

The involution kernel and the dual potential for functions in the Walters family

L. Y. Hataishi and Lopes, A. O.

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Instituto de Matemática e Estatística, UFRGS - Porto Alegre, Brasil.

Abstract

First, we set a suitable notation. Points in $\{0, 1\}^{\mathbb{Z}-\{0\}} = \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} = \Omega^- \times \Omega^+$, are denoted by $(y|x) = (\dots, y_2, y_1|x_1, x_2, \dots)$, where $(x_1, x_2, \dots) \in \{0, 1\}^{\mathbb{N}}$, and $(y_1, y_2, \dots) \in \{0, 1\}^{\mathbb{N}}$. The bijective map $\hat{\sigma}(\dots, y_2, y_1|x_1, x_2, \dots) = (\dots, y_2, y_1, x_1|x_2, \dots)$ is called the bilateral shift and acts on $\{0, 1\}^{\mathbb{Z}-\{0\}}$. Given $A : \{0, 1\}^{\mathbb{N}} = \Omega^+ \rightarrow \mathbb{R}$ we express A in the variable x , like $A(x)$. In a similar way, given $B : \{0, 1\}^{\mathbb{N}} = \Omega^- \rightarrow \mathbb{R}$ we express B in the variable y , like $B(y)$. Finally, given $W : \Omega^- \times \Omega^+ \rightarrow \mathbb{R}$, we express W in the variable $(y|x)$, like $W(y|x)$. By abuse of notation, we write $A(y|x) = A(x)$ and $B(y|x) = B(y)$. The probability μ_A denotes the equilibrium probability for $A : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$.

Given a continuous potential $A : \Omega^+ \rightarrow \mathbb{R}$, we say that the continuous potential $A^* : \Omega^- \rightarrow \mathbb{R}$ is the dual potential of A , if there exists a continuous $W : \Omega^- \times \Omega^+ \rightarrow \mathbb{R}$, such that, for all $(y|x) \in \{0, 1\}^{\mathbb{Z}-\{0\}}$

$$A^*(y) = [A \circ \hat{\sigma}^{-1} + W \circ \hat{\sigma}^{-1} - W](y|x).$$

We say that W is an involution kernel for A . It is known that the function W allows to define a spectral projection in the linear space of the main eigenfunction of the Ruelle operator for A . Given A , we describe explicit expressions for W and the dual potential A^* , for A in a family of functions introduced by P. Walters. Denote by $\theta : \Omega^- \times \Omega^+ \rightarrow \Omega^- \times \Omega^+$ the function $\theta(\dots, y_2, y_1|x_1, x_2, \dots) = (\dots, x_2, x_1|y_1, y_2, \dots)$. We say that A is symmetric if $A^*(\theta(x|y)) = A(y|x) = A(x)$. We present conditions for A to be symmetric and to be of twist type. It is known that if A is symmetric then μ_A has zero entropy production.

Key words: Involution kernel, dual potential, symmetric potential, Walters' family, entropy production, twist condition.

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Email of Lopes, A. O. is arturoscar.lopes@gmail.com

Email of Lucas Y. Hataishi is lucas.hataishi@gmail.com

1 Introduction

The set $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ represents the one-dimensional unilateral lattice. Denote $\Omega = \Omega^+ = \{0, 1\}^{\mathbb{N}}$. The set $\{0, 1\}$ is the set of symbols (or spins) of the symbolic space Ω . Points in Ω are denoted by (x_1, x_2, x_3, \dots) , $x_j \in \{0, 1\}$, $j \in \mathbb{N}$. In some specific models in Statistical Mechanics the symbol 0 can represent the spin $-$ and the symbol 1 can represent the spin $+$.

The natural metric on $\{0, 1\}^{\mathbb{N}}$ is such that $d(x, y) = 2^{-j}$, where $j \in \mathbb{N}$ is the smallest one such that $x_j \neq y_j$.

We denote by \mathbb{Z}_*^- the set such that $\mathbb{Z}_*^- \cup \mathbb{N} = \{\dots, -3, -2, -1, 1, 2, 3, \dots\} = \mathbb{Z} - \{0\}$, which represents the one-dimensional bilateral lattice.

Define $\Omega^- := \{0, 1\}^{\mathbb{Z}_*^-}$, and endow Ω^- with a metric space structure analog to the metric which was defined for Ω .

The cartesian product $\Omega^- \times \Omega = \Omega^- \times \Omega^+$ is denoted by $\hat{\Omega}$, and a general element described by the ordered pair $((x_n)_{n \in \mathbb{Z}_*^-}, (x_n)_{n \in \mathbb{N}})$. We use the symbol $|$ for a better notation, and pairs can be written as $((x_n)_{n \in \mathbb{Z}_*^-} | (x_n)_{n \in \mathbb{N}})$.

It is also natural to identify Ω^- with $\Omega := \Omega^+$ and we will do this without mention. $\hat{\Omega}$ is in some sense a version of $\{0, 1\}^{\mathbb{Z} - \{0\}} \sim \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} = \Omega \times \Omega$. Under the such point of view we prefer the following notation: given $(x_1, x_2, x_3, \dots) \in \Omega$ and $(y_1, y_2, y_3, \dots) \in \Omega$, then a general point in $\hat{\Omega}$ is written as

$$(y|x) = (\dots, y_3, y_2, y_1 | x_1, x_2, x_3, \dots) \in \hat{\Omega} = \Omega^- \times \Omega^+.$$

We denote by $\mathcal{C}(\Omega)$ the set of continuous functions on Ω taking real values. Cylinder sets in Ω are denoted by $[a_1, a_2, \dots, a_n]$, $a_j \in \{0, 1\}$, $j = 1, 2, \dots, n$.

The *natural extension* of a potential $A \in \mathcal{C}(\Omega) = \mathcal{C}(\Omega^+)$ to $\hat{\Omega}$ is the potential $\hat{A} : \hat{\Omega} \rightarrow \mathbb{R}$ given by

$$\hat{A}(y|x) = A(x), \quad \forall (y|x) \in \hat{\Omega}. \quad (1)$$

When $B(y|x)$ does not depend on y we will use sometimes the simplified expression $B(y|x) = B(x)$.

The bijective map

$$(\dots, y_2, y_1 | x_1, x_2, \dots) \rightarrow \hat{\sigma}(\dots, y_2, y_1 | x_1, x_2, \dots) = (\dots, y_2, y_1, x_1 | x_2, \dots)$$

is called the bilateral shift and acts on $\{0, 1\}^{\mathbb{Z} - \{0\}}$.

The map $\sigma : \Omega \rightarrow \Omega$, such that

$$(x_1, x_2, \dots) \rightarrow \sigma(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

is called the unilateral shift and acts on $\Omega^+ = \Omega$.

Given $y = (y_1, y_2, y_3, \dots)$, we will also consider the function

$$x = (x_1, x_2, x_3, \dots) \rightarrow \tau_y(x) = (y_1, x_1, x_2, x_3, \dots),$$

for $\tau_y : \Omega \rightarrow \Omega$. The function τ_y is sometimes called an inverse branch.

Definition 1.1. Given a continuous function $A : \Omega \rightarrow \mathbb{R}$, if there exist continuous functions $W : \hat{\Omega} \rightarrow \mathbb{R}$ and $A^* : \Omega^- \rightarrow \mathbb{R}$, such that, for any $(y|x) \in \Omega$,

$$A^*(y) = [A \circ \hat{\sigma}^{-1} + W \circ \hat{\sigma}^{-1} - W](y|x), \quad (2)$$

then we say that W is an **involution kernel** for A , and that A^* is the **dual potential** of A relatively to W .

Above we could write \hat{A} instead of A . (see (1))

Given $A : \Omega^+ = \Omega \rightarrow \mathbb{R}$, the involution kernel W is not unique. One can show that in the case A is Hölder there exist W and a Hölder function $A^* : \Omega^- \rightarrow \mathbb{R}$ satisfying (2) (for a proof see [1], [14], [23], [40], [30] and [41]). We will consider here a large class of functions such that some of them are not of Hölder class, but despite that, there exist W and A^* satisfying (2).

Given A we are interested in explicit expressions for the involution kernel W and for the dual potential A^* .

The involution kernel was introduced in [1] where it was shown that a natural way to obtain an involution kernel W for A is via expression (10) (or (11)). The main inspiration for considering such concept in [1] was due to [7] which - in a different setting - considers a similar concept, but related to distributions, the Helgason kernel and eigenvalues of the Laplacian for surfaces of constant negative curvature. A mathematical rigorous formulation of [7] was later presented in [26].

Given a continuous potential $U : \Omega^+ = \Omega \rightarrow \mathbb{R}$, we say that the shift-invariant probability μ_U on Ω^+ is the equilibrium probability for U , if maximizes

$$\sup_{\mu \text{ shift invariant probability}} \{h(\mu) + \int U d\mu\}, \quad (3)$$

where $h(\mu)$ is the entropy of μ . The maximal value (3) is called the Pressure of U . The study of properties of μ_U is the main topic of Thermodynamic Formalism, and the text [36] describes the main properties of this theory. If U is Hölder continuous the probability of attaining the maximal value in (3) is unique. If U is just continuous there exist probabilities attaining the maximal value but maybe not unique. This phenomena is associated to phase transition (see [22], [32], [16], [9], [20])

In Statistical Mechanics, if $H : \Omega^+ \rightarrow \mathbb{R}$ is a Hamiltonian and the physical system under analysis is considered under temperature T , the probability $\mu_{-\frac{1}{T}H}$ corresponds to the observed equilibrium probability.

Note that in the case W is an involution kernel for A , then, given $\beta \in \mathbb{R}$, we get that βW is an involution kernel for βA . The value β corresponds in Thermodynamic Formalism (and Statistical Mechanics) to $\frac{1}{T}$, where T is temperature. In the case A is Hölder, using this property and (6), large deviation properties for the family of equilibrium probabilities $\mu_{\beta A}$, when $\beta \rightarrow \infty$ (that is, when the temperature goes to zero) are obtained in [1]. More precisely, [1] (see also [34]) address the following problem: given a Borel set K in Ω^+ , estimate the asymptotic value

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log(\mu_{\beta A}(K)).$$

Questions related to the selection of probability, that is the existence of a limit probability $\lim_{\beta \rightarrow \infty} \mu_{\beta A}$, when the temperature goes to zero appear in [4] and [2].

Denote by $\theta : \Omega^- \times \Omega^+ \rightarrow \Omega^- \times \Omega^+$ the function such that

$$\theta(\dots, y_2, y_1 | x_1, x_2, \dots) = (\dots, x_2, x_1 | y_1, y_2, \dots). \quad (4)$$

We say that A is **symmetric** if $A^*(\theta(x|y)) = A(y|x) = A(x)$.

Note that considering the set of symbols $\{-1, 1\}$ instead of $\{0, 1\}$ does not change much the above definitions.

Example 1.2. (Taken from Section 5 in [8]) Consider the alphabet $\{-1, 1\}$ and the symbolic space $\{-1, 1\}^{\mathbb{N}}$. In this case $\hat{\Omega}$ is $\{-1, 1\}^{\mathbb{Z}^0}$. We will define a potential $A : \{-1, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ which is symmetric. Indeed, consider a sequence $a_n > 0$, $n \geq 1$, such that $\sum_{i \geq 1} \sum_{j > i} a_j < \infty$, and for $x = (x_1, x_2, x_3, \dots) \in \{-1, 1\}^{\mathbb{N}}$, we define $A(x) = \sum_{n=1}^{\infty} a_n x_n$ (it is called a product type potential). Consider $W : \{-1, 1\}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbb{R}$, given by

$$W(y|x) = \sum_{i=1}^{\infty} [(x_i + y_i) (\sum_{j>i} a_j)] = \sum_{i=1}^{\infty} (x_i + y_i) (a_{i+1} + a_{i+2} + \dots).$$

Then, one can show that W is an involution kernel and A is symmetric. A particular example is when $a_n = 2^{-n}$, $n \geq 1$, in which case A is of Hölder class.

It is known that eigenfunctions for the Ruelle operator for an Hölder potential $A : \Omega^+ \rightarrow \mathbb{R}$ and eigenprobabilities for the dual of the Ruelle operator for $A^* : \Omega^- \rightarrow \mathbb{R}$ are related via the involution kernel (see [1] or [23]), which plays the role of a dynamical integral kernel (see expression (6)).

We elaborate on that: \mathcal{L}_A denotes the Ruelle operator for $A : \Omega^+ = \Omega \rightarrow \mathbb{R}$, which is the linear operator acting on continuous function $f_0 : \Omega^+ \rightarrow \mathbb{R}$, such that,

$$f_0 \rightarrow f_1(z) = \mathcal{L}_A(f_0)(z) = \sum_{\sigma(x)=z} e^{A(x)} f_0(x).$$

When looking for equilibrium probabilities on $\hat{\Omega}$ (which are invariant for $\hat{\sigma}$) for potentials $\hat{A} : \hat{\Omega} \rightarrow \mathbb{R}$ properties of the Ruelle operator, as defined above, are quite useful (for more details see [36] or Appendix in [31]).

The Ruelle theorem for an Hölder function A claims the existence of a positive eigenfunction φ_A for the operator \mathcal{L}_A and an associated to the eigenvalue $\lambda_A > 0$ (see [36] for a proof). φ_A is called the main eigenfunction of \mathcal{L}_A . The dual operator for A is denoted \mathcal{L}_A^* and acts on finite measures.

Given a Hölder potential A , we say that the probability ν_A on $\Omega = \Omega^+$ is the eigenprobability for the dual of the Ruelle operator \mathcal{L}_A^* , if $\mathcal{L}_A^*(\nu_A) = \lambda_A \nu_A$ (see [36] for a proof).

The Ruelle theorem helps to identify the equilibrium probability for the potential A (see [36]). More precisely

$$\mu_A = \varphi_A \nu_A. \quad (5)$$

Given A^* , the Ruelle operator \mathcal{L}_{A^*} acts on functions $f_0 : \Omega^- \rightarrow \mathbb{R}$.

Note that from [1] (or [23]) the main eigenvalue of the Ruelle operator for A and the main eigenvalue of the Ruelle operator for its dual A^* are the same. The same thing for \mathcal{L}_A^* and \mathcal{L}_{A^*} . That is $\lambda_A = \lambda_{A^*}$. We denote by ν_{A^*} the eigenprobability for the dual operator \mathcal{L}_{A^*} . The probability ν_{A^*} is defined on $\Omega^- = \Omega$.

Denote by $W = W_A$ the involution for A , then,

$$\int e^{W(y|x)} d\nu_{A^*}(y) = \varphi_A(x) \quad (6)$$

is the main eigenfunction of the Ruelle operator \mathcal{L}_A (see, for instance [23], for a proof). The involution kernel allows us to define a spectral projection in the linear space of the main eigenfunction of the Ruelle operator \mathcal{L}_A (see page 482 in [29]). Expression (6) describes one of the main reasons for the interest in the involution kernel.

Note that in equation (2) for the involution kernel, the eigenvalue λ_A does not appear; this is a property that is eventually quite useful when trying to get the main eigenfunction in the study of a particular example of potential A .

An interesting fact is that for a general continuous potential $A : \Omega \rightarrow \mathbb{R}$ an eigenprobability always exists but a continuous positive eigenfunction does not always (see [12], [11] and [10]).

Knowing the involution kernel for the potential A , other eigenfunctions for \mathcal{L}_A (not strictly positive) can be eventually obtained via eigendistributions for \mathcal{L}_{A^*} (see [18]).

When A is symmetric, it follows from (6) that

$$\int e^{W(y|x)} d\nu_A(y) = \varphi_A(x). \quad (7)$$

[25] shows that the involution kernel appears as a natural tool for the investigation of entropy production of μ_A . If the potential A is symmetric then the entropy production of μ_A is zero (see Section 7 in [25]). General results for entropy production in Thermodynamic Formalism appears in [37], [38], [39], [17], [33] and [21]. The case of symbolic spaces where the alphabet is a compact metric space is considered in [25].

In Example 2 (product type potentials) and Example 3 (Ising type potentials) in Section 5 in [8] the authors present examples of potentials A in $\{-1, 1\}^{\mathbb{N}}$ that are symmetric by exhibiting the explicit expression of the involution kernel (see our Example 1.2). From this property and results in [25] it follows that the equilibrium probabilities μ_A for such potentials have zero entropy production. Explicit expressions for the involution kernel (and the dual potential) were obtained in several sources as in Proposition 9 in [1], Section 5 in [5], in Remarks 6 and 7 in [28] and in Section 13.2 in [15].

In the works [28], [14] and [29], the involution kernel $W : \Omega^- \times \Omega \rightarrow \mathbb{R}$ is considered as a dynamical cost in an Ergodic Optimal cost Problem. In the classical setting, a lot of nice results on Optimal Transport are proved under the condition of convexity of the cost function. One can define the Twist Condition (see Definition 6.2) for an involution kernel W (of a potential A) and this plays the role of a form of convexity in Ergodic Transport. We will address this issue in Section 6. The twist condition is sometimes called the supermodular condition (see section 5.2 in [35]), which a natural hypothesis in optimization problems (see [6]), and Aubry-Mather theory (see [14], [29] and [28]).

Here, we investigate the existence of the involution kernel $W : \Omega^- \times \Omega \rightarrow \mathbb{R}$ and we study duality and symmetry issues for potentials $g : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ in a certain class of continuous potentials g (see Sections 4 and 5) to be defined next. We present explicit expressions.

We will consider a potential $g : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ which is at least continuous. Potentials on the so-called Walters family were introduced in [43] where some explicit results for the main eigenfunction of the Ruelle operator were obtained. For this family of potentials in [13] the authors study a certain class of Spectral Triples, and in [2] the authors present explicit expressions for subactions in Ergodic Optimization.

Definition 1.3. A potential $g : \Omega \rightarrow \mathbb{R}$ is in the Walters family, if and only if, there exist convergent sequences $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, $(c_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$, such that, for any $x \in \Omega$,

- $g(0^{n+1}1x) = a_{n+1}$;
- $g(01^n0x) = b_n$;
- $g(1^{n+1}0x) = c_{n+1}$;

- $g(10^n 1x) = d_n$,

for all $n \in \mathbb{N}$.

The set of such continuous potentials is described by $R(\Omega)$.

Some of these functions g are not of Hölder class.

We denote by a, b, c, d , respectively, the limits of the sequences a_n, b_n, c_n, d_n .

Theorem 1.1 in [43] describes the condition for the potential g (on the Walters family) to be in Bowen's class or the Walters' class (do not confuse this set with the class of Walters potentials described by Definition 1.3). Theorem 3.1 in [43] presents conditions on g which will imply that the Ruelle Theorem (on the existence of eigenvalues and eigenfunctions for the Ruelle operator) is true (see [36] for details).

If

$$a_n - a, b_n - b, c_n - c \text{ and } d_n - d \tag{8}$$

converge to zero exponentially fast to zero, then, it is easy to see that the potential g is of Hölder class. In this case, the equilibrium states are unique (no phase transition) and the pressure function is differentiable.

The family described by Definition 1.3 contains a subfamily of potentials called the Hofbauer potentials (see [20], [22], [32], [16], [24] and [9]) which are not of Hölder class. For this subclass, in some cases, there exists more than one equilibrium state (phase transition happens) and the pressure function may not be differentiable. The Hofbauer example corresponds to the following case: take $1 < \gamma$, then consider $c_n = -\gamma \log(\frac{n+1}{n})$, $n \geq 2$, $d_n = -\gamma \log(2)$, and $a_{n+1} = d_n = \log(\zeta(\gamma))$, where $\zeta(\gamma)$ is the Riemann zeta function. Questions related to renormalization for this class of potentials appear in [3] and [24]; the Hofbauer potential is a fixed point for the renormalization operator.

The variety of possibilities of the functions on the Walters' family of potentials is so rich that given a certain sequence c_n , $n \in \mathbb{N}$, describing a possible decay of correlations (under some mild assumptions), one can find a potential in the family such the equilibrium probability for this potential has this decay of correlation (see [24] for proof).

The potential of Example 1.2 is not in the Walters' family.

A positive continuous function $g : \Omega \rightarrow (0, 1)$, satisfying $g(0x) + g(1x) = 1$, for all $x \in \Omega$, is called a g -function. Let $G(\Omega)$ denote the set of all g -functions on the Walters family. Corollary 2.3 in [43] (see also our Corollary 8.2) implies that, the equilibrium probability μ_A of the continuous potential $A = \log g$, where $g \in G(\Omega)$ is on the Walters family, satisfies $\forall n \in \mathbb{N}$, and each pair of cylinders $[x_1, x_2, \dots, x_{n-1}, x_n]$ and $[x_n, x_{n-1}, \dots, x_2, x_1]$

$$\mu_A([x_1, x_2, \dots, x_{n-1}, x_n]) = \mu_A([x_n, x_{n-1}, \dots, x_2, x_1]). \tag{9}$$

This symmetry on the measure of cylinders means zero entropy production (see [21] and Section 8) and is related to the concept of A being

symmetric via the involution kernel (see [25]). It follows from the reasoning of [25] (using [21] and (9)) that given an Hölder potential A on the Walters family, there exists an involution kernel W such that makes A symmetric.

Given $x' = (x'_1, x'_2, \dots, x'_n, \dots) \in \Omega$ fixed, if

$$W(y|x) = \sum_{n \in \mathbb{N}} \left[\hat{A} \circ \hat{\sigma}^{-n}(y|x) - \hat{A} \circ \hat{\sigma}^{-n}(y|x') \right] \quad (10)$$

is convergent for any $(y|x) \in \hat{\Omega}$, it is shown in Section 5 that W is an involution kernel for A (see also [23]). It is called the **involution kernel** of A based on $x' \in \Omega$.

Given a continuous potential $A : \Omega \rightarrow \mathbb{R}$, a more simple expression for the involution kernel W for A is

$$W(y|x) = \sum_{n \geq 1} A(y_n, \dots, y_1, x_1, x_2, \dots) - A(y_n, \dots, y_1, x'_1, x'_2, \dots), \quad (11)$$

where $x = (x_1, x_2, \dots, x_n, \dots)$ and $y = (y_1, y_2, \dots, y_n, \dots)$.

In the case (11) converges an expression for a dual potential A^* associated with such involution kernel is given by (29).

If A is of Hölder class the above sum (11) always converges.

An outline of our main results: we investigate the existence and explicit expressions for the involution kernel W (see Section 4) and the dual potential A^* (see Tables (1), (2) and (3) in Subsection 5.1), for potentials A in the class $R(\Omega)$. Note that in order to show that (2), we just require convergence (of a sum of functions which will define W) and not much regularity (like Hölder, or Lipschitz continuity). We give a sufficient and necessary condition for the existence of W (see Theorem 3.1). For results about the symmetry of the potential A see expression (33) and Theorem 5.3.

For a potential g in the Walters' family we present here in Section 6 conditions for a potential of such type to satisfy the relaxed twist condition (see Definition 6.2 and Theorem 6.3).

Questions related to the characterization of normalized potentials on $G(\Omega)$ are described in Section 7 (see expression (38)).

In Section 8 we consider the sets $\mathcal{M}(\sigma)$ and $\mathcal{M}(\hat{\sigma})$, which are, respectively, the set of Borel invariant probability measures for $\sigma : \Omega \rightarrow \Omega$ and $\hat{\sigma} : \hat{\Omega} \rightarrow \hat{\Omega}$. The reasoning we followed above deals with the Equilibrium Statistical Mechanics of the lattice \mathbb{N} , which results in probabilities on $\mu_A \in \mathcal{M}(\sigma)$. On the other hand, the study of equilibrium probabilities on the lattice $\mathbb{Z} - \{0\}$ result in probabilities $\mu_{\hat{A}}$ on $\mathcal{M}(\hat{\sigma})$, where $\hat{A} : \hat{\Omega} = \{0, 1\}^{\mathbb{Z} - \{0\}} \rightarrow \mathbb{R}$. The relation between these two frameworks is the topic of Section 8 (see also appendix in [31]).

2 Ruelle's Theorem

We briefly describe some properties of Ruelle's Operator Theorem before addressing the main issues of our paper.

Definition 2.1. A continuous function $A : \Omega \rightarrow \mathbb{R}$ is an element of $\Xi(\Omega)$ if, and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, $\forall n \in \mathbb{N}$ and $\forall x \in \Omega, y \in B(x, n, \delta)$ implies

$$\left| \sum_{j=0}^{n-1} A(\sigma^j(x)) - \sum_{j=0}^{n-1} A(\sigma^j(y)) \right| < \epsilon ,$$

The set $\Xi(\Omega)$ is the set of potentials with Walters Regularity.

The Hofbauer potential does not have Walters Regularity.

In [43] is proved the following result:

Theorem 2.2. $A \in R(\Omega) \cap \Xi(\Omega) \iff \sum_{n \in \mathbb{N}} (a_n - a)$ and $\sum_{n \in \mathbb{N}} (c_n - c)$ are both convergent. If $A \in R(\Omega) \cap \Xi(\Omega)$ then Ruelle's Operator Theorem holds for A and it has a unique equilibrium state.

It follows from our calculations (in the next sections) the following result:

Theorem 2.3. Let $A \in \mathbb{R}(\Omega)$. If, for some $x' = (x'_n)_{n \in \mathbb{N}} \in \Omega$, the series

$$[A(y_1 x) - A(y_1 x')] + \sum_{n=2}^{\infty} [A(y_n \dots y_1 x) - A(y_n \dots y_1 x')]$$

converges absolutely, then it converges absolutely for any $x' \in \Omega$. Also, the convergence of the above series implies $A \in \Xi(\Omega)$, and A has a unique equilibrium state.

This result is related to expressions (11) and (12).

3 Existence of the Involution kernel

We are going to calculate formally in Subsection 4 the exact expression for the involution kernel (based on a certain point x') of a given potential in $R(\Omega)$. The result also gives a sufficient and necessary condition for the existence of the involution kernel based on a certain point x' . We analyze this problem considering Definition 1.1.

After the calculations which will be presented in Section 4, the final result is:

Theorem 3.1. For the involution kernel as in (10) to exist, it is necessary and sufficient that the potential is of Walters regularity.

We investigate next the existence of involution kernels of the form (10) for the potentials in $R(\Omega)$. We will give a sufficient and necessary condition for its existence and calculate when this condition is satisfied, the exact expression of the involution kernel.

Definition 3.2. We call W be the involution kernel of A based at the point $x' \in \Omega$, the function such that, for any ordered pair $(y|x) \in \hat{\Omega}$,

$$W(y|x) = \sum_{n=1}^{\infty} [A \circ \hat{\sigma}^{-n}(y|x) - A \circ \hat{\sigma}^{-n}(y|x')], \quad (12)$$

in this case, the sum converges.

See (16), (24), and (28) for affirmative cases with explicit expressions.

Definition 3.3. If W is an involution kernel for $A \in R(\Omega)$, based at x' , we define the dual potential $A^* : \Omega \rightarrow \mathbb{R}$ to be the potential given by

$$A^*(y) := A^*(y|x) = A(\tau_y(x)) + W \circ \hat{\sigma}^{-1}(y|x) - W(y|x), \quad (13)$$

(in the case the sum converges) which one can show does not depend on x (see (30) and (31)).

Conditions for symmetry for the Walters' family are given by (33).

One can show that when A is of Hölder class, then expressions (12) and (13) are well defined and they are, respectively, an involution kernel and a dual potential for A (see a proof for instance [23]).

4 Calculation of the Involution Kernel for the Walters' family

We calculate formally an exact expression for the involution kernel of a potential in $R(\Omega)$ via expression (12). The result gives a sufficient and necessary condition for the existence of the involution kernel on the base point x' . We analyze different choices of x' (each one can produce different versions of the involution kernel).

Let $A \in R(\Omega)$ be defined by the following convergent sequences of real numbers

$$(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}} \text{ and } (d_n)_{n \in \mathbb{N}}.$$

4.1 First Case: $x' = 0^\infty$

With this choice of base point, $x = 0^\infty \implies W(\cdot|x) \equiv 0$. For $y = 0^\infty$ and $x \in [0^k 1]$, with $k \in \mathbb{N}$,

$$A(y_n \dots y_1 x) - A(y_n \dots 1 x') = A(0^{k+n} 1 \dots) - A(0^\infty) = a_{k+n} - a$$

$$W(0^\infty|[0^k 1]) = \sum_{n \in \mathbb{N}} (a_{n+k} - a) . \quad (14)$$

If $y = 0^\infty$ and $x = 1^\infty$, then, since $A(01^\infty) - A(0^\infty)$ and $A(0^n 1^\infty) - A(0^\infty) - a_n - a$ for natural numbers $n > 1$,

$$W(0^\infty|1^\infty) = b - a + \sum_{n=2}^{\infty} (a_n - a) . \quad (15)$$

If $y = 0^\infty$ and $x \in [1^k 0]$, $k \in \mathbb{N}$, then

$$W(y|x) = b_k - a + \sum_{n=2}^{\infty} (a_n - a) , \quad (16)$$

since $A(01^k 0 \dots) - A(0^\infty) = b_k - a$ and $A(0^n 1^k 0 \dots) - A(0^\infty) = a_n - a$ if $n > 1$. We obtained the expressions for the involution kernel when the first coordinate is the point $y = 0^\infty$.

Consider now $y \in [0^l 1]$, with $l \in \mathbb{N}$. If $x \in [0^k 1]$. If $1 \leq n \leq l$,

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(0^{n+k} 1 \dots) - A(0^\infty) = a_{n+k} - a .$$

Now

$$A(y_{l+1} \dots y_1 x) - A(y_{l+1} \dots y_1 x') = A(10^{l+k} 1 \dots) - A(10^\infty) = d_{l+k} - d .$$

The next terms are null, so

$$W([0^l 1]|[0^k 1]) = \sum_{n=1}^l (a_{n+k} - a) + d_{l+k} - d . \quad (17)$$

Still for $y \in [0^l 1]$, let $x = 1^\infty$. In this case, $A(y_1 x) - A(y_1 x') = A(01^\infty) - A(0^\infty) = b - a$. If $1 < n \leq l$,

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(0^n 1^\infty) - A(0^\infty) = a_n - a .$$

$$A(y_{l+1} \dots y_1 x) - A(y_{l+1} \dots y_1 x') = A(10^l 1^\infty) - A(10^\infty) = d_l - d ,$$

and all the other terms are null. Therefore,

$$W([0^l 1]|1^\infty) = (b - a) + \sum_{n=2}^l (a_n - a) + d_l - d . \quad (18)$$

If $y \in [0^l 1]$ and $x \in [1^k 0]$, then

$$A(y_1 x) - A(y_1 x') = A(y_1 x') = A(01^k 0 \dots) - A(0^\infty) = b_k - a$$

For $1 < n \leq l$,

$$A(y_n \dots y_1 x) - A(y_n \dots y_1) = A(0^n 1^k 0 \dots) - A(0^\infty) = a_n - a .$$

$$A(y_{l+1} \dots y_1 x) - A(y_{l+1} \dots y_1 x') = A(10^l 1^k 0 \dots) - A(10^\infty) = d_l - d .$$

Since the other terms are zero, we conclude

$$W([0^l 1] | [1^k 0]) = b_k - a + \sum_{n=2}^l (a_n - a) + d_l - d . \quad (19)$$

The above equation closes the case $y \in [0]$. We proceed in a similar way to calculate it for $y \in [1]$.

If $y = 1^\infty$ and $x \in [0^k 1]$, with $k \in \mathbb{N}$, then

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(1^n 0^k 1 \dots) - A(1^n 0^\infty) = \begin{cases} d_k - d, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases} .$$

First, consider $y = x = 1^\infty$.

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(1^\infty) - A(1^n 0^\infty) = \begin{cases} c - d, & \text{if } n = 1 \\ c - c_n, & \text{if } n > 1 \end{cases} ,$$

which implies

$$W(1^\infty | 1^\infty) = c - d + \sum_{n=2}^{\infty} (c - c_n) \quad (20)$$

If $y = 1^\infty$ and $x \in [1^k 0]$, then

$$A(y_1 x) - A(y_1 x') = A(1^{1+k} 0 \dots) - A(10^\infty) = c_{k+1} - d ,$$

and

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(1^{n+k} 0 \dots) - A(1^n 0^\infty) = c_{n+k} - c_n ,$$

for $n > 1$. We have, therefore,

$$W(1^\infty | [1^k 0]) = c_{k+1} - d + \sum_{n=2}^{\infty} (c_{n+k} - c_n) . \quad (21)$$

Consider now the case $y \in [1^l 0]$ and $x \in [0^k 1]$, for $l, k \in \mathbb{N}$.

$$A(y_1 x) - A(y_1 x') = A(10^k 1 \dots) - A(10^\infty) = d_k - d .$$

If $1 < n \leq l$, then

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(1^n 0^k 1 \dots) - A(1^n 0^\infty) = c_n - c_n = 0 ,$$

so

$$W([1^l 0][0^k 1]) = d_k - d . \quad (22)$$

If $y \in [1^l 0]$ and $x = 1^\infty$,

$$A(y_1 x) - A(y_1 x') = A(1^\infty) - A(10^\infty) = c - d .$$

For $1 < n \leq l$, then

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(1^\infty) - A(1^n 0^\infty) = c - c_n .$$

$$A(y_{l+1} \dots y_1 x) - A(y_{l+1} \dots y_1 x') = A(01^\infty) - A(01^l 0^\infty) = b - b_l .$$

The other terms are zero, so

$$W([1^l 0][0^k 1]) = c - d + \sum_{n=2}^l (c - c_n) + b - b_l . \quad (23)$$

Let $y \in [1^l 0]$ and $x \in [1^k 0]$.

$$A(y_1 x) - A(y_1 x') = A(1^{k+1} 0 \dots) - A(10^\infty) = c_{k+1} - d .$$

For $1 < n \leq l$,

$$A(y_n \dots y_1 x) - A(y_n \dots y_1 x') = A(1^{n+k} 0 \dots) - A(1^n 0^\infty) = c_{n+k} - c_n .$$

$$A(y_{l+1} \dots y_1 x) - A(y_{l+1} \dots y_1 x') = A(01^{l+k} 0 \dots) - A(01^l 0^\infty) = b_{l+k} - b_l .$$

The other terms are zero, so

$$W([1^l 0][1^k 0]) = c_{k+1} - d + \sum_{n=2}^l (c_{n+k} - c_n) + b_{l+k} - b_l \quad (24)$$

Equations (14) - (24) give the involution kernel, and the convergence conditions are needed only in equations (14), (15), (16), (21). They are

$$\exists \sum_{n \in \mathbb{N}} |a_n - a| , \exists \sum_{n \in \mathbb{N}} |c_n - c| , \exists \sum_{n \in \mathbb{N}} |c_{n+k} - c_n| \forall k \in \mathbb{N} . \quad (25)$$

Observe, however, that the third convergence condition is implied by the second, since $|c_{n+k} - c_n| \leq |c_{n+k} - c| + |c - c_n|$.

All the calculations were made with the involution kernel based on $x' = 0^\infty$. Thus, it is important to investigate if the convergence hypothesis changes if the other point x' is fixed as a base point. It is sufficient to consider $x' = [0^\alpha 1]$ for some natural number α . The other cases are obtained from these two by permutation of symbols $a \leftrightarrow c$ and $b \leftrightarrow d$.

4.2 Second case: $x' \in [0^\alpha 1]$

We present below the calculations of the involution kernel based on a point $x' \in [0^\alpha 1]$.

$$\begin{aligned} W(10^l|0^k 1) &= \sum_{j=1}^l \left[A(0^{j+k} 1 \dots) - A(0^{j+\alpha} 1 \dots) \right] + A(10^{l+k} 1 \dots) - A(10^{j+\alpha} 1 \dots) \\ &= \sum_{j=1}^l (a_{j+k} - a_{j+\alpha}) + d_{l+k} - d_{l+\alpha} . \end{aligned}$$

$$\begin{aligned} W(10^l|1^k 0) &= \left[A(01^k 0 \dots) - A(0^{\alpha+1} 1 \dots) \right] + \sum_{j=2}^l \left[A(0^j 1^k 0) - A(0^{j+\alpha} 1 \dots) \right] + \\ &\quad + \left[A(10^l 1^k 0 \dots) - A(10^{l+\alpha} 1 \dots) \right] \\ &= b_k - a_{\alpha+1} + \sum_{j=2}^l (a_j - a_{j+\alpha}) + d_l - d_{l+\alpha} . \end{aligned}$$

$$\begin{aligned} W(01^l|0^k 1) &= A(10^k 1 \dots) - A(10^\alpha 1 \dots) + \sum_{j=2}^l \left[A(1^j 0^k 1 \dots) - A(1^j 0^\alpha 1 \dots) \right] \\ &= A(10^k 1 \dots) - A(10^\alpha 1 \dots) \\ &= d_k - d_\alpha . \end{aligned}$$

$$W(01^l|1^k 0) = A(1^{k+1} 01 \dots) - A(10^\alpha 1 \dots) + \sum_{j=2}^l \left[A(1^{k+j} 0 \dots) - A(1^j 0^\alpha 1 \dots) \right] + \quad (26)$$

$$+ A(01^{k+l} 0 \dots) - A(01^l 0^\alpha 1 \dots) \quad (27)$$

$$= c_{k+1} - d_\alpha + \sum_{j=2}^l (c_{k+j} - c_j) + (b_{k+l} - b_l). \quad (28)$$

All other remaining cases are obtained from the previous one by limits. The results are summarized in Table 2.

We denote the involution kernel based at $x \in [0^\alpha 1]$ by W_α . From all those equations, we can conclude two things:

- (1) there is no change in the convergence hypothesis needed to assert the existence of the involution kernel;

- (2) the sequence of functions $(W_\alpha)_{\alpha \in \mathbb{N}}$, assuming that they exist (equivalently, if one of them exists) converges uniformly to W , the involution kernel based at $x' = 0^\infty$.

5 The dual Potential

Let $A \in R(\Omega)$ be a potential that admits involution kernel.

In [1] (see also [23]) it was shown that

Proposition 5.1. *Given a continuous potential $A : \Omega \rightarrow \mathbb{R}$ and $x' \in \Omega$, if A^* is well defined when given by*

$$\begin{aligned} A^*(y) &= A^*(y_1, y_2, \dots) = \\ &A(y_1, x'_1, x'_2, \dots) + [A(y_2, y_1, x'_1, x'_2, \dots) - A(y_2, x'_1, x'_2, \dots)] + \dots \\ &[A(y_n, \dots, y_2, y_1, x'_1, x'_2, \dots) - A(y_n, \dots, y_2, x'_1, x'_2, \dots)] + \dots, \end{aligned} \quad (29)$$

then A^* is a dual potential for A (considering the involution kernel W given by (12)).

We are interested here in explicit expressions for the dual potential A^* (which will be presented in Section 5.1). Let W be the involution kernel of A based on the point $x' \in \Omega$. Then, for any ordered pair $(y|x) \in \hat{\Omega}$,

$$\begin{aligned} &A(\tau_y(x)) + W \circ \hat{\sigma}^{-1}(y|x) - W(y|x) = \\ &A(\tau_y(x)) + \sum_{n=1}^{\infty} [\hat{A} \circ \hat{\sigma}^{-n}(\sigma(y)|\tau_y(x)) - \\ &\hat{A} \circ \hat{\sigma}^{-n}(\sigma(y)|x')] - \sum_{n=1}^{\infty} [\hat{A} \circ \hat{\sigma}^{-n}(y|x) - \hat{A} \circ \hat{\sigma}^{-n}(y|x')] = \\ &A(\tau_y(x)) + \lim_{N \rightarrow \infty} \sum_{n=1}^N [\hat{A} \circ \hat{\sigma}^{-(n+1)}(y|x) - \hat{A} \circ \hat{\sigma}^{-n}(\sigma(y)|x') - \\ &\hat{A} \circ \hat{\sigma}^{-n}(y|x) + \hat{A} \circ \hat{\sigma}^{-n}(y|x')] , \end{aligned}$$

since $\sum \alpha_n + \sum \beta_n = \sum(\alpha_n + \beta_n)$ if the series are convergent. But, for any natural number N ,

$$\begin{aligned} &A(\tau_y(x)) + \sum_{n=1}^N [\hat{A} \circ \hat{\sigma}^{-(n+1)}(y|x) - \hat{A} \circ \hat{\sigma}^{-n}(\sigma(y)|x') - \hat{A} \circ \hat{\sigma}^{-n}(y|x) + \hat{A} \circ \hat{\sigma}^{-n}(y|x')] = \\ &A(\tau_{y, N+1}(x)) + \sum_{n=1}^N [\hat{A} \circ \hat{\sigma}^{-n}(y|x') - \hat{A} \circ \hat{\sigma}^{-n}(\sigma(y)|x')] . \end{aligned}$$

Therefore, if there is an involution kernel based on x' , the limit

$$\lim_{N \rightarrow \infty} A(\tau_{y, N+1}(x)) + \sum_{n=1}^N \left[\hat{A} \circ \hat{\sigma}^{-n}(y|x') - \hat{A} \circ \hat{\sigma}^{-n}(\sigma(y)|x') \right] \quad (30)$$

must exist and, by continuity of A , it does not depend on x . This shows that the function $\hat{A}^* : \hat{\Omega} \rightarrow \mathbb{R}$, given by

$$A^*(y) = \hat{A}^*(y|x) := A(\tau_y(x)) + W \circ \hat{\sigma}^{-1}(y|x) - W(y|x), \quad (31)$$

is independent of x and is a dual potential for A .

5.1 Calculation of the Dual Potential for the Walters' family

Let W be an involution kernel for $A \in R(\Omega)$ and A^* be the dual potential of A associated with W .

The tables (1), (2) and (3), to follow, show our results in the calculation of dual potentials and helps our future analysis of the conditions for symmetry (see for instance Theorem 5.3).

Definition 5.2. The potential A is **symmetrized** by W if, and only if, $A^* = A \circ \theta$. If W is the involution kernel based at x' and A is symmetrized by W , we say that A is symmetric relatively to x' .

x	A	$A^* \circ \theta^{-1}$
0^∞	a	a
$0^{n+1}1z$	a_{n+1}	a
01^n0z	b_n	a
01^∞	b	a
1^∞	c	c
$1^{n+1}0z$	c_{n+1}	$c_{n+1} + b_{n+1} - b_n$
10^n1z	d_n	$(b_1 - a) + \sum_{j=2}^n (a_j - a) + d_n$
10^∞	d	$d + (b_1 - a) + \sum_{j=2}^\infty (a_j - a)$

Table 1: Dual Potential associated to the Involution Kernel-based at $x' = 0^\infty$

- Conditions for symmetry: The condition for symmetry is that the second column equals the third. The first two lines show that $b = b_n = a_{n+1} = a$ for all natural numbers n . If these conditions hold, the other equations are trivially satisfied.

As should be expected, the first table is obtained from the second, making $\alpha \rightarrow \infty$. The first result of the above calculations is that the dual of Walter's potential is also Walter's potential.

x	A	$A^* \circ \theta^{-1}$
0^∞	a	a
$0^{n+1}1z$	a_{n+1}	$a_{\alpha+n+1} + (d_{\alpha+n+1} - d_{\alpha+n})$
01^n0z	b_n	$a_{\alpha+1} + (d_{\alpha+1} - d_\alpha)$
01^∞	b	$a_{\alpha+1} + (d_{\alpha+1} - d_\alpha)$
1^∞	c	c
$1^{n+1}0z$	c_{n+1}	$c_{n+1} + b_{n+1} - b_n$
10^n1z	d_n	$d_\alpha + (b_1 - a_{\alpha+1}) + \sum_{j=2}^n (a_j - a_{\alpha+j}) + (d_n - d_{\alpha+n})$
10^∞	d	$d_\alpha + (b_1 - a_{\alpha+1}) + \sum_{j=2}^\infty (a_j - a_{\alpha+j})$

Table 2: Dual Potential associated to the Involution Kernel-based at $x' = 0^\alpha 1w$

• Conditions for symmetry: The condition for symmetry is that the second column equals the third. Then

$$b_n = a_{\alpha+1} + (d_{\alpha+1} - d_\alpha) = b ,$$

that is, $b_n = b$ for all $n \in \mathbb{N}$, is a necessary condition.

$$a_{n+1} = a_{\alpha+n+1} + d_{\alpha+n+1} - d_{\alpha+n} \iff a_{\alpha+n+1} - a_{n+1} = d_{\alpha+n} - d_{\alpha+n+1} , \forall n \in \mathbb{N} .$$

Also, for all $n \in \mathbb{N}$

$$d_\alpha + (b_1 - a_{\alpha+1}) + \sum_{j=2}^n (a_j - a_{\alpha+j}) = d_{\alpha+n} ,$$

and

$$d = d_\alpha + b - a_{\alpha+1} + \sum_{j=2}^\infty (a_j - a_{\alpha+j}) . \quad (32)$$

Then the conditions for symmetry, in this case, is the validity of the system of equation

$$\begin{cases} a_{\alpha+n+1} - a_{n+1} = d_{\alpha+n} - d_{\alpha+n+1} \\ b_n = a_{\alpha+1} + (d_{\alpha+1} - d_\alpha) = b \\ d_{\alpha+n} = d_\alpha + (b_1 - a_{\alpha+1}) + \sum_{j=2}^n (a_j - a_{\alpha+j}) \\ d = d_\alpha + b - a_{\alpha+1} + \sum_{j=2}^\infty (a_j - a_{\alpha+j}) \end{cases} , \quad (33)$$

$\forall n \in \mathbb{N}$.

x	A	$A^* \circ \theta^{-1}$
0^∞	a	a
$0^{n+1}1z$	a_{n+1}	$a_{n+1} + (d_{n+1} - d_n)$
01^n0z	b_n	$b_\alpha + (d_1 - c_{\alpha+1}) + \sum_{j=2}^n (c_j - c_{j+\alpha}) + (b_n - b_{n+\alpha})$
01^∞	b	$b_\alpha + (d_1 - c) + \sum_{j=2}^\infty (c_j - c)$
1^∞	c	c
$1^{n+1}0z$	c_{n+1}	$c_{\alpha+n+1} + (b_{\alpha+n+1} - b_{\alpha+n})$
10^n1z	d_n	$c_{\alpha+1} + (b_{\alpha+1} - b_\alpha)$
10^∞	d	$c_{\alpha+1} + (b_{\alpha+1} - b_\alpha)$

Table 3: Dual Potential associated to the Involution Kernel-based at $x' = 1^\alpha 0w$

x	A	$A^* \circ \theta^{-1}$
0^∞	a	a
$0^{n+1}1z$	a_{n+1}	$a_{n+1} + (d_{n+1} - d_n)$
01^n0z	b_n	$b + (d_1 - c) + \sum_{j=2}^n (c_j - c) + (b_n - b)$
01^∞	b	$b + (d_1 - c) + \sum_{j=2}^\infty (c_j - c)$
1^∞	c	c
$1^{n+1}0z$	c_{n+1}	c
10^n1z	d_k	c
10^∞	d	c

Table 4: Dual Potential associated to the Involution Kernel-based at $x' = 1^\infty$

$$\begin{cases} c_{\alpha+n+1} - c_{n+1} = b_{\alpha+n} - b_{\alpha+n+1} \\ d_n = c_{\alpha+1} + (b_{\alpha+1} - b_\alpha) = d \\ b_{\alpha+n} = b_\alpha + (d_1 - c_{\alpha+1}) + \sum_{j=2}^n (c_j - c_{\alpha+j}) \\ b = b_\alpha + d - c_{\alpha+1} + \sum_{j=2}^\infty (c_j - c_{\alpha+j}) \end{cases}, \quad (34)$$

$\forall n \in \mathbb{N}$.

In analogy with the first case, the conditions for symmetry here are $c_{n+1} = d_n = c \forall n \in \mathbb{N}$.

The following Theorem summarizes our conclusions:

Theorem 5.3. *Suppose that $A \in R(\Omega)$ admits involution kernel. If $b_n = a_{n+1} = a$ for all $n \in \mathbb{N}$, then A is symmetric relatively to the involution*

kernel based at $x' = 0^\infty$. Let $\alpha \in \mathbb{N}$. If

$$\begin{cases} a_{\alpha+n+1} - a_{n+1} = d_{\alpha+n} - d_{\alpha+n+1} \\ b_n = a_{\alpha+1} + (d_{\alpha+1} - d_\alpha) = b \\ d_{\alpha+n} = d_\alpha + (b_1 - a_{\alpha+1}) + \sum_{j=2}^n (a_j - a_{\alpha+j}) \\ d = d_\alpha + b - a_{\alpha+1} + \sum_{j=2}^\infty (a_j - a_{\alpha+j}) \end{cases}$$

holds $\forall n \in \mathbb{N}$, then A is symmetric relatively to $x' = 0^\alpha 1z$. If $d_n = c_{n+1} = c$ for all natural numbers n then A is symmetric relatively to the point $x' = 1^\infty$. If

$$\begin{cases} c_{\alpha+n+1} - c_{n+1} = b_{\alpha+n} - b_{\alpha+n+1} \\ d_n = c_{\alpha+1} + (b_{\alpha+1} - b_\alpha) = d \\ b_{\alpha+n} = b_\alpha + (d_1 - c_{\alpha+1}) + \sum_{j=2}^n (c_j - c_{\alpha+j}) \\ b = b_\alpha + d - c_{\alpha+1} + \sum_{j=2}^\infty (c_j - c_{\alpha+j}) \end{cases}$$

holds for all $n \in \mathbb{N}$ then A is symmetric relatively to the point $x' = 1^\alpha 0z$.

The above Theorem allows us to show that there exist potentials in $R(\Omega)$ for which there is no involution kernel, **accordingly to Definition 1.1**, which makes it symmetric. In fact, take four distinct real number a, b, c and d , and define A with the constant sequences $a_n = a$, $b_n = b$, $c_n = c$ and $d_n = d$. By Theorem 2.3, A has an involution kernel based at any point $x' \in \Omega$. For any $n, \alpha \in \mathbb{N}$,

$$\begin{cases} b_n = b \neq a = a_n \implies A \text{ is not symmetric relatively to } 0^\infty \\ b_n \neq a_{\alpha+1} = a_{\alpha+1} + (d_{\alpha+1} - d_\alpha) \implies A \text{ is not symmetric relatively to } 0^\alpha 1z \\ d_n = d \neq c = c_{n+1} \implies A \text{ is not symmetric relatively to } 1^\infty \\ d_n \neq c_{\alpha+1} = c_{\alpha+1} + (b_{\alpha+1} - b_\alpha) \implies A \text{ is not symmetric relatively to } 1^\alpha 0z \end{cases}$$

so A cannot be symmetric relative to some point.

Remark 5.4. The results in this paper are restricted to a specific class of involution kernels; those which come from a limit as in (10). Thus, our results have some restrictions, which we illustrate with an example. It comes from a simple observation: If A_1 and A_2 are potentials, W_1 is an involution kernel for A_1 and W_2 is an involution kernel for A_2 , then $W_1 + W_2$ is an involution kernel for $A_1 + A_2$. Also, if $k \in \mathbb{R} \setminus \{0\}$, kW_1 is an involution kernel for kA_1 .

Suppose that $A : \Omega \rightarrow \mathbb{R}$ is a potential in $R(\Omega)$ depending only on the two first coordinates. Given a point $x' = (x'_n)_{n \in \mathbb{N}} \in \Omega$,

$$\sum_{n \in \mathbb{N}} [\hat{A} \circ \hat{\sigma}^{-n}(y|x) - \hat{A} \circ \hat{\sigma}^{-n}(y|x')] =$$

$$A(y_1x_1\dots) - A(y_1x'_1\dots) + A(y_2y_1x_1\dots) - A(y_2y_1x'_1\dots) = A(y_1x_1\dots) - A(y_1x'_1),$$

and this implies that

$$\hat{A}^*(y|x) = A(y_2y_1x) + [A(y_1x') - A(y_2x')].$$

Suppose now that A is a indicator function $\chi_{[a_0a_1]}$ of a cylinder $[a_0a_1]$. The above expression shows that, taking $x' \notin [a_1]$,

$$\hat{\chi}_{[a_0a_1]}^*(y|x) = \chi_{[a_0a_1]}(y_2, y_1x) = \chi_{(a_1a_0)}(y),$$

i.e., it is possible to choose x' in such a way that $\chi_{[a_0a_1]}^* = \chi_{(a_1a_0)}$.

Given a 2×2 line stochastic matrix $\mathcal{P} = [p_{ij}]$, the associated stationary Markovian measure over Ω can be obtained via Thermodynamic Formalism by considering the potential $A = p_{11}\chi_{[00]} + p_{12}\chi_{[01]} + p_{21}\chi_{[10]} + p_{22}\chi_{[11]}$. The potential A thus defined is in $R(\Omega)$. For each $\chi_{[a_0a_1]}$ we can choose $x'(a_0a_1) \in \Omega$ such that $\chi_{[a_0a_1]}^*$, the dual with respect to the involution kernel based at $x'(a_0, a_1)$, is $\chi_{(a_1a_0)}$. Considering the observation made at the beginning of the remark, we conclude that

$$B := p_{11}\chi_{(00)} + p_{12}\chi_{(10)} + p_{21}\chi_{(01)} + p_{22}\chi_{(11)}$$

is a dual potential for A (see Section 5 in [5]). If \mathcal{P} is row-stochastic, we have an associate Markov process in Ω^- defined by the column-stochastic matrix \mathcal{P}^T .

Besides B is a dual potential for A , it may not be obtained by an involution kernel of the form (10): for each indicator function $\chi_{[a_0a_1]}$ we choose a $x'(a_0a_1)$ to obtain the kernel, and B is obtained through the linear combination of these kernels. The sum of two kernels of type (10) may not be of the same type unless the respective base points are the same. Thus, our results do not apply to B as it does not apply, in general, to dual potential obtained by linear combinations of involution kernels of type (10).

The important result which is announced in Corollary 7.6 has also the same restriction

6 Twist Condition

In the works [28] and [29], the involution kernel appears related to an Ergodic Transport Problem. The involution kernel can be seen as a natural cost function on $\hat{\Omega}$. In the symbolic setting, the Twist Condition plays the role of convexity.

Definition 6.1. Considering the lexicographic order $<$ in Ω and Ω^- , the involution kernel W is said to satisfy the Twist Condition if and only if $y < y'$ and $x < x'$ implies

$$W(y|x) + W(y'|x') < W(y|x') + W(y'|x).$$

For the involution kernels in this paper, the twist condition cannot be satisfied, as we show now. Let $a \in [1^s 0]$, $a' \in [0^{s+p} 1]$, $b \in [0^{k+q} 1]$ and $b' \in [0^k 1]$. Then, if the involution kernel based at $x' = 0^\alpha 1 z$ satisfies the twist condition,

$$\begin{aligned} W(\dots 01^s | 0^{k+q} 1 \dots) + W(\dots 01^{s+p} | 0^k 1 \dots) < \\ W(\dots 01^s | 0^k 1 \dots) + W(\dots 01^{s+p} | 0^{k+q} 1 \dots) \end{aligned}$$

This is equivalent to

$$(d_{k+q} - d_\alpha) + (d_k - d_\alpha) < (d_k - d_\alpha) + (d_{k+q} - d_\alpha) . \quad (35)$$

If we change the base point of the involution kernel to be $x' = 0^\infty$, the above inequality changes only by the substitution $d_\alpha \rightarrow d$. To consider basis points in the cylinder [1] would lead to a similar contradiction: considering $b \in [1^s 0]$, $b' \in [1^{s+p} 0]$, $a \in [0^{k+q} 1]$ and $a' \in [0^k 1]$ and the involution kernel based at $x' = 1^\alpha 0 z$, it holds

$$\begin{aligned} W(\dots 10^{k+q} | 1^s 0 \dots) + W(\dots 10^k | 1^{s+p} 0 \dots) = \\ b_{k+q} - b_\alpha = W(\dots 10^{k+q} | 0^{s+p} 1 \dots) + W(\dots 10^k | 1^s 0 \dots). \end{aligned}$$

Therefore, the twist condition cannot be satisfied.

Definition 6.2. A cost function $c : \hat{\Omega} \rightarrow \mathbb{R}$ is said to satisfy a relaxed twist condition if, for any pair $(a, b), (a', b') \in \hat{\Omega}$ with $a < a'$ and $b < b'$,

$$c(a, b) + c(a', b') \leq c(a, b') + c(a', b) .$$

We can define a class of Walters Potentials for which the involution kernel satisfies the relaxed twist condition. Indeed, let W be the involution kernel based at the point $0^\alpha 1 z$ of a Walters potential A defined by the sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$.

Define, for each $n \in \mathbb{N}$, the sequences $(\Delta_i^n)_{i \in \mathbb{N}}$ and $(\Gamma_i^n)_{i \in \mathbb{N}}$ via $\Delta_i^n := d_{n+i} - d_i$, $\Gamma_i^m := b_{n+i} - b_i$.

Theorem 6.3. *If $(a_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$ and, for each $m \in \mathbb{N}$, $(\Delta_n^m)_{n \in \mathbb{N}}$ and $(\Gamma_n^m)_{n \in \mathbb{N}}$ are decreasing, and if $(d_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are sub-additive, the involution kernel based at x satisfies the relaxed twist condition, for any $x \in \Omega$.*

7 Normalized Potentials on $G(\Omega)$

Suppose A is normalized. If A^* is the dual potential associated to the point $0^\alpha 1z$, then for it to be normalized, accordingly to Table 2, the following equations must be satisfied:

$$\exp \left[d_\alpha + b_1 - a_{\alpha+1} + \sum_{j=2}^n (a_j - a_{\alpha+j}) + d_n - d_{n+\alpha} \right] =$$

$$1 - \exp (a_{\alpha+n+1} + d_{\alpha+n+1} - d_{\alpha+n}) , \quad (36)$$

$$\exp (a_{\alpha+1} + d_{\alpha+1} - d_\alpha) = 1 - \exp (c_{n+1} + b_{n+1} - b_n) . \quad (37)$$

The normalization condition of A reduces its degree of freedom; the sequences $(b_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$ becomes entirely defined by the sequences $(a_{n+1})_{n \in \mathbb{N}}$ and $(c_{n+1})_{n \in \mathbb{N}}$. We show that normalization of A^* defines $(c_{n+1})_{n \in \mathbb{N}}$ in terms of $(a_{n+1})_{n \in \mathbb{N}}$.

Proposition 7.1. *If A and A^* are normalized, then*

$$e^{c_{n+2}} = 1 - \left(1 - e^{a_{\alpha+1} + d_{\alpha+1} - d_\alpha} \right) \cdot (e^{-c_{n+1}} - 1) .$$

Proposition 7.2. *If A and A^* are normalized, then*

$$1 - e^{c^2} = \frac{1 - e^{a_{\alpha+n+1} + a_{\alpha+n+2}}}{(e^{-a_{\alpha+1}} - 1)(1 - e^{a_{n+1}}) \exp \left[\sum_{j=2}^n (a_j - a_{\alpha+j}) \right]} , \quad \forall n \in \mathbb{N} .$$

Corollary 7.3. *If A and A^* are normalized, then the sequence*

$$\frac{1 - e^{a_{\alpha+n+1} + a_{\alpha+n+2}}}{(e^{-a_{\alpha+1}} - 1)(1 - e^{a_{n+1}}) \exp \left[\sum_{j=2}^n (a_j - a_{\alpha+j}) \right]}$$

is constant.

Then we can reach the following conclusion: there are normalized potentials $A \in R(\Omega)$ for which there is no dual normalized potential. In fact, A is normalized if and only if the system

$$\begin{cases} d_n = \log(1 - e^{a_{n+1}}) \\ b_n = \log(1 - e^{c_{n+1}}) \end{cases} \quad \forall n \in \mathbb{N} , \quad (38)$$

holds and there exists $r \in (0, 1)$ such that $e^{a_{n+1}} \in (c, 1)$ and $e^{c_{n+1}} \in (0, 1 - c)$ for all $n \in \mathbb{N}$

For any choice of sequences $(a_{n+1})_{n \in \mathbb{N}}$ and $(c_{n+1})_{n \in \mathbb{N}}$ satisfying these respective constraints, with

$$\sum_{n \in \mathbb{N}} (a - a_n) \text{ and } \sum_{n \in \mathbb{N}} (c - c_n)$$

convergent, defining b_n and d_n by the above equations, we get a normalized potential, which may not satisfy the conclusion of Proposition 7.1.

Example 7.4. Let $a_n = d_n = \log 1/2$ for all $n \in \mathbb{N}$. Let $c_n = \log 2/3$ and $b_n = \log 1/3$, for all $n \in \mathbb{N}$. Since $1/2 > 1/3$, $1/3 < 2/3$, and the equations (38) are satisfied, the potential derived from these sequences is normalized. We show that Proposition 7.1 does not hold.

$$\begin{aligned} 1 - \left(1 - e^{a_{\alpha+1} + d_{\alpha+1} - d_{\alpha}}\right) \cdot (e^{-c_{n+1}} - 1) &= 1 - \left(1 - \frac{1}{2}\right) \left(\frac{3}{2} - 1\right) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \\ &\neq \frac{2}{3} \\ &= e^c. \end{aligned}$$

On the other hand, if A satisfies Ruelle's Operator Theorem, its dual also satisfies it.

Let (λ, h) be a positive eigenpair for the Ruelle Operator associated with A , and let (λ^*, h^*) be a positive eigenpair for the Ruelle Operator associated with A^* . Moreover, take λ and λ^* as the greatest eigenvalue of \mathcal{L}_A and \mathcal{L}_{A^*} , respectively. We know that $\lambda = \lambda^*$.

Given a Holder potential A and its equilibrium probability μ_A , there exists a unique positive function $J : \Omega \rightarrow (0, 1)$, such that,

$$\mathcal{L}_{\log J}^*(\mu_A) = \mu_A.$$

For proof see [36]. We call J the Jacobian of the probability μ_A .

We study the relation between h and h^* and the Jacobians of the equilibrium states of the potentials A and A^* . Fixing $A \in R(\Omega)$ with involution kernel W and denoting A^* the corresponding dual relatively to the involution kernel based at the point $0^\delta 1w$, we prove

Theorem 7.5. *Let μ and μ^* be the equilibrium states of A and A^* , respectively. Then the Jacobians of J of μ and J^* of μ^* satisfies $J = J^* \circ \theta^{-1}$.*

Corollary 7.6. *Let θ_*^{-1} be the pull-back of μ^* by θ^{-1} . In the above conditions, $\mu = \theta_*^{-1}\mu^*$.*

Proof. Since we are supposing that the series $\sum(a_n - a)$ and $\sum(c_n - c)$ converges, A and A^* has unique equilibrium states; μ and μ^* , respectively. Let \bar{A} and \bar{A}^* be the normalized potentials associated to A and A^* , respectively. As described in [43], for any continuous functions $f : \Omega \rightarrow \mathbb{R}$,

$$\mathcal{L}_{\bar{A}}^n f \rightarrow \int f d\mu, \text{ and } \mathcal{L}_{\bar{A}^*}^n f \rightarrow \int f \circ \theta^{-1} d\mu^* = \int f d(\theta_*^{-1}\mu^*),$$

in the uniform topology (the right side of the arrows denotes, with language abuse, constant functions which assume those values). Since A and A^* has the same Jacobians, $\mathcal{L}_{\bar{A}} = \mathcal{L}_{\bar{A}^*} \circ \theta^{-1}$, and thus $\int f d\mu = \int f d(\theta_*^{-1}\mu^*)$ for any continuous function $f : \Omega \rightarrow \mathbb{R}$. From Riesz - Markov Representation Theorem, it follows that $\mu = \theta_*^{-1}\mu^*$. \square

Remark 7.7. Corollary 7.6 can be generalized: in the class $R(\Omega)$ if the equilibrium states of two potentials have the same Jacobians, then the equilibrium states are equal.

8 A generalization

Let $\mathcal{M}(\sigma)$ and $\mathcal{M}(\hat{\sigma})$ be the set of Borel invariant probability measures for σ and $\hat{\sigma}$, respectively. We show the existence of a bijection between these sets, which preserves entropy. Given $\mu \in \mathcal{M}(\sigma)$, define $\hat{\mu}$, the **extension of μ to $\hat{\Omega}$** , as being the measure in $\mathcal{M}(\hat{\sigma})$ satisfying

$$\hat{\mu}(\langle a_n \dots a_1 | b_1 \dots b_m \rangle) = \mu([a_n \dots a_1 b_1 \dots b_m]),$$

for any $(a_n, \dots, a_1, b_1, \dots, b_m) \in \bigcup_{k \in \mathbb{N}} \mathcal{A}^{(k)}$. Now, there is a natural inclusion i of Borel σ -algebra $\mathcal{B}(\Omega)$ into the Borel σ -algebra $\mathcal{B}(\hat{\Omega})$. It satisfies $i([a_1 \dots a_n]) = |a_1 \dots a_n \rangle$ for all words in the alphabet \mathcal{A} . Given $\hat{\mu} \in \mathcal{M}(\hat{\sigma})$, let μ be the pull-back by the map i of the restriction of $\hat{\mu}$ to $i(\mathcal{B}(\Omega))$. The map $\hat{\mu} \mapsto \mu$ thus defined is the inverse of the map $\mu \mapsto \hat{\mu}$ defined previously.

The natural extension of a σ -invariant probability on Ω to a $\hat{\sigma}$ -invariant probability on $\hat{\Omega}$ is unique. Results related to this section appear in Section 7 in [25].

Proposition 8.1. *For all $\hat{\mu} \in \mathcal{M}(\hat{\sigma})$, $h_{\hat{\mu}}(\hat{\sigma}) = h_{\mu}(\sigma)$.*

Proof. Let $\hat{\mathcal{P}} = \{|a \rangle; a \in \mathcal{A}\}$ and $\mathcal{P} = \{[a]; a \in \mathcal{A}\}$. $\hat{\mathcal{P}}$ is a bilateral generating partition for $(\hat{\Omega}, \hat{\sigma})$ (i.e., $\bigcup_{n \in \mathbb{N}} \hat{\mathcal{P}}^{\pm n} = \bigcup_{n \in \mathbb{N}} \bigvee_{j=-n}^n \hat{\sigma}^j(\hat{\mathcal{P}})$ generates the Borel σ -algebra of $\hat{\Omega}$), while \mathcal{P} is a unilateral generating partition for

(Ω, σ) ($\bigcup_{n \in \mathbb{N}} \mathcal{P}^n = \bigcup_{n \in \mathbb{N}} \bigvee_{j=0}^{n-1} \sigma^{-j}(\mathcal{P})$ generates de Borel σ -algebra of Ω).
By Kolmogorov-Sinai's Theorem,

$$h_{\hat{\mu}}(\hat{\sigma}) = h_{\hat{\mu}}(\hat{\sigma}, \hat{\mathcal{P}}) \text{ and } h_{\mu}(\sigma) = h_{\mu}(\sigma, \mathcal{P}) .$$

Invariance of the measures under the respective dynamics, and the relation between μ and $\hat{\mu}$, implies $h_{\hat{\mu}}(\hat{\sigma}, \hat{\mathcal{P}}) = h_{\mu}(\sigma, \mathcal{P})$. \square

If $P : \mathcal{C}(\Omega) \rightarrow \mathbb{R}$ and $\hat{P} : \mathcal{C}(\hat{\Omega}) \rightarrow \mathbb{R}$ are the topological pressures, Proposition 8.1 implies that $\hat{P}(\hat{A}) = P(A)$ for all $A \in \mathcal{C}(\Omega)$. Thus, the restriction of any equilibrium state of \hat{A} to Ω is an equilibrium state for A . The map $\hat{\mu} \mapsto \mu$ is a bijection. So, if A has a unique equilibrium state, \hat{A} also has a unique equilibrium state.

Analogous results hold for the pair \hat{A}^* and A^* . But, since

$$\int \hat{A}^* d\hat{\mu} = \int \left[\hat{A} \circ \hat{\sigma}^{-1} + W \circ \hat{\sigma}^{-1} - W \right] d\hat{\mu} = \int \hat{A} d\hat{\mu} \quad \forall \hat{\mu} \in \mathcal{M}(\hat{\sigma}) ,$$

these two potentials have the same equilibrium states. Therefore, the uniqueness of the equilibrium state for A implies the uniqueness of the equilibrium state for \hat{A} , which implies the uniqueness of the equilibrium state for \hat{A}^* , which implies uniqueness of equilibrium state for A^* . Also,

$$\hat{\mu}_{A^*} = \hat{\mu}_A .$$

Corollary 8.2. *If $\mu_A([a_1 \dots a_n]) = \mu_{A^*}([a_n \dots a_1])$ for all $(a_1, \dots, a_n) \in \bigcup_{m \in \mathbb{N}} \mathcal{A}^m$, then $\theta_*^{-1} \mu_{A^*} = \mu_A$.*

Corollary 2.3 in [43] claims that equilibrium probabilities for normalized potentials on the Walters' family satisfy the hypothesis of the above corollary. It follows from [21] that the entropy production, in this case, is 0.

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