on systems other than the lattices $\mathbb{Z}$ or $\mathbb{N}$. We also show that the long range Ising model on $\mathbb{N}$ can be studied using the Ruelle operator, at least when the interaction is of the form $1/r^{\alpha}$ with $\alpha > 2$. In these cases, we have proved that the unique Gibbs measure of this model satisfies $\mathcal{G}^{DLR}(\Phi) = \mathcal{G}^{TL}(\Phi) = \mathcal{G}^{*}(f)$, but on the other hand it is not clear how to treat the cases $1 < \alpha \leq 2$ by using the Ruelle operator and what kind of information is obtainable through this approach. It is worth pointing out that treating this model with the DLR approach is fairly standard, so the connection made here suggests that more understanding of the DLR Specification theory can shed light on more general spaces where one can efficiently use the Ruelle Operator. Another important feature of the DLR-measure Theory is that it is also readily appliable to standard Borel spaces, which includes compact and non-compact spaces [17]. Thus the results obtained here can be extended to compact spaces, but some measurability issues have to be taken into account and some convergence theorems have to be completely rewritten, although the main ideas are contained here. We will approach this issue in the near future.

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