Weight-balanced measures and free energy for one-dimensional dynamics

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Abstract. In this paper we consider Thermodynamic Formalism properties of one dimensional maps. We consider the existence of weight-balanced measures and large deviation properties of the Free-Energy of the Jacobian of measures. We show that a weight-balanced measure exists under the hypotheses that the map is piecewise-homeomorphc and the weights piecewise constant.

We consider also a certain class of measures with the property that the Free-Energy of the Jacobian is differentiable by parts. For measures in this class we show that a certain measure is a maximized entropy measure if and only if the Free-Energy of the Jacobian is linear. The result holds from general properties of Large-Deviation Theory and does not use the more classical approach of Thermodynamic Formalism.

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0. Introduction

In this article we consider a dynamical system consisting of a continuous function $f$ on an interval $[a, b]$ with the following properties:

1. $f([a, b]) = [a, b]$.

2. There exist points $a = c_0 < c_1 < \ldots < c_k = b$, $d \in \mathbb{N}$, such that $f$ is homeomorphic on $[c_i, c_{i+1}]$, $i \in \sigma = \{0, 1, \ldots, d - 1\}$, but $f$ is not one-to-one in any neighborhood of $c_i$, $i \in \sigma - \{0\}$. For example, if $f(x) = \lambda x (1 - x)$, $\sqrt{\lambda} + 1 < \lambda \leq 4$, $f$ satisfies these hypotheses if we take $d = 2, c_1 = 1/2, b = c_2 = f(c_1)$, and $a = c_0 = f(c_2)$.

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We say that a map $f$ is expanding if $f$ is continuously differentiable and there exist $a > 0, 2 > 1$ such that \((f^n)'(x) > a^n\) for all $n \in \mathbb{N}$ and all $x$ in the nonwandering set (see Mañé [15] for definitions). The nonwandering set is a Cantor set in this case.

In some aspects, a real quadratic map has more obscure dynamics than a complex quadratic map. A major difficulty is that not all points have the same number of preimages under a real quadratic map. While the entropy of a real quadratic map is given by the exponential rate of growth of the number of preimages of the critical point (see Misiurewicz [16] and Misiurewicz and Slenk [17]), finding the measure if maximal entropy is more difficult in the real case. Moreover, Hofbauer [8] showed that while a real quadratic map has a unique measure of maximal entropy, a real cubic map can have more than one measure of maximal entropy.

This contrast with the case of complex rational maps of degree $d > 1$, for which the entropy is always given by $\log d$ and the measure of maximal entropy always exists uniquely.

The maximal entropy measure is the weak limit as $n$ goes to infinity of the measure giving mass $d^{-n}$ to each of the $d^n$ points $g^{-n}(x_0)$. Here $x_0$ can be any fixed point of $g$ with at most two exceptions. The Jacobian of this measure is simply $1/d$. Indeed, the Jacobian of a maximal entropy measure for a real polynomial map is much more difficult.

If we try to proceed in an analogous way for a real quadratic polynomial, assigning equal masses to the preimages of the critical point, we do not even obtain an invariant measure in the limit. Again, this is because various points may have one, two, or even infinitely many preimages under $f$. This problem arises from the problem of assigning masses to the preimages of a point in order to obtain a maximal-entropy measure in the limit. The procedure defined a measure by assigning mass $n$ to preimages of a point is useful in determining a functional equation satisfied by the moment-generating function of the measure. The solution to this problem is still unknown, even in the case where $f$ is expanding.

An invariant measure created by assigning different masses to the preimages of a given point is essentially a weight-balanced measure. Barnsley and several other authors [2], [3], [6], [12] consider weight-balanced measures that are essentially measures obtained from balanced probabilities (see [5] for references). The balanced probabilities are related to the Jacobian of such measures ([12]). The definition of the Jacobian of a measure is also given in Section 2.

An important problem in the theory of weight-balanced measures is to find a measure with a predefined Jacobian. This type of measure is also sometimes called a $g$-measure. Several interesting results in this direction have been obtained by [2], [5], [6]. Barnsley, Elton, Dzomko, and Geronimo [2] is a general reference about the subject. Most of these results assume the map has a fixed number of preimages. In this article we study properties of weight-balanced measures for (not necessarily polynomial) maps satisfying hypotheses (i) and (ii) given at the beginning of the section. This study differs from previous work on weight-balanced measures in allowing the number of preimages to vary from point to point.

In Section 1 we consider the existence of weight-balanced measures for such maps.
1. The existence of a weight-balanced measure

We consider continuous functions \( f \) on an interval \([a, b]\) with the following properties:

(i) \( f([a, b]) = [a, b] \).

(ii) There exist points \( a = c_0 < c_1 < \ldots < c_{d-1} = b \) such that \( f \) is homeomorphic on \([c_{i-1}, c_i] \) for \( i = 1, \ldots, d-1 \), but \( f \) is not homeomorphic on any neighborhood of \( c_i \).

We define \( C \) to be the set \( \{c_1, \ldots, c_{d-1}\} \). We set

\[ E = f(C) \cap [a, b] \]

\[ F = \bigcup_{i=0}^{d-1} f^i(E). \]

The set \( E \) has no more than \( d-1 \) elements.

For each \( i \in \sigma \), if \( f \) is a homeomorphism when restricted to \( I_i = [c_i, c_{i+1}] \), we denote the inverse of the restriction of \( f \) to \( I_i \) by \( \beta_i : f(I_i) \to I_i \).

(iii) We also consider functions \( p_i : [a, b] \to [0, 1] \), \( i \in \sigma \) such that \( p_i(x) > 0 \) if and only if \( x \in f(I_i) \), \( p_i \) is piecewise constant with discontinuities only at points of \( E \), and

\[ \sum_{i \in \sigma} p_i(x) = 1. \]

We set \( p = (p_0, \ldots, p_{d-1}) \).

We define a Borel measure \( \mu \) on \([a, b]\) to be \( p \)-balanced if

\[ \mu(f(X)) = \frac{1}{p} \int_X p(s) d\mu(s) \]

for all \( i \in \sigma \) and Borel subsets \( X \) of \([a, b]\). Equivalently, \( \mu \) is \( p \)-balanced if

\[ \int_X d\mu = T_i \]

for all continuous \( \phi : [a, b] \to \mathbb{R} \), where we define the operator \( T \) by

\[ T \phi(x) = \sum_{i \in \sigma} p_i(x) \phi(\beta_i(x)). \]

Since \( p_i(x) = 0 \) unless \( x \) is in the domain of \( \beta_i \), \( T \) is well defined.

We say \( \mu \) is invariant under \( f \) if \( \mu \) is \( \mu \)-invariant for any Borel subset \( A \) of \([a, b] \). Note that weight-balanced measures are always invariant and it is also true that

\[ \int f \phi(x) d\mu(x) = \int \phi(x) d\mu(x) \]

for any integrable function \( \phi \) and any invariant measure \( \mu \).

We denote by \( \Phi \) the operation of concatenation from \( \sigma^* \times \sigma^* \) to \( \sigma^* \times \sigma^* \),

\[ (i_1, \ldots, i_d)(k_1, \ldots, k_d) = (i_1, \ldots, i_{d-1}, k_{d-1}). \]

Weight-balanced measures

We can then write

\[ T^k \phi(x) = \sum_{i \in \sigma^k} p_i(x) \phi(\beta_i(x)). \]

where \( p_i \) and \( \beta_i \) are defined recursively by \( \beta_{i+1}(x) = \beta_i(\beta_i(x)) \), and \( \beta_0(x) = x \).

Note that \( p_i(x) > 0 \) if and only if \( x \) is in the domain of \( \beta_i \).

We assume one additional hypothesis:

(i) There exists \( K \) such that

\[ p = \sup_{x \in \sigma^k} \sup_{i \in \sigma^k} p_i(x) < 1. \]

This is equivalent to saying that each \( x \in [a, b] \) has at least two preimages under \( f^k \).

Our goal in this section is to prove the existence and uniqueness of a \( p \)-balanced probability measure for a map \( f \) and probabilities \( p_i \) satisfying hypotheses (i)-(iv).

1.1 Lemma. Let

\[ \text{disc} p_i(x) = \left| p_i(x) - \lim_{x \to x^+} p_x(x) \right| + \left| p_i(x) - \lim_{x \to x^-} p_x(x) \right| \]

be the discontinuity of \( p_i \) at \( x \). Then

\[ \sum_{i \in \sigma^*} \text{disc} p_i(x) = 0 \]

when \( x = f^k(x) \) for some \( k \in \mathbb{N} \) and \( 0 \leq n < k \). Moreover, there exists a constant \( M \) independent of \( k \) such that \( \sum_{x \in E} \text{disc} p_i(x) \leq M \).

Proof. To prove the first part of the lemma, we use induction on \( k \). The case \( k = 1 \) is immediately obtainable. Now assume the first part of the lemma holds up to \( k-1 \). For \( i \in \sigma^* \), we have \( \beta_i(x) = p_i(x) \beta_i(\beta_i(x)) \). If \( \text{disc} p_i(x) = 0 \), then \( \beta_i(x) = f^i(x) \) for some \( e \in E \) and \( 0 \leq n < k - 1 \). Therefore, \( x = f^k(x) \) then the first part of the lemma holds because \( x \in E \).

To prove the second part of the lemma, we let \( F \) be the set of all points \( x \) of \( F \) such that \( x \in E \) or the set \( f^{-1}(x) \cap F \) has more than one point. Note that, while \( F \) may be
infinite, $F_t$ must be finite. Let $k_t$ be large enough that, for any $x \in F_t$, there exists $e \in E$ and $n < k_t$ such that $x = f^n(e)$. Set

$$ M = \sup_{x \in E} \sup_{k \leq k_t} \sup_{e \in E} \sum_{i=0}^{k-1} \text{disc} p_i(f^i(e)). $$

This ensures that the second part of the lemma holds for all $k \leq k_t$ and in particular holds for all cases where $x \in E$ or $f^{-1}(x) \cap F$ contains more than one point. We now work again by induction. Suppose the second part of the lemma holds up through $k-1 \leq k_t$. For $i \in \sigma_k$, we have $p_{k_t}(x) = p_{k_t}(\beta_i(x))$. Thus,

$$ \text{disc} p_{k_t}(x) \leq \text{disc} (\beta_i(x)) \leq \text{disc} p_i(\beta_i(x)) + p_i(\beta_i(x)) \text{disc} p_k(x). $$

The second part of the lemma follows immediately from the first unless $x = f^n(e)$ for some $e \in E$ and $0 \leq n < k_t$; let us therefore assume that this is so. Moreover, our choice of $M$ yields the desired conclusion if $x \in E$ or $f^{-1}(x) \cap F$ has more than one point; let us therefore assume that $x \notin E$, and that $f^{-1}(x) \cap F$ has only one element. Then $p_{k_t}(x) = 0$, and

$$ \text{disc} p_{k_t}(x) \leq \text{disc} p_i(\beta_i(x)). $$

Thus

$$ \sum_{i \in \sigma_{k-1}} \text{disc} p_i(x) = \sum_{i \in \sigma_{k-1}} \sum_{e \in E} \text{disc} p_i(\beta_i(x)) $$

$$ = \sum_{i \in \sigma_{k-1}} \sum_{e \in E} \sum_{i \in \sigma_{k-1}} \text{disc} p_i(\beta_i(x)). $$

Let $m_{k-1} \in E$ be chosen so that $f^{-1}(x) = \beta_{m_{k-1}}(f^0(x))$. Since we are assuming that $f^{-1}(x) \cap F$ has only one element, and the first part of the lemma ensures that all $\text{disc} p_i$ is uniformly bounded on $F$, we can apply the above inequality to $x$ with $e = m_{k-1}$. This implies

$$ \sum_{i \in \sigma_{k-1}} \text{disc} p_i(x) = \sum_{i \in \sigma_{k-1}} \sum_{i \in \sigma_{k-1}} \text{disc} p_i(\beta_{m_{k-1}}(x)). $$

By assumption,

$$ \sum_{i \in \sigma_{k-1}} \text{disc} p_i(\beta_{m_{k-1}}(x)) = \sum_{i \in \sigma_{k-1}} \text{disc} p_i(f^{i-1}(x)) \leq M \prod_{j=1}^{k-1} p_{m_{k-1}}(f^{j}(e)). $$

Thus

$$ \sum_{i \in \sigma_{k-1}} \text{disc} p_i(x) \leq M \prod_{j=1}^{k-1} p_{m_{k-1}}(f^{j}(e)). $$

1.2 Lemma. If

$$ D = \prod_{i=0}^{\infty} f^{-i}(C) $$

is dense in $[a, b]$, then the family

$$ \{ \phi(\beta_i; i \in \sigma_i, k = 1, 2, \ldots) \} $$

is uniformly equicontinuous for any continuous $\phi$ on $[a, b]$.  \hspace{1cm} \Box

Proof. Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$ |\phi(\beta_i(x)) - \phi(\beta_i(y))| < \varepsilon $$

for all $|x - y| < \delta$ and all $i \in \sigma_i$. Choose $\theta > 0$ so that $|\phi(x) - \phi(y)| < \varepsilon$ for $|x - y| < \theta$. Choose $m_i$ sufficiently large that any point in $[a, b]$ is within $1/2$ of $\beta_i(x)$ for some $e \in E$. If $k \leq m_i$, then (2C) holds for any $k > m_i$, any $e \in \sigma_i$, and any $y \in [a, b]$ for the following reason. If $k > m_i$, then the range of $\beta_i$ contains $\delta$ by our assumption. Thus $\text{diam} \beta_i([a, b]) \leq \theta$. The family

$$ \{ (\beta_i; i = 1, \ldots, m, j \in \sigma_i) \} $$

is uniformly equicontinuous; we can thus choose $\delta$ so that (1C) holds for all members of this family. \hspace{1cm} \Box

1.3 Theorem. If $D$ as defined in Lemma 1.2 is dense in $[a, b]$, then there exists a $\mu$-balanced probability measure $\mu$ for $f$.  \hspace{1cm} \Box

Proof. We show that $f^* \phi$ converges uniformly for any continuous real-valued $\phi$ on $[a, b]$. The $\mu$-balanced measure $\phi$ can then be defined by

$$ \phi(x) = \lim_{k \to \infty} f^k \phi(x) $$

for some fixed $x \in [a, b]$. The first step is to prove that

$$ \sum_{i=1}^{\infty} \prod_{j=1}^{i} p_{m_{k-1}}(f^{j}(e)) < \infty $$

for all $e \in E$, where $m_{k-1}$ is chosen so that $f^{k-1}(e) = \beta_{m_{k-1}}(f^{k}(e))$. Note that

$$ \prod_{j=1}^{i} p_{m_{k-1}}(f^{j}(e)) - p_{m_{k-1}}(f^{i}(e)). $$

From hypothesis (iv), we have $p_k(x) \neq P_k$ for $i \in \sigma_i$. Thus the sum in (1D) is dominated by a geometric series and hence converges.

We would like to point out that the above theorem can be stated in a more general form with the only assumption that (1.D) is true.
For \( x \in \mathbb{R} \), define:
\[
\eta(x) = \inf \left\{ 1 \pi_{n+1}(f^*(y)) \right\},
\]
where \( y \in E \) and \( n \geq 0 \) such that \( x = f^*(y) \). We define mappings \( \pi_{n+1}, \pi_n, \) and \( \pi_{1/2} \) on \([a, b] \) by
\[
\pi_{n+1}(x) = x + \sum_{r \in \mathbb{N}, r > n} 2q(r),
\]
\[
\pi_n(x) = x + \sum_{r \in \mathbb{N}, r > n} 2q(r),
\]
and
\[
\pi_{1/2}(x) = (\pi_{-1}(x) + \pi_1(x))/2.
\]
Let
\[
\Omega = \bigcup_{n \in \mathbb{N}} \{0, b\} \cup \mathbb{N}_0 \{0, b\} \cup \mathbb{N}_1 \{0, b\}.
\]
Then
\[
\Omega = \left\{ a, b + 2 \sum_{r \in \mathbb{N}} q(r) \right\} - \bigcup_{n \in \mathbb{N}} (\pi_{n+1}(x), \pi_n(x)) - \bigcup_{n \in \mathbb{N}} (\pi_0(x), \pi_1(x));
\]
thus \( \Omega \) is compact. The mappings \( \pi_{-1}, \pi_0, \) and \( \pi \), have a common left inverse \( \pi^{-1}: \Omega \to [a, b] \), so that \( \pi^{-1} \pi(n(x)) = x \) for \( n \in \mathbb{N} \) and \( i \in \{-1, 0, 1\} \). Note \( \pi^{-1} \) is increasing and nonexpansive.

We define functions \( \beta_i: \Omega \to [0, 1], i \in \mathbb{N}_0, k = 1, 2, \ldots \), by
\[
\beta_i(\pi_n(x)) = \beta_i(x),
\]
\[
\beta_i(\pi_{-1}(x)) = \lim_{y \to \pi_n^{-1}(x)} p_i(y),
\]
and
\[
\beta_i(\pi_1(x)) = \lim_{y \to \pi_n^{-1}(x)} p_i(y).
\]

Lemma 1.1 and the summability condition (1.1) together imply that these one-sided limits exist. Note that the functions \( \beta_i \) are continuous on \( \Omega \) and
\[
\sum_{i \in \mathbb{N}_0} \beta_i = 1.
\]
We define mappings \( \tilde{\beta}_i: \Omega \to \Omega, i \in \mathbb{N}_0, k = 1, 2, \ldots \), by
\[
\tilde{\beta}_i(\pi_n(x)) = \pi_n(\beta_i(x)),
\]
and extending \( \tilde{\beta}_i \) continuously to \( \pi_{-1}(\{a, b\}) \) and \( \pi_1(\{a, b\}) \). We then have
\[
\tilde{\beta}_{i+1} - \tilde{\beta}_i = \pi^{-1}.
\]
Thus

$$|| T^k \phi^* - T^k \phi^*_0 || \leq \lambda^n (1 - \lambda^{-1})^n \mu + M || \phi^* ||. $$

This shows that the family $\{ T^k \phi^* \} \text{ uniformly equicontinuous on } \Omega$. By the Arzelà-Ascoli Theorem, it therefore has a uniformly convergent subsequence. Denote the limit of this sequence by $\phi^*$. It is easy to show that $|| T P^k || \leq || P^k ||$ for any function $P^k \text{ on } \Omega$; thus

$$|| T^{k+1} \phi^* - T^k \phi^* || = || T^{k+1} \phi^* - T^k \phi^* || \leq || T^k \phi^* - T^k \phi^* || = || T^k \phi^* - T^k \phi^* ||;$$

and $T^k \phi^*$ converges uniformly to $\phi^*$.

Since $T^k \phi^* - (T^k \phi^*_0 - \pi_\delta) \to (T^k \delta - T^k \phi^*_0)$ converges uniformly to $\phi^* - \pi_\delta$. \qed

1.4 Lemma. Let $X$ be a nonempty closed set in $[a, b]$ with $f^{-1}(X) \cup Z = X$, where $Z$ is finite. Then there exists a nonempty perfect subset $Y$ of $X$ with $f^{-1}(Y) = Y$.

Proof. Note that $X \cup Z$ contains the closure of the preimage of any given $x \in Y$ under iterations of $f$. Using this fact and hypothesis (iv) it can be shown that $X$ must be uncountable. Let $Y$ be the set of points where $f$ is not injective. Choose a point $y$ such that $f(y) = y$; then $Y$ is nonempty. It is straightforward to show that $Y$ is a perfect set and $f^{-1}(Y) = Y$. \qed

1.5 Lemma. Let $X$ be a nonempty closed set in $[a, b]$ with $f^{-1}(X) \cup Z = X$, where $Z$ is finite. Then there exists a continuous increasing map $h$ on $[a, b]$ with $h(X) = [h(a), h(b)]$ such that

$$y = f \circ h^{-1}$$

and

$$\int_0^1 f(y) \, dy = \int_0^1 f(h(x)) \, dx$$

satisfy hypotheses (i)-(iv) on $h([a, b])$.

Proof. By Lemma 1.4, $X$ contains a nonempty perfect set $Y$ with $f^{-1}(Y) = Y$. We construct $h : [a, b] \to [\inf Y, \sup Y]$ which maps $Y$ onto $[\inf Y, \sup Y]$ as follows. Define $h(x) = \inf Y \text{ for } x \leq \inf Y$ and $h(x) = \sup Y \text{ for } x \geq \sup Y$. For each maximum open interval $(a, b)$ in $[a, b]$, set $h(x) = (a + b)/2 \text{ for } x \in (a, b)$. We then extend $h$ continuously to the boundary of $Y$ and linearly to intervals contained in $Y$.

It is then straightforward to show that $f$ and $h$ satisfy hypotheses (i)-(iv). One easily shows that $f^{-1}(Y)$ is a perfect set and that $h$ maps each interval intersecting $Y$ to a single point. \qed
2. Free energy and the maximal-entropy measure

In this section we show an invariant measure has maximal entropy if and only if its free energy associated with its Jacobian is linear. Our result generalizes in some sense one of K. Zimmer in [28]. Our techniques, however, differ from those used in [28]; we will use mainly ideas from Large-Deviation Theory to prove our main theorem.

We also show in the expanding case that a measure has maximal entropy if and only if its Jacobian has zero asymptotic variance. This fact was already known, but our proof follows from the results previously obtained in the beginning of this paragraph.

We first review the definition of a maximal-entropy measure. We denote by \( M(f) \) the set of all probability measures invariant under \( f \). We denote by \( h(f) \) the entropy of \( f \) under \( f \). See Mañe [15] for definitions and general references on entropy.

We further define the entropy of \( f \), denoted \( h(f) \), by

\[
h(f) = \sup_{\nu \in M(f)} h(\nu, f).
\]

A most useful formula gives \( h(f) \) as the asymptotic growth rate of the number of preimages of a point:

\[
h(f) = \lim_{n \to \infty} \frac{1}{2^n} \log \# f^{-n}(x).
\]

If \( \mu \in M(f) \) and \( h(\mu) = h(f) \), we call \( \mu \) a maximal-entropy measure for \( f \).

Newhouse and Yi [26] showed that a maximal-entropy measure exists in any \( C^\infty \) map. Hofbauer [8] showed that in the case \( d = 2 \) (minimal map) the maximal-entropy measure is unique. Polynomial maps of higher degree may have more than one maximal-entropy measure.

Let \( \nu \) be a Borel measure on \([a, b]\). We define the Jacobian of \( \nu \) to be a function \( J_\nu : [a, b] \to [0, \infty] \) given by

\[
J_\nu(x) = \lim_{h \to 0} \frac{\nu([x, x+h] \cap [a, b])}{\nu([x, x+h])}.
\]

Alternatively, \( J_\nu \) is the Radon-Nikodym derivative of \( \nu \cdot f \) with respect to \( \nu \); this shows that \( J_\nu(x) \) is defined \( \nu \)-almost all \( x \) in \([a, b]\).

If \( \nu \in M(f) \) then \( J_\nu \) must be positive \( \nu \)-almost everywhere; moreover,

\[
\sum_{j \in \mathbb{Z}} J_\nu(j) = 1.
\]

We show that \( h(f) > 0 \) if and only if \( \nu \cdot f \) is expanding; this implies an upper bound on the entropy, which is also related to the existence of a unique equilibrium state for the system.

3. Theorem 1.7: there exist \( \lambda > 0 \), a continuous function \( h \), and a probability measure \( \nu \) on \([a, b]\) such that \( t \mapsto h(t \cdot f) = \lambda h(t) \). Then \( h(\nu, f) = 1 \), and the sequence of functions \( h^{-1} \) \( \nu \)-converges uniformly to \( h \) for any continuous function \( \alpha \) on \([a, b]\).

The condition \( \sum p_i = 1 \) further implies \( \lambda = 1 \) and \( h = 1 \). Thus \( \nu \) is the required \( \rho \)-balanced measure. \( \square \)

For \( \nu \)-almost all \( x \) in \([a, b]\). Moreover, if \( \nu \) is a weight-balanced measure associated with probabilities \( (p_i : i \in \mathbb{Z}) \), and \( (f_i : i \in \mathbb{Z}) \) are the corresponding inverse branches of \( f \), then we have

\[
P_i(x) = \frac{1}{f_i(\nu_i(x))} \int |f_i(\nu_i(x))| \, d\nu_i(x)
\]

for \( \nu \)-almost all \( x \) in the domain of \( f_i \). If \( f_i(\nu_i(x)) > 0 \) for \( \nu \)-almost all \( x \) then one can show

\[
\nu_i = \frac{1}{\log f_i(\nu_i(x))} \int f_i(\nu_i(x)) \, d\nu_i(x)
\]

as in [11].

We consider \( \nu \in M(f) \) with Jacobian \( J_\nu \). Let \( f_1, \ldots, f_k \) denote the inverse branches of \( f \). We define functions \( p_1, \ldots, p_k \) by (2.2) and then recursively

\[
P_i(x) = \frac{1}{f_i(\nu_i(x))} \int \log f_i(\nu_i(x)) \, d\nu_i(x)
\]

as in (11).

We define functions \( w_n : n = 1, 2, \ldots, i \in \mathbb{Z} \) by \( w_n(x) = f_i^n(\nu_i(x)) \), where \( i \in \mathbb{Z} \) is chosen so that \( x = f_i^n(\nu_i(x)) \). It is easy to verify that the functions \( w_n \) satisfy the following relations:

\[
w_n(x) = w_{n+1}(f_i^n(x)) w_n(f_i^n(x)), \quad 1 \leq k \leq n;
\]

\[
w_n(x) = w_n(x) \prod_{k=1}^{n-1} J_k(f_i^n(x))^{-r_i}.
\]

21 Definition. For \( \nu \in M(f) \), we define the free energy of \( \nu \) to be a function \( c(\nu, \infty) \) given by

\[
c(\nu, \infty) = \lim_{k \to \infty} \frac{1}{k} \log \int w_k(\nu)_k(x) \, d\nu(x)
\]

\( \in \mathbb{R} \) is such that

\( \in \mathbb{R} \)

We say that a measure \( \nu \) is in \( M(f)^* \) if the above limit exists and the three conditions below are satisfied:

1. \( \nu \) is balanced, that is for any \( \phi \) continuous function and any \( \mu \) probability measure, we have

\[
\int |\phi| \, d\mu = \lim_{n \to \infty} \int |\nu^n| \, d\mu.
\]

2. \( \nu \) is a \( C^\infty \) map. Hofbauer [8] showed that in the case \( d = 2 \) (minimal map) the maximal-entropy measure is unique. Polynomial maps of higher degree may have more than one maximal-entropy measure.

3. \( \nu \) is differentiable by parts and right and left derivatives exist (for the values of \( \nu \) where \( c(\nu, \infty) \) is not differentiable). We also assume \( c(\nu, \infty) \) is differentiable at \( t = 0 \).

For each value of \( t \) the Ruelle-Perron-Frobenius Operator associated to the map \( w_k(\nu)_k(x) \) is uniformly equicontinuous.

We point out that all the above properties 1), 2), 3) are true, and the above limit exists in the case where the Jacobian is Hölder-continuous and \( f \) is expanding. The function \( c(\nu, \infty) \) in this case is in fact real analytic.

Examples of situations where the free-energy (also called pressure) is not differentiable but differentiable by parts appear in ([11] page 95), [13] (page 402),
Remark. In the paper of E. Hofbauer-"Examples for the Nonuniqueness of the Equilibrium State"-Transactions AMS vol. 238, 1977, pp. 223–241) the shift in the symbols with a non-Hölder potential is analyzed and the author shows that for the Ruelle-Perron-Frobenius Operator property 3) above is true in the second paragraph from bottom to top on page 239. A large class of examples are presented in the above-mentioned paper and in some of them, two equilibrium states can coexist (see table page 239). If one considers for example, the class of examples satisfying a), b) and c) on page 238 in the paper of Hofbauer, then one can find a large class of situations where two equilibrium states exist, one is a Dirac-Delta and the other is a nonextensive measure denoted by μ. In [8] (bottom of page 468) is presented the Jacobian of the above measure μ. In this case the potential given by this Jacobian produces the same property 3) above. The graph of the function ρ(0) is presented in fig. 1 in page 402; the last mentioned paper and it satisfies 2) above. Therefore the hypothesis of Theorem 2.2 below cover cases of weight-balanced measures not represented only by the expanding case with Hölder continuous potential (or Jacobian) and we can ask a balanced measure of this kind can be the maximal measure. The answer is no because ρ(0) is not linear.

The purpose of presenting all the above examples is to stress the fact that in several situations the free-energy (or pressure) is not so nice as in the expanding maps or Hölder-continuous case. Nevertheless under some suitable assumptions (to be in $M(f)$) we will be able to show in Theorem 2.2 some properties for $c_i(0)$.

In [11], [19], [23] several examples of the pathologies of the non-expanding case are considered.

We do not use in our proof properties like Quasi-Compactness of Operators (F1), (F2), or (F3), or approach of [22].

The concept of free energy comes from Large-Deviation Theory. The main concepts of this theory are typically defined for all objects of interest, but in a dense subset thereof. In this case we require $x \in M(f)^*$ rather than $x \in M(f)$. In [4] is a general reference for Large-Deviation Theory, including free energy and the Legendre-Fenchel transform.

Our main goal in this section is the following:

2.2 Theorem. Let $v \in M(f)^*$, that is:

1) $v$ is balanced, that is, for any continuous function $\phi$ and any probability measure $\nu$ we have

$$\int \phi dv = \lim_{n \to \infty} T^{-n} \phi d\nu.$$ 

2) $c_i(0)$ is differentiable by parts and there exist right and left derivatives at the values of $i$ where $c_i(0)$ is not differentiable. We also assume $c_i$ is differentiable at $i = 0$.

3) For each value of $i$ the Ruelle-Perron-Frobenius Operator associated with $\log v^{-1}$ is uniformly equicontinuous.

Then $h(v, f) = h(f)$ if and only if $c_i$ is a linear function.

Before the proof of the theorem we need a lemma that relates pressure and free energy (see [22] for the concept of pressure and about Ruelle-Perron-Frobenius operators).

1.3 Lemma. For fixed $x \in [a, b]$, we have

$$c_i(t) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{i=0}^{k-1} p_i(x)^{-t}.$$ 

This limit converges and hence $c_i(t)$ is defined for all real $t$.

Note: It should be understood here that the sum includes only those terms with $|t| = 0$, as we consider cases with $t \neq 0$.

Proof. Let the operator $T$ be defined as in (1,A). As the measure is balanced

$$\int \phi dv = \lim_{n \to \infty} T^n \phi d\nu$$

for any fixed probability measure $\mu$ (not necessarily invariant) and any continuous function $\phi$. See also the definition of an attractive measure in [2]. We consider the particular case where $\phi(x) = w_{\mu}(x)$ and $\mu$ is a unit point mass at $x$. Then we have

$$\int w_{\mu}(x) d\mu(x) = \lim_{n \to \infty} T^n w_{\mu}(x).$$

Thus

$$c_i(t) = \lim_{n \to \infty} \frac{1}{n} \log \lim_{n \to \infty} T^n w_{\mu}(x).$$

We assume $n > m$ without loss of generality. Recalling (1,A), we have

$$T^n w_{\mu}(x) = T^{-m} T^m w_{\mu}(x) = \sum_{\beta_1 \in \beta_m} p_{\beta_1}(x) (T^m w_{\mu})(\beta_1(x)).$$

From (2.B), we have

$$(T^m w_{\mu})(\beta_1(x)) = \sum_{\beta_2 \in \beta_m} w_{\mu}(\beta_2)(\beta_2(x)) p_{\beta_2}(x)$$

$$= \sum_{\beta_2 \in \beta_m} w_{\mu}(\beta_2)(\beta_2(x)) p_{\beta_2}(x).$$

Therefore

$$\int \phi dv = \lim_{n \to \infty} T^n \phi d\nu.$$
since from the definition of \( w' \), we have \( p_i(\beta_\delta(x)) = w'_\delta(\beta_\delta(x)) \). Using (2.1), and setting \( \psi(x) - \log w'_\delta(x) \), we find

\[
(T^x w'_\delta)(x) = \sum_{k=0}^{\infty} \mu_{k-i}(x) = L^\delta_i(1)(x),
\]

where \( L_i \) is the Ruelle-Perron-Frobenius operator associated with \( \psi \). Because of the uniform equicontinuity of iterates of \( L^\delta \), we can find a constant \( K > 0 \) such that

\[
e^{-K} L^\delta_i(1)(x) < L^\delta_i(1)(\beta(x)) < e^K L^\delta_i(1)(x).
\]

We can interpret this as

\[
e^{-K} T^x w'_\delta(x) < T^x w'_\delta(x) < e^K T^x w'_\delta(x).
\]

Since \( \sum p_i(x) = 1 \), we have finally

\[
e^{-K} T^x w'_\delta(x) < T^x w'_\delta(x) < e^K T^x w'_\delta(x).
\]

Equation (2.3) now easily transforms to

\[
c_i(t) = \lim_{\delta \to 0} \frac{1}{\delta} \log T^x w'_\delta(x)
\]

\[
= \lim_{\delta \to 0} \frac{1}{\delta} \log \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x)
\]

\[
= \lim_{\delta \to 0} \frac{1}{\delta} \log \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x),
\]

using once more the definition of \( w'_\delta \).

To show now convergence of the limit, we write \( i e^x \) as a concatenation \( i = fl \), for some \( f \in \mathbb{R}^+ \) and \( k \in \mathbb{N}^+ \). Thus

\[
\sum_{x \in \mathbb{R}^+} p_i(x) \log \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x)
\]

\[
= \sum_{x \in \mathbb{R}^+} p_i(x) \log \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x),
\]

using (1.3). Note now that (2.2) can also be interpreted as

\[
e^{-K} \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x) > e^{K+x} \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x),
\]

Thus

\[
e^{-K} \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x) > e^{K+x} \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x),
\]

or

\[
\left| \log \sum_{x \in \mathbb{R}^+} p_i(x) \mu_{k+i}(x) - \log \sum_{k=0}^{\infty} p_i(x) \mu_{k+i}(x) \right| < K.
\]
But
\[ \frac{d}{dz}(z - C(z)) = 1 \]
for \( z = h(v) \), so this is not a local maximum for \( z - C(z) \). Thus
\[ h(v) = -\log(1 - C(v)) < h(f). \]

Now we assume \( c_\varepsilon \) is linear and show that \( h(v) \geq h(f) \). We fix a point \( x \in [a, b] \) and define a sequence of probabilities \( v_n, n = 0, 1, \ldots \)
by
\[ \int \phi \, d\nu_n(x) = T^*\phi(x) \]
for any continuous function \( \phi \) on \([a, b]\). Note that \( \nu_n \) is supported on \( f^{-n}(x) \) and for any subset \( S \) of \( f^{-n}(x) \) we have
\[ \nu_n S = \sum_{y \in S} \nu_n(y). \]

Consider a certain value \( c_\varepsilon \) fixed. Let \( \varepsilon > 0 \) be given. We define a sequence of closed sets
\[ B_n : n = 1, 2, 3, \ldots \]
by
\[ B_n = \left\{ x \in f^{-n}(x) \mid \frac{1}{n} \log \nu_n(x) - h(v) \geq \varepsilon \right\}. \]

Define also \( A_n \) by \( A_n = f^{-n}(x) - B_n \).

Our chief tool in this argument is [4, Theorem II.6.1], which says that
\[ \limsup_{n \to \infty} \nu_n B_n \geq \inf_{x \in [-\infty, \infty]} C_\varepsilon(x). \]
We now establish
\[ \lim_{n \to \infty} \frac{1}{n} \log \# A_n = h(f). \]

Since \( c_\varepsilon \) is linear, \( c_\varepsilon(z) = c_\varepsilon(1) \cdot \delta - c_\varepsilon(0) = h(f) \), thus \( C_\varepsilon(h(f)) = 0 \) while \( C_\varepsilon(z) = \infty \) if \( z \neq h(f) \). Thus from (2.E) we have
\[ \lim_{n \to \infty} \frac{1}{n} \log \# A_n = -\infty. \]

By assumption, \( J_\varepsilon \) is bounded from (2.E) we find that
\[ \nu_n B_n = \sum_{n=1}^{\infty} \nu_n(y) \geq \left( \sup_{x \in f^{-n}} \nu_n(x) \right)^{-\infty}. \]
Thus
\[ \lim_{n \to \infty} \frac{1}{n} \log \# A_n = -\infty. \]

Note that as the cardinal of \( B_n \) is a natural number, the only possibility is that \( B_n \) is eventually empty.

Hence
\[ \lim_{n \to \infty} \frac{1}{n} \log \# A_n = \lim_{n \to \infty} \frac{1}{n} \log \left( \# A_n + \# B_n \right) = \lim_{n \to \infty} \frac{1}{n} \log \# f^{-n}(x) - h(f). \]

Note that \( w^2_J(x) > e^{-\theta t} + \varepsilon \) for \( J \in A_n \). Thus
\[ \# A_n \leq \theta \varepsilon^{-t} w^2_J \leq \theta e^{t} \varepsilon^{-t}. \]

Thus we have
\[ h(f) = \lim_{n \to \infty} \frac{1}{n} \log \# A_n \leq h(v) + \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, this proves the theorem. \( \Box \)

Suppose now that \( f \) is expanding. We will show a proof of a result already known [1, 2, 23, 21].

14 Lemma. \( c_\varepsilon(t) \) is a \( C^1 \) function of \( t \).

Proof. This follows from the same type of argument as those used by Mañé [15, Corollary 1.4] or Pollicott [20] for the pressure function. We conjugate \( f \) to a Markov shift of finite type and consider a complexification of \( t \) in the integrand \( w^*(t) \) in the definition of \( c_\varepsilon \). The hypotheses that \( f \) is expanding and \( J \) is Hölder-continuous are important here.

15 Definition. Let \( \nu \in M(f) \) and \( \phi : \mathbb{R} / \mathbb{Z} \to \mathbb{R} \) be continuous. We define the asymptotic variance of \( \phi \), denoted \( e^2_{\phi} \), by
\[ e^2_{\phi} = \lim_{n \to \infty} \frac{1}{n} \left( \int \phi(x) \, dx - \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) \right)^2 \, dx. \]

Przytycki, Urbanski and Zdunik [21, Lemma 1] showed that this limit exists and is finite if \( e^2_{\phi} > 0 \) if and only if \( f \) is cohomologous to zero.

16 Theorem. Let \( \nu \in M(f) \). Then
\[ e_{\nu, f}^2 = 0 \]
if and only if \( h(f) = h(f \circ f). \)
Proof. By Lemma 2.4, \( c_n \) can be differentiated under the limit; thus we obtain
\[
c'_n(t) = \lim_{n \to \infty} \left[ -\frac{n^{-1}(x) \log w_n(x) dx}{n} \right] w_n(x) dx(x)
\]
and
\[
c'_n(t) = \lim_{n \to \infty} \left[ \frac{w_n(x) dx(x)}{n} \left( \frac{w_n(x) \log w_n(x) dx(x)}{n} \right)^2 \right] dx(x) - \left( \frac{w_n(x) \log w_n(x) dx(x)}{n} \right)^2
\]
Thus, using (2.8) and the invariance of \( v \), we find
\[
c'_n(t) = \lim_{n \to \infty} \left[ \frac{1}{2} \left( \int \log w_n(x) dx(x) - \left[ \int \log w_n(x) dx(x) \right]^2 \right) \right] \]
\[
= \lim_{n \to \infty} \left[ \frac{1}{2} \left( \sum_{j=0}^{n-1} \log f_j(x) \right)^2 \right] \]
From the theory of large deviations it is known that \( c'_n(0) = 0 \) if and only if \( c_n \) is linear. From Theorem 2.2, this occurs only if \( h(v) = h(f) \).

3. Entropy of a weight-balanced measure: an example

For a map with essentially the same number of preimages for each point, the maximal-entropy measure can be easily described as a weight-balanced measure corresponding to equal weights on all branches (with multiplicity) of the inverse. When the number of preimages varies from point to point, there is no known explicit formula for the weights which yield a measure of maximal entropy. In this section, we present an example of such a map whose entropy is explicitly calculable. This example shows that maximal entropy is not necessarily attained by assigning equal weight to each branch of the inverse map when the number of branches depends on the point.

We define \( f : [0, 1] \to [0, 1] \) by
\[
f(x) = \begin{cases} 
2x + \frac{1}{4} & \text{if } 0 \leq x \leq \frac{3}{8}, \\
\frac{7}{4} - 2x & \text{if } \frac{3}{8} \leq x \leq \frac{3}{4}, \\
1 - x & \text{if } \frac{3}{4} \leq x \leq 1.
\end{cases}
\]
Note that Lebesgue measure on \([0, 1]\) is an invariant probability measure for this map.

We set \( X = (0, 1/4), Y = (1/4, 3/4), \) and \( Z = (3/4, 1). \) Any point in \( X \) has just one preimage in \( Z. \) Any point in \( Y \) has one preimage in \( Y \) and one preimage in \( Z. \) Any point in \( Z \) has two preimages in \( Y. \) Thus, if \( x \in X \cup Y \cup Z, \) and we write \( \xi_n = \#(f^{-n}(x) \cap X), \eta_n = \#(f^{-n}(x) \cap Y), \) and \( \zeta_n = \#(f^{-n}(x) \cap Z), \) then we have the formula

\[
\begin{pmatrix}
\xi_n + 1 \\
\eta_n + 1 \\
\zeta_n + 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 2 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\xi_n \\
\eta_n \\
\zeta_n
\end{pmatrix}
\]

The asymptotic growth rate of the number of preimages of a point is thus given by the largest eigenvalue of the matrix in the equation; this is
\[
\lambda = \frac{(224 + 244/74)^{1/2} + (224 - 244/74)^{1/2} + 2}{6} \approx 1.6956.
\]
The entropy of \( f \) is then given by \( \log \lambda = 0.32805. \)

Lebesgue measure is an invariant measure for \( f; \) its Jacobian under \( f \) is equal to \( 1 \) in \((0, 3/4)\) and equal to \( 2 \) on \((3/4, 1). \) The corresponding weights are \( 1/2 \) on each branch when there are two inverse branches, and \( 1 \) (necessarily) when there is one inverse branch.

We can calculate the entropy of \( f \) explicitly as \( h(m) = \int \log f_\nu(x) d\nu(x). \)

We find \( h(m) = \log 2 \cdot 3/4 = 0.51938 < h(f). \)

References

Metric properties of positively ordered monoids

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Abstract. We introduce here an intrinsic (quasi-) metric on each positively ordered monoid (POM), which is defined in terms of the evaluation map from the given POM to its dual and for which POM-homomorphisms are continuous. Moreover, we find a class of refinement POM's, which, equipped with the canonical metric, are complete metric spaces; this class includes the class of weak cardinal algebras, but also most cases of completions of certain kind (so-called "strongly reduced products") of POM's, and of which a prototype has been used as a previous paper for the description of the evaluation map of a given refinement POM. This must can also be viewed as a wide generalization of the non-linearly ordered case (for example weak cardinal algebras) of the Cauchy-completeness of the real line.

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4. Introduction: basic definitions and notations

We recall here the definition of positively ordered monoids, as it appears in [15].

Definition. A positively ordered monoid (from now on a POM) is a structure \((A, +, 0, \leq)\) where \((A, +, 0)\) is a commutative monoid and \(\leq\) is a preordering of \(A\) such that \((A, +, 0, \leq)\) satisfies the following:

1. \((\forall a, b, c)(a + b = a + c \Rightarrow b = c),\)
2. \((\forall a)(a \geq 0).\)

In particular, when \((A, +, 0, \leq)\) is a commutative monoid, then we can define a preordering (not always antisymmetric) \(\leq\) on \(A\) by

\[(\forall x, y \in A)(x \leq y \iff (3 \in A)(x + z = y)).\]