Comput. & Graphics Vol. 16, No. 1, pp. 15–23, 1992
Printed in Great Britain.
0097–8493/92 $5.00 + .00

Chaos and Graphics

ON THE DYNAMICS OF REAL POLYNOMIALS ON THE PLANE

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Abstract—We analyze the dynamics of the map $f(z) = z^2 - s\bar{z}$ where $s \in \mathbb{C}$ is a constant and $z \in \mathbb{C}$ is a variable. For some values of s, we can have invariant measures of density with respect to the two-dimensional Lebesgue measure. For other values of s we can have fractal repellers or the X-trange attractor. This problem is related to a Triple Point Phase Transition Model (Potts Model).

1. INTRODUCTION

We will analyze computer experiments and present several conjectures on the dynamics of real polynomials on the plane of the form $f(x, y) = (x^2 - y^2 - ax - by, 2xy - bx + ay)$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are constants. In a compact form f can be written as $f(z) = z^2 - s\bar{z}$, where $z \in \mathbb{C}$ and $s = a + bi$. Some strange attractors appear for some values of the parameter s. In one of these cases, the attractor has the geometrical shape of the letter X. For other values of the parameters, there exist an invariant measure with density with respect to the two-dimensional Lebesgue measure. We use the computer to analyze the bifurcation set of parameters and the dynamics of these maps.

The physical motivation for analyzing such classes of maps is related to a dynamical system model recently introduced for understanding an old problem: triple point phase transition (Potts Model). In our case the model applies to a semi-infinite one dimensional spin lattice $\mathbb{N}$ with four spin components in each site of the lattice.

The Yang-Lee zeros are part of a very important area of study of concrete physical problems related to sudden magnetization of ferromagnetic systems. In the dynamical system model we consider here, we will exhibit the locus of points of the set that is the analog of the set of Yang-Lee zeros. This is relevant for the phenomena of triple point phase transitions where it is useful to know for each values of s whether the maps f_s are expanding or not.

2. REAL POLYNOMIALS IN THE PLANE

In this note we will consider the family of real polynomials on the plane of the form

$$f(x, y) = (x^2 - y^2 - ax - by, 2xy - bx + ay), \quad a \in \mathbb{R}, \quad b \in \mathbb{R}.$$

In complex coordinates such map can be written in the form

$$f(z) = z^2 - s\bar{z}, \quad \text{where} \quad s = a + bi.$$

* Partially supported by AFOSR.

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For the parameter $s = 2$, the dynamic of this map is very well understood. This map was first considered by M. E. Hofman and W. D. Withers in [8] and [15], and is also known as the Generalized Chebyshev Polynomial on the plane (see also [12]). In this case the map f has an invariant measure, absolutely continuous with respect to the 2-dimensional Lebesgue measure. This measure is the measure of maximal entropy. The support of this invariant measure is the interior of the deltoid curve given by

$$4(z^3 + \bar{z}^3) - (z\bar{z})^2 - 18z\bar{z} + 27 = 0.$$

The picture of the deltoid region is shown in Fig. 1(a). The analytical expression of the density is

$$\frac{3}{\pi^2}(4(z^3 + \bar{z}^3) - (z\bar{z})^2 - 18z\bar{z} + 27)^{-1/2}.$$

All these results are presented in [8] and [15]. We refer the reader to these two papers for other interesting properties of such map.

The real polynomial on the real line $g(x) = x^2 - 2x$ also has an invariant measure absolutely continuous with respect to the one-dimensional Lebesgue measure. This map is conjugated with the map $1 - 2x^2$.

The family of real polynomials $g_c(x) = 1 - cx^2$ was analyzed by several authors (see [2] for references).

There exists a value $\tilde{a}$ such that the set A of parameters c such that the family g_c has an absolutely invariant measure, is contained in $(\tilde{a}, 2)$. The value $\tilde{a}$ is known as the Feigenbaum point[5].

M. Yakobson showed that the set A has positive Lebesgue measure in the set of real parameters[9].

For values of c larger than 2, the critical point goes to ∞ under iterations of g_c. In this case the map g_c is expanding, and the nonwandering set has a Cantor set structure (see [2]).

It is conjectured that the set of parameters values c, where g_c is expanding is dense in $(\tilde{a}, 2)$. An important result about the bifurcation set of the family g_c was obtained by Feigenbaum[5].

The analysis of the family $f_s(z) = z^2 - s\bar{z}$ on the plane is a natural extension of the problem considered above for the family g_c on the real line. Another natural

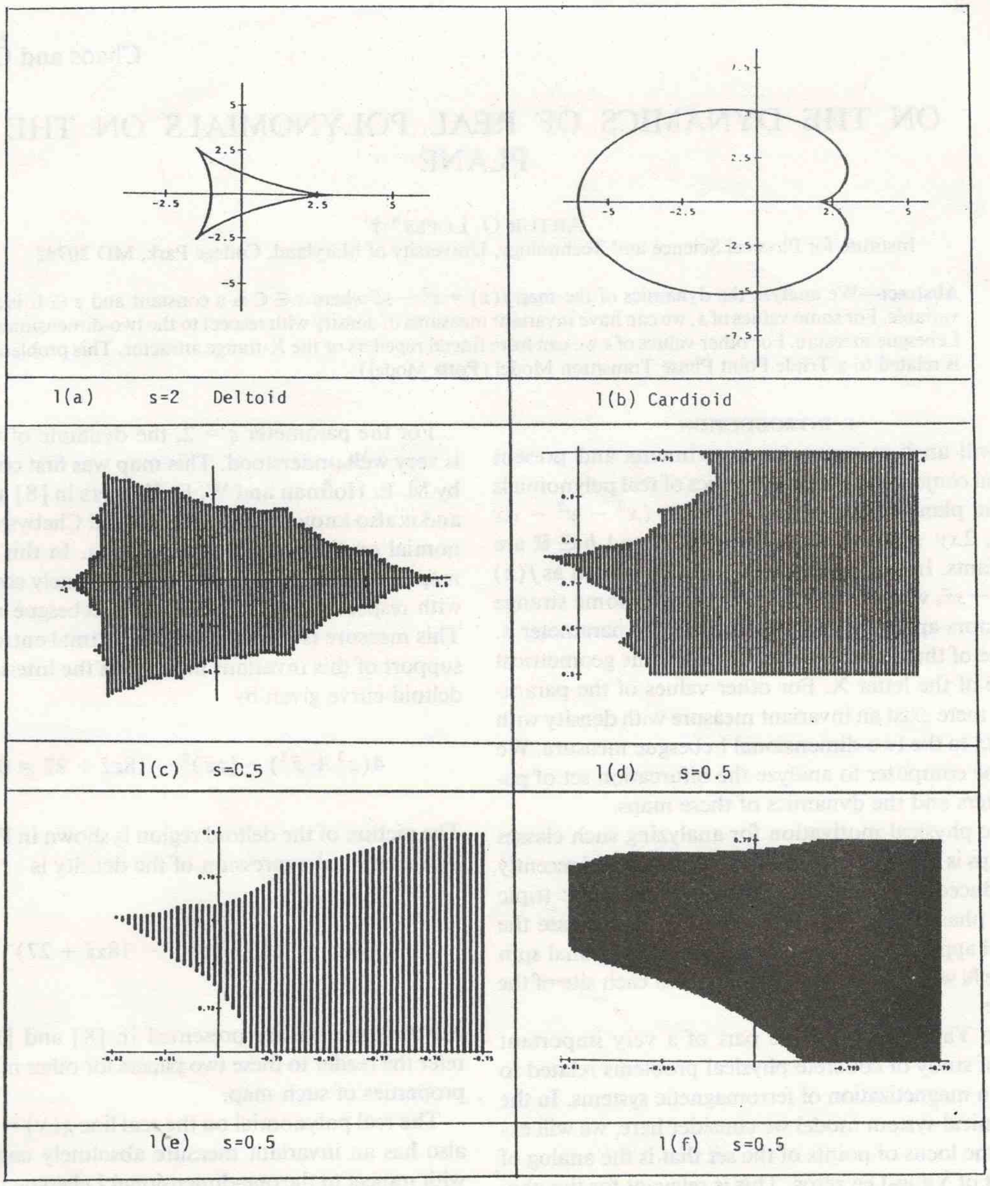

Fig. 1. Portraits of dynamics. (a) $s = 2$ Deltoid, (b) Cardioid, (c) $s = 0.5$, (d) $s = 0.5$, (e) $s = 0.5$, (f) $s = 0.5$.

extension for the plane of the family g_c is the family of maps of the form

$$v_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

In this case the bifurcation set is also known as the Mandelbrot set [13].

In recent years several papers on the dynamics of polynomials on the plane appeared in the literature, [1] and [6]. Most of these papers are related to the Henon map [7].

We became interested in the dynamics and the bifurcation set of the family $f_s(z) = z^2 - s\bar{z}$ because this is related to a model associated with a triple point phase transition and Yang-Lee zeros. In this model the value $s = 2$ corresponds to the point of triple point phase transition. Other values of s should correspond to different external magnetic fields. We will explain more carefully now the physical problem to which our mathematical model is related.

It is well known that certain materials present magnetic properties at low temperatures. In first order transition, the transition from nonmagnetic state to the magnetic state is noncontinuous. In fact, there exist a certain transition value of the parameter temperature where suddenly the magnetization occurs. For the physics literature on this subject, we refer the reader to [9]. Note that the free energy (or pressure) is continuous with the temperature t. For each t there exist an equilibrium state (sometimes more than one) also known as Gibbs State. Suppose now we decrease the temperature t of a ferromagnetic material. Until we reach a certain transition value t_o, equilibrium states are unique. For this value t_o, there exist more than one equilibrium state. This means a discontinuity (in the set of probabilities) of the equilibrium state with the variation of t. In the Ising Model, two equilibrium measures can coexist in the transition temperature (double point transition). In the Potts Model, three equilibrium measures can coexist in the transition temperature (triple point transition).

In [11] we present a mathematical model for double transition and in [12] we show that $f(z) = z^2 - 2\bar{z}$ represents a model for triple transition. Suppose now we also want to change the magnetic field and not only the temperature around the bifurcation point of triple transition.

The Yang-Lee zeros appears in the concrete physical problem as the locus of points where coming from very large values of a "*complex*" magnetic field, there exist transition from one equilibrium state to more than one equilibrium state[13]. It is well known that for expanding systems equilibrium states are unique[10, 11].

The existence of more than one equilibrium state in "thermodynamic formalism" terms (see [11]) is related with nonexpansive maps. This happens, for instance, for $f(z) = z^2 - 2\bar{z}$ where for a certain value of the external parameter t, three equilibrium states coexist.

We will explain now in a more rigorous way what we mean by equilibrium state. Consider a certain fixed map f and $M(f)$ the set of invariant probabilities for f. Consider now t an external parameter (t plays the role of temperature) and for each t we will be interested in finding the probability μ_t that attains the supremum for the following variational problem:

$$\sup_{v \in M(f)} \{h(v) - t \int \log|\det(Df(z))|\, dv(z)\}.$$

We will call the probabilities that attain such supremum of equilibrium states. The term $h(v)$ is the entropy of v (the kinetic energy term) and the term $\int \log \# \det|D(f(z))|\, dv(z)$, is the Liapunov number of the probability v (the potential energy term).

For expanding systems such equilibrium states are unique for every t. Suppose now we decrease the value t in a continuous fashion. For some nonexpanding systems as, for example, $f(z) = z^2 - 2\bar{z}$ (see [12]), there exist a unique equilibrium state μ_t, until we reach a transition value of parameter $t = t_o$ where there exist three equilibrium states. This is the phenomena of triple point transition.

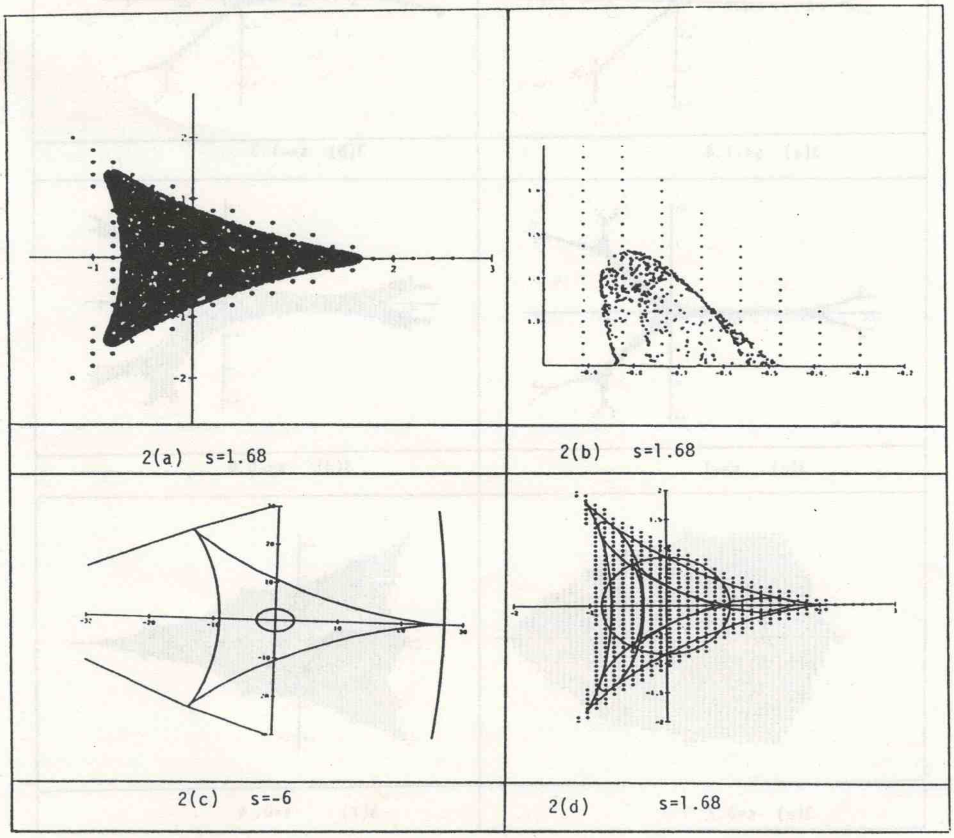

Fig. 2. (a) $s = 1.68$, (b) $s = 1.68$, (c) $s = -6$, (d) $s = 1.68$.

Now we have to analyze the other parameter $s \in \mathbb{C}$ and consider the maps f_s and the equilibrium probabilities μ_t associated with each of these maps $f_s(z) = z^2 - s\bar{z}$ for s close to 2. In a neighbourhood of 2, some of the s are such that f_s is expanding and for other values of s the map f_s is not expanding. In the first case, equilibrium states μ_t are unique for all t, and in the second case, perhaps for some values of t, equilibrium states are not unique (see [12]).

The natural question is the following: Is there, in the parameter space, $s \in \mathbb{C}$, an analytical curve representing the locus of points where there exist the transition from one case (one equilibrium state for every t) to the other case (more than one equilibrium state for some value of t)? In this case we think that different values of s represent different values of "*complex*" magnetic fields.

Using the computer we were able to obtain the cardioid curve shown in Fig. 1(b) as the natural candidate to be the Yang-Lee zeros set related to our model.

For values of s close to 2 and outside the cardioid curve, the system seems to be expanding. The reason is that the critical set goes to infinity (and also geometrical aspects observed in the pictures). We refer the reader to [10–12] for references related to the Ising and Potts model of statistical mechanics.

Another justification for the study of polynomial and rational maps on the variable z and $\bar{z}$ is related to convergent algorithms of the Newton type [14].

We will present here several pictures obtained in the computer that we believe are worthwhile for a better understanding of the topological dynamics of such maps. We hope these pictures can stimulate other people for a rigorous mathematical analysis of the problem. We will present several open questions. The family of such maps presents a very rich dynamics and a certain analogy with real quadratic polynomials in the line.

The critical set of f_s is, by definition, the set of points of the plane where the determinant Jacobian of f_s is zero. This set is the circle of center zero and radius $\frac{\|s\|}{2}$. For values of s such that $\|s\| < 1$, the point (0,

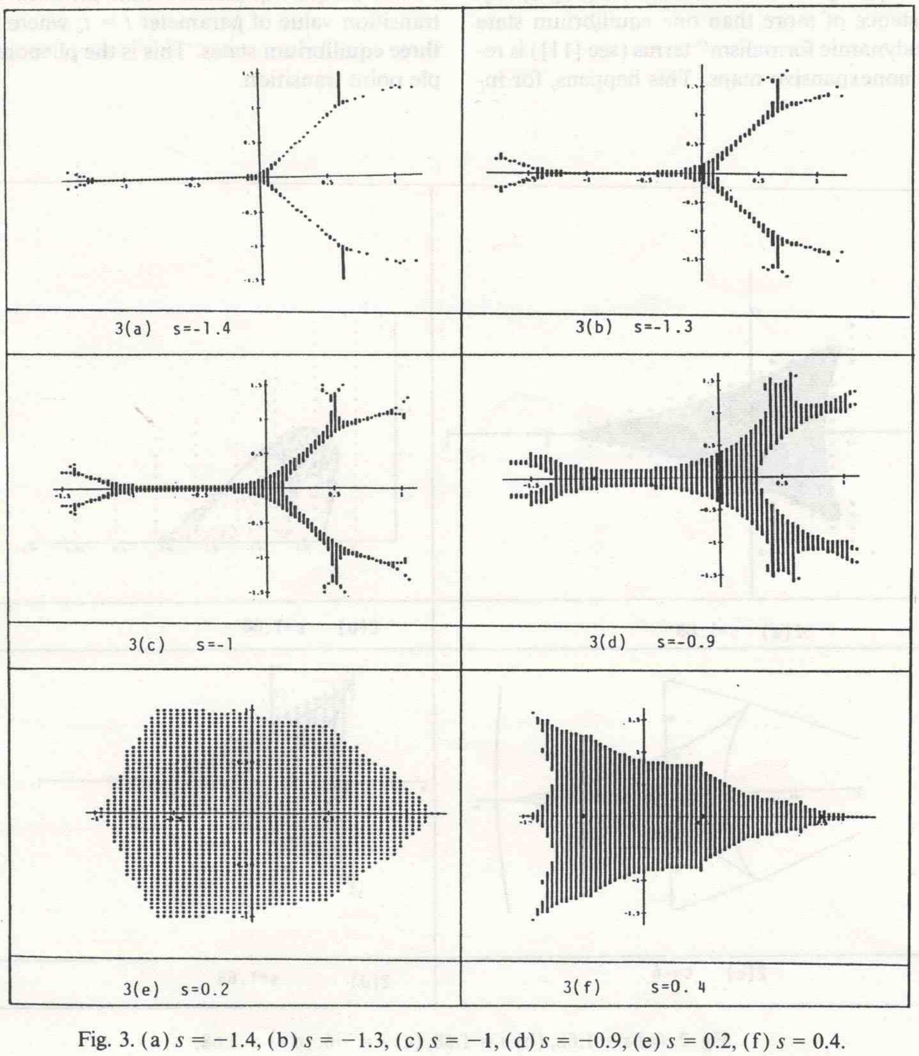

Fig. 3. (a) $s = -1.4$, (b) $s = -1.3$, (c) $s = -1$, (d) $s = -0.9$, (e) $s = 0.2$, (f) $s = 0.4$.

0) is attracting. For all other values of s, the point (0, 0) is repelling. For values of s with large modulus, the critical set goes to ∞ under iterations of f_s.

The nonwandering set in this case has a Cantor set structure as shown in Fig. 4(g) and 4(h).

The set of bifurcation points in the s-parameter space (coming from ∞) is the cardioid with extremes in -6 an 2, shown in Fig. 1(b). For values of s outside the interior of the cardioid the critical set goes to ∞. For values of s inside the cardioid, some critical points may not go to ∞ under iterations of f_s.

In Fig. 2(c) we show the picture for $s = -6$ of the critical set (a small circle), the image of the critical set (a deltoid shape figure) and part of the second image of the critical set (the three curves that are outside the deltoid). This situation is a limit one, for $s < -6$ the image of the critical set will not intersect the second image, and for $s > -6$ the opposite will happen.

The *filled-in set* of f_s is, by definition, the set of points z that does not go to ∞ under iterations of f_s. In the pictures presented here the filled-in set is always shown with a uniform pattern of dots. The evolution of the filled-in set is shown in the sequence of pictures shown in Fig. 3. For values of s larger than 0.3, the filled-in set looks like a delta-wing airplane (see Fig. 1(c)). The boundary of the filled-in set seems to have a fractal nature, but for the value $s = 0.5$, a closer and closer look shows that this property perhaps is not true. The pictures 1(d) to 1(f) show closer and closer views of part of the left side of the "wing."

Open-Problem. Is the boundary of the filled-in set a fractal for values of $s \in \mathbb{R}$ different from 0 and 2?

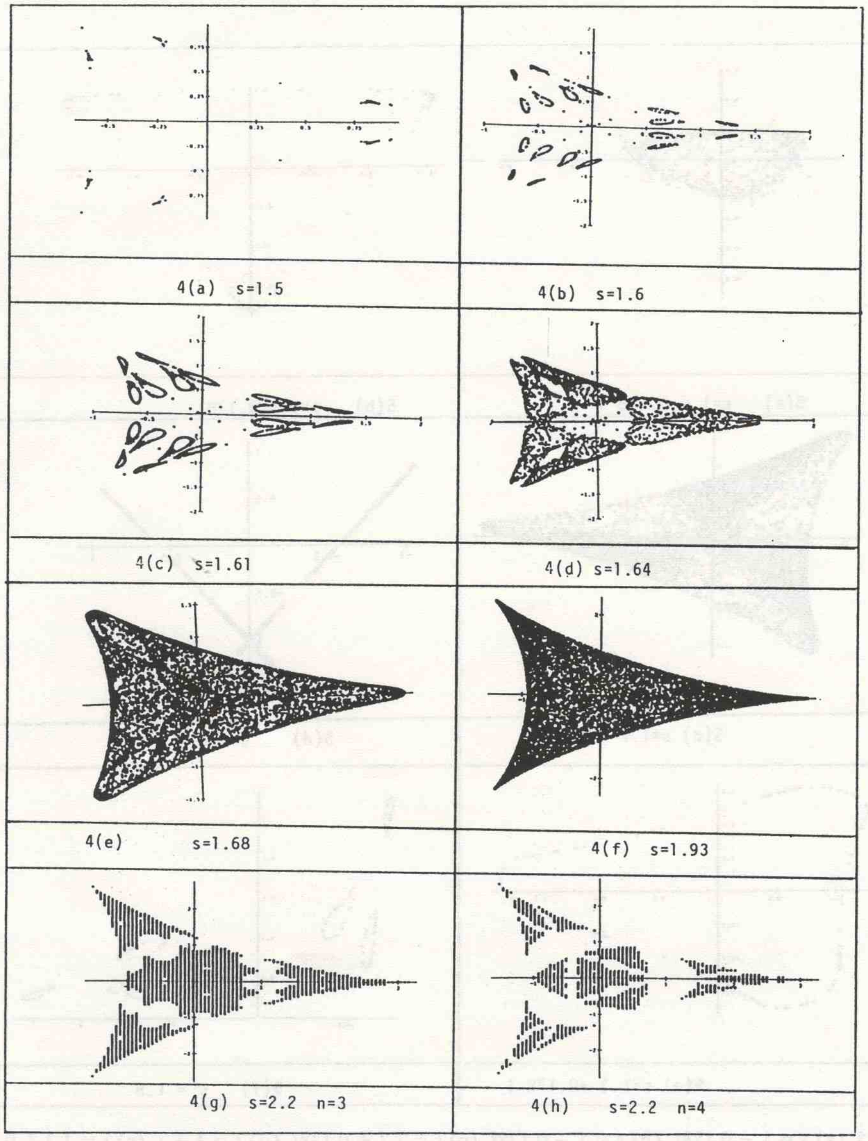

Fig. 4. (a) $s = 1.5$, (b) $s = 1.6$, (c) $s = 1.61$, (d) $s = 1.64$, (e) $s = 1.68$, (f) $s = 1.93$, (g) $s = 2.2$, $n = 3$, (h) $s = 2.2$, $n = 4$.

For small values of s (close to zero), the filled-in set is a topological disk. All points in the filled-in set seems to converge to zero. This situation looks similar to the one presented for the complex polynomial family $\ell_s(z) = z^2 - sz$ [3].

In Fig. 3, we show the evolution of the filled-in set with the changing of the parameter s. For values of s with modulus larger than one, a nontrivial attractor appears in the filled-in set. For values of s close to 2, it seems to exist an open region attracting all points of the filled-in set. For $s = 2$, the attractor is equal to the filled-in set.

In Fig. 2(a), we show that for $s = 1.68$, the attracting region is strictly contained in the filled-in set. Attractors here will be shown with a random pattern of dots. Figure 2(b) shows a closer look of the upper-left side of Fig. 2(a). In Fig. 2(d), we show the critical set, the first and second images of the critical set, and also the filled-in set.

Conjecture. For a set of positive measure of real values of s close to 2, there exist an invariant-measure absolutely continuous with respect to the two-dimensional Lebesgue measure.

For values of s between 1 and 1.641, it seems to exist nontrivial attractors with two-dimensional Lebesgue measure zero.

Conjecture. For values of s between 1.4 and 1.61, the attractor set has dimension 1.

The evolution of the attractor with the changing of the parameter s is shown in Fig. 4(a) to 4(f). The evolution of part of the attractor for values of s between 1.4 and 1.6 is shown in Fig. 6.

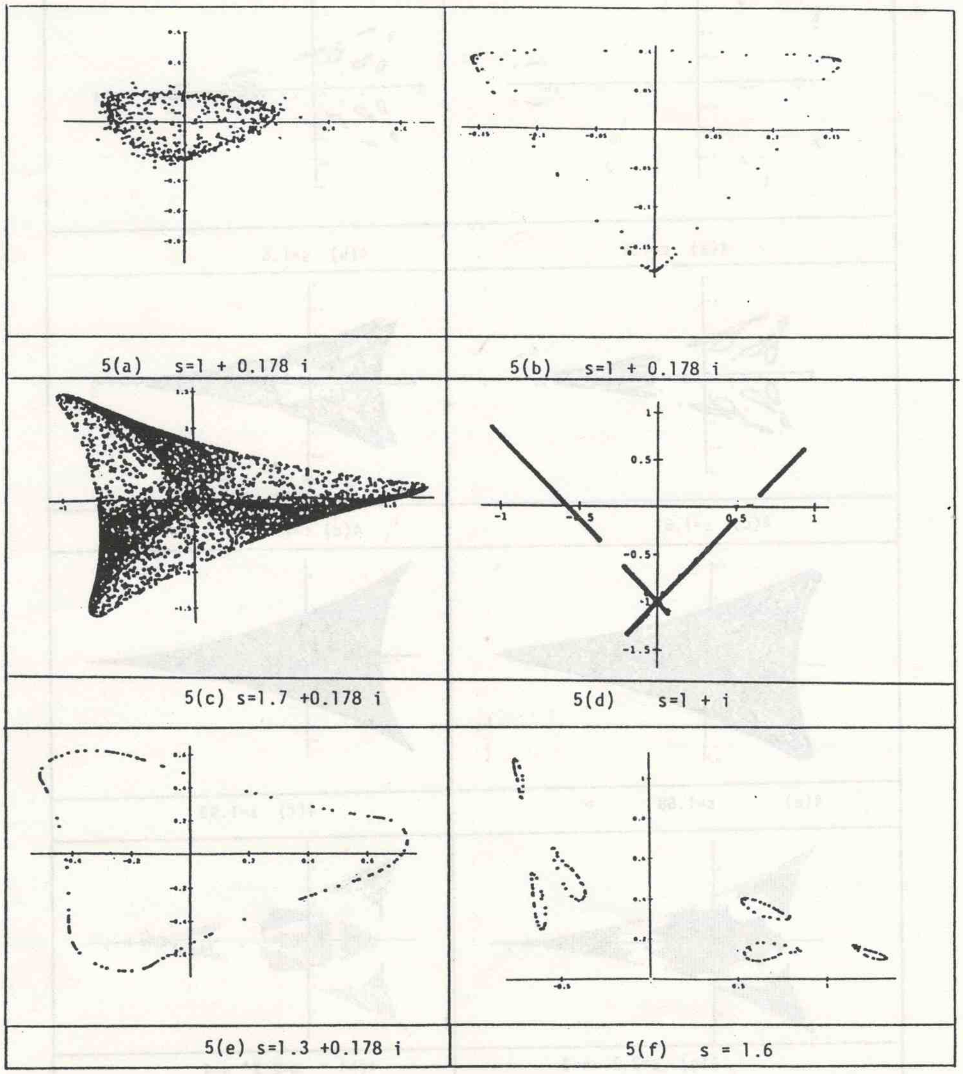

Fig. 5. (a) $s = 1 + 0.178i$, (b) $s = 1 + 0.178i$, (c) $s = 1.7 + 0.178i$, (d) $s = 1 + i$, (e) $s = 1.3 + 0.178i$, (f) $s = 1.6$.

First the attractor can be a point, but around $s = 1.5$ looks like a curve that turns out in a distorted hypotrocoid. For larger value of $s \approx 1.6$, it seems to exist distorted circles that attract the points of the filled-in set.

The pictures of Fig. 6 are produced by magnifying part of the attractor. In fact, some of these curves are in an orbit of period 2 (see Fig. 5(f)).

For values of s larger than two, the nonwandering set seems to have a fractal nature and a Cantor-set structure. For $s = 2.2$ the nonwandering set has an appolonian packing shape (see [4]).

In Fig. 4(g) and 4(h) the shaded areas show the points that remain in a square centered in (0, 0) with size 8 after 3 and 4 iterates, respectively.

Conjecture. For values of s close to 2 and outside the cardioid shown in Fig. 1(b) the nonwandering set is expanding. Note that Fig. 4(g) and 4(h) seems to indicate the existence of a Cantor set with expanding dynamics. The above question is related with the triple point phase transition and Yang-Lee zeros problem mentioned in [12].

Suppose now that $s = a + bi$ with b different from zero. For small values of b a similar pattern of the case $b = 0$ seems to happen.

We followed the evolution of a, with a fixed value of $b = 0.178$. For values of a around 1, it seems to exist three points that attract the all filled-in set. It seems to exist invariant curves connecting such points. The attracting periodic points are shown in the points of accumulation of dots in Fig. 5(b). The curves connecting these points are shown in the accumulation of dots in Fig. 5(a). We believe there exist three other saddle periodic points such that the unstable manifolds of such points are the curves mentioned above.

For larger values of a, a similar situation like the one shown in Fig. 6 happens. For values of a around 1.7 the attractor has dimension two. The heavy dot-line areas in Fig. 5(c) show a much larger frequency

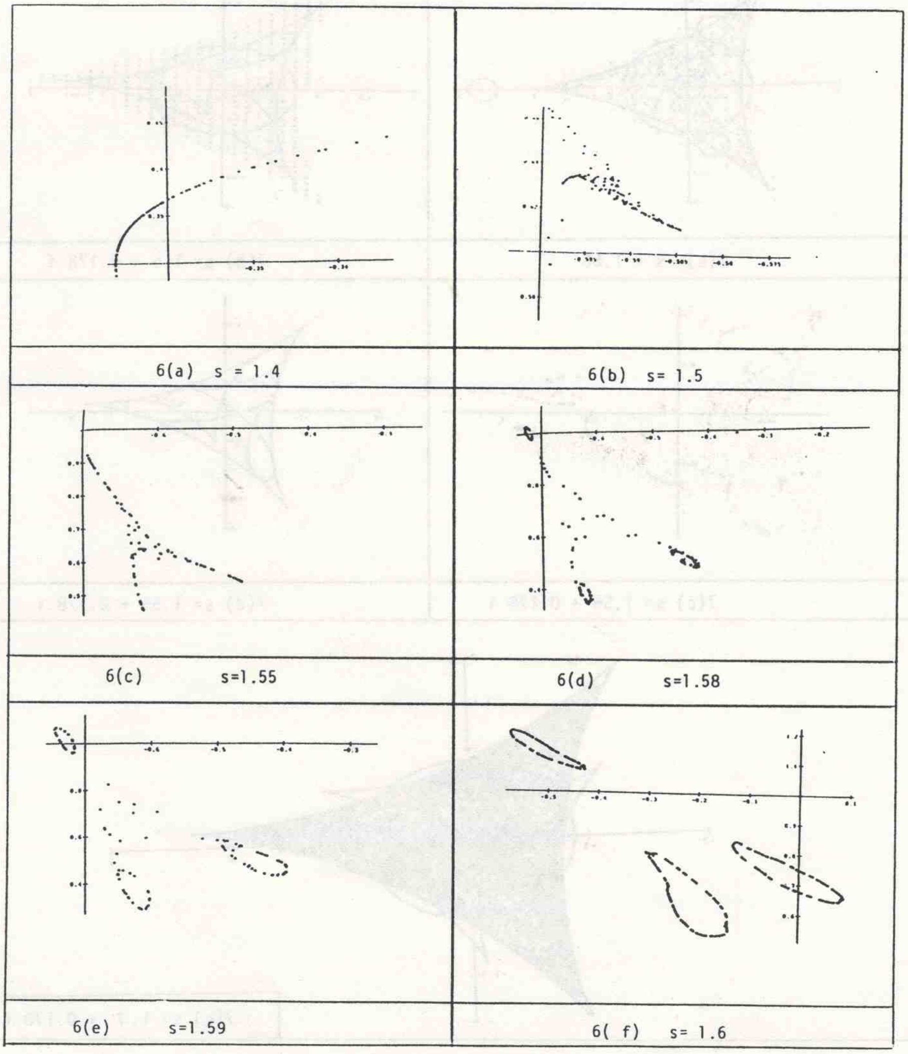

Fig. 6. (a) $s = 1.4$, (b) $S = 1.5$, (c) $S = 1.55$, (d) $S = 1.58$, (e) $S = 1.59$, (f) $S = 1.6$.

of iterates in some parts of the attractor of the initial point chosen in the filled-in set.

We also show the attractor for $s = 1 + i$ in Fig. 5(d). The area without dots is probably related with stable manifolds of saddle periodic points mentioned before.

It seems to exist a relation of the attractors with the position of the iterates of the critical set. In Fig. 7(b), we show the critical set, the first and second iterates of the critical set, the filled-in set, and some attractors (of the kind in Fig. 6(c)) presented for the value $s = 1.5 + 0.178i$.

Figure 7(a) shows the filled-in set, the critical set, the image of the critical set, the attractor, and the position of the four periodic fixed points centered in small ellipses. Figure 7(d) shows the attractor and the second iterate of the critical set. Figure 7(e) shows the attractor, the critical set, and the first and second iterates of the critical set.

3. AN X-TRANGE ATTRACTOR

In Fig. 8(d) we show the attractor set for f_s when $s = 1 + 1.05i$. The attractor has the shape of the letter X. We also show in Fig. 8 other kind of attractors for values of s very close to $s = 1 + i$. This value is a bifurcation parameter for the family, as it is shown in Fig. 8(b), 8(c), and 8(d).

In Fig. 8(a), the two line segments are in an orbit of period 2. The other X-trange attractor has period 1.

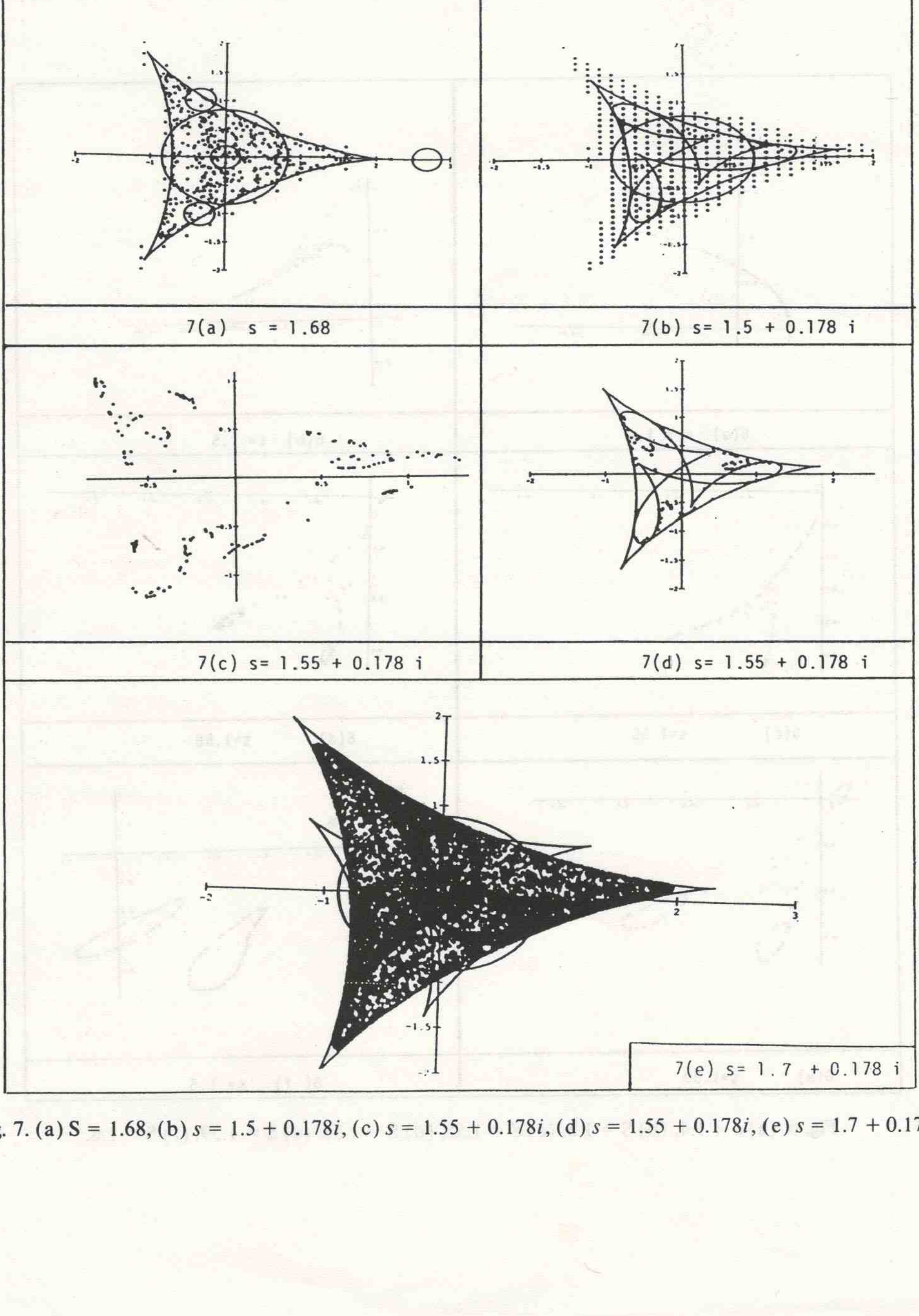

Fig. 7. (a) S = 1.68, (b) $s = 1.5 + 0.178i$, (c) $s = 1.55 + 0.178i$, (d) $s = 1.55 + 0.178i$, (e) $s = 1.7 + 0.178i$.

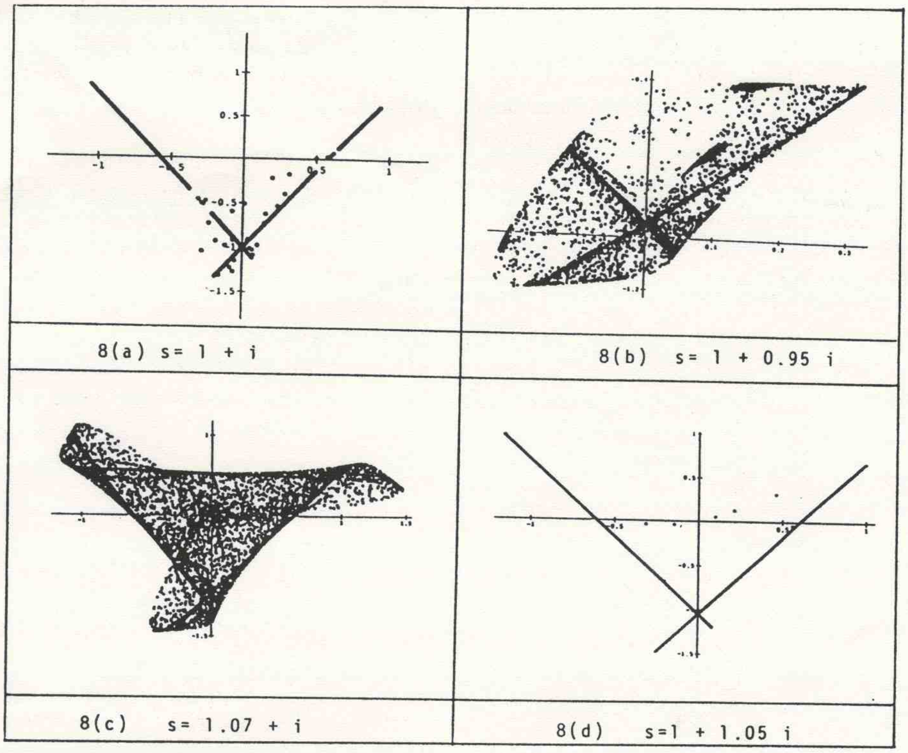

Fig. 8. (a) $s = 1 + i$, (b) $s = 1 + 0.95i$, (c) $s = 1.07 + i$, (d) $s = 1 + 1.05i$.

Acknowledgement—I would like to thank C. Leme for assistance in plotting the pictures presented in this paper.

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