Grand-canonical Thermodynamic Formalism via IFS: volume, temperature, gas pressure and grand-canonical topological pressure

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Abstract

We consider here a dynamic model for a gas in which a variable number of particles $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ can be located at a site. This point of view leads us to the grand-canonical framework and the need for a chemical potential. The dynamics is played by the shift acting on the set of sequences $\Omega := \mathcal{A}^{\mathbb{N}}$, where the alphabet is $\mathcal{A} := \{1, 2, ..., r\}$. Introducing new variables like the number of particles N and the chemical potential μ, we adapt the concept of grand-canonical partition sum of thermodynamics of gases to a symbolic dynamical setting considering a certain family of potentials $(A_N)_{N\in\mathbb{N}_0}$, $A_N : \Omega \to \mathbb{R}$, satisfying at least a Dini condition. Our main results will be obtained from adapting well-known properties of the Thermodynamic Formalism for IFS with weights to our setting. We introduce the grand-canonical-Ruelle operator: $\mathcal{L}_{\beta,\mu}(f) = g$, when, $\beta > 0, \mu < 0$, where

$$g(x) = \mathcal{L}_{\beta,\mu}(f)(x) = \sum_{N\in\mathbb{N}_0} e^{\beta\,\mu\,N} \sum_{j\in\mathcal{A}} e^{-\,\beta\,A_N(jx)} f(jx).$$

We show the existence of the main eigenvalue, an associated eigenfunction, and an eigenprobability for $\mathcal{L}^*_{\beta,\mu}$. When the A_N, $N \in \mathbb{N}$, satisfy a Lipschitz condition, we can show the analytic dependence of the eigenvalue on the grand-canonical potential. Considering the concept of entropy for holonomic probabilities on $\Omega\times\mathcal{A}^{\mathbb{N}_0}$, we relate these items with the variational problem of maximizing grand-canonical pressure. In another direction, in the appendix, we briefly digress on a possible interpretation of the concept of topological pressure as related to the gas pressure of gas thermodynamics.

Keywords: Particles of a gas, symbolic spaces, grand-canonical partition, IFS Thermodynamic Formalism, Ruelle operator, holonomic probabilities, entropy, grand-canonical entropy, grand-canonical pressure.

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1 Introduction

The study of the thermodynamics of gases with a non-specified number of particles is a classical topic in Mathematical Physics (see Section 1.6 in [Nau11] or Section 11.2.4 in [VL]). Here we will investigate this type of problem from a dynamic perspective. That is the search for statistical properties that can be obtained with the help of a generalization of the Ruelle operator (which corresponds to the transfer operator of Statistical Mechanics) of Thermodynamic Formalism (in the sense of [PP90] or [FL99]). Concepts like volume, temperature, entropy, and gas pressure arise naturally in thermodynamics when we introduce the number of particles N as a variable (see [Cal14] or Section 5.6 in [Bena]). The introduction of a negative constant μ, called the chemical potential, plays an important role in the convergence of the grand-canonical partition sum (see (1.34) in [Nau11]). We analyze such kinds of problems from a mathematical perspective and we leave the question of physical relevance for a posterior investigation. The discussion in Section 3.2.4 in [VL] on the topic of probabilities for particle distributions is quite enlightening. We would like to emphasize that the postulates of equilibrium thermodynamics of gases are an issue subject to controversy (see Section 4).

In classical Thermodynamic Formalism, in general, results avoid taking into account for this variable number of particles.

We are interested in the mathematical formulation of physical problems in equilibrium from a dynamic perspective. Time does not occur as a variable in thermodynamic equations. When we allude *a mathematical formulation in a dynamical setting*, by this, we mean problems related to the action of the shift σ on the symbolic space $\{1, 2, ..., r\}^{\mathbb{N}}$; this is associated with translation on the one-dimensional lattice and is not related to time.

We are interested in the statistics of the number of particles: any number $N \in \mathbb{N}_0$ particles can be in one site. Therefore, in principle, it is natural to consider an IFS with a countable number of functions (and with weights), but it is possible (for part of the results we consider) to translate it, after some work, to the case of a finite one (and then results from [LO09] can be used).

We will consider a family of potentials $A_N : \{1, 2, ..., r\}^{\mathbb{N}} \to \mathbb{R}$, $N \in \mathbb{N}_0$, and we are interested in equilibrium states. In the IFS setting instead of shift-invariant probabilities it is natural to consider holonomic probabilities, as described in [LO09] (see Definition 9). All this will be carefully described in Section 3.

We introduce what we call the grand-canonical-Ruelle operator (see (1)) and we show a version of the Ruelle Theorem (about eigenfunctions and eigenvalues), which is presented as our main result in Theorem 12. The main eigenvalue will be called the grand-canonical eigenvalue. We will assume some mild conditions for potentials A_N, $N \in \mathbb{N}_0$, in order to control the behavior of the grand-canonical operator. The family of potentials A_N can growth, for instance, like N (but not necessarily like that). We present an example at the end of the paper.

The grand-canonical potential

$$\psi(y) := \ln \left(\sum_{N \in \mathbb{N}_0} e^{-\beta[A_N(y) - \mu N]} \right),$$

will play an important role in our reasoning (in Section 2 we present appropriate conditions on the A_N). It will be required that ψ is at least Dini continuous (see (11) and (12) for the definition).

In Corollary 13 we will show an analytical dependence of the grand-canonical eigenvalue in ψ assuming more regularity on ψ.

In Section 2 we recall some classical results in thermodynamic formalism and we introduce the dynamical canonical ensemble in this context. Later, we will analyze the main properties of the grand-canonical-Ruelle operator, the concepts of entropy for holonomic probabilities (see Definitions 9 and 10), and also the grand-canonical topological pressure (see (18) and item a) in Theorem 12 and also (22)).

Given a family of potentials $A_N : \Omega = \{1, ..., r\}^{\mathbb{N}} \to \mathbb{R}$, $N \in \mathbb{N}_0$, $\beta > 0$, and $\mu < 0$, satisfying a Dini condition, the grand-canonical-Ruelle operator $f \to \mathcal{L}_{\beta,\mu}(f) = g$, is given by

$$g(x) = \mathcal{L}_{\beta,\mu}(f)(x) = \sum_{N \in \mathbb{N}_0} e^{\beta\,\mu\,N} \sum_{j \in \mathcal{A}} e^{-\,\beta\,A_N(jx)} f(jx). \tag{1}$$

We denote by $\Phi = (A_N)_{N \in \mathbb{N}_0}$ the family of potentials.

Note that the points of the form $j\,x$, $j \in \mathcal{A}$, describe the set of solutions y of $\sigma(y) = x$. Then, the operator $\mathcal{L}_{\beta,\mu}$ is **dynamically** defined; it corresponds to the classical transfer operator of Statistical Mechanics but for a dynamical setting.

For the benefit of the mathematical reader, we will briefly describe some basic properties of the thermodynamics of *ideal* gases in Section 4. Reading this section is not necessary for understanding the mathematical reasoning followed in the previous sections. The objective is only to show the motivation that led us to analyze the problems that were proposed.

In Remark 22 we will investigate a possible interpretation of the terminology topological pressure in a comparison with the concept of gas pressure, which originated from the postulates of the theory that analyzes gases confined under certain variable walls and at a certain temperature.

In a related work, the authors consider in [LR22] non-equilibrium and the second law of thermodynamics in Thermodynamic Formalism. In [LW] it is presented a brief account of Thermodynamics, Statistical Physics, and their relation to the Thermodynamic Formalism of Dynamical Systems.

The study of Thermodynamic Formalism for symbolic spaces with an infinite countable alphabet (the set $\mathcal{A} = \mathbb{N}$) is the topic of [Sarig], [BBE] and [FV]; but a different class of problems is considered there.

Results for IFS using conformal branches appear in [Mih22] but it is also a different setting compared to ours.

Conclusion: The classical study of the grand-canonical partition sum in the thermodynamics of gases considers an indefinite number of particles N, a Hamiltonian A_N, and the chemical potential μ, which is negative in order to ensure convergence of the associated sum.

We consider the corresponding problems on the symbolic dynamical setting considering a Dini family of potentials $(A_N)_{N \in \mathbb{N}_0}$, $A_N : \Omega \to \mathbb{R}$. We introduce the grand-canonical-Ruelle operator: $\mathcal{L}_{\beta,\mu}$, $\beta > 0, \mu < 0$ (as defined in (1)), and we can get concepts like discrete-time entropy (and also the pressure problem associated with such entropy). Our main results will be obtained from adapting well-known properties of the Thermodynamic Formalism for IFS with weights to our dynamical setting. The naturally associated Gibbs probability is not shift invariant (it is holonomic). One of our main results is Theorem 7, which shows the existence of eigenfunctions and eigenprobabilities (a key step for analyzing questions related to maximizing pressure). In the variational problem of grand-canonical topological pressure (see Theorem 12) the holonomic probabilities play an important (and natural) role due to the structure of the IFS setting; indeed, it is required a special (and natural) concept of entropy which is described by Definition 10.

We follow two different lines of reasoning, the first one is modeling the problem via a finite IFS with weights see Section 3.1). Alternatively, we consider an infinitely countable IFS setting (see Section 3.2). Note that in Section 3.1 we get probabilities on $\Omega \times \mathcal{A}^{\mathbb{N}_0}$ and in Section 3.2 we get probabilities on Ω.

It is well known that a Lipschitz function also satisfies the Dini condition.

2 A brief review of Classical Thermodynamic Formalism

Consider a finite alphabet $\mathcal{A} := \{1, ..., r\}$ and the *shift map* $\sigma((x_n)_{n\in\mathbb{N}}) := (x_{n+1})_{n\in\mathbb{N}}$ acting on the *symbolic space* $\Omega := \mathcal{A}^{\mathbb{N}}$ which is equipped with the metric (which makes $\mathrm{diam}(\Omega) < 1$)

$$d(x, y) := \begin{cases} 2^{-\min\{n\in\mathbb{N}: x_n \neq y_n\}}, & x \neq y; \\ 0, & x = y. \end{cases}$$

The dynamical system (Ω, σ) is widely known in the mathematical literature as the *full-shift* on the alphabet $\mathcal{A}$. We denote the set of *continuous functions* from Ω into $\mathbb{R}$ by $\mathrm{C}(\Omega)$ and we use the notation $\mathrm{C}^+(\Omega)$ for the corresponding cone of *positive continuous functions*. We also denote the set of *Lipschitz continuous functions* from Ω into $\mathbb{R}$ by $\mathrm{Lip}(\Omega)$ and we use the notation $\mathrm{Lip}(f)$ for the *Lipschitz constant* of $f \in \mathrm{Lip}(\Omega)$. Besides that, we denote the set of *Borel probability measures* on Ω by $\mathcal{M}_1(\Omega)$ and we use the notation $\mathcal{M}_\sigma(\Omega)$ for the set of Borel σ*-invariant probability measures* on Ω.

Here, we consider a system describing the dynamical behavior of a classical gas one-dimensional lattice composed of N particles at temperature T, which are contained in a region with volume V. It is natural to introduce a parameter β in such a way that satisfies relation

$$\beta := \frac{1}{k_B T} , \tag{2}$$

where $k_B \sim 1.38066 \times 10^{-23} J/K$ is the so-called *Boltzmann's constant* (see (1.2) in [Nau11] for details).

We assume that the number of particles ranges on the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and we consider a potential $A : \Omega \times \mathbb{N}_0 \to \mathbb{R}$ which is Lipschitz continuous w.r.t. the first variable. It is not difficult to check that the potential A induces a family of potentials $\Phi = (A_N)_{N\in\mathbb{N}_0}$, where $A_N := A(\cdot, N)$ for each $N \in \mathbb{N}_0$. In fact, the last assumption guarantees that $A_N \in \mathrm{Lip}(\Omega)$.

We consider first the case where N, the number of particles, is a fixed natural number. This corresponds to just considering a classical Ruelle operator (as in [PP90]). Given $N \in \mathbb{N}_0$ and some $\beta > 0$ satisfying the expression in (2), we consider the *Ruelle operator* $\mathcal{L}_{N,\beta}$ associated to a Dini potential A_N, as the one given by the equation

$$\mathcal{L}_{N,\beta}(f)(x) := \sum_{\sigma(y)=x} e^{-\beta A_N(y)} f(y) = \sum_{j\in\mathcal{A}} e^{-\beta A_N(jx)} f(jx), \forall x \in \Omega. \tag{3}$$

It is well known (when A_N satisfy a Dini condtion) that for each pair N, β, there are a main *eigenvalue* $\lambda_{N,\beta} > 0$ and an *eigenfunction* $f_{N,\beta} \in \mathrm{Lip}(\Omega)$ for the operator $\mathcal{L}_{N,\beta}$ (see for instance [FL99] or [Fan0]).

Given a continuous potential $A_N : \Omega \to \mathbb{R}$ and $\beta > 0$, we can define the dual operator $\mathcal{L}^*_{N,\beta}$ acting on the space of the Borel finite measures, as the operator that sends a measure v to the measure $\mathcal{L}^*_{N,\beta}(v)$, defined by

$$\int \psi \, d\mathcal{L}^*_{N,\beta}(v) = \int \mathcal{L}_{N,\beta}(\psi)\, dv \,, \tag{4}$$

for any continuous function $\psi : \Omega \to \mathbb{R}$. This is well defined by the Riesz Theorem.

We denote by $\nu_{N,\beta} \in \mathcal{M}_1(\Omega)$ the *eigenprobability* of the operator $\mathcal{L}^*_{N,\beta}$ associated to $\lambda_{N,\beta}$ and by $\rho_{N,\beta} \in \mathcal{M}_\sigma(\Omega)$ the *equilibrium state* for the potential $-\beta A_N$ which, up to a normalization, is of has the form $\rho_{N,\beta} = f_{N,\beta}\, \nu_{N,\beta}$ (see [PP90] or [Lop1] for details).

Given $x \in \Omega$ and $N \in \mathbb{N}_0$, we define the *N-canonical partition* for the iterate $n \in \mathbb{N}$ calculated at the point $x \in \Omega$ by

$$Z^n_N(\beta)(x) := \mathcal{L}^n_{N,\beta}(1)(x) \,, \tag{5}$$

where $\mathcal{L}^{n+1}_{N,\beta}(f) = \mathcal{L}_{N,\beta}(\mathcal{L}^n_{N,\beta}(f))$ for each $n \in \mathbb{N}$.

The pointwise limit $\lim_{n\to\infty} \frac{1}{n} \log(Z^n_N(\beta)(x))$, which is independent of x (see next Lemma), plays here the role of the so-called configurational partition sum appearing at (1.17) on page 7 of [Nau11].

The study of the properties of an individual Transfer operator $\mathcal{L}_{N,\beta}$, for N and β fixed, it is not suitable for the case where the number of particles N ranges in the set of natural numbers (which is the goal of the next section).

The next lemma is well-known in Thermodynamical Formalism and we will not present a proof (see [PP90]).

Lemma 1. *The pointwise limit* $\lim_{n\to\infty} \frac{1}{n} \log(Z^n_N(\beta)(x))$ *exists and it is equal to* $\log(\lambda_{N,\beta})$. *In particular, it is independent of the choice of* $x \in \Omega$.

We call *N-Topological Pressure* for β to the observable satisfying

$$P_N(\beta) = P(-\beta A_N) := \lim_{n\to\infty} \frac{1}{n} \log(Z^n_N(\beta)) = \log(\lambda_{N,\beta}) \,.$$

In this way, by Lemma 1 we obtain that $Z^n_N(\beta) \sim \lambda^n_{N,\beta}$. Moreover, one can show that the following expression holds true

$$P_N(\beta) = P(-\beta A_N) = \sup_{\rho \in \mathcal{M}_\sigma(\Omega)} \Big\{ h(\rho) - \beta \int A_N \, d\rho \Big\}, \tag{6}$$

where $h(\rho)$ is the *Kolmogorov-Sinai entropy* of ρ and $\mathcal{M}_\sigma(\Omega)$ denotes the set of σ-invariant probabilities (for details see [PP90]).

For a grand-canonical version of topological pressure, we will need a different version of (6) due to the fact that we have to consider probabilities such that the concept of Kolmogorov-Sinai entropy does not apply (see expression (17)).

The above computation implies that $\lim_{N\to\infty} P_N(\beta) = -\infty$ and $\lim_{N\to\infty} \lambda_{N,\beta} = 0$. Furthermore, in [PP90] (see also [GKLM18]), the authors prove that

$$\begin{aligned} \frac{\partial}{\partial\beta} \log \lambda_{N,\beta}|_{\beta=\beta_0} &= \frac{\partial}{\partial\beta} P_N(\beta)|_{\beta=\beta_0} \\ &= \frac{\partial}{\partial\beta} P(-\beta A_N)|_{\beta=\beta_0} = -\int A_N d\rho_{N,\beta_0} \ . \end{aligned}$$

So, by the above formula, we get

$$\frac{1}{\lambda_{N,\beta_0}} \frac{\partial \lambda_{N,\beta}}{\partial\beta}|_{\beta=\beta_0} = -\int A_N d\rho_{N,\beta_0} \ . \tag{7}$$

On the other hand (see for instance Section 9 in [CL17]), it is also well known that for any continuous function $A : \Omega \to \mathbb{R}$ and each $x \in \Omega$, we have

$$\lim_{n\to\infty} \frac{\mathcal{L}^n_{N,\beta}(A)(x)}{\mathcal{L}^n_{N,\beta}(1)(x)} = \int A \, d\nu_{N,\beta},$$

where $\nu_{N,\beta}$ is the eigenprobability for the Ruelle operator of the potential $-\beta A_N$ (see [PP90]).

Above we described the classical dynamical properties of the individual transfer operator $\mathcal{L}_{N,\beta}$. In the next section, we will use properties of IFS Thermodynamical Formalism to address the analogous issue for the case of a variable number of particles, where it is necessary to consider a countable number of classical Ruelle operators (each one indexed by the number N of particles). We believe the material presented on the present section will help the reader to understand the reasoning of the next one.

3 A grand-canonical Thermodynamic Formalism

Here we consider a variable number of particles. In order to do that, we consider linear operators involving the new variable N (describing the number of particles), which is defined in the following way: given a family of potentials

$\Phi = (A_N)_{N\in\mathbb{N}_0}$ (which play the role of Hamiltonians), a chemical potential $\mu < 0$ and a value $\beta > 0$ satisfying the expression in (2), the *grand-canonical-Ruelle operator* $\mathcal{L}_{\beta,\mu}$ is defined as the operator assigning to each $f \in \mathrm{C}(\Omega)$ the function

$$\mathcal{L}_{\beta,\mu}(f)(x) := \sum_{N\in\mathbb{N}_0} e^{\beta\,\mu\,N} \sum_{a\in\mathcal{A}} e^{-\,\beta\,A_N(ax)} f(ax) = \sum_{N\in\mathbb{N}_0} e^{\beta\,\mu\,N}\,\mathcal{L}_{N,\beta}(f)(x), \quad (8)$$

for any $x \in \Omega$.

In order to get convergence in the above sum we need some hypotheses: for fixed μ we will assume that the family of potentials A_N, $N \in \mathbb{N}$, is admissible, that is, we assume that for any $x \in \Omega$ the sums

$$\sum_{N\in\mathbb{N}_0} e^{\beta\,n\,\mu} \sum_{a\in\mathcal{A}} e^{-\beta\,A_N(a\,x)} < \infty.$$

Then, it follows for any f that

$$||\mathcal{L}_{\beta,\mu}(f)||_\infty \leq ||f||_\infty \sum_{N\in\mathbb{N}_0} e^{\beta\,n\,\mu} \sum_{a\in\mathcal{A}} e^{-\beta\,A_N(a\,x)} < \infty.$$

Once (8) is well defined, we can ask about the existence of eigenvalues, eigenfunctions for $\mathcal{L}_{\beta,\mu}$, and also holonomic probabilities for the IFS pressure. Our main goal is to represent (after some work) the operator $\mathcal{L}_{\beta,\mu}$ as the transfer operator of a standard IFS with weights for which the thermodynamic formalism is already known from the literature. One can follow two different lines of reasoning, the first one is modeling the problem via a finite IFS with weights (see Section 3.1). We have to show that our model fits the hypothesis of [LO09] and [FL99]. Alternatively, one could use an infinitely countable IFS (see Section 3.2), this is fine as we will see, but it brings technical difficulties and some limitations as will be explained.

3.1 Transferring the problem to the case of a finite IFS with weights.

In this section, we introduce an IFS with weights in such a way that its associated transfer operator coincides with the grand-canonical-Ruelle operator $\mathcal{L}_{\beta,\mu}$. By showing that the weights satisfy the necessary regularity conditions we will use Fan's Theorem (see [FL99], Theorem 1.1) to obtain a positive eigenfunction for $\mathcal{L}_{\beta,\mu}$. Note that the weights are not periodic. Once we have this positive eigenfunction associated with the spectral radius of $\mathcal{L}_{\beta,\mu}$ we can introduce the thermodynamical formalism based on holonomic measures according to [LO09] or [CO17]. We recall the ideas of variational entropy and topological pressure based on holonomic probabilities. Finally, we will show that is possible to build a variational principle and show the existence of holonomic equilibrium states.

Given $r \geq 2$, consider the *IFS* $\mathcal{R} := (\Omega, \phi_j)_{j\in\mathcal{A}}$ where $\mathcal{A} := \{1, ..., r\}$, $\phi_j(x) = jx$ is the *mnemonic representation* for the sequence $(j, x_1, x_2, ...)$, where $x = (x_1, x_2, ...) \in \Omega = \mathcal{A}^{\mathbb{N}}$. Of course, this IFS is contractive w.r.t. the distance introduced in Ω. Moreover, $\mathrm{Lip}(\phi_j) = \frac{1}{2}$ for all $j \in \mathcal{A}$.

Given a family of continuous functions $q_j : \Omega \to \mathbb{R}$, $j \in \mathcal{A}$ we say that $\mathcal{R} := (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ is an *IFS with weights*. In the particular case where $q_j(x) \geq 0$ and $\sum_{j\in\mathcal{A}} q_j(x) = 1$, for all $x \in \Omega$, it is called an *IFS with probabilities*. According to [FL99], an IFS with weights where the maps are contractions and the weights are non-negative is called a *contractive system*.

In this setting the *transfer operator* associated to $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ is a map $B_q : \mathrm{C}(\Omega) \to \mathrm{C}(\Omega)$ given by:

$$B_q(g)(x) := \sum_{j\in\mathcal{A}} q_j(x) g(\phi_j(x)), \forall x \in \Omega, \tag{9}$$

for any $g \in \mathrm{C}(\Omega)$.

The next lemma shows how to pick the right weights q_j in order to obtain the equality $B_q = \mathcal{L}_{\beta,\mu}$.

Let $\psi : \Omega \to \mathbb{R}$ be the grand-canonical potential

$$\psi(y) := \ln \left(\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(y) - \mu N]} \right), \tag{10}$$

where the family of potentials A_N satisfies some prescribed conditions so that the formal series converges.

We recall that for a function$f : \Omega \to \mathbb{R}$ the modulus of continuity of f is

$$\omega_f(t) = \sup_{d(x,y)\leq t} f(x) - f(y). \tag{11}$$

The function f is called Dini-continuous if

$$\int_0^1 \frac{\omega_f(t)}{t} dt < \infty.$$

An equivalent condition (see [Ste01]) is the following: for some $c \in (0, 1)$,

$$\sum_{i=1}^{\infty} \omega_f\left(c^i\right) < \infty \tag{12}$$

(since Ω is compact and has diameter equal to 1). It is easy to see that f is Lipschitz (resp. α-Hölder) if, and only if $\omega_f(t) \leq \mathrm{Lip}(f)\, t$ (resp. $\omega_f(t) \leq \mathrm{Hol}(f)\, t^{\alpha}$). Thus, both classes are contained in the class of Dini continuity.

In our reasoning, it will be required to assume the hypothesis assuring that (10) is at least Dini continuous (see condition b) in the next Lemma).

Lemma 2. *Consider the family of potentials $\Phi = (A_N)_{N\in\mathbb{N}_0}$ and the weights*

$$q_j(x) := e^{\psi(\phi_j(x))} > 0,\ j \in \mathcal{A}.$$

a) If Φ satisfy

$$\liminf_{N\to\infty} \frac{A_N(x)}{N} > \mu,\ \forall x \in \Omega,$$

then the contractive system $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ is well defined and $B_q(g) = \mathcal{L}_{\beta,\mu}(g)$, for any $g \in \mathrm{C}(\Omega)$;

b) Suppose that

$$\exists\, \varepsilon > 0, \delta \geq 0 \text{ s.t. } A_N(x) > (\mu+\varepsilon)N + \delta,\ \forall x \in \Omega, \forall N \in \mathbb{N}_0. \quad (13)$$

If each A_N is Dini continuous and

$$\limsup_{i\to\infty} \left(\sum_{N\in\mathbb{N}_0} \omega_{A_N}(c^i)\ (e^{-\beta\,\varepsilon})^N \right)^{1/i} < 1,$$

then ψ is Dini continuous. In particular, if the family of potentials is uniformly Lipschitz, that is, $\mathrm{Lip}(A_N) \leq M$, then $\mathrm{Lip}(\psi) \leq \beta\, M$ (and so $\mathrm{Lip}(\ln(q_j)) \leq \frac{\beta\, M}{2}$).

Proof. (a) The proof follows easily from the commutativity of the summation in the formula for $\mathcal{L}_{\beta,\mu}$ if we prove that for each $j \in \mathcal{A}$ the positive series $\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(jx)-\mu N]}$ is convergent. Consider the root test:

$$\limsup_{N\to\infty} \sqrt[N]{e^{-\beta\,[A_N(jx)-\mu(N)]}} = e^{-\beta\ \liminf_{N\to\infty}[\frac{1}{N}A_N(jx)-\mu]} < 1$$

if and only if $\liminf_{N\to\infty} \frac{1}{N} A_N(x) > \mu$, which is our hypothesis.

(b) For the second part, we notice that our assumption $A_N(x) > (\mu + \varepsilon)N + \delta$ ensures that $\liminf_{N\to\infty} \frac{A_N(x)}{N} > \mu$, $\forall x \in \Omega$. Thus, from (a), ψ and $q_j(x)$ are well defined.
For each $x \in \Omega$ we define a probability ν_x over $\mathbb{N}_0$ by the formula

$$\int_{\mathbb{N}_0} g(N) d\nu_x(N) := \frac{1}{e^{\psi(x)}} \sum_{N\in\mathbb{N}_0} g(N)\ e^{-\beta[A_N(x)-\mu N]},$$

for any continuous function $g : \mathbb{N}_0 \to \mathbb{R}$.
A consequence from our assumption (13), is that for any $x \in \Omega$ we have

$$A_N(x) > (\mu + \varepsilon)N + \delta \Leftrightarrow e^{-\beta[A_N(x)-\mu N]} < e^{-\beta[\varepsilon N+\delta]}.$$

Moreover,

$$e^{\psi(x)} = \sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(x)-\mu N]} > e^{-\beta[A_0(x)-\mu 0]} = e^{-\beta A_0(x)} > e^{-\beta\|A_0\|_0} = \gamma > 0,$$

thus,

$$\int_{\mathbb{N}_0} g(N)d\nu_x(N) < \frac{1}{\gamma} \sum_{N\in\mathbb{N}_0} g(N)\ e^{-\beta[\varepsilon N+\delta]}$$

or

$$\sup_{x\in\Omega} \int_{\mathbb{N}_0} g(N)d\nu_x(N) < \frac{1}{\gamma e^{\beta\delta}} \sum_{N\in\mathbb{N}_0} g(N)\ (e^{-\beta\,\varepsilon})^N \tag{14}$$

for any continuous function $g : \mathbb{N}_0 \to \mathbb{R}$.
We notice that, for any $c \in (0,1)$ and $x, y \in \Omega$ with $d(x,y) < c$ we have

$$\psi(x) - \psi(y) = -\ln\left(\frac{\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(y)-\mu N]}}{\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(x)-\mu N]}}\right) =$$

$$= -\ln\left(\frac{\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(y)-A_N(x)]}\ e^{-\beta[A_N(x)-\mu N]}}{e^{\psi(x)}}\right) =$$

$$= -\ln\left(\int_{\mathbb{N}_0} e^{-\beta[A_N(y)-A_N(x)]} d\nu_x(N)\right) \le \int_{\mathbb{N}_0} -\ln\left(e^{-\beta[A_N(y)-A_N(x)]}\right) d\nu_x(N) =$$

$$= \beta \int_{\mathbb{N}_0} [A_N(y) - A_N(x)] d\nu_x(N) \le \beta \int_{\mathbb{N}_0} \omega_{A_N}(c) d\nu_x(N).$$

In the above inequality, we used the fact that $-\ln(\cdot)$ is a convex function and ν_x is a probability, so Jensen's inequality holds.
Thus

$$\sum_{i=1}^{\infty} \omega_\psi\left(c^i\right) \le \beta \sum_{i=1}^{\infty} \sup_{x\in\Omega} \int_{\mathbb{N}_0} \omega_{A_N}(c^i) d\nu_x(N).$$

From (14) we get

$$\sup_{x\in\Omega} \int_{\mathbb{N}_0} \omega_{A_N}(c^i) d\nu_x(N) < \frac{1}{\gamma e^{\beta\delta}} \sum_{N\in\mathbb{N}_0} \omega_{A_N}(c^i)\ (e^{-\beta\,\varepsilon})^N,$$

thus
$$\sum_{i=1}^{\infty}\omega_{\psi}\left(c^{i}\right) \leq \frac{\beta}{\gamma e^{\beta\delta}}\sum_{i=1}^{\infty}\left(\sum_{N\in\mathbb{N}_0}\omega_{A_N}(c^{i})\ (e^{-\beta\,\varepsilon})^{N}\right).$$
By the root test, this non negative series is convergent if
$$\limsup_{i\to\infty}\left(\sum_{N\in\mathbb{N}_0}\omega_{A_N}(c^{i})\ (e^{-\beta\,\varepsilon})^{N}\right)^{1/i} < 1$$
which is a pretty easy condition to fulfill, since for each N we have $\omega_{A_N}(c^i) \to 0$ when $i \to \infty$, because A_N is Dini continuous, we just need to have some controlled growing with respect to N (see remarks after).
In order to conclude our proof we notice that, if $\mathrm{Lip}(A_N) \leq M$ then we get $\omega_{A_N}(c^i) \leq Mc^i, \forall N \in \mathbb{N}_0$, then from the previous computations
$$\omega_{\psi}\left(c^{i}\right) \leq \beta\sup_{x\in\Omega}\int_{\mathbb{N}_0}\omega_{A_N}(c^{i})d\nu_x(N) \leq \beta\sup_{x\in\Omega}\int_{\mathbb{N}_0}Mc^{i}d\nu_x(N) = \beta\,M\,c^{i},$$
that is $\mathrm{Lip}(\psi) \leq \beta\,M$. □

Remark 3. *We notice that in Lemma 2, item (a), the condition*
$$\liminf_{N\to\infty}\frac{1}{N}A_N(x) > \mu$$
is only sufficient. As a matter of fact, if we assume that $(A_N)_{N\in\mathbb{N}_0}$, is an increasing sequence of functions then we can use Dalembert's convergence test:
$$\limsup_{N\to\infty}\frac{e^{-\beta[A_{N+1}(jx)-\mu(N+1)]}}{e^{-\beta[A_N(jx)-\mu N]}} = \limsup_{N\to\infty}e^{-\beta[A_{N+1}(jx)-A_N(jx)-\mu]} < 1,$$
if and only if $\limsup_{N\to\infty}A_{N+1}(jx) - A_N(jx) - \mu > 0$, which is the case because $-\mu > 0$ and $A_{N+1}(jx) > A_N(jx)$ by hypothesis.

Example 4. *We choose $A_N(x) := N\,E(x) \geq 0$ for any $x \in \Omega$, where the energy $E : \Omega \to \mathbb{R}$ is Lipschitz (or even Dini) continuous. Taking $\delta = 0$, $0 < \varepsilon < \min E(x) - \mu$ we get*
$$A_N(x) > (\mu+\varepsilon)N + \delta,\ \forall x\in\Omega, \forall N\in\mathbb{N}_0.$$
Moreover, each A_N is Dini continuous and $\omega_{A_N}(c^i) \leq N\omega_E(c^i) \leq N\ \mathrm{Lip}(E)\,c^i$ thus
$$\limsup_{i\to\infty}\left(\sum_{N\in\mathbb{N}_0}\omega_{A_N}(c^{i})\ (e^{-\beta\,\varepsilon})^{N}\right)^{1/i} \leq$$

$$\leq c\ \limsup_{i\to\infty}\left(\mathrm{Lip}(E)\ \sum_{N\in\mathbb{N}_0} N\,(e^{-\beta\,\varepsilon})^N\right)^{1/i} < 1.$$

Thus, from Lemma 2, item (b), ψ is Dini continuous.

Example 5. *In this example, we provide a different construction of a Dini continuous grand-canonical potential ψ, where each potential $A_N : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is Lipschitz continuous (but with unbounded Lipschitz constant).*

We choose $A_N(x) := \theta(N)\sum_{j=1}^{+\infty}\frac{x_j}{2^j}$ for any $x \in \Omega$, where $\theta(N)$ is a given sequence of nonnegative real numbers. Notice that $A_N(x) = B_N(t) = (B_N \circ \pi)(x)$, where $t = \pi(x) := \sum_{j=1}^{+\infty}\frac{x_j}{2^j}$ and $B_N(t) := \theta(N)\,t,\ t \in [0,1]$. As π preserves distance we just need to check if

$$\psi(t) := \ln\left(\sum_{N=0}^{\infty} e^{-\beta[B_N(t)-\mu N]}\right) = \ln\left(\sum_{N=0}^{\infty} e^{-\beta[\theta(N)\,t-\mu N]}\right),\ t \in [0,1]$$

is Dini (off course, $\psi(x) := \psi(\pi(x))$, for any $x \in \Omega$, if there is no risk of confusion).
Applying our criteria and the fact that $\omega_{A_N}(t) \leq \theta(N)\,t$, we obtain

$$\limsup_{i\to\infty}\left(\sum_{N\in\mathbb{N}_0}\omega_{A_N}(c^i)\ (e^{-\beta\,\varepsilon})^N\right)^{1/i} \leq \limsup_{i\to\infty}\left(\sum_{N\in\mathbb{N}_0}\theta(N)\,c^i\ (e^{-\beta\,\varepsilon})^N\right)^{1/i} =$$

$$c\ \limsup_{i\to\infty}\left(\sum_{N\in\mathbb{N}_0}\theta(N)\,(e^{-\beta\,\varepsilon})^N\right)^{1/i} < 1,$$

provided that $\sum_{N\in\mathbb{N}_0}\theta(N)\,(e^{-\beta\,\varepsilon})^N < \infty$. By the root test we must have

$$\limsup_{N\to\infty}\,(\theta(N))^{1/N} < e^{\beta\,\varepsilon}$$

to ensure that ψ is Dini continuous. Particular choices would be $\theta(N) := \ln(1+N)$ or $\theta(N) := N^{\alpha}, \alpha > 0$. In these cases $\limsup_{N\to\infty}(\theta(N))^{1/N} = 1 < e^{\beta\,\varepsilon}$ because $\beta\,\varepsilon > 0$.

The next example shows that we may have a Dini continuous grand-canonical potential ψ which is not Lipschitz or Hölder continuous. This shows that the generality of our results cannot be reduced to a direct application of the classical Ruelle Theorem, which is well known for the case of Lipschitz (or Hölder) potentials (note that our IFS is contractive as in [FL99]).

Example 6. *We follow the notation of Theorem 2.1 in [Wal07]. We will present an example where ψ is Dini but not Lipschitz. Consider the Bernoulli space $\Omega = \{0,1\}^{\mathbb{N}}$ and we are going to define a function $u : \Omega \to \mathbb{R}$. We set for $0 < t < 1$ fixed,*

$$u(0^p 1z) = t\, e^{p^{-2-\epsilon}} = \gamma_p = \delta_p = u(1^p 0z),$$

and in the other points we define u in such way that $u(0z) + u(1z) = 1$, for all z. That is, we take

$$u(10^p 1z) = 1 - \gamma_{p+1} = 1 - \delta_{p+1} = u(01^p 0z).$$

We set $t < 1$, in such that there exist a $c \in (0,1)$ such that $c \leq \gamma_p$ and $\gamma_p \leq (1-c)$ (see paragraph before Theorem 2.1 in [Wal07]).
The normalized potential $\log u$ *is Walters (also Dini) according to Theorem 2.1 (ii). Indeed, it follows from Lemma 2.1 in [Wal07] that the variation of* $\log u$ *on a cylinder of size p (in our example) is of order $p^{-2-\epsilon}$. Therefore, $\psi = \log u$ is not Lipschitz (or Hölder).*

Now we are going to define a family of potentials $A_n : \Omega \to \mathbb{R}$.

For fixed p, and for each $N = 0, 1, 2, \ldots$, define

$$B_{N,p} = B_N(0^p 1z) = \gamma_{p,N} = \delta_{p,N} = B_N(1^p 0z) = t\, \frac{1}{N!} [p^{-2-\epsilon}]^N.$$

In this way $\sum_{N=0}^{\infty} B_N(0^p 1z) = t\, e^{p^{-2-\epsilon}} = \sum_{N=0}^{\infty} B_N(1^p 0z)$. We also take

$$B_N(10^p 1z) = B_N(01^p 0z),$$

in such a way that

$$\sum_{N=0}^{\infty} B_N(10^p 1z) = 1 - t\, e^{p^{-2-\epsilon}} = \sum_{N=0}^{\infty} B_N(01^p 0z).$$

Now taking $A_{N,p}(0^p 1z) = -\ln B_{N,p} + N\mu$, it follows that $e^{-[A_N(0^p 1z) - \mu N]} = B_{N,p}$. In this way the normalized potential

$$\psi(x) = \ln u(x) = \ln \left(\sum_{N \in \mathbb{N}_0} e^{-\beta [A_N(x) - \mu N]} \right)$$

is a Dini potential which is not Lipschitz (or Hölder).

Now we can state our main result in this section, the existence of a positive eigenfunction for the grand-canonical-Ruelle operator $\mathcal{L}_{\beta,\mu}$ and an eigenprobability for $(\mathcal{L}_{\beta,\mu})^*$.

Theorem 7. *Consider* $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ *the contractive system in Lemma 2. If the sequence* $\Phi = (A_N)_{N\in\mathbb{N}_0}$*, satisfies*

$$\exists\, \varepsilon > 0, \delta \geq 0 \text{ s.t. } A_N(x) > (\mu + \varepsilon)N + \delta,\ \forall x \in \Omega, \forall N \in \mathbb{N}_0, \tag{15}$$

each A_N *is Dini continuous and, for some* $0 < c < 1$,

$$\limsup_{i\to\infty} \left(\sum_{N\in\mathbb{N}_0} \omega_{A_N}(c^i)\ (e^{-\beta\,\varepsilon})^N \right)^{1/i} < 1,$$

then there exists a unique continuous function $h : \Omega \to \mathbb{R}$, $h > 0$ *and a unique probability measure* ν *on* Ω *such that*

$$\mathcal{L}_{\beta,\mu}(h) = \lambda h, (\mathcal{L}_{\beta,\mu})^*(\nu) = \lambda\nu \text{ and } \nu(h) = 1$$

where $\lambda > 0$ *is the spectral radius of* $\mathcal{L}_{\beta,\mu}$*. Moreover, for any* $g \in C(\Omega)$ *the sequence of functions* $\lambda^{-n}(\mathcal{L}_{\beta,\mu})^n(g)$ *converges uniformly to* $\nu(g)h$ *and for any probability measure* θ *the sequence* $\lambda^{-n}(\mathcal{L}^*_{\beta,\mu})^n(\theta)$ *converges weakly to* $\theta(h)\nu$.

Proof. From Lemma 2 a) we get $\mathcal{L}_{\beta,\mu} = B_q$, thus we can derive our theorem directly from Ruelle-Perron-Frobenius theorem for IFS (see [FL99], Theorem 1.1), by showing that $\log q_j$ is Dini continuous for any $j \in \mathcal{A}$.
Since $q_j(x) := e^{\psi(\phi_j(x))} > 0$, $j \in \mathcal{A}$. The above condition is equivalent to $\log q_j = \psi(\phi_j(x))$ to be Dini continuous, which is the case because ϕ_j is Lipschitz continuous and, under our hypothesis, we get from Lemma 2 (b) that ψ is Dini continuous.

We recall that a contractive IFS always has attractors, that is, a unique compact set $K \subseteq \Omega$ such that $K = \bigcup_{j\in\mathcal{A}} \phi_j(K)$. Although due to the very particular structure of our maps ϕ_j, we obtain $\Omega = \bigcup_{j\in\mathcal{A}} \phi_j(\Omega)$, thus we will ignore this feature when applying Fan's theorem. □

Now we can translate all the results from the thermodynamical formalism for an IFS with weights $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ following [LO09] and its improvement given in [CO17] (a different approach is given by [Mih22] considering measures invariant w.r.t. a skew product).

The next lemma generalizes the result obtained in Lemma 1 for the setting of grand-canonical-Ruelle operators.

Lemma 8. *If the family of potentials* $\Phi = (A_N)_{N\in\mathbb{N}_0}$*, satisfy the hypothesis from Theorem 7, then the following limit exists*

$$\lim_{n\to\infty} \frac{1}{n} \log\left((\mathcal{L}_{\beta,\mu})^n(1)(x)\right) = \log\lambda\ , \tag{16}$$

the convergence is uniform in $x \in \Omega$ *and* λ *is the spectral radius of* $\mathcal{L}_{\beta,\mu}$ *acting on* $C(\Omega)$*. We call* (16) *the log of the grand-canonical eigenvalue* λ.

The expression (16) also represents the dynamical partition function and is the dynamical analogous of the microcanonical partition function (25).

Proof. Consider $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ the contractive system in Lemma 2. So, we get $\mathcal{L}_{\beta,\mu} = B_q$ and, thus, we can derive our theorem directly from [CO17], Lemma 1, and from Theorem 7 who gives us a positive eigenfunction for B_q, with eigenvalue λ. $\square$

Definition 9. *A holonomic probability $\hat{\nu}$ with respect to $\mathcal{R}$ on $\Omega \times \mathcal{A}_0^{\mathbb{N}}$, is a probability such that*

$$\int_{\Omega\times\mathcal{A}_0^{\mathbb{N}}} g\left(\phi_{w_1}(x)\right) d\hat{\nu}(x,w) = \int_{\Omega} g(x) d\nu(x),$$

for all $g : \Omega \to \mathbb{R}$ continuous, where ν is the projection on the first coordinate of $\hat{\nu}$ (i.e., $\int g(x)d\nu(x) := \int g(x)d\hat{\nu}(x,w)$). The set of all holonomic probability measures with respect to $\mathcal{R}$ is denoted $\mathcal{H}(\mathcal{R})$.

Definition 10 ([LO09, CO17])**.** *Let $\mathcal{R} = (\Omega, \phi_j)_{j\in\mathcal{A}}$ be an IFS and $\hat{\nu} \in \mathcal{H}(\mathcal{R})$. The variational entropy of $\hat{\nu}$ is defined by*

$$h_v(\hat{\nu}) \equiv \inf_{g\in \mathrm{C}^+(\Omega)} \left\{ \int_{\Omega} \log \frac{B_1(g)(x)}{g(x)} d\nu(x) \right\} \tag{17}$$

where $B_1(g)(x) = \sum_{j\in\mathcal{A}} g(\phi_j(x))$.

Call such variational entropy the grand-canonical entropy when applied to the case we consider here.

From [LO09], Proposition 19 (see also [CO17], Theorem 10), we know that $0 \le h_v(\hat{\nu}) \le r = \sharp(\mathcal{A})$ for any $\hat{\nu} \in \mathcal{H}(\mathcal{R})$.

Inspired by [CO17], Definition 11, we can now introduce the concept of grand-canonical topological pressure.

Definition 11. *Consider the IFS with weights $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$. The grand-canonical topological pressure of $Q = (q_j)_{j\in\mathcal{A}}$, is defined by the following expression*

$$P(Q) := \sup_{\hat{\nu}\in\mathcal{H}(\mathcal{R})} \inf_{g\in C^+(\Omega)} \left\{ \int_{\Omega} \log \frac{\mathcal{L}_{\beta,\mu}(g)}{g} d\nu \right\}, \tag{18}$$

assuming $\mathcal{L}_{\beta,\mu} = B_q$.

Theorem 12. *Consider $(\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$ the contractive system in Lemma 2. Assume that the family of potentials $\Phi = (A_N)_{N\in\mathbb{N}_0}$, satisfy the hypothesis from Theorem 7 and $\mathcal{L}_{\beta,\mu} = B_q$.*

Denote by $\psi(y) := \ln\left(\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(y)-\mu N]}\right)$, $y \in \Omega$, the grand-canonical potential. Then:

a) $P(Q) = P(\psi) := \sup_{\hat{\nu}\in\mathcal{H}(\mathcal{R})} \left\{ h_v(\hat{\nu}) + \int_\Omega \psi(y)\, d\nu(y) \right\}.$

b) The set of equilibrium states, that is, holonomic measures $\hat{\mu}$ satisfying $P(\psi) = h_v(\hat{\mu}) + \int_\Omega \psi\, d\mu$, is not empty;

c) $P(Q) = \log(\lambda)$, *where λ is the grand-canonical eigenvalue given by Theorem 7.*

Proof. a) In the general setting presented in [CO17], Definition 11, one can consider the homogeneous case, that is, when $\psi : \Omega \to \mathbb{R}$ is a positive continuous function and $\mathcal{R} = (\Omega, \phi_j, q_j)_{j\in\mathcal{A}}$, with $q_j = \psi \circ \phi_j$ for each $j \in \mathcal{A}$. In this case, the topological pressure of ψ is alternatively given by

$$P(\psi) = \sup_{\hat{\nu}\in\mathcal{H}(\mathcal{R})} \left\{ h_v(\hat{\nu}) + \int_\Omega \psi\, d\nu \right\}.$$

Actually, this is the case for the weights q_j obtained for a family $\Phi = (A_N)_{N\in\mathbb{N}_0}$ as in Lemma 2. Indeed, given $q_j(x) := \sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(jx)-\mu N]} > 0$, $j \in \mathcal{A}$ we can choose the function

$$\psi(y) := \ln\left(\sum_{N\in\mathbb{N}_0} e^{-\beta[A_N(y)-\mu N]} \right), \tag{19}$$

which is well defined. It follows immediately that $q_j = e^{(\psi\circ\phi_j)}$, $j \in \mathcal{A}$. b) We say that the holonomic measure $\hat{\mu}$ is an equilibrium state for ψ if $h_v(\hat{\mu}) + \int_\Omega \psi(x)\, d\mu(x) = P(\psi)$. From [CO17], Theorem 13, we know that for such IFS and $\psi : \Omega \to \mathbb{R}$ a continuous function, so that e^ψ is positive, the set of equilibrium states for ψ is not empty.
c) Is a direct consequence of [LO09], Theorem 22, and Theorem 7. □

The next corollary highlights the importance of the results about the existence of an eigenfunction in our version of Ruelle's Theorem. Under the validity of the hypothesis of Theorem 7 we will show the analytic dependence of the eigenfunction λ as a function of ψ in (19).

Corollary 13. *If the family $\Phi = (A_N)_{N\in\mathbb{N}_0}$, satisfies the hypothesis of Theorem 7 and the family of potentials is uniformly Lipschitz, that is,* $\mathrm{Lip}(A_N) \leq M$*, for some fixed m, then, the eigenfunction varies analytically on the grand-canonical potential ψ given by* (19)*. From this follows the analyticity of the pressure (and also of the eigenvalue λ) as a function of ψ.*

Proof. First note that from our hypothesis we get, from Lemma 2 (b), that the weights of our IFS are Lipschitz continuous.

As seen in [BCLMS23, Remark 2], whose reasoning is similar to [PP90], the positiveness of the transfer operator (and the fact that the attractor is Ω) means that the dimension of the eigenspace associated to the spectral radius λ is one. Also, since the weights are Lipschitz continuous according to Lemma 2, we know from [PP90] that the essential spectrum of the operator is contained in a disc of radius strictly smaller than λ. For a general version of this result (which contemplates the case of the Grand-canonical potential ψ considered by us) we refer the reader to [Henion] and [Ye]. This implies that the main eigenvalue is isolated. Using a standard argument of complex analysis (Cauchy's integral formula for bounded operators on Banach spaces) we get the analyticity (see for instance Theorem 5.1 in [Mane], [SilvaFe], or Proposition 35 and 36 in [Lop1]); the reasoning here should follow exactly the same procedures: take a circle path around the eigenvalue on the complex plane, etc, and we leave the details for the reader. □

3.2 An alternative setting via an infinite countable IFS with weights.

The idea in this section is to choose an infinite countable IFS with weights whose transfer operator matches the grand-canonical-Ruelle operator $\mathcal{L}_{\beta,\mu}$ (that appears in (8)), so that the thermodynamical formalism for $\mathcal{L}_{\beta,\mu}$ can be derived from the well-known thermodynamical formalism for that kind of system, for details see [HMU02]. We have to show that our model fits the hypothesis of [HMU02].

In this section, the emphasis will be on results for the grand-canonical-dual-Ruelle operator $\mathcal{L}^*_{\beta,\mu}$ and not so much for the grand-canonical-Ruelle operator $\mathcal{L}_{\beta,\mu}$. We will be able to obtain the same claims as presented in Theorem 12 but without the need to show the existence of an eigenfunction for the operator $\mathcal{L}_{\beta,\mu}$.

We start by setting up the appropriate IFS and choosing maps and weights. After showing that the transfer operator for that IFS coincides with $\mathcal{L}_{\beta,\mu}$ we prove that the IFS satisfies the regularity requirements from [HMU02]. Finally, we introduce results on partition functions and topological pressure characterizing the topological pressure through the eigenvalue of the dual operator $\mathcal{L}^*_{\beta,\mu}$. In this framework, we do not provide a version of the Ruelle theorem for $\mathcal{L}_{\beta,\mu}$ because it is not known of any result about the existence of eigenfunctions in the IFS literature, to the best of our knowledge.

To simplify the notation, we consider now the alphabet $\mathcal{A} := \{0, ..., r-1\}$

instead of $\mathcal{A} := \{1, ..., r\}$. In particular, $\Omega := \mathcal{A}^{\mathbb{N}} = \{0, ..., r-1\}^{\mathbb{N}}$ is our symbolic metric space endowed with the metric

$$d(x, y) := \begin{cases} 2^{-\min\{n \in \mathbb{N} : x_n \neq y_n\}}, & x \neq y; \\ 0, & x = y. \end{cases}$$

Consider the *countable IFS* $\mathcal{R} := (\Omega, \phi_j)_{j \in I}$ on $I := \mathbb{N}_0$, where $\phi_j(x) = (j \mod r)x$ is the *mnemonic representation* for $(j \mod r, x_1, x_2, ...)$ and $x = (x_1, x_2, ...) \in \Omega$. This IFS is obviously contractive w.r.t. the distance introduced in Ω, that is, $\mathrm{Lip}(\phi_j) = \frac{1}{2}$ for all $j \in \mathbb{N}_0$. Note that the sequence of maps is formed by repetitions, $\phi_0(x) = 0x$, $\phi_1(x) = 1x$, ..., $\phi_{r-1}(x) = (r-1)x$, $\phi_r(x) = (r \mod r)\, x = 0x$, $\phi_{r+1}(x) = (r+1 \mod r)x = 1x$, $\phi_{r+2}(x) = (r+2 \mod r)x = 2x$, and so on.

Here we follow [HMU02], where only positive weights are allowed in the Ruelle operator. This is achieved by considering a countable *IFS with weights* $\mathcal{R} := (\Omega, \phi_j, q_j)_{j \in \mathbb{N}_0}$, where $Q := \{q_j : \Omega \to \mathbb{R},\ j \in \mathbb{N}_0\}$ is family of continuous functions. The new *transfer operator* $B_q : \mathrm{C}(\Omega) \to \mathrm{C}(\Omega)$ is given by

$$B_q(g)(x) = \sum_{j \in \mathbb{N}_0} e^{q_j(x)} g(\phi_j(x)),$$

for any $g \in \mathrm{C}(\Omega)$.

Lemma 14. *Suppose that* $\Phi = (A_N)_{N \in \mathbb{N}_0}$ *is an admissible sequence of potentials satisfying* $\mathrm{Lip}(A_N) \leq M$*, for some* $M > 0$*. Consider the function* $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ *defined by*

$$\xi(j) = \frac{j - (j \mod r)}{r}.$$

If we choose the weights $q_j(x) := -\beta\, (A_{\xi(j)}(\phi_j(x)) - \xi(j)\, \mu)$, $j \in \mathbb{N}_0$ *and construct the IFS with weights* $\mathcal{R} := (\Omega, \phi_j, q_j)_{j \in \mathbb{N}_0}$ *then*

$$B_q(g)(x) = \mathcal{L}_{\beta,\mu}(g)(x)$$

for any $g \in \mathrm{C}(\Omega)$*. In particular* $(q_j)_{j \in \mathbb{N}_0}$ *is a sequence of uniformly Lipschitz continuous functions with* $\sup_{j \in \mathbb{N}_0} \mathrm{Lip}(q_j) \leq \frac{\beta M}{2}$.

Proof. The proof follows easily from a computation. In order to obtain a representation of the operator $\mathcal{L}_{\beta,\mu}$ as the transfer operator of an IFS we must find a suitable family of weights. To do that we observe that

$$\mathcal{L}_{\beta,\mu}(f)(x) := \sum_{N \in \mathbb{N}_0} e^{\beta\, \mu\, N} \sum_{j \in \mathcal{A}} e^{-\beta\, A_N(jx)} f(jx) =$$

$$= e^{\beta\,\mu\,0}[e^{-\,\beta\,A_0(0x)}f(0x) + e^{-\,\beta\,A_0(1x)}f(1x) + \cdots + e^{-\,\beta\,A_0((r-1)x)}f((r-1)x)] +$$
$$+ e^{\beta\,\mu\,1}[e^{-\,\beta\,A_1(0x)}f(0x) + e^{-\,\beta\,A_1(1x)}f(1x) + \cdots + e^{-\,\beta\,A_1((r-1)x)}f((r-1)x)] + \cdots =$$
$$= e^{-\,\beta\,(A_0(0x)-\mu\,0)}f(0x) + e^{-\,\beta\,(A_0(1x)-\mu\,0)}f(1x) + \cdots + e^{-\,\beta\,(A_0((r-1)x)-\mu\,0)}f((r-1)x) +$$
$$+ e^{-\,\beta\,(A_1(0x)-\mu\,1)}f(0x) + e^{-\,\beta\,(A_1(1x)-\mu\,1)}f(1x) + \cdots + e^{-\,\beta\,(A_1((r-1)x)-\mu\,1)}f((r-1)x) + \cdots .$$

In the first line we replace $0x = \phi_0(x)$, $1x = \phi_1(x)$, ..., $(r-1)x = \phi_{(r-1)}(x)$ and after, in the second line, we replace $0x = \phi_r(x)$, $1x = \phi_{r+1}(x)$, ..., $(r-1)x = \phi_{(2r-1)}(x)$, and so on.

From this choice for the coefficients we get:

$$q_0(x) := -\,\beta\,(A_0(0x) - \mu\,0),\ q_1(x) := -\,\beta\,(A_0(1x) - \mu\,0), ...,$$
$$q_{r-1}(x) := -\,\beta\,(A_0((r-1)x) - \mu\,0),$$
$$q_r(x) := -\,\beta\,(A_1(0x) - \mu\,1),\ q_{r+1}(x) := -\,\beta\,(A_1(1x) - \mu\,1), ...,$$
$$q_{2r-1}(x) := -\,\beta\,(A_1((r-1)x) - \mu\,1),$$
$$q_{2r}(x) := -\,\beta\,(A_2(0x) - \mu\,2),\ q_{2r+1}(x) := -\,\beta\,(A_2(1x) - \mu\,2), ...,$$
$$q_{3r-1}(x) := -\,\beta\,(A_2((r-1)x) - \mu\,2), ...$$

which obviously satisfy $q_j(x) := -\,\beta\,(A_{\xi(j)}(\phi_j(x)) - \xi(j)\,\mu),\ j \in \mathbb{N}_0$, proving our claim. $\square$

Despite the fact that the maps ϕ_j, $j \in \mathbb{N}$, repeat themselves periodically, the weights do not. Thus, $\mathcal{R} = (\Omega, \phi_j, q_j)$ is a genuine countable IFS with weights.

To further developments we need to introduce some notation on countable IFSs taken from [HMU02].

A word of length n in $I^{\mathbb{N}}$ is an element $w := (w_1, ..., w_n)$ of $I_n := I^n$ and $\sigma(w) := (w_2, ..., w_n) \in I_{n-1}$. Given $x \in \Omega$ the iterate $\phi_w(x)$ is the point

$$y := \phi_{w_1}(\cdots(\phi_{w_n}(x)) \in \Omega.$$

A family of continuous functions $Q := \{q_j : \Omega \to \mathbb{R},\ j \in I\}$ is α-Hölder if

$$V_\alpha(Q) := \sup_{n\geq 1} V_n(Q) < \infty$$

where

$$V_n(Q) := \sup_{w\in I_n}\sup_{x\neq y} |q_{w_1}(\phi_{\sigma(w)}(x)) - q_{w_1}(\phi_{\sigma(w)}(y))| e^{\alpha(n-1)}.$$

Additionally, if $B_q(1) \in C(\Omega)$ then we say that the family Q is strongly α-Hölder or, according to [ARU18], *summable.*

Lemma 15. *Suppose that $\Phi = (A_N)_{N\in\mathbb{N}_0}$ is a family of potentials satisfying* $\mathrm{Lip}(A_N) \le M$*, for some $M > 0$ and*

$$A_N(x) > \left(\mu + \frac{\log(r)}{\beta}\right) N,\ \forall x \in \Omega, \forall N \in \mathbb{N}_0. \tag{20}$$

Then, the family Q is α-Hölder for $\alpha := \log(2)$ and summable, that is $\mathcal{L}_{\beta,\mu}(1) \in C(\Omega)$.

Proof. Consider $w \in I_n$ then

$$|q_{w_1}(\phi_{\sigma(w)}(x)) - q_{w_1}(\phi_{\sigma(w)}(y))| \le \mathrm{Lip}(q_{w_1})\frac{1}{2^{n-1}}\, d(x,y) \le$$
$$\le \frac{1}{2^{n-1}}\frac{\beta\, M}{2}\, d(x,y) = \frac{\beta\, M}{2^n}\, d(x,y),$$

because we get from Lemma 14 that $\sup\limits_{j\in\mathbb{N}_0} \mathrm{Lip}(q_j) \le \dfrac{\beta\, M}{2}$. Thus,

$$V_n(Q) \le \sup_{w\in I_n}\sup_{x\neq y} \frac{\beta\, M}{2^n}\, d(x,y)e^{\alpha(n-1)} \le$$
$$\le \beta\,\frac{M}{2}\,\mathrm{diam}(\Omega)\frac{e^{\alpha(n-1)}}{e^{\log(2)(n-1)}} \le \beta\,\frac{M}{2}\,\mathrm{diam}(\Omega),$$

for $\alpha = \log(2)$.

To see the second part we recall that from Lemma 14 we get

$$\mathcal{L}_{\beta,\mu}(1)(x) = B_q(1)(x) = \sum_{N\in\mathbb{N}_0}\sum_{j\in\mathcal{A}} e^{-\,\beta\,(\,A_N(jx)-\mu\, N\,)}.$$

Thus $\mathcal{L}_{\beta,\mu}(1) \in C(\Omega)$ if and only if the **positive series above** is convergent. The root test claims that it is sufficient to prove that

$$\limsup_{N\to\infty} \sqrt[N]{\sum_{j\in\mathcal{A}} e^{-\,\beta\,(\,A_N(jx)-\mu\, N\,)}} < 1.$$

Recall that for nonnegative real numbers $t_1, \ldots., t_k$ we have $\sqrt[n]{\sum_{i=1}^k t_i} \le \sum_{i=1}^k \sqrt[n]{t_i}$, thus

$$\limsup_{N\to\infty} \sqrt[N]{\sum_{j\in\mathcal{A}} e^{-\,\beta\,(\,A_N(jx)-\mu\, N\,)}} \le \limsup_{N\to\infty} \sum_{j\in\mathcal{A}} \sqrt[N]{e^{-\,\beta\,(\,A_N(jx)-\mu\, N\,)}} \le$$
$$\le \sum_{j\in\mathcal{A}} e^{\limsup_{N\to\infty}(-\,\beta)\left(\,\frac{1}{N}A_N(jx)-\mu\right)} < \sum_{j\in\mathcal{A}} e^{\limsup_{N\to\infty}(-\,\beta)\,\left([\frac{\log(r)}{\beta}+\mu]\,-\mu\right)} = 1,$$

because $A_N(y) > N\left(\dfrac{\log(r)}{\beta} + \mu\right)$, for any $y \in \Omega$. $\square$

It follows from the above that we can apply the results from [HMU02] regarding entropy and topological pressure for IFS.

Lemma 16. *Suppose that $\Phi = (A_N)_{N\in\mathbb{N}_0}$ is a family of potentials satisfying* $\mathrm{Lip}(A_N) \leq M$*, for some $M > 0$ and*

$$A_N(x) > \left(\mu + \frac{\log(r)}{\beta}\right) N, \ \forall x \in \Omega, \forall N \in \mathbb{N}_0. \tag{21}$$

The number $\lambda := \mathcal{L}^*_{\beta,\mu}(\nu_q)(1)$ is an eigenvalue of the dual operator $\mathcal{L}^*_{\beta,\mu}$ associated to an eigenmeasure ν_q, that is $\mathcal{L}^*_{\beta,\mu}(\nu_q) = \lambda\nu_q$.

Proof. As $\mathcal{L}_{\beta,\mu} = B_q$ from Lemma 14, the result goes as follows. First we notice that $B_q^* : C(\Omega)^* \to C(\Omega)^*$ is well defined because the family Q is summable ($B_q^*(\nu)(1) < \infty$) and the operator $\nu \to \frac{B_q^*(\nu)}{B_q^*(\nu)(1)}$ has a fixed point by Schauder-Tychonoff's Theorem (see [HMU02] for details). Lets say that the probability measure ν_q is that fixed point, then $\frac{B_q^*(\nu_q)}{B_q^*(\nu_q)(1)} = \nu_q$ or equivalently $B_q^*(\nu_q) = \lambda\nu_q$, where $\lambda := B_q^*(\nu_q)(1) \in \mathbb{R}$. $\square$

Remark 17. *As we pointed out before a contractive countable IFS has an attractor $K \subseteq \Omega$ satisfying $K = \bigcup_{j\in I}(K)$ which is not necessarily compact, unless I is a finite set. However in our case $\phi_j(x) = (j \mod r)x$, $j \in \mathbb{N}_0$ is actually a finite family and $K = \Omega$ which is compact, by construction. Thus Lemma 2.5 from [HMU02], claiming that $\nu_q(K) = 1$, says only that ν_q has full support in Ω.*

Then, we define the partition function associated with the family Q, previously defined

$$Z_n(Q) := \sum_{w\in I_n} \|e^{\sum_{j=1}^n q_{w_j}(\phi_{\sigma^j(w)})}\|_0 = \sum_{w\in I_n} e^{\sup_{x\in\Omega}\sum_{j=1}^n q_{w_j}(\phi_{\sigma^j(w)}(x))}.$$

Since the function, $\log(Z_n(Q))$ is subadditive the topological pressure of a α-Hölder summable family Q can be defined analogously to the classical theory

$$P(Q) := \lim_{n\to\infty} \frac{1}{n}\log(Z_n(Q)) = \inf_{n\geq 1} \frac{1}{n}\log(Z_n(Q)). \tag{22}$$

A useful result from [HMU02] is the following one.

Proposition 18 ([HMU02], Proposition 2.3)**.** *The function $Q \to P(Q)$ is lower semicontinuous on the space of all α-Hölder summable families w.r.t. the topology of the uniform convergence.*

We also have the fundamental characterization of the eigenvalue $\lambda = (\mathcal{L}_{\beta,\mu})^*(\nu_q)(1)$ in terms of the pressure:

Lemma 19. *Suppose that $\Phi = (A_N)_{N\in\mathbb{N}_0}$ is a family of potentials satisfying $\mathrm{Lip}(A_N) \leq M$, for some $M > 0$ and*

$$A_N(x) > \left(\mu + \frac{\log(r)}{\beta}\right) N, \ \forall x \in \Omega, \forall N \in \mathbb{N}_0. \tag{23}$$

The eigenvalue λ of the dual operator $(\mathcal{L}_{\beta,\mu})^*$ is given by

$$\lambda = e^{P(Q)}.$$

Proof. From [HMU02], Lemma 2.4, the eigenvalue λ of the dual operator B_q^* is given by $\lambda = e^{P(Q)}$. As $B_q = \mathcal{L}_{\beta,\mu}$, from Lemma 16, the result follows. □

4 Appendix - A brief account on Thermodynamics of ideal gases

This section does not primarily have a dynamical system content; our goal is to present a brief description of concepts and phenomena occurring in the physical world, aimed at an audience of readers who are mathematically oriented. Here, we are interested in physical systems which are in thermodynamical equilibrium. For simplicity, we mainly consider the case of ideal gases (i.e., systems of non-interacting point-like particles). Thermodynamics is the branch of physics that organizes systematically the empirical laws referring to the thermal behavior of the macroscopic world. It is one of our intentions to explain the meaning of this statement. For the mathematical reader who tries to understand the content of some texts in Physics, an initial difficulty is the jargon used there; all this will be translated here into a more formal context. At the end of this Appendix, we will mention a possible interpretation of topological pressure as related to gas pressure (see Remark 22).

Our goal is to draw a parallel between concepts of thermodynamics of gases with similar ones in the mathematical theory of Thermodynamic Formalism in the sense of [PP90]. We believe that this can provide an enrichment of the class of questions that can be raised and proposed in Thermodynamic Formalism.

4.1 The case of a definite number of particles

Consider a classical gas with N particles (initially N is fixed) at *temperature* T in a region with *volume* V. Actually, we assume that $T > 0$ satisfies (2). Denote by p the *gas pressure* of the system, by S the *entropy*, and by μ the *chemical potential*. The *total energy* U is a function $U = U(S, V, N)$, of the variables S, V, N. It is important to point out that we are assuming that the macroscopic variables T, V, N, p can be measured. In fact, the values of these variables S, V, N, are not so important in themselves, but their variations $\delta S, \delta V, \delta N$ are.

For the benefit of the reader, we will briefly describe below some basic properties of the thermodynamics of gases (intertwined with Statistical Mechanics). For more details see [Nau11], [Sal01], [Sch89], [Cal14], [Zu] or Section 6 in [LR22]. In this way, we think that some of the future (and also past) definitions that we will present here will look natural.

The fundamental relations concerning the above-described variables can be found in (3.6) on page 42 on [Sal01]:

$$\begin{aligned} T &= \frac{\partial U}{\partial S}; \\ p &= -\frac{\partial U}{\partial V}; \\ \mu &= \frac{\partial U}{\partial N} \,. \end{aligned}$$

Another fundamental relation in thermal physics (of ideal gases) is the following

$$p\,V = k_B\,N\,T \,. \tag{24}$$

Recall that k_B denotes the Boltzmann constant. The above equation is called the equation of state of the ideal gas. Other kinds of systems (like interacting gases, liquids, solids, etc.) have their own equations of state.

The above can be understood in several different ways from a physical point of view. For instance, consider a piston at temperature T, where the piston chamber has volume V and contains N particles. When the volume V and the temperature T are fixed, the pressure p becomes a linear function of the number of particles N. On the other hand, when the volume V and the number of particles N are fixed, it follows that the pressure p is linear with respect to the temperature T. Later, we will be interested in the case where the number N of particles is an unknown value ranging on $\mathbb{N}_0$ (the grand canonical case).

We point out that the relation (24) is valid under the quasi-static regime (this means that the thermodynamic processes we consider are such that the changes are slow enough for the system to remain in equilibrium).

A variable is called *extensive (intensive)*, if it is proportional (not proportional) to the volume V, like energy and number of particles (energy and particle densities). More precisely, it is important to recall that the definitions of extensive and intensive variables make sense only in the thermodynamic limit, i.e., extensive (intensive) variables are proportional (not proportional) to the volume V, when V tends to infinity. For instance, if the ratio N/V tends to some constant ϱ, then the particle density, which is precisely ϱ, is an intensive quantity, whereas the number of particles N is the associated extensive quantity. In a similar way as for intensive quantities, one also defines so-called molar quantities, which are, by definition, quantities that are proportional to the number of particles N. For instance, the total energy per particle is a molar quantity if the ratio E/N has a limit, as $N \to \infty$. Frequently, the ratio N/V is set to be constant, that is, the particle density is fixed. In this particular situation, molar and intensive quantities are, of course, equivalent notions.

We point out that the mathematical formalism for Gas Thermodynamics and Statistical Mechanics is not exactly the same, but certain general principles are common in both theories. Particles of a gas are displayed in a random way, as well as spins on a lattice. The randomness of the particles of gas (for instance each position and velocity) should be described by a probability. Note that a probability (a law that is assigned to Borel sets values in $[0,1]$) is not a physical entity; it is a tool to predict (or to explain) - values that are measured in the Physics of the real world - in circumstances of lack of complete knowledge (for a discussion relating Physics to Information Theory see for instance [Cat08] and [Bri22]).

For practical purposes (aiming the reader familiar with probability) one can identify *microstates* with points in $\{1,2,..,d\}^{\mathbb{N}}$, *ensembles* with probabilities in $\{1,2,..,d\}^{\mathbb{N}}$, and finally *macrostates* with continuous functions $A : \{1,2,..,d\}^{\mathbb{N}} \to \mathbb{R}$. Given a probability (an ensemble) ρ in $\{1,2,..,d\}^{\mathbb{N}}$, the value

$$< A >_\rho = \int A d\rho,$$

is considered a *macroscopical quantity.* Macroscopic variables are easier to measure.

What are the probabilities (ensembles) ρ which are relevant when an isolated system is governed by a certain Hamiltonian (a macrostate) A and it is at temperature $T = \frac{1}{k_B\,\beta}$? Typically one is interested in a minimization (or maximization) problem-related to free energy - where an illustration of this problem is given in (32) - (or the MaxEnt method, where an exemplification is given in (30)), and this requires the addition of the concept of entropy (to be introduced soon in the comments to the Second Postulate of Thermodynamics). Equilibrium in isolated systems occur according to the principle of entropy maximization (the Second Law of Thermodynamics).

Related to the above-mentioned variational problems, it is worthwhile to consider the concept of canonical distribution. Given $\beta > 0$, a macrostate A, and an *a priori* measure ρ on $\{1, 2, .., d\}^{\mathbb{N}}$, the *microcanonical partition function* is

$$Z(\beta) = \int e^{-\beta A(x)} d\rho(x) < \infty, \tag{25}$$

and the *canonical distribution* $\mu_{A,\rho,\beta}$ is given by the law

$$B \to \mu_{A,\rho,\beta}(B) = \frac{\int_B e^{-\beta A(x)} d\rho(x)}{Z(\beta)}. \tag{26}$$

We call $\mu_{A,\rho,\beta}$ the *microcanonical distribution* (or else, *microcanonical ensemble*) for A, β and the *a priori* measure ρ.

The importance of this class of probabilities is due to the fact that they are the solutions to certain kinds of variational problems (see Remark 20) which are related to the Second Law of Thermodynamics. Therefore, $\mu_{A,\rho,\beta}$ describes an equilibrium ensemble (state).

When ρ is fixed, all these probabilities $\mu_{A,\rho,\beta}$ are absolutely continuous with respect to each other.

When ρ is the counting measure on $\{1, 2, ..., d\}$ and $A : \{1, 2, ..., d\} \to \mathbb{R}$, the *microcanonical partition function* is

$$Z(\beta) = \sum_j e^{-\beta A(j)},$$

and the *canonical distribution* $\mu = \mu_{A,\beta}$ is the probability such that

$$\int f d\mu_{A,\beta} = \frac{\sum_j f(j) e^{-\beta A(j)}}{Z(\beta)}. \tag{27}$$

The $\mu_{A,\beta}$-probability of j_0 is

$$\frac{e^{-\beta A(j_0)}}{\sum_j e^{-\beta A(j)}}. \tag{28}$$

All the above does not have a *dynamical* content. Entering in a dynamically context we can say that an *equilibrium ensemble* can be seen as a shift-invariant probability on $\{1, 2, .., d\}^{\mathbb{N}}$. In Statistical Mechanics the dynamics of the shift describe translation on the one-dimensional lattice.

Note that ergodic probabilities are singular with respect to each other; therefore, the dynamical point of view presents some conceptual differences when compared with the above.

Energy, volume and the number of particles N are called the *macroscopic extensive parameters.* The *microscopical variables* refer to probabilities (states), and the main issue (see Remark 21) is to establish a connection between them and the visible variables of the macroscopic world (that is, with thermodynamics).

We observe, from a historical perspective, that initially the thermodynamics of gases was developed without the knowledge that gas was constituted by a large number of particles.

Now we will describe the *postulates of equilibrium thermodynamics* in a simplified way (for more details see pages 40-41 in [Sal01] or Section 14.4 in [Bai12]). These laws describe the interplay between microscopical and macroscopical variables. They are rules of Nature, and their validity derives from the fact that when assuming them, the consequences that are inferred are in accordance with the observed reality.

First Postulate: The macroscopic state is completely characterized by the internal energy U, the volume V, and the number of particles N. The total energy of an isolated system (for which energy and matter transfer through the system boundary are not possible) is conserved.

Comment: In Thermodynamic Formalism, the potential $A = -H$, where H plays the role of the Hamiltonian and corresponds to the concept of internal energy. From [Cal14]: "Energy U is transferred between systems in two forms: energy in the form of work W, and disordered energy in the form of heat Q... Note that heat Q and work W are not themselves functions of the state of a system; they are measures of the amount of energy $\delta U = \delta W + \delta Q$ that is transferred between systems in different forms."

Relations between heat Q, work W and energy U in a dynamical setting are described in Section 7 on [LR22].

Second Postulate: There is a function of the extensive parameters U, V, N called entropy, denoted by $S = S(U, V, N)$, that is defined for all states of equilibrium. If we remove an internal constraint, in the new state of equilibrium the extensive parameters of the system assume a set of values that maximize the entropy. The entropy, as a function of the extensive parameters, is a fundamental equation of the system. It contains all the thermodynamic information about this system. For isolated systems, entropy never decreases.

Comment: When considering probabilities $p = (p_1, p_2, ..., p_d)$ on the set $\{1, 2, ..., d\}$, the entropy $h(p) = -\sum_{j=1}^{d} \log p_j \, p_j \geq 0$. The above postulate is *one of the forms of the second law of thermodynamics (see [Sch89] or Section 6 in [LR22] either), a topic which is discussed in the dynamical setting in Sections 3 and 4 in [LR22]. This is related to the principle of maximization of entropy when the mean energy (or other variables) is fixed (see* (30) *for a particular case). Expression (87) in Section 7 in [LR22] illustrates the Max-Ent principle in Thermodynamic Formalism. Entropy plays a fundamental role in the search for equilibrium via the variational problem associated with the Second Law. This makes the theoretical model in consonance with what is physically observed.*

The probability with maximal entropy can be seen as the one that contains the maximum amount of uncertainty, or, else, contains the minimum amount of information.

Entropy S, temperature T, and internal energy U are related via

$$\frac{dS}{dU} = \frac{1}{T}. \tag{29}$$

The corresponding expression in Thermodynamic Formalism is Proposition 42 in [LR22].

Another related version of the Second Law can be described in the following way according to [Cal14]:

"Time does not occur as a variable in thermodynamic equations... However, we are crucially interested in the direction of time, in the sense of the distinction between the past and the future. The second law of thermodynamics says that other variables being held constant heat always flow from an object of higher temperature to an object of lower temperature, and never the other way around."

There is a discussion in the physics community if the Boltzmann's entropy and the Gibbs' entropy are in fact the same (see the interesting discussion of the subject presented in the recent book [Bri22]).

The meaning of equilibrium can be tricky: in the case of glass, recently, researchers contradicted the flowing glass window claim, by determining that the glass in medieval windows only succumbs to gravity, after very long geological time scales. Glass can be seen as in equilibrium, or not, depending on the scale of time (see [Bri22] for more details).

An expression of the relation of temperature with the variation of entropy in Thermodynamic Formalism is described in Proposition 42 in [LR22].

Third Postulate: The entropy of a composite system is additive over each one of the individual components; the so-called extensive property of entropy. Entropy is a continuous and monotonically increasing function of energy.

Comment: From a dynamical point of view, the notion of entropy in the above postulate is aimed at a system with discrete time shift-invariance (stationary in $\mathbb{N}$)). However, in Statistical Mechanics, this shift represents rather a shift in space, typically the shift of spins in a chain. Thus, physically, entropy usually refers to translation invariant states. Hence, we can also talk about the entropy of states out of equilibrium, which, in particular, may depend on time $t \in \mathbb{R}$. By contrast, usually, the Shannon-Kolmogorov entropy is defined for invariant probabilities (stationary in $\mathbb{N}$) and is a concept independent on time $t \in \mathbb{R}$. The claim about additivity over a composite system is in the sense that the entropy associated with a system described by two independent systems is the sum of the entropy of each component; and this is true for such kind of entropy. The so-called non-extensive point of view of Statistical Mechanics considers other concepts of entropy that are not additive.

Fourth Postulate: (Nernst law) The entropy vanishes when there is only one equilibrium state at the absolute zero of temperature.

Comment: this postulate corresponds in some sense to thermodynamic formalism to the property that for a generic Hölder potential A, the ground state (a maximizing probability) is realized by a unique probability with support in a periodic orbit, which, of course, has zero entropy (see [Con16], [CLT01] and [ILM18]). The Nernst Law is sometimes called the Third Law.

After the above, we believe it is appropriate now to briefly describe the point of view of the MaxEnt Method.

Remark 20. *Given $A : \{1, 2, ..., d\} \to \mathbb{R}$, and $\alpha \in \mathbb{R}$, the (A, α)-MaxEnt solution is the the vector of probability $\bar{p} = (\bar{p}_1, \bar{p}_2, ..., \bar{p}_d)$ maximizing*

$$\max_{p=(p_1,p_2,...,p_d)} \{- \sum_{j=1}^{d} \log p_j \, p_j \mid \sum_{j=1}^{d} p_j A(j) \, = \, \alpha\} =$$

$$\max_{p=(p_1,p_2,...,p_d)} \{h(p) \mid \sum_{j=1}^{d} p_j A(j) \, = \, \alpha\}. \tag{30}$$

Via Legendre transform one can show that there exists $\beta \in \mathbb{R}$, such that, $\bar{p} = (\bar{p}_1, \bar{p}_2, ..., \bar{p}_d)$ also maximizes the variational problem

$$\max_{p=(p_1,p_2,...,p_d)} \{- \sum_{j=1}^{d} \log p_j \, p_j + \beta \, \sum_{j=1}^{d} p_j A(j)\} =$$

$$\max_{p=(p_1,p_2,...,p_d)} \{h(p) + \beta \, \sum_{j=1}^{d} p_j A(j)\}.. \tag{31}$$

The above is equivalent to considering the variational problem (minimizing free energy)

$$\min_{p=(p_1,p_2,...,p_d)} \{\frac{1}{\beta}\sum_{j=1}^{d} p_j \, \log p_j \, + \, \sum_{j=1}^{d} p_j A(j)\} =$$

$$\min_{p=(p_1,p_2,...,p_d)} \{ \sum_{j=1}^{d} p_j A(j) \, - \, \frac{1}{\beta} h(p) \, \}. \tag{32}$$

Taking ρ as the counting measure in $\{1, 2, ..., d\}$, one can show (see for instance Proposition 7.5 in [Bri22]) that the (A, α)-MaxEnt solution probability $\mu_{A,\alpha}$ in $\{1, 2, ..., d\}$ (the $\bar{p}$ maximal solution of (30)) can be written on the canonical distribution form

$$B \subset \{1, 2, ..., d\} \to \mu_{A,\alpha}(B) = \frac{\int_B e^{-\beta A(x)} d\rho(x)}{\int_B e^{-\beta A(x)} d\rho(x)} = \frac{\sum_{j\in B} e^{-\beta A(j)} p_j}{\sum_{j=1}^{d} e^{-\beta A(j)} p_j}. \tag{33}$$

For dynamical counterparts of the above see [LR22], [Lal87] and [CL00].

The postulates of equilibrium thermodynamics are an issue subject to controversy. Indeed, we quote G. Gour in [Gou22].

"Thermodynamics is one of the most prevailing theories in physics with vast applications spreading from its early days focused on steam engines to modern applications in biochemistry, nanotechnology, and black hole physics, just to name a few. Despite the success of this field, the foundations of thermodynamics remain controversial even today. Not only is there persistent confusion over the relation between the macroscopic and microscopic laws, in particular their reversibility and time-symmetry, there is not even a consensus on how best to formulate the second law. Indeed, as the Nobel laureate Percy Bridgman remarked in 1941 "there are almost as many formulations of the Second Law as there have been discussions of it" and the situation hasn't improved much since then."

Remark 21. *The natural variational problem in thermodynamics corresponds to find the state (ensemble) ρ_U minimizing the Helmholtz free energy (see (3.51) page 51 in [Sal01])*

$$F(T, V, N) := U - T\,S\;. \tag{34}$$

The equilibrium probability (ensemble) ρ_U for the Hamiltonian U at temperature T is the one minimizing the integral

$$\int (U - T\,S)\, d\rho. \tag{35}$$

For an illustration of this kind of variational problem see (32).

For T fixed (or, equivalently for β fixed), this corresponds to find ρ maximizing the integral of

$$-\frac{F}{T} = -k_B\,\beta\,F = -k_B\,\beta U + S\ . \tag{36}$$

For an illustration of this kind of variational problem see (31).

The rule of Nature determining a minimization of the expression (35) can be seen as the connection of the microscopical variables with the macroscopic variables. Actually, when U is the Hamiltonian, to maximize expression (36) corresponds in Thermodynamic Formalism to maximize topological pressure for the potential $-k_B\,\beta U$ among σ-invariant probabilities.

From the physical point of view, it is natural to introduce the chemical potential μ and the macrostate variable N, which describes the number of particles. The number of particles N can range in the set of natural numbers $\mathbb{N}$.

The *grand-canonical thermodynamic potential* (see (3.55) page 52 in [Sal01]), is given by the expression

$$U(T,\mu) := U - TS - \mu N\ , \tag{37}$$

where μ is the chemical potential.

In this case, the rule of Nature determines equilibrium via the minimization of

$$\int (U - TS - \mu N)\,d\rho, \tag{38}$$

among probabilities ρ.

Physical experiments in the laboratory indicate that in several cases the probability that the number of particles N is large is very small; the term $-\mu N$ in the above equation is in consonance with this claim.

4.2 Grand-canonical systems - indefinite number of particles

Now, we will outline a simplified version (suitable for us) of the main issues on the topic of the grand-canonical partition as presented in chapter 1.6 in [Nau11] (see also Section 8 in [Tsch]); here we consider a certain number of indistinguishable particles N ranging in $\mathbb{N}_0 = \{0,1,2,...,N,...\}$. In dealing with particle densities, it makes sense to consider the number of particles (per unit volume) as being] a real non-negative number but we will avoid this issue here (see [LW] for a more detailed account).

Given $\beta > 0$, and a sequence $A_N > 0$, $N \in \mathbb{N}_0$, we denote

$$Z_N(\beta) := e^{-\beta A_N} \ .$$

So, for $\mu < 0$, the *grand-canonical partition sum* associated to the family $\Phi = (A_N)_{N\in\mathbb{N}_0}$ is

$$Z(\beta,\mu) := \sum_{N\in\mathbb{N}_0} e^{\beta\, N\, \mu} Z_N(\beta) = \sum_{N\in\mathbb{N}_0} e^{\beta\, N\, \mu} e^{-\beta A_N} \ . \tag{39}$$

Moreover, given $N \in \mathbb{N}_0$, the probability $P_{N,\beta,\mu}$ of the number of particles to be N at temperature $T = \frac{1}{k_B \beta}$, is given by the value

$$P_{N,\beta,\mu} := \frac{e^{\beta\, N\, \mu} e^{-\beta A_N}}{Z(\beta,\mu)} \ . \tag{40}$$

Compare (40) with (28).

Besides that, important information is given by the partial derivatives

$$\frac{\partial}{\partial \beta} \log(Z(\beta,\mu))|_{\beta=\beta_0} = \mu \sum_{N\in\mathbb{N}_0} N \, \frac{e^{\beta_0\, N\, \mu} e^{-\beta_0 A_N}}{Z(\beta_0,\mu)} - \sum_{N\in\mathbb{N}_0} A_N \, \frac{e^{\beta_0\, N\, \mu} e^{-\beta_0 A_N}}{Z(\beta_0,\mu)} =$$

$$\mu \sum_{N\in\mathbb{N}_0} < N >_{P_{N,\beta_0,\mu}} - \sum_{N\in\mathbb{N}_0} < A_N >_{P_{N,\beta_0,\mu}} \ . \tag{41}$$

and

$$\frac{\partial}{\partial \mu} \log(Z(\beta,\mu))|_{\mu=\mu_0} = \beta \sum_{N\in\mathbb{N}_0} N \, \frac{e^{\beta\, N\, \mu_0} e^{-\beta A_N}}{Z(\beta,\mu_0)} = \beta \sum_{N\in\mathbb{N}_0} < N >_{P_{N,\beta,\mu_0}} \ . \tag{42}$$

The two above expressions are analogous to the ones appearing in equations (1.37) and (1.38) in [Nau11].

Now, we want to trace a parallel with the case of finite particles (comparing with (27) etc,...)

Given $\beta, A : \{1,2,...,d\} \to \mathbb{R}$, the *microcanonical partition function (see* (27)*) is*

$$Z(\beta) = \sum_{j=1}^{d} e^{-\beta A(j)}. \tag{43}$$

As we mentioned before, the *canonical distribution* $\mu^{can} = \mu^{can}_{A,\beta}$ is the probability $\mathfrak{Q}$ such that

$$j_0 \to \mathfrak{Q}(j_0) = \mu^{can}(j_0) = \frac{e^{-\beta A(j_0)}}{\sum_{j=1}^d e^{-\beta A(j)}} = \frac{e^{-\beta A(j_0)}}{Z(\beta)}. \tag{44}$$

Now we present the dynamical version of (27) and (44): given a Lipschitz function $f : \Omega \to \mathbb{R}$, and $x_0 \in \Omega$, denote

$$\lambda = e^{P(-\beta A)}.$$

It is known (see [PP90]) that

$$\lim_{n\to\infty} \frac{\mathcal{L}^n_{-\beta A}(f)(x_0)}{\lambda^n} = \varphi(x_0) \int f d\nu,$$

where φ is the eigenfunction and ν the eigenprobability of the Ruelle operator $\mathcal{L}_{-\beta A}$.

Then, the $Z(\beta)$ in expression (27) correspond here to $\lambda = e^{P(-\beta A)}$.

In our setting it is natural to denote $Z(\beta) := e^{P(-\beta A)}$.

Then, $P(\beta) := \log Z(\beta)$ (see also(49)) $\Leftrightarrow$ Topological Pressure $P(-\beta A)$.

We believe we made more clear to the reader the relationship between the corresponding concepts under the two possible settings (more details in Remark 22).

In the case of the statistics of a countable number of particles, it is a different setting: we consider a number of indistinguishable particles N ranging in $\mathbb{N}_0$.

Remember that $\mu < 0$ is called the chemical potential. Therefore, it is natural in our dynamical setting to adapt the reasoning which derived (27).

Given $\beta > 0$, $\mu < 0$ and a sequence $A_N \geq 0$, $N \in \mathbb{N}_0$, the *grand-canonical partition sum* (compare with (43)) is

$$Z(\beta, \mu) := \sum_{N\in\mathbb{N}_0} e^{\beta\, N\, \mu} e^{-\beta A_N} \ . \tag{45}$$

An example: if each particle has energy $E > 0$, then, take $A_N = N\, E$.

Moreover, given $N \in \mathbb{N}_0$, the probability $P_{N,\beta,\mu}$ of the number of particles to be N at temperature $T = \frac{1}{k_B \beta}$, is (see (40))

$$P_{N,\beta,\mu} := \frac{e^{\beta\, N\, \mu} e^{-\beta A_N}}{Z(\beta, \mu)} \ \text{(compare with (44)}\ \mu^{can}(j_0) = \frac{e^{-\beta A(j_0)}}{Z(\beta)}). \tag{46}$$

In consonance with (16) and (27) we call

$$P(\beta, \mu) = \log Z(\beta, \mu)) \tag{47}$$

grand-canonical asymptotic pressure.

Remark 22. *Fixing the volume V, the chemical potential μ, the temperature T, and the grand-canonical pressure, we get that the grand-canonical gas pressure p satisfies (see page 12 in [Nau11]).*

$$p := \frac{k_B\, T\, P(\beta, \mu)}{V} \quad \textit{(compare with (24)} \;\; p = \frac{k_B\, T\, N}{V}). \tag{48}$$

If V and T are fixed, then p is linear on $P(\beta, \mu)$, or considering the case of the statistics of just one particle, $\mu = 0$, $V = 1$ and $T = 1$, we get

$$p \sim k_B P(\beta, \mu) \sim \; k_B P(\beta). \tag{49}$$

In this way, the origin of the terminology topological pressure can be seen as related in some way to the pressure of a gas.

Below, we will consider a non-dynamical example:

Example 23. *Consider a chemical potential $\mu < 0$, assume that $E > 0$ represents the energy of a particle, and, under the assumption of non-interaction between particles, we take the Hamiltonian A_N of the form $A_N = NE$, for each $N \in \mathbb{N}_0$. In this case, we have*

$$Z(\beta, \mu) = \sum_{N \in \mathbb{N}_0} e^{\beta\, N\, \mu} e^{-\beta N\, E} = \sum_{N \in \mathbb{N}_0} e^{N\, \beta\, (\mu - E)} = \frac{1}{1 - e^{\beta(\mu - E)}} > 0\,.$$

Moreover, under the above assumptions, it follows that

$$\begin{aligned}\frac{\partial}{\partial \beta} \log(Z(\beta, \mu))|_{\beta = \beta_0} &= \frac{1}{Z(\beta_0, \mu)} \sum_{N \in \mathbb{N}} N(\mu - E) e^{N\, \beta_0\, (\mu - E)} \\ &= \frac{(\mu - E) e^{\beta_0(\mu - E)}}{Z(\beta_0, \mu)(1 - e^{\beta_0(\mu - E)})^2}\,,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial \mu} \log(Z(\beta, \mu))|_{\mu = \mu_0} &= \frac{1}{Z(\beta, \mu_0)} \sum_{N \in \mathbb{N}} N\, \beta e^{N\, \beta\, (\mu_0 - E)} \\ &= \frac{\beta e^{\beta(\mu_0 - E)}}{Z(\beta, \mu_0)(1 - e^{\beta(\mu_0 - E)})^2}\,.\end{aligned}$$

$\diamondsuit$

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