Invariant Measures for Gauss Maps Associated with Interval Exchange Maps

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ABSTRACT. An explicit formula for an ergodic σ-finite measure invariant by the Gauss map associated to a new induction on the interval exchange maps is given. The techniques developed allow another proof of Keane's conjecture which was first shown to be true by Veech and Mazur.

1. Introduction. In this paper we study the induction defined in [7] for interval exchange maps from the metrical point of view.

In this induction we take $\mathbf{T} = \mathbf{T}(\pi, \alpha): [0,1) \to [0,1)$ an exchange of $m \geq 1$ intervals and $n > 0$ a critical iterate of $\mathbf{T}$ (this means that $\mathbf{T}^n(0)$ is closer to a discontinuity of $\mathbf{T}$ than any iterate $\mathbf{T}^k(0)$, $0 \leq k < n$) and stack the intervals $[\mathbf{T}^k(0), \mathbf{T}^\ell(0)]$, $0 \leq k$, $l \leq n$, free of $\mathbf{T}$-iterates of 0 up to the order n in its interior. This stacking is done upward up to the first discontinuity of $\mathbf{T}$ and downward down to the first discontinuity of $\mathbf{T}^{-1}$. In this way we get a finite number of towers of intervals which are in bijective correspondence with the points of the Farey cell of order n around $\mathbf{T}$, $\mathscr{F}_n = \mathscr{F}_n(\mathbf{T})$. $\mathscr{F}_n(\mathbf{T})$ is the equivalence class of $\mathbf{T}$ under the relation $\sim = \overset{n}{\sim}$ defined on the space of interval exchange maps by $\mathbf{T} \sim \mathbf{S}$ iff the itineraries of 0 under $\mathbf{T}$ and $\mathbf{S}$ on the respective permuted intervals are the same up to the n-th iterate. These classes will define a sequence of partitions of the space of interval exchange maps that get finer and finer as n grows and for most exchanges $\mathbf{T}$ (in the sense of Lebesgue measure) the sequence of atoms around $\mathbf{T}$ converges to $\mathbf{T}$.

To parametrize these towers and the corresponding Farey cells $\mathscr{F}_n$ we use a finite set of disjoint polyhedra $\mathscr{C}_\gamma \subseteq \mathbf{R}^{2m-2}$, $\gamma \in \mathscr{A}(\pi)$, which we call abstract Farey cells. This parametrization is a dynamically defined projective isomorphism. On $\mathscr{C} = \sum \mathscr{C}_\gamma$ we have naturally defined a locally projective map $\mathscr{G} = \mathscr{G}(\pi)$, the Gauss map, which takes a given set of towers associated to a

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critical iterate to the next one. Using the dynamics of $\mathscr{G}$ it is possible to capture the set of $\mathbf{T}$-invariant measures and therefore the uniquely ergodic ones, [7].

Now we come to the main results of this paper. We exibit an explicit formula

$$d\mu = \prod_{i=0}^{m-1} \frac{1}{L_i + R_{f(i)}} d\lambda$$

for an ergodic σ-finite $\mathscr{G}$-invariant measure. In this formula $f = f(\pi)$ is a bijection $\{0, 1, \dots, m-1\} \to \{1, 2, \dots, m\}$ depending only on the permutation π defining the space of interval exchange maps, $L_0, L_1, \dots, L_{m-1}$ and $R_1, R_2, \dots, R_m$ are the dynamically defined variables used to parametrize the cells and $d\lambda$ is the Lebesgue measure on $\mathscr{C}$.

We close the paper using the techniques developed to construct the measure $d\mu$ to give another proof of Keane's conjecture. This conjecture was first shown to be true by Veech [9] and Mazur [4]. See also Kerkhoff [3] and Rees [5]

The paper is organized as follows: in the next two sections we recall the induction introduced in [7]; in Section 2 we give examples and, in order to ilustrate the main features of our method, consider the cases of two and three intervals. In Section 3 we recall the general formalism of the induction and show that $d\mu$ is $\mathscr{G}$-invariant. In Section 4 we present the procedure used to get $d\mu$. This is the same procedure abstracted from Veech [9] by Arnoux-Nogueira and used in [2]. This construction will be useful in the next section when we show that $d\mu$ is conservative. The technical lemma needed in this section we pospone to the Appendix. Finally, in Section 6 we give a proof of the egodicity of $d\mu$ and another proof of Keane's conjecture.

2. Examples. We will illustrate the procedure sketched above considering the cases of interval exchange maps of respectively two and three intervals.

The case of two intervals.

In the case of just two permuted intervals we will denote by $\beta = \beta_1 = \alpha_1$ the discontinuity of $\mathbf{T}$. In this case the map is given by just one parameter, namely β.

Let's consider the particular example given by the map $\mathbf{T}$ described in Fig. 1. In this case if one follows the orbit of zero by $\mathbf{T}$ we see that $n = 4$ is a critical iterate. The location of the orbit of zero up to the 4-th iterate is presented in the x-axis of Figure 1. This order is:

$$0 < \mathbf{T}^2(0) < \beta < \mathbf{T}^4(0) < \mathbf{T}^1(0) < \mathbf{T}^3(0) < 1 \tag{2.1}$$

Consider the right and left intervals defined by the closest aproach to β given by $L = [\mathbf{T}^2(0), \beta)$ and $R = [\beta, \mathbf{T}^1(0)]$, respectively. In this particular example the value $\mathbf{T}^4(0)$ is in the interval R. In Figure 2 the interval $L \cup R$ is shown,

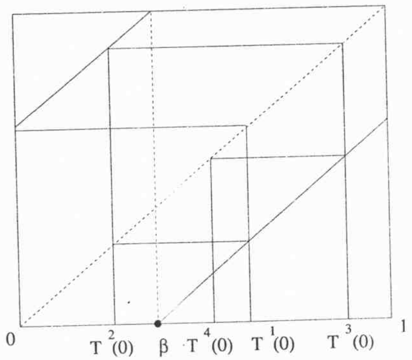


FIGURE 1

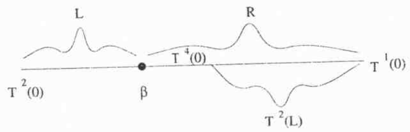


FIGURE 2

and we point out to the reader that the length of the interval $[\mathbf{T}^2(0), \beta)$ is equal to the length interval $[\mathbf{T}^4(0), \mathbf{T}^1(0))$. This fact is important in order to see that the next critical iterate is $\mathbf{T}^7(0)$ an that the new set of right and left intervals is $L^* \cup R^*$ where $L^* = [\mathbf{T}^2(0), \beta)$ and $R^* = [\beta, \mathbf{T}^4(0)]$. In the particular case we are considering here the critical iterate $\mathbf{T}^7(0)$ is in the interval L^*, but it could also happen that $\mathbf{T}^7(0)$ be in R^*.

To see the truth of these assertions stack the intervals defined by the iterates of (2.1) as described in the introduction. In the present example the stacks associated to the critical iterate $\mathbf{T}^4(0)$ are shown in the two stacks of Figure 3 a). Note that the full dynamical information about the map $\mathbf{T}$ is contained in this picture since each interval is mapped by $\mathbf{T}$ on the interval that is placed on the top of it in the stack and the top intervals the stacks join to form $L^* \cup R^*$ and each of these intervals is mapped to the botton of the opposite stack.

The two stacks associated to the next critical iterate, which is $\mathbf{T}^7(0)$, is shown in Figure 3 b). This can be easily understood as follows: move the stack that do not contain the discontinuity β in its top to the bottom in such way the property "each interval is mapped into the interval that is on top of it" is mantained. In this way we obtain the next stack given in Figure 3 b). The fact that the new critical iterate is determined by the previous critical iterate and has the stated properties is now transparent (see Figure 3).

The procedure is always the same, each critical iterate will determine the

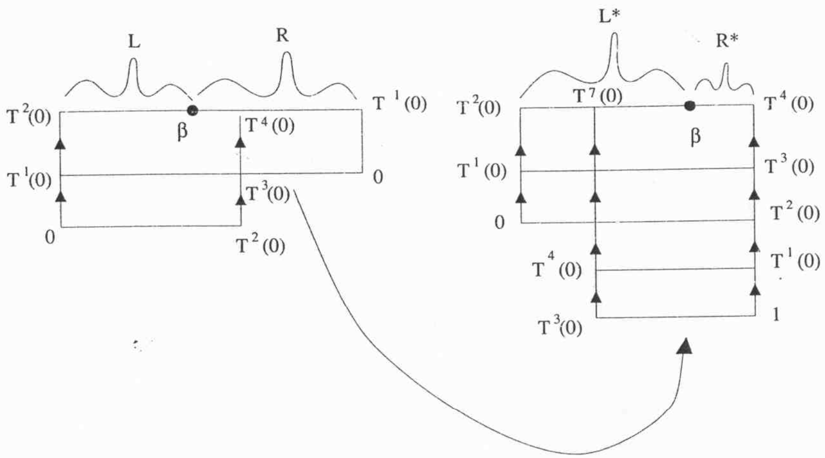


FIGURE 3(A) FIGURE 3(B)

next one. The previous critical iterate will be one of the extremals of the new interval $L^* \cup R^*$ containing the next critical iterate.

Note that the stacks of Figure 3 b) also describe the full dynamics of the same exchange map $\mathbf{T}$, but now with a different height and witdth of the stacks.

Now we want the analytical expression for the lengths of the new left and right intervals, L^* and R^*, obtained from the lengths of the preceding intervals, L and R. The new values for L^* and R^* will depend on the position of the critical iterate: if it is in R or in L. The two possibilities are shown in Figure 4. For example in Figure 4 b) the critical iterate is in R as in the example we considered in the beginning.

In order to simplify the notation we will denote the size of the intervals L, R, L^* and R^* by the same letters L, R, L^* and R^*, respectively. If we normalize these variables by requiring that $L+R=1$, we have in fact just one free variable. We choose to work with $x=L$. The Gauss map $\mathscr{G}$ at x, $\mathscr{G}(x)$, will express the value of the new L^* in a normalized form, that is $\mathscr{G}(x) = L^*/(L^*+R^*)$. The Gauss map in the present situation is defined from $[0,1]$ to $[0,1]$. The abstract Farey cell in this case has just one piece, namely $[0,1]$. Note the very important fact that the critical iterate is always in the larger of the intervals L or R (the reader should convince himself of this fact by looking at the several possibilities of the graph of $\mathbf{T}$). If $L<R$ then $x<1/2$ and $L^*=L$ (the critical iterate was in R). The interval R^* is equal to $R-L$ (see Figure 2). Therefore $L^*+R^* = L+R-L = 1-x$. The new normalized L^* is $\mathscr{G}(x) = L^*/(L^*+R^*) = x/(1-x)$.

In case $L>R$ we have $x>1/2$ and the new L^* is $L-R = x-(1-x) = 2x-1$ (the critical iterate was in L). Then $R^*=R$, and $L^*+R^* = L-R+R = L = x$. Therefore the normalized L^* is given by $\mathscr{G}(x) = (2x-1)/x$.

In this case $\mathscr{G}$ is also known as the backward continued fraction map, [1],

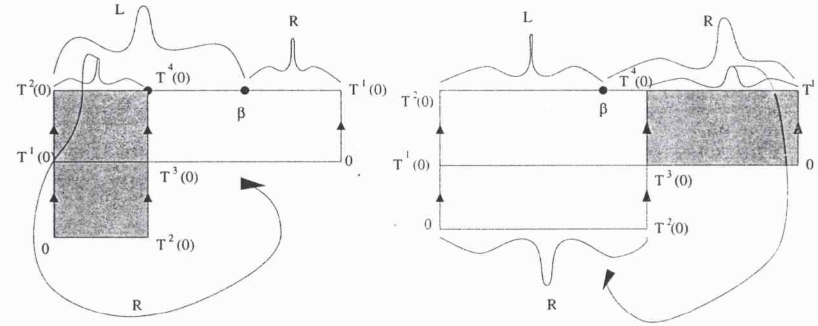


FIGURE 4

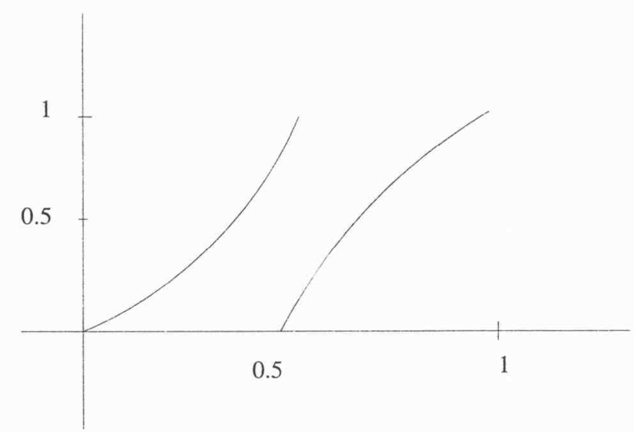


FIGURE 5

and its graph is shown in Figure 5. The map $\mathscr{G}$ is not expanding due to the fixed points 0 and 1 that have eigenvalue 1. This map $\mathscr{G}$ leaves invariant a σ-finite invariant measure given by $dx/(x(1-x)$. Measures with infinite mass will appear in all cases of Gauss maps which we will consider here. One of the purposes of this paper is to show explicit formulas for infinite measures (equivalent to Lebesgue measure) which are invariant by the Gauss maps in the case of interval exchange maps with more than three intervals.

The case of three intervals. We will denote by $\beta_1 = \alpha_1$ the first discontinuity and $\beta_2 = \alpha_1 + \alpha_2$ the second discontinuity of $\mathbf{T}$, where α_1, α_2 and α_3 are the lengths of the intervals permuted by $\mathbf{T}$.

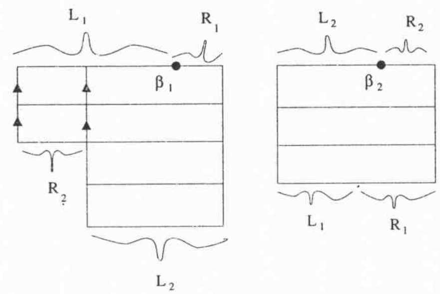
Fix a critical iterate n of $\mathbf{T}$ and denote by L_1, R_1 and L_2, R_2 the left and right intervals around the discontinuities β_1 and β_2. We get these intervals by taking the points in the $\mathbf{T}$-orbit of 0 up to the $n-1$-th iterate closest to the left and right of the respective discontinuities. The critical iterate $\mathbf{T}^n(0)$ can possibily be in any one of these four intervals.

As we did before we will denote an interval and its length by the same symbol and consider the normalizing condition $L_1 + R_1 + L_2 + R_2 = 1$. In fact there is just two independent variables (because there exists always one more relation among L_1, R_1, L_2, R_2 as we will see in a moment) which we choose to be R_1 and R_2. We will be interested in finding the new R_1^* and R_2^* using the procedure of going from one critical iterate to the next critical iterate. This procedure is analogous to the previous one and is also described by moving the stacks in such way that the old critical iterate will turn out to be a new extremal of one of the intervals $L_1^* \cup R_1^*$ and $L_2^* \cup R_2^*$.

It is not difficult for the reader to convince himself that in the present situation, the only possibilities for the stacks are the ones schematically shown in Figures 6 a), 7 a) and 8 a). There are three towers, two of them always coming together in a discontinuity and the third with its top at the other discontinuity of $\mathbf{T}$.

To simplify the notation we will denote the intervals $\mathbf{T}(L_1), \mathbf{T}(R_1), \mathbf{T}(L_2)$ and $\mathbf{T}(R_2)$, which are at the bottom of the stacks by L_1, R_1, L_2 and R_2, respectively. Note however that the sizes of these intervals are correspondingly equal.

In the first case Figure 6 a), (denoted by cell I), the critical iterate can be in L_1 or in R_1. These two cases will have to be considered when we define the Gauss map. Before doing that, however, we will describe the Farey cells (Figures 6 b), 7 b) and 8 b)). Note that in Figure 6 a) the right tower give us the relation $L_1 + R_1 = L_2 + R_2$. As $1 = L_1 + R_1 + L_2 + R_2 = 2(L_1 + R_1) = 2(L_2 + R_2)$, then $L_1 = 1/2 - R_1$ and $L_2 = 1/2 - R_2$. Therefore the possible values of (R_1, R_2) are in the square $[0, 1/2] \times [0, 1/2]$ (see Figure 6 b)). The upper triangle of the square correspond to $L_1 < R_1$ (in this case the critical iterate is in R_1) and the lower triangle of the square correspond to $L_1 > R_1$ (in this case the critical iterate is in L_1). The two possibilities are shown in the top towers of Figures 10 and 11.



$$\frac{1}{R_2(1-2R_2)}\, dR_1\, dR_2$$

FIGURE 6(A)

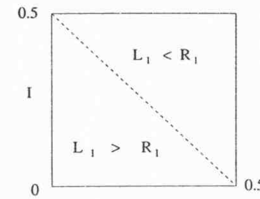
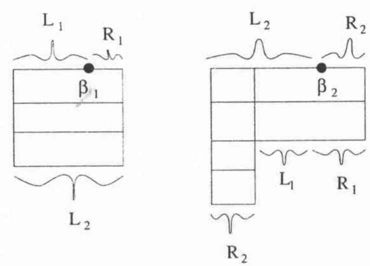


FIGURE 6(B)



$$\frac{1}{R_2(1-R_2)^2}\, dR_1\, dR_2$$

FIGURE 7(A)

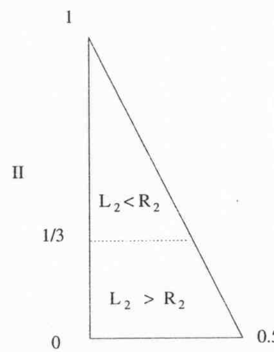
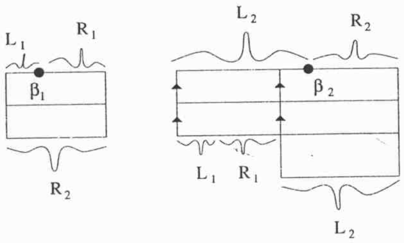
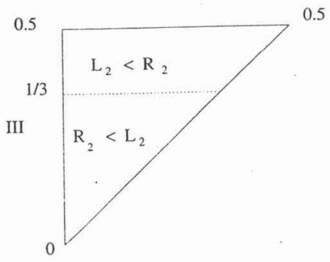


FIGURE 7(B)





$$\frac{1}{2R_2^2(1-2R_2)^2}\, dR_1\, dR_2$$

FIGURE 8(A) FIGURE 8(B)

Now we consider Figure 7 b). In this case from the left tower of Figure 7 a) we get the relation $L_2 = L_1 + R_1$. As $1 = L_1 + R_1 + L_2 + R_2 = 2L_1 + 2R_1 + R_2$, then $L_1 = 1/2 - (R_1 + 1/2R_2)$. The possible values of (R_1, R_2) are in the right triangle with height 1 and width 1/2 (see Figure 7 b)). The dotted horizontal line is at heigth 1/3. The upper triangle is given by the condition $R_2 > L_2$ (the critical iterate is in R_2) and the lower quadrilateral is given by the condition $R_2 < L_2$ (the critical iterate is in L_2). The two possibilities are shown in the two top towers of Figures 12 and 13. We will denote such cell by II.

Finally we will analyze Figure 8 b). The left tower from Figure 8 a) gives us the relation $L_1 = R_2 - R_1$. But as $L_1 + R_1 + L_2 + R_2 = 1$ we have $L_2 = 1 - 2R_2$. The values (R_1, R_2) are then in the right isoceles triangle with equal sides of lenght 1/2 shown in Figure 8 b).The dotted horizontal line is at height 1/3. The upper quadrangle contained in the triangle is given by the condition $L_2 < R_2$ (the critical point in R_2) and the lower triangle is given by the condition $L_2 > R_2$ (the critical point in L_2). The two possibilities are shown in the two top towers of Figures 14 and 15. Denote such cell by III.

Now that we defined the Farey cells, our next goal is to compute the Gauss map $\mathscr{G}$. This map is defined from the disjoint union of the three Farey cells to itself. It will be a two to one map. Note that each Farey cell have two subpieces described in Figures 6 b) 7 b) and 8 b). Each subtriangle or subquadrilateral will be mapped to one of the full Farey cells via a projective isomorphism in such way that triangles will go to triangles and quadrilaterals to the square I. The diagram of the Gauss map and its analytical expression is shown in Figure 9. Our next purpose is to show that the analytical expressions shown in this picture are correct. In other words, given the values R_1 and R_2, we want to know the new normalized values $(r_1, r_2) = \mathscr{G}(R_1, R_2)$, $r_1 = R_1^*/(L_1^* + R_1^* + L_2^* + R_2^*)$ and $r_2 = R_2^*/(L_1^* + R_1^* + L_2^* + R_2^*)$, where we denoted the new values of L_1, L_2, R_1 and R_2 by L_1^*, L_2^*, R_1^* and R_2^*, respectively. These new values are to be obtained

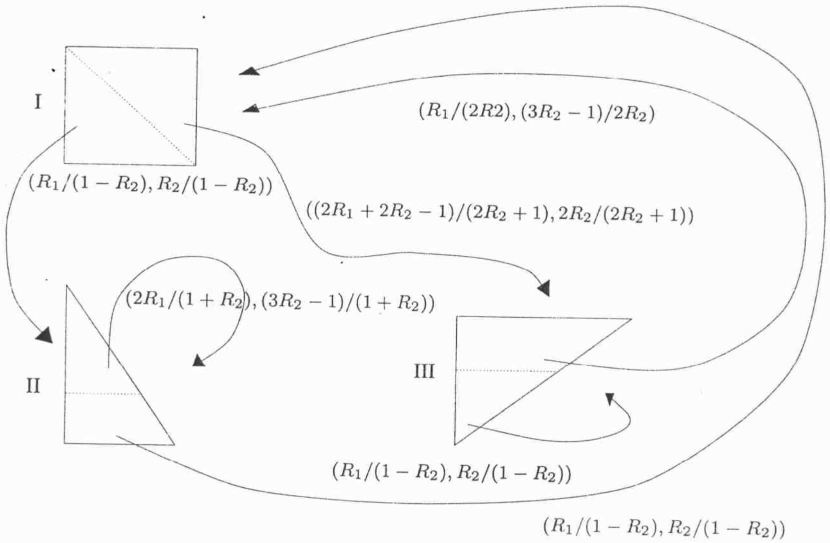


FIGURE 9

by the procedure of moving stacks associated to one critical iterate to the next one.

In order to define the Gauss map for (R_1, R_2) in case I, we have to analyze two possibilities:

(1) $L_1 > R_1$ (corresponding to the lower subtriangle of I) and
(2) $L_1 < R_1$ (corresponding to the upper subtriangle of I).

I(1) If $L_1 > R_1$, the moving stacks procedure lead us to map (L_1, R_1, L_2, R_2) to $(L_1^*, R_1^*, L_2^*, R_2^*) = (L_1 - R_2, R_1, L_2, R_2)$ (see Figure 10). This is so because, in the new towers, only the value of L_1 change. Note that the new stacks are of class II. This explains the arrow in the diagram of Figure 9 going from the lower triangle of I to II. From the sum $L_1^* + R_1^* + L_2^* + R_2^* = (L_1 - R_2) + R_1 + L_2 + R_2 = 1 - R_2$ we get the normalized values $(r_1, r_2) = (R_1/(1 - R_2), R_2/(1 - R_2))$. It is not difficult to see that the map taking (R_1, R_2) to (r_1, r_2) is one to one and map the lower triangle of I onto the full triangle II.

I(2) If $L_1 < R_1$, then the moving stacks procedure gives $(L_1^*, R_1^*, L_2^*, R_2^*) = (L_1, R_1 - L_2, L_2, R_2)$ (see Figure 11). The new towers are of class III. That is why the diagram of Figure 9 indicates that the upper subtriangle of I goes to III. We have $L_1^* + R_1^* + L_2^* + R_2^* = L_1 + (R_1 - L_2) + L_2 + R_2 = 1 - L_2 = R_2 + 1/2$. The last equality was obtained using the fact that, from the right top tower in Figure 11, $L_1 + R_1 = L_2 + R_2$, therefore as $L_1 + R_1 + L_2 + R_2 = 1$, then $L_2 + R_2 = 1/2$. From this fact also follows that $(L_1^*, R_1^*, L_2^*, R_2^*) = (L_1, R_1 + R_2 - 1/2, L_2, R_2)$.

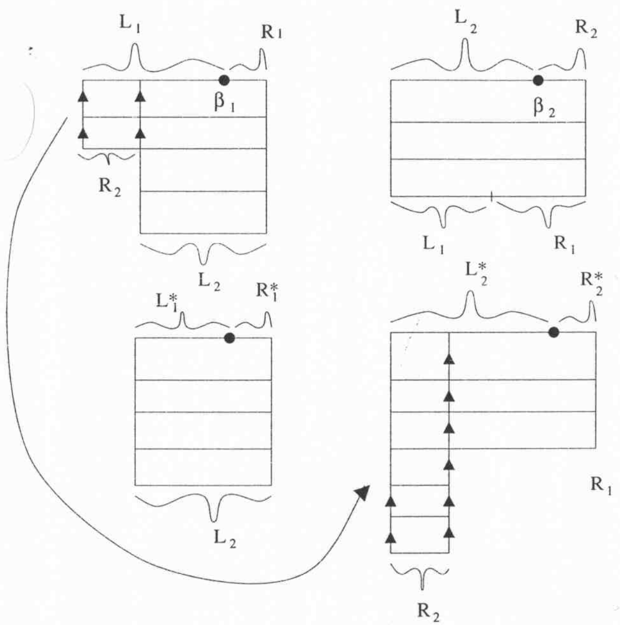


Figure 10

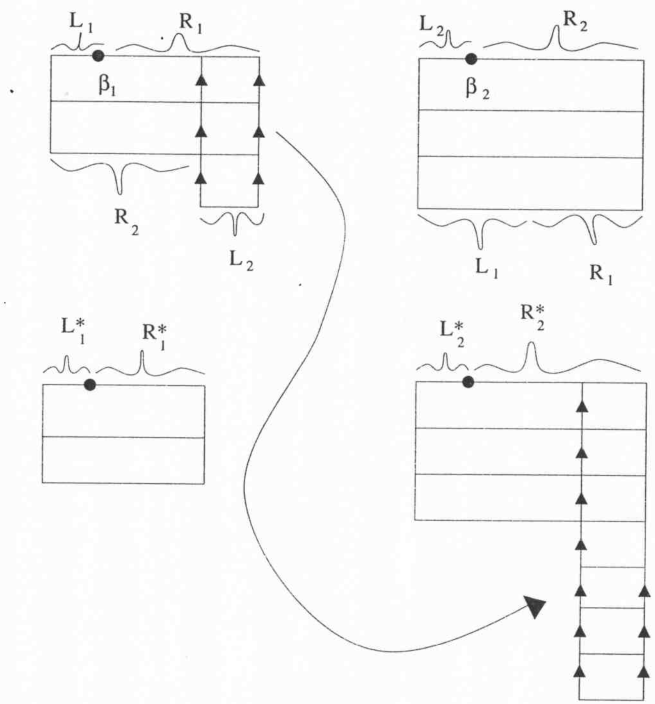


Figure 11

After normalization we obtain $(r_1, r_2) = ((2R_1+2R_2-1)/(2R_2+1), 2R_2/(2R_2+1))$. It is easy to see that $\mathcal{G}$ is one to one and onto from the upper subtriangle of I to the triangle III.

Now we will define the Gauss map for (R_1, R_2) in the triangle II. We have again two possibilities:

(1) $R_2 < L_2$ (corresponding to the subquadrangle of II) and
(2) $R_2 > L_2$ (corresponding to the subtriangle of II).

II(1) If $R_2 < L_2$, (L_1, R_1, L_2, R_2) goes to $(L_1, R_1, L_2 - R_2, R_2)$ as can be seen in Figure 12. The sum $L_1^* + R_1^* + L_2^* + R_2^* = 1 - R_2$ gives the normalizing condition. Therefore $(r_1, r_2) = (R_1/(1-R_2), R_2/(1-R_2))$. In this case II goes to I bijectively as indicated in Figure 9.

II(2) If $R_2 > L_2$, the moving stacks procedure associates (L_1, R_1, L_2, R_2) to $(L_1, R_1, L_2, R_2 - L_2)$ (see Figure 13). The normalization factor is $L_1^* + R_1^* + L_2^* + R_2^* = 1 - L_2$. After a simple calculation (as $L_2 = L_1 + R_1$ in the left top tower of Figure 14 and $L_1 + R_1 + L_2 + R_2 = 1$, then $2L_2 + R_2 = 1$) we obtain

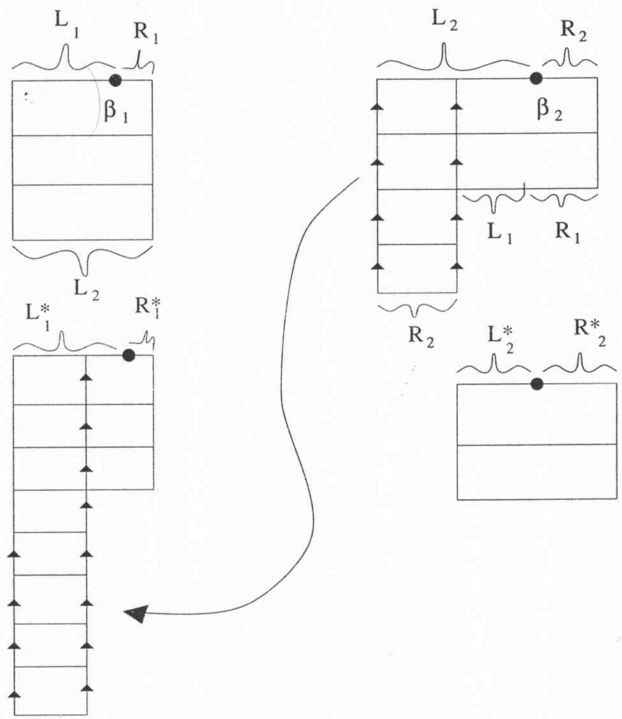


FIGURE 12

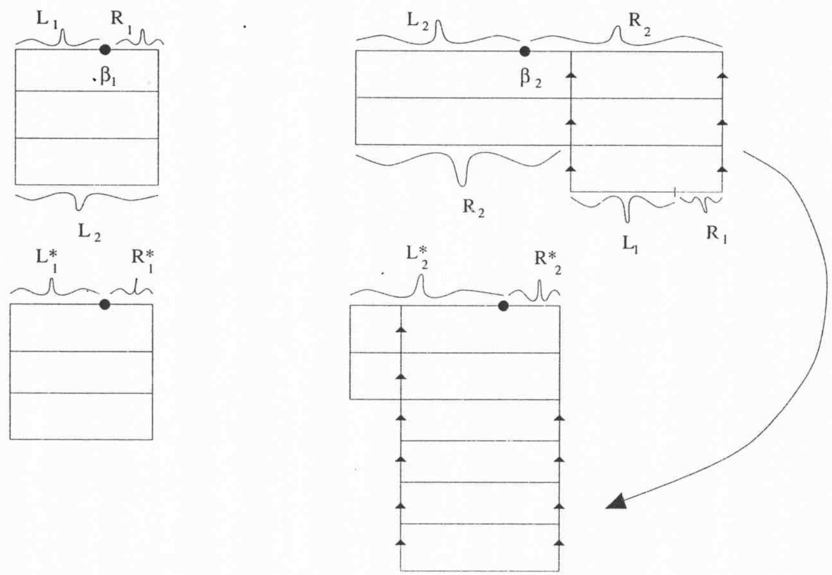


FIGURE 13

$1-L_2 = 1-(1/2-1/2R_2) = 1/2+1/2R_2$ and therefore the Gauss map is given by $(r_1, r_2) = (2R_1/(1+R_2), (3R_2-1)/(1+R_2))$. In this case II goes to II by the Gauss map (see Figure 13).

To define the Gauss map in the triangle III we have two possibilities

(1) $L_2 > R_2$ (corresponding to the subtriangle of III) and
(2) $L_2 < R_2$ (corresponding to the subquadrangle of III).

III(1) If $L_2 > R_2$, the procedure (see Figure 14) associates (L_1, R_1, L_2, R_2) to (L_1, R_1, L_2-R_2, R_2). The sum $L_1^*+R_1^*+L_2^*+R_2^* = 1-R_2$ will give the normalizing factor. The Gauss map $\mathcal{G}(R_1, R_2) = (r_1, r_2) = (R_1/(1-R_2)), R_2/(1-R_2))$ will map the subtriangle of III onto III.

III(2) If $L_2 < R_2$, (L_1, R_1, L_2, R_2) will be taken to (L_1, R_1, L_2, R_2-L_2). As $L_1+R_1+L_2+R_2 = 1$ and $L_1+R_1 = R_2$ (see the left top tower of Figure 15), we conclude that $L_2 = 1-2R_2$. Therefore the sum $L_1^*+R_1^*+L_2^*+R_2^* = 1-L_2 = 2R_2$ will determine the normalization condition. In this case the Gauss map is $\mathcal{G}(R_1, R_2) = (r_1, r_2) = (R_1/(2R_2), (3R_2-1)/2R_2)$, and map the subquadrangle of III into the square I.

The diagram and analytical expressions given in Figure 9 are thus justified.

To finish this section let us point out the formulas for a $\mathcal{G}$-invariant measure in this particular case we are considering of three intervals permuted.

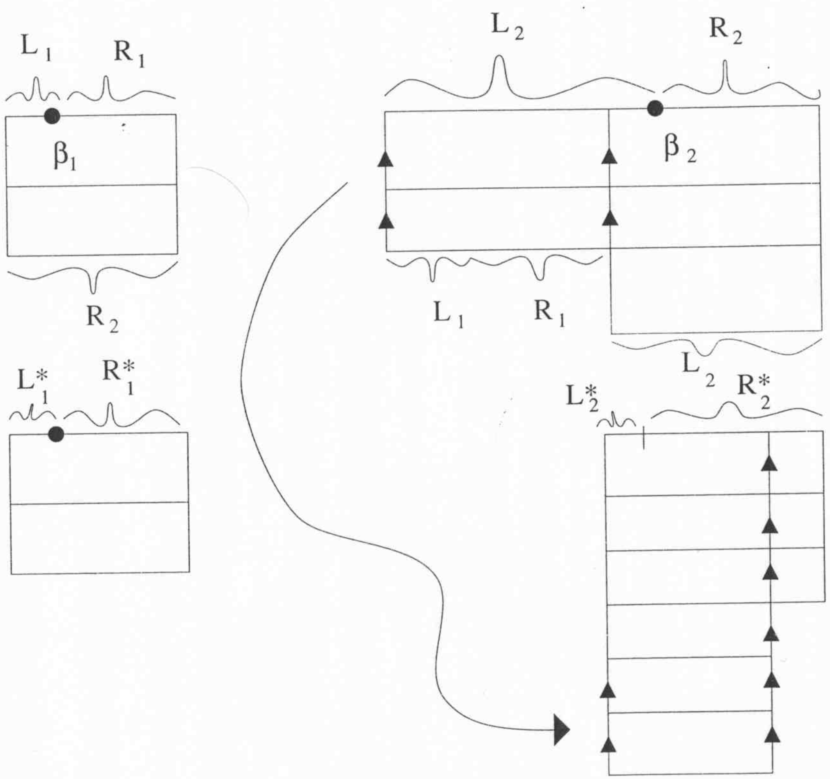


FIGURE 14

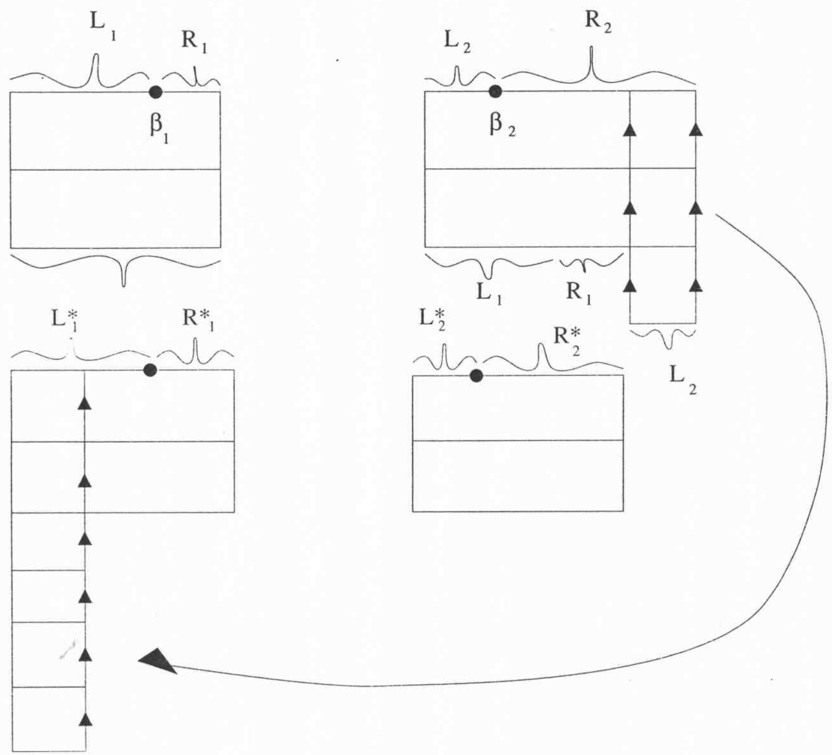


FIGURE 15

They are given explicitely by:

(i) $(R_2(1-2R_2))^{-1}dR_1dR_2$ in cell I,
(ii) $(R_2(1-R_2)^2)^{-1}dR_1dR_2$ in cell II and
(iii) $(2R_2^2(1-2R_2))^{-1}dR_1dR_2$ in cell III.

It is a measure absolutely continuous with respect to the Lebesgue measure which has infinite mass.

3. The invariant measure. In this section we recall the general formalism for the induction introduced in [7] and show the $\mathscr{G}$-invariance of $d\mu$.

Given π a permutation of $\{1,...,m\}$ irreducible and discontinuous, define:

$$f = f(\pi): \{0,...,m-1\} \to \{1,...m\}$$

by:

$$f(j) = \begin{cases} \pi^{-1}(1) - 1, & \text{if } j = 0; \\ m, & \text{if } j = \pi^{-1}(m); \\ \pi^{-1}(\pi(j)+1) - 1, & \text{otherwise.} \end{cases}$$

if $\pi(m) + 1 = \pi(1)$ and

$$f(j) = \begin{cases} \pi^{-1}(1) - 1, & \text{if } j = 0; \\ m, & \text{if } j = \pi^{-1}(\pi(1)-1); \\ \pi^{-1}(\pi(m)+1) - 1, & \text{if } j = \pi^{-1}(m); \\ \pi^{-1}(\pi(j)+1) - 1, & \text{in the remaining cases.} \end{cases}$$

if $\pi(m) + 1 \neq \pi(1)$.

It is easy to see that f is bijective.

Now, using f define the set $\mathscr{A} = \mathscr{A}(\pi)$ of pairs $\gamma = (g, G)$ where:

$$g: \{0,...,m-1\} \to \{1,...,m-1\}$$

and

$$G: \{1,...,m\} \to \{1,...,m-1\}$$

satisfy:

(i)

(3.2) $$g = G \circ f$$

(ii)

(3.3) $$\{g(0), g^2(0), ..., g^{m-1}(0)\} = \{1, 2, ...m-1\}$$
$$= \{G(m), G^{\,2}(m), ..., G^{\,m-1}(m)\} \text{ and } f(g^{m-1}(0)) \neq G^{\,m-1}(m)$$

(iii) C_γ, the convex subset of $\mathbf{R}^{2(m-1)} = \{0\} \times \mathbf{R}^{m-1} \times \mathbf{R}^{m-1} \times \{0\} \subseteq \mathbf{R}^{2m}$ given by the column matrices $(L_0, L_1, ..., L_{m-1}, R_1, R_2, ..., R_m)^t$ satisfying:

(a)

(3.4) $$L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)};\; i = 1, ..., m-1$$

(b) $$L_i > 0 \text{ and } R_i \geq 0\;;\; i = 1, ..., m-1$$

(c)

(3.5) $$\sum_{i=1}^{m-1} (L_i + R_i) = 1$$

has dimension $m-1$.

We call the convex set C_γ the abstract Farey cell of type γ.

It follows from (3) that g and G are onto and there is precisely one $i_0 \in \{1,...,m-1\}$ such that $\#g^{-1}(i_0) = \#G^{-1}(i_0) = 2$. We say that i_0 is the type of γ or, by abuse of language, the type of g (or G).

Note that we can also write (3.4a) as:

$$L_i + R_i = \sum_{k \in G^{-1}(i)} L_{f^{-1}(k)} + R_k\;;\; i = 1, ..., m-1$$

or, more simetrically :

$$L_i + R_i = L_{g^{-1}(i)} + R_{G^{-1}(i)}\;;\; i = 1, ..., m-1 \text{ and } i \neq i_0$$

$$L_{i_0} + R_{i_0} = L_{g^{-1}(i_0)} + R_{G^{-1}(i_0)} + L_{g^{m-1}(0)} + R_{G^{m-1}(m)}$$

Where $g^{-1}(G^{-1})$ is the unique right inverse of g (resp. G) which misses $g^{m-1}(0)$ (resp. $G^{m-1}(m)$) in its image.

Now we define the Gauss map

$$\mathscr{G} = \mathscr{G}(\pi): \mathscr{C} \to \mathscr{C}$$

where $\mathscr{C}$ = disjoint union of $\mathscr{C}_\gamma$, $\gamma \in \mathscr{A}$.

Before doing that, however, we recall that a map

$$\mathbf{S}: \mathbf{P} \cap \mathbf{A} \cap \mathscr{V} \to \mathbf{P} \cap \mathbf{A} \cap \mathscr{W}$$

where $\mathscr{V}$ and $\mathscr{W}$ are $m-1$-dimensional subspaces of $\mathbf{R}^{\mathbf{n}}$, is said to be projective if

$$\mathbf{S}(x) = \frac{Mx}{\mid Mx \mid} \tag{3.6}$$

for $x \in \mathscr{C} \cap \mathscr{A} \cap \mathscr{V}$ and M an $\mathbf{n} \times \mathbf{n}$ matrix with non-negative entries and whose restriction to $\mathscr{V}$ has determinant ± 1.

By $x \geq 0$ we mean that all entries of the $\mathbf{n}$ rows column matrix x are non-negative, $\mid x \mid = \sum_{k=1}^{\mathbf{n}} x_k$, $\mathbf{P} = \{x \geq 0 \mid x \in \mathbf{R}^{\mathbf{n}}\}$ and $\mathbf{A} = \{x \mid \mid x \mid = 1\}$.

It is clear that the inverse and composite of projective maps are projective. Since we will need the jacobian of a projective map, the following lemma from p.248 of Veech [8] is handy.

Lemma 3.1. *If $\mathbf{S}$ is a projective map as above and we take the Lebesgue measure on $\mathbf{A} \cap \mathscr{V}$ we have for $x \in \mathbf{P} \cap \mathbf{A} \cap \mathscr{V}$ that*

$$\Delta(x) = \text{Jacobian of } \mathbf{S} \text{ at } \mathrm{x} = \frac{1}{(\mid Mx \mid)^m} \tag{3.7}$$

We start by defining two maps $\mathscr{L}$ and $\mathscr{R}: \mathscr{A} \to \mathscr{A}$ as follows $\mathscr{L}(\gamma) = \gamma^{\mathscr{L}}$ where $\gamma = (g, G)$ and $\gamma^{\mathscr{L}} = (g^{\mathscr{L}}, G^{\mathscr{L}})$ is given by:

$$g^{\mathscr{L}}(j) = \begin{cases} g(j), & \text{if } \#g^{-1}(g(j)) = 1 \text{ or } j = g^{m-1}(0); \\ g^2(j), & \text{otherwise.} \end{cases}$$

and $G^{\mathscr{L}} = g^{\mathscr{L}} \circ f^{-1}$. As to the definition of $\mathscr{R}$ we have $\mathscr{R}(\gamma) = \gamma^{\mathscr{R}}$, where $\gamma = (g, G)$ and $\gamma^{\mathscr{R}} = (g^{\mathscr{R}}, G^{\mathscr{R}})$ is given by:

$$G^{\mathscr{R}}(j) = \begin{cases} G(j), & \text{if } \#G^{-1}(G(j)) = 1 \text{ or } j = G^{m-1}(m); \\ G^2(j), & \text{otherwise.} \end{cases}$$

and $g^{\mathscr{R}} = G^{\mathscr{R}} \circ f$. It is easily seen that $\gamma^{\mathscr{L}}$ and $\gamma^{\mathscr{R}}$ satisfy (3.2) and (3) above.

Now, fix $\gamma \in \mathscr{A}$ and consider the hyperplane $R_{i_0} = L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}$ where i_0 is the type of γ. This hyperplane divides the polyhedron $\mathscr{C}_\gamma$ into two polyhedra:

$$\mathscr{C}_\gamma^{\mathscr{R}} = \{R_{i_0} \geq L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}\} \cap \mathscr{C}_\gamma$$

$$\mathscr{C}_\gamma^{\mathscr{L}} = \{R_{i_0} < L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}\} \cap \mathscr{C}_\gamma$$

with non-empty interiors.

Restricting ourselves to $(L, R) \in \mathscr{C}_\gamma^{\mathscr{L}}$ and defining $L_i^{\mathscr{L}}$ and $R_i^{\mathscr{L}}$ by

$$R^{\mathscr{L}} = R_i \text{ for } i = 1, \ldots, m \tag{3.8}$$

and:

$$L_i^{\mathscr{L}} = \begin{cases} L_{i_0} - (L_{g^{-1}(i_0)} + R_{f(g^{-1}(i_0))}), & \text{if } i = i_0; \\ L_i, & \text{otherwise.} \end{cases} \tag{3.9}$$

we have that $\mathscr{L}(\gamma)$ is in $\mathscr{A}$ and the projective map induced by $\mathbf{L}(\gamma): (L, R) \mapsto (L^{\mathscr{L}}, R^{\mathscr{L}})$ is an isomorphism between $\mathscr{C}_\gamma^{\mathscr{L}}$ and $\mathscr{C}_{\mathscr{L}(\gamma)}$. Similarly $\mathscr{R}(\gamma)$ is in $\mathscr{A}$ and $\mathbf{R}(\gamma): (L, R) \mapsto (L^{\mathscr{R}}, R^{\mathscr{R}})$ given by:

$$L^{\mathscr{R}} = L_i \text{ for } i = 1, \ldots, m-1 \tag{3.10}$$

and:

$$R_i^{\mathscr{R}} = \begin{cases} R_{i_0} - (L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}), & \text{if } i = i_0; \\ R_i, & \text{otherwise.} \end{cases} \tag{3.11}$$

induces an isomorphism between $\mathscr{C}_\gamma^{\mathscr{R}}$ and $\mathscr{C}_{\mathscr{R}(\gamma)}$.

The Gauss map $\mathscr{G}$ is defined by $\mathscr{G} \mid_{\mathscr{C}_\gamma^{\mathscr{L}}} = \mathbf{L}(\gamma)$ and $\mathscr{G} \mid_{\mathscr{C}_\gamma^{\mathscr{R}}} = \mathbf{R}(\gamma)$ for $\gamma \in \mathscr{A}$.

On $\mathscr{C}$ take the σ-finite measure μ which has, on each $\mathscr{G}_\gamma$, $\gamma \in \mathscr{A}$, a density with respect to the Lebesgue measure $d\lambda$ given by

$$\Delta_\gamma = \prod_{i=0}^{m-1} \frac{1}{L_i + R_{f(i)}} \tag{3.12}$$

Proposition 3.1. *$d\mu$ is $\mathscr{G}$-invariant.*

Proof. All we have to do is check that the Perron-Frobenius equation

$$\Delta_\gamma(L, R) = \Delta_{\gamma_1}(L^1, R^1) \left| \frac{d\mathbf{L}^{-1}}{d\lambda}(L, R) \right| + \Delta_{\gamma_2}(L^2, R^2) \left| \frac{d\mathbf{R}^{-1}}{d\lambda}(L, R) \right| \tag{3.13}$$

holds, where $\gamma = (g, G) \in \mathscr{A}$, $(L, R) \in \mathscr{C}_\gamma$, $\gamma_1 = (g_1, G_1) = \mathscr{L}^{-1}(\gamma)$, $\gamma_2 = (g_2, G_2) = \mathscr{R}^{-1}(\gamma)$, $\mathbf{L}(L^1, R^1) = (L, R)$ and $\mathbf{R}(L^2, R^2) = (L, R)$.

By the definition of the Gauss map we have

$$L_i^1 = \begin{cases} \dfrac{L_{g^{m-1}(0)} + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}}{1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}}, & \text{if } i = g^{m-1}(0); \\[2ex] \dfrac{L_i}{1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}}, & \text{otherwise.} \end{cases}$$

$$R_i^1 = \frac{R_i}{1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}}$$

and

$$L_i^2 = \frac{L_i}{1 + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}}$$

$$R_i^2 = \begin{cases} \dfrac{R_{G^{m-1}(m)} + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}}{1 + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}}, & \text{if } i = G^{m-1}(m); \\[2ex] \dfrac{R_i}{1 + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}}, & \text{otherwise.} \end{cases}$$

for $i = 1, 2, \dots, m-1$.

Using (1.) we have

$$\left| \frac{d\mathbf{L}^{-1}}{d\lambda}(L, R) \right| = \frac{1}{(1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m-1)})^m}$$

and

$$\left| \frac{d\mathbf{R}^{-1}}{d\lambda}(L, R) \right| = \frac{1}{(1 + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)})^m}$$

The expressions above give

$$\Delta_{\gamma_1}(L^1, R^1) \left| \frac{d\mathbf{L}^{-1}}{d\lambda}(L, R) \right| + \Delta_{\gamma_2}(L^2, R^2) \left| \frac{d\mathbf{R}^{-1}}{d\lambda}(L, R) \right| =$$

$$= \frac{\prod_{i=0, i \neq g^{m-1}(0)}^{m-1} (L_i + R_{f(i)})^{-1}}{L_{g^{m-1}(0)} + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + R_{G^{-1}(i_0)}}$$

$$\frac{\prod_{i=0, i \neq g^{-1}(i_0)}^{m-1} (L_i + R_{f(i)})^{-1}}{L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}}$$

$$= \frac{\prod_{i=0, i \neq g^{m-1}(0), g^{-1}(i_0)}^{m-1} (L_i + R_{f(i)})^{-1}}{L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}}$$

$$\left(\frac{1}{L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}} + \frac{1}{L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}} \right)$$

$$= \prod_{i=0}^{m-1} \frac{1}{L_i + R_{f(i)}} = \Delta_\gamma(L, R)$$

which proves the proposition. □

4. The construction. In this section we describe the procedure that lead us to define the density [3.12] and justify the $\mathscr{G}$-invariance of $d\mu$.

Let V be an m-dimensional vector real space and denote by $\bigwedge^r = \bigwedge^r(V)$ the space of exterior r-forms over V, $0 \le r \le m$. Take $F_1, F_2, \dots, F_k$, $1 \le k \le m$, a linearly independent set in $\bigwedge^1$ and $0 \ne \Omega \in \bigwedge^m$. Although there are several ways in which we can factor Ω as an exterior product $\Omega = F_1 \wedge F_2 \wedge \dots \wedge F_k \wedge \omega$, $\omega \in \bigwedge^{m-k}$, it is easy to see that $\omega\,|_K$ is uniquely determined, where K is the kernel of the linear map $F\colon V \to \mathbf{R}^k$ with components F_i. We call ω the volume induced on K by Ω and $F_1, F_2, \dots, F_k$.

Globalizing this result for $k = 1$ we see that if M^m is a differentiable manifold (here and in what follows manifolds and maps are C^∞), Ω is a volume form on M^m and $f\colon M^m \to \mathbf{R}$ is a function then Ω induces a volume, ω, on $S = f^{-1}(r)$, where $r \in \mathbf{R}$ is a regular value of f. It is clear that if ψ is a diffeomorphism preserving Ω and f then the induced diffeomorphism in S preserves ω.

Now take M^m, Ω, f, r and S as above, ψ a diffeomorphism and φ_t, $t \in \mathbf{R}$, a one parameter group of diffeomorphism of M^m. Suppose ψ and φ_t comute and preserve Ω. If each orbit of φ_t intercepts S exactly once we can define a map $\Psi\colon S \to S$, $\Psi(s) = s'$, where s' is the only point in S in the φ_t orbit of $\psi(s)$. If X, the infinitesimal generator of φ_t, is transversal to S then the following lemma, whose proof is a simple calculation, holds:

Lemma 4.1. *Ψ is a diffeomorphism and preserves the $m-1$-form $\imath_X\Omega$ restricted to S, where $\imath_X\Omega$ is the inner product of X and Ω. Moreover, if we write $\Omega = df \wedge \omega$ as above, with $\imath_X\omega = 0$, we have:*

$$\imath_X\Omega\big|_{T_p(S)} = df_p(X_p)\ \omega\big|_{T_p(S)}$$

for $p \in S$.

To see how the above construction lead us to the density (3.12) we start by introducing a new set of variables $\ell_0, \ell_1, \ldots, \ell_{m-1}$ and $r_1, r_2, \ldots, r_m$ which will play the role of the heights of the stacks associated to the abstract Farey cell C_γ, $\gamma = (g, G) \in \mathscr{A}$, in such a way that:

(1) ℓ_i is the height that the stack with botton $L_i^\flat + R_{f(i)}^\flat$ has above the interval $L_i^\flat$ and
(2) r_j is the height that the stack with botton $L_{f^{-1}(j)}^\flat + R_j^\flat$ has above the interval $R_j^\flat$.

From these definitions we are led to the relations

$$\ell_i = r_{f(i)} \tag{4.14}$$

$i = 0, 1, 2, \ldots, m-1$ which shows that we can retain only the r_j's as a set of independent variables.

Now for each $\gamma = (g, G) \in \mathscr{A}$ take a copy of

$$\mathbf{R}^{3m-2} = \mathbf{R}^{m-1} \times \mathbf{R}^{m-1} \times \mathbf{R}^m$$

$\mathbf{R}_\gamma^{3m-2}$, with coordinates $(L_1, \ldots, L_{m-1}, R_1, \ldots, R_{m-1}, r_1, \ldots, r_m)$. and decompose $\mathbf{R}_\gamma^{3m-2}$ in two open cones, $\widetilde{\mathscr{C}_\gamma^{\mathscr{R}}}$ and $\widetilde{\mathscr{C}_\gamma^{\mathscr{L}}}$, given, respectively, by

$$R_{i_0} > L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}$$

and

$$L_{i_0} > L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}$$

where i_0 is the type of γ. On these cones define the maps

(1) $\tilde{\mathbf{R}} = \tilde{\mathbf{R}}(\gamma)$: $\widetilde{\mathscr{C}_\gamma^{\mathscr{R}}} \to \mathbf{R}^{3m-2}$, $\tilde{\mathbf{R}}(L, R, r) = (\tilde{L}^{\mathscr{R}}, \tilde{R}^{\mathscr{R}}, \tilde{R}^{\mathscr{R}})$, given by

$$\tilde{L}_i^{\mathscr{R}} = L_i \text{ for } i = 1, \ldots, m-1,$$

$$\tilde{R}_i^{\mathscr{R}} = \begin{cases} R_{i_0} - (L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}), & \text{if } i = i_0; \\ R_i, & \text{otherwise.} \end{cases}$$

$$\tilde{R}_i^{\mathscr{R}} = \begin{cases} r_{G^{-1}(i_0)} + r_{i_0}, & \text{if } i = G^{-1}(i_0); \\ r_i, & \text{otherwise.} \end{cases}$$

and

(2) $\tilde{\mathbf{L}} = \tilde{\mathbf{L}}(\gamma)$: $\widetilde{\mathscr{C}_\gamma^{\mathscr{L}}} \to \mathbf{R}^{3m-2}$, $\tilde{\mathbf{L}}(L, R, r) = (\tilde{L}^{\mathscr{L}}, \tilde{R}^{\mathscr{L}}, \tilde{R}^{\mathscr{L}})$, given by

$$\tilde{R}_i^{\mathscr{L}} = R_i \text{ for } i = 1, \ldots, m-1,$$

$$\tilde{L}_i^{\mathscr{L}} = \begin{cases} L_{i_0} - (L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}), & \text{if } i = i_0; \\ L_i, & \text{otherwise.} \end{cases}$$

$$\tilde{R}_i^{\mathscr{L}} = \begin{cases} r_{G^{m-1}(m)} + r_{f(i_0)}, & \text{if } i = G^{m-1}(m); \\ r_i, & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{\mathbf{R}}$ is a diffeomorphism onto the cone of $\mathbf{R}^{3m-2}_{\mathscr{R}(\gamma)}$ given by

$$r_{G^{-1}(i_0)} > r_{G^{m-1}(m)} \tag{4.15}$$

and $\tilde{\mathbf{L}}$ is a diffeomorphism onto the cone of $\mathbf{R}^{3m-2}_{\mathscr{L}(\gamma)}$ given by

$$r_{G^{m-1}(m)} > r_{G^{-1}(i_0)} \tag{4.16}$$

and that this set of maps define a diffeomophism ψ of the manifold $\mathbf{M} = \sum_{\gamma \in \mathscr{A}} \mathbf{R}_\gamma^{3m-2}$. Note that G in (4.15) and (4.16) above refers to $\mathscr{R}(\gamma)$ and $\mathscr{L}(\gamma)$ respectively. To be precise this diffeomorphism is not well defined on a finite set of hiperplanes but, since this is a set of zero measure, this little imprecision will not matter in what follows.

Finally define the flux φ_t on $\mathbf{M}$ by

$$\varphi_t(L, R, r) = (\exp(t)L, \exp(t)R, \exp(-t)r)$$

whose infinitesimal generator is $X(L, R, r) = (L, R, -r)$. It is clear that φ_t comutes with ψ.

On $\mathbf{M}$ take the volume element given, on each $\mathbf{R}_\gamma^{3m-2}$, by $\Omega = dL_1 \wedge \ldots \wedge dL_{m-1} \wedge dR_1 \wedge \ldots \wedge dR_{m-1} \wedge dr_1 \wedge \ldots \wedge dr_m$.

Given $\gamma = (g, G) \in \mathscr{A}$ define the subspace of $\mathbf{R}_\gamma^{3m-2}$ $K_\gamma = \bigcap_{i=1}^{m-1} \mathrm{KerF_i}$ where $F_i = L_i + R_i - \sum_{j\in g^{-1}(i)} L_j + R_{f(j)}$ for $i = 1, \ldots, m-1$ and on K_γ take the volume ω induced by Ω and the functionals F_i. We can write

$$\omega = dL_{g^{m-1}(0)} \wedge dR_1 \wedge \ldots \wedge dR_{m-1} \wedge dr_1 \wedge \ldots \wedge dr_m =$$

$$dL_1 \wedge \ldots \wedge dL_{m-1} \wedge dR_{G^{m-1}(m)} \wedge dr_1 \wedge \ldots \wedge dr_m$$

It is clear that φ_t and ψ go down to $\sum K_\gamma$, preserve this volume and permute the positive cones of the spaces K_γ. We denote the disjoint union of these cones by $\mathbf{K}$.

For each $\gamma \in \mathscr{A}$ take the total area of the sacks associated to γ,

$$A_\gamma = \sum_{i=1}^{m-1} \ell_i L_i + \sum_{j=1}^{m-1} r_j R_j \tag{4.17}$$

Using [4.14] we have

$$A_\gamma = \sum_{i=0}^{m-1} r_{f(i)}(L_i + R_{f(i)}) = \sum_{j=1}^{m} r_j(L_{f^{-1}(j)} + R_j)$$

On the hypersurface $A_\gamma = 1$, ω induces a volume element which we still call ω. This volume can be written as

$$\frac{\pm 1}{L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}} dL_{g^{m-1}(0)} \wedge dR_1 \wedge \cdots dR_{m-1} \wedge dr_1 \wedge \cdots \wedge \widehat{dr_{G^{-1}(i_0)}} \wedge \cdots \wedge dr_m$$

$$= \frac{\pm 1}{L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}} dL_1 \wedge \cdots dL_{m-1} \wedge dR_{G^{m-1}(m)} \wedge dr_1 \wedge \cdots \wedge \widehat{dr_{G^{m-1}(m)}} \wedge \cdots \wedge dr_m$$

where the superscript $\widehat{\ }$ indicates omission and i_0 is the type of γ. Since ψ and φ_t preserve ω it is clear that ψ and φ induce diffeomorphisms on A_γ and preserve the induced volume form.

Consider now the normalizing map N given on each C_γ, $\gamma = (g, G) \in \mathscr{A}$, by

$$N_\gamma = \sum_{i=1}^{m-1} L_i + R_i = \sum_{i=0}^{m-1} L_i + R_{f(i)} \tag{4.18}$$

Each orbit of φ_t intercepts the hypersurface $N_\gamma = 1$ exactly once and the hypothesis of lemma 1. are met thus showing that we have a diffeomorphism $\Psi\colon \mathbf{K}' \to \mathbf{K}'$, where $\mathbf{K}' = \mathbf{K} \cap \{N_\gamma = 1\}$, preserving the volume

$$\frac{\pm 1}{L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}} dL_{g^{m-1}(0)} \wedge dR_1 \wedge \cdots \wedge \widehat{dR_{G^{m-1}(m)}} \wedge \cdots \wedge dR_{m-1} \wedge dr_1 \wedge \cdots \wedge \widehat{dr_{G^{-1}(i_0)}} \wedge \cdots \wedge dr_m$$

$$= \frac{\pm 1}{L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}} dL_1 \wedge \cdots \wedge \widehat{dL_{g^{m-1}(0)}} \wedge \cdots \wedge dL_{m-1} \wedge dR_{G^{m-1}(m)} \wedge dr_1 \wedge \cdots \wedge \widehat{dr_{G^{m-1}(m)}} \wedge \cdots \wedge dr_m .$$

It is easy to see that Ψ covers $\mathscr{G}$ in the sense that $\pi \circ \Psi = \mathscr{G} \circ \pi$ where π is the projection $\pi(L, R, r) = (L, R)$. If we push the measure of $\mathbf{K}'$ by this projection we get, integrating in the fibers, that the volume form

$$\pm\left(\prod_{j=1}^{m} \frac{1}{L_{f^{-1}(j)} + R_j}\right) dL_{g^{m-1}(0)} \wedge dR_1 \wedge \cdots \wedge \widehat{dR_{G^{m-1}(m)}} \wedge \cdots \wedge dR_{m-1}$$

$$= \pm\left(\prod_{i=0}^{m-1} \frac{1}{L_i + R_{f(i)}}\right) dL_1 \wedge \cdots \wedge \widehat{dL_{g^{m-1}(0)}} \wedge \cdots \wedge dL_{m-1} \wedge dR_{G^{m-1}(m)}$$

is invariant by $\mathscr{G}$ (Each fiber is a simplex with volume a fraction depending only on m of the volume of the spanned paralelepiped.) This form induces a measure on each C_γ which, up to a constant, has the density (3.12) with respect to the Lebesgue measure.

5. $d\mu$ is conservative. In this section we show that $\mathscr{G}$ is conservative. This means that there is no wandering set of positive measure or, what is the same, that $\mathscr{G}$ induces a first return map on each subset of positive measure of $\mathscr{C}$. It is here that we will use that the construction of the preceding section gives the measure $d\mu$ which, as we know from the beginning, is $\mathscr{G}$-invariant.

Proposition 5.1. *$\mathscr{G}\colon(\mathscr{C}, \mu) \to (\mathscr{C}, \mu)$ is conservative.*

Proof. To get a contradiction, suppose that there is a $\mathscr{G}$-wandering subset of positive measure of $\mathscr{C}$. Taking the pull-back of this set by $\pi\colon \mathbf{K}' \to \mathscr{C}$ we get a Ψ-wandering subset of positive measure of $\mathbf{K}'$, $\mathscr{U}$. Since $\mathscr{U}$ has positive measure, the positive φ_t saturated of this set, $\mathscr{X}$, has infinite measure in $\mathbf{K}$. On the other hand, since $\mathscr{U}$ is Ψ-wandering and ψ and φ_t comute, we can write $\mathscr{X}$

as a disjoint union

$$\mathscr{X} = \bigcup_{n=1}^{\infty} \psi^n(\mathscr{D} \cap \{\varphi_t \text{ saturated of } \Psi^{-n}(\mathscr{U})\})$$

where $\mathscr{D}$ is the fundamental domain of the action of ψ on $\mathbf{K}$ given by

$$\mathscr{D} = \{\varphi_t(s) \mid N(s) = 1 \text{ and } 0 \le t \le \tau(s)\}$$

and $\tau(s)$ is the time needed to flow back to $\{N = 1\}$ from $\psi(s)$, $s \in \{N = 1\}$. Now, since ψ preserves measure, we get the contradiction that finishes the proof of the proposition if we show that $\mathscr{D}$ has finite volume since the sets $\mathscr{D} \cap \{\varphi_t \text{ saturated of } \Psi^{-n}(\mathscr{U})\}$ are disjoint. □

Lemma 5.1. *$\mathscr{D}$ has finite measure.*

Proof. It is enough to show that, for each $\gamma = (g, G) \in \mathscr{A}$, the measure of the set $\mathscr{D}_\gamma$ which is the intersection of $\mathscr{D}$ with the positive cone of F_γ is finite. In fact we will show that $\mathscr{D}_\gamma^{\mathscr{L}}$, the intersection of $\mathscr{D}_\gamma$ with the cone

$$r_{G^{m-1}(m)} > r_{G^{-1}(i_0)}$$

has finite measure. The proof that $\mathscr{D}_\gamma^{\mathscr{R}}$, the intersection of $\mathscr{D}_\gamma$ with the cone

$$r_{G^{-1}(i_0)} > r_{G^{m-1}(m)}$$

has finite volume is similar and will be left to the reader. □

$\mathscr{D}_\gamma^{\mathscr{L}}$ is the set of $2m-1$-column row matrices

$$\left(xL_1 \cdots xL_{m-1} \; xR_{G^{m-1}(m)} \; \frac{r_1}{x} \cdots \widehat{\frac{r_{G^{m-1}(m)}}{x}} \cdots \frac{r_m}{x} \right)$$

with entries satisfying

$$\begin{gathered}
L_i,\ R_i > 0, \quad i = 1, \ldots, m-1, \\
r_1 > 0, \ldots, r_m > 0, \\
L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)} \quad i = 1, \ldots, m-1,
\end{gathered}$$

$$r_{G^{m-1}(m)} > r_{G^{-1}(i_0)}$$

$$1 = \sum_{i=0}^{m-1} r_{f(i)}(L_i + R_{f(i)}) = \sum_{j=1}^{m} r_j(L_{f^{-1}(j)} + R_j), 1 = \sum_{i=1}^{m-1} L_i + R_i = \sum_{i=0}^{m-1} L_i + R_{f(i)},$$

and

$$1 \ge x \ge \frac{1}{1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}}.$$

If we eliminate x in the above expressions we get that $\mathscr{D}_\gamma^{\mathscr{L}}$ is the set of matrices

$$(L_1 \cdots L_{m-1} \; R_{G^{m-1}(m)} \; r_1 \cdots \widehat{r_{G^{m-1}(m)}} \cdots r_m)$$

with entries satisfying

$$\begin{gathered}
L_i,\ R_i > 0,\ i = 1, \ldots, m-1, \\
r_1 > 0, \ldots, r_m > 0\,, \\
L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)}\ ;\ i = 1, \ldots, m-1\,, \\
r_{G^{m-1}(m)} > r_{G^{-1}(i_0)}\,, \\
1 = \sum_{i=0}^{m-1} r_{f(i)}(L_i + R_{f(i)}) = \sum_{j=1}^{m} r_j(L_{f^{-1}(j)} + R_j)\,, \\
1 \ge \sum_{j=1}^{m} L_{f^{-1}(j)} + R_j\,,
\end{gathered}$$

and

$$L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + \sum_{j=1}^{m} L_{f^{-1}(j)} + R_j \ge 1\,.$$

We have to show that the integral

$$\int_{\mathscr{D}_\gamma^{\mathscr{L}}} \frac{dL_1 \cdots dL_{m-1} dR_{G^{m-1}(m)} dr_1 \cdots \widehat{dr_{G^{m-1}(m)}} \cdots dr_m}{L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}}$$

is finite. Integrating in the r's we get that the above integral is, up to a constant, equal to

$$\int \frac{dL_1 \cdots dL_{m-1} dR_{G^{m-1}(m)}}{(L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}) \prod_{j=1, \neq G^{-1}(i_0)}^{m} L_{f^{-1}(j)} + R_j}$$

over the set of matrices

$$(L_1 \cdots L_{m-1}\ R_{G^{m-1}(m)})$$

with entries satisfying

$$L_i, R_i > 0,\ i = 1, \dots, m-1$$

$$L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)}\ ;\ i = 1, ..., m-1$$

$$1 \geq \sum_{j=1}^{m} L_{f^{-1}(j)} + R_j$$

and

$$L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + \sum_{j=1}^{m} L_{f^{-1}(j)} + R_j \geq 1$$

Pull-back the above integral to the cone with vertex the origin and spanned by $\mathscr{C}_{\mathscr{L}^{-1}(\gamma)}$, using the linear map that induces $\mathscr{G}$, $\mathbf{L}(\mathscr{L}^{-1}(\gamma))$. We get the integral

$$\int \frac{dL_1 \cdots dL_{m-1} dR_{G^{m-1}(m)}}{\prod_{j=1}^{m} L_{f^{-1}(j)} + R_j}$$

over the set of matrices

$$(L_1 \cdots L_{m-1}\ R_{G^{m-1}(m)})$$

with entries satisfying

$$L_i, R_i > 0,\ i = 1, \dots, m-1$$

$$L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)} \quad i = 1, ..., m-1$$

$$1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} \geq \sum_{j=1}^{m} L_{f^{-1}(j)} + R_j$$

$$\sum_{j=1}^{m} L_{f^{-1}(j)} + R_j \geq 1$$

and

$$L_{i_0} \geq L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}$$

where now $(g, G) = \mathscr{L}^{-1}(\gamma)$.

In this integral we make the change of variables given by the formulae

$$L_i = tL_i',\ R_i = tR_i'\ ;\ i = 1, ..., m-1$$

and

$$1 = \sum_{j=1}^{m} L'_{f^{-1}(j)} + R'_j$$

If we trade the variable $R'_{G^{m-1}(m)}$ for the variable t in the integral thus obtained and integrate with respect to t we finally get, up to a constant, the integral

$$\int \frac{\ln(1 + L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}) dL_1 \cdots dL_{m-1}}{\prod_{j=1}^{m} L_{f^{-1}(j)} + R_j}$$

where for simplicity we dropped the primes. This integral is over the set of matrices

$$(L_1 \cdots L_{m-1})$$

with entries satisfying

$$L_i, R_i > 0,\ i = 1, \dots, m-1$$

$$L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)}\ ;\ i = 1, ..., m-1$$

$$\sum_{j=1}^{m} L_{f^{-1}(j)} + R_j = 1$$

and

$$L_{i_0} \geq L_{g^{-1}(i_0)} + R_{G^{m-1}(m)}$$

This integral, in its turn, is finite or infinite with the integral

$$\int \frac{dL_1 \cdots dL_{m-1}}{\prod_{j=1, \neq G^{m-1}(m)}^{m} L_{f^{-1}(j)} + R_j}$$

over the same set. This set is a polyhedron and can be decomposed as a union of simplexes. Using theorem 1. in the Appendix we see that the proof of lemma 1. is complete once we prove the next lemma.

Lemma 5.2. *Given $\gamma = (g, G) \in \mathscr{A}$ with type i_0 and a point*

$$P = (L_1 \cdots L_{m-1})$$

in the polyhedron given as above by

$$L_i, R_i \geq 0, \ i = 1, \dots, m-1$$

$$L_i + R_i = \sum_{j \in g^{-1}(i)} L_j + R_{f(j)} \ ; \ i = 1, ..., m-1 \tag{5.19}$$

$$\sum_{j=1}^{m} L_{f^{-1}(j)} + R_j = 1$$

and

$$L_{i_0} \geq L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} \tag{5.20}$$

the number of factors of the product

$$\prod_{j=1, \neq G^{m-1}(m)}^{m} L_{f^{-1}(j)} + R_j = \prod_{i=0, \neq g^{-1}(i_0)}^{m-1} L_i + R_{f(i)} \tag{5.21}$$

which are zero at P is less than the maximal number of linearly independent equations of the set

$$L_i = 0, R_i = 0; \ i = 1, \dots, m-1$$

which are satisfied by P

Proof. Since

$$L_i + R_i = L_i + R_{f(i)}; \ i = 1, \dots, m-1, i \neq i_0$$

the factors of (5.21) are $L_i + R_i$ for $i = 1, \dots, m-1$, $i \neq i_0$ and $L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}$.

We have several cases to consider depending on which factors of (5.21) are zero at P.

(1) If $L_{g^{m-1}(0)} + R_{G^{-1}(i_0)} = 0$ at P then $L_{g^{m-1}(0)} = R_{G^{-1}(i_0)} = 0$ and $R_{i_0} = 0$ at P since (5.20) implies $L_{g^{m-1}(0)} + R_{G^{-1}(i_0)} \geq R_{i_0}$. If $L_{g^{m-1}(0)} + R_{g^{m-1}(0)} > 0$ at P the lemma follows since each factor $L_i + R_i$ which vanishes gives one equation $L_i = 0$ and the factor $L_{g^{m-1}(0)} + R_{G^{-1}(i_0)} = 0$ the two equations $L_{g^{m-1}(0)} = R_{i_0} = 0$. If $L_{g^{m-1}(0)} + R_{g^{m-1}(0)} = 0$ we consider two cases

(a) $g^{m-1}(0) = i_0$ and

(b) $g^{m-1}(0) \neq i_0$.

In the first case $L_{g^{m-1}(0)} + R_{g^{m-1}(0)}$ is not a factor of (5.21) and the argument just made holds. In the second case take $k > \ell$, for ℓ such that $g^{\ell}(0) = i_0$, the last iterate of g, starting from above, $g^{m-1}(0)$, and going down, for which we have the equality $L_{g^k(0)} + R_{g^k(0)} = 0$. In this case each factor $L_i + R_i$ which vanishes at P gives one equation $L_i = 0$ and we have one extra equation, $L_i = 0$, satisfied besides $R_{i_0} = 0$, since either $k = l+1$ and then $L_{i_0} = 0$ for $L_{g(i_0)} + R_{g(i_0)} = L_{i_0} + R_{f(i_0)}$ or $k > \ell + 1$ and then using $0 = L_{g^k(0)} + R_{g^k(0)} = L_{g^{k-1}(0)} + R_{f(g^{k-1}(0))}$ we get $L_{g^{k-1}(0)} = 0$ which is again an extra equation since $L_{g^{k-1}(0)} + R_{g^{k-1}(0)} \neq 0$. This finishes the case $L_{g^{m-1}(0)} + R_{G^{-1}(i_0)} = 0$.

(2) If $L_{g^{m-1}(0)} + R_{G^{-1}(i_0)} > 0$ at P. take k, $k \in \{1, \dots, m-1\}$, the greatest iterate of g for which we have the equality $L_{g^k(0)} + R_{g^k(0)} = 0$ at P. If $k > \ell$, where $g^{\ell}(0) = i_0$, the lemma follows by repeating the argument we just made. We suppose then that $k < \ell$ and $L_{g^r(0)} + R_{g^r(0)} > 0$ for $r > k$. If $L_{g^s(0)} + R_{g^s(0)} > 0$ for some $s < k$ we can still get an extra equation $L_i = 0$ by the same argument. The only possibility left is $L_{g^s(0)} + R_{g^s(0)} = 0$ for $s \leq k$. We can write the equations (5.19) as:

$$\begin{aligned}
L_{g(0)} + R_{g(0)} &= R_{f(0)} \\
L_{g^2(0)} + R_{g^2(0)} &= L_{g(0)} + R_{f(g(0))} \\
L_{g^3(0)} + R_{g^3(0)} &= L_{g^2(0)} + R_{f(g^2(0))} \\
&\vdots \\
L_{g^{m-1}(0)} + R_{g^{m-1}(0)} &= L_{g^{m-2}(0)} + R_{f(g^{m-2}(0))}
\end{aligned}$$

where the ℓ-th equation, corresponding to $g^{\ell}(0) = i_0$, is missing. This equation,

$$L_{i_0} + R_{i_0} = L_{g^{-1}(i_0)} + R_{G^{m-1}(m)} + L_{g^{m-1}(0)} + R_{G^{-1}(i_0)}$$

is a linear combination of [5.22]. If some of the R's appearing at the right side of these equations do not show up in the left side we have $k+1$ vanishing R's and, as these equations are linearly independent, we are done. On the other hand, it is not possible that any R appearing at the right side of these equations appear also at the left side. In fact, suming the first k equations of [5.22], we have $L_{g^k(0)} = 0$, which contradicts the fact that $\mathscr{C}_\gamma$ has dimension $m-1$.

The proof of the lemma is now complete. □

6. Ergodicity and Keane's conjecture. In this section we show that $d\mu$ is ergodic under the action of $\mathscr{G}$ and give another proof to Keane's conjecture.

We start by recalling some results of Rényi's [6] which we will need. Let $(\Omega, \mathscr{B}, \nu)$ be a measure space and let $\mathscr{F}: \Omega \to \Omega$ be a measurable non-singular map. We say $\mathscr{F}$ admits a Markov partition $(C(i))_{i\in I}$, if $C(i)$ is a measurable partition of Ω, I is countable or finite and

$$\mathscr{F}(C(i)) = \sum_{j\in I(i)} C(j) \quad \text{for} \quad I(i) \subseteq I$$

Define the transition matrix $\mathscr{T} = (\mathscr{T}_{ij})_{i,j\in I}$ associated to this Markov partition by

$$\mathscr{T}_{ij} = \begin{cases} 1, & \text{if } \mathscr{F}(C(i)) \supseteq C(j); \\ 0, & \text{if } \mathscr{F}(C(i)) \cap C(j) = \emptyset. \end{cases}$$

for i, $j \in I$.

A sequence of indices $i_1, i_2, \ldots, i_n$, $n \geq 1$, is called admissible if $\mathscr{T}_{i_k i_{k+1}} = 1$ for $k = 1, 2, \ldots, n-1$. In the cases we will be considering $\mathscr{T}$ is irreducible, which means that given indices i and j there is an admissible sequence $i_1, i_2, \ldots, i_n$ starting at $i = i_1$ and ending at $j = i_n$.

We suppose that for each $i \in I$ there is a measurable and non-singular map

$$\mathscr{H}(i): \mathscr{F}(C(i)) \to C(i)$$

which is the inverse to $\mathscr{F}|_{C(i)}$. In other words $\mathscr{F} \circ \mathscr{H}(i) = \mathbf{Id}_{\mathscr{F}(C(i))}$ and $\mathscr{H}(i) \circ \mathscr{F} = \mathbf{Id}_{C(i)}$.

Given $i_1, i_2, \ldots, i_n$ an admissible sequence define

$$\mathscr{H}(i_1, i_2, \ldots, i_n): \mathscr{F}(C(i_n)) \to C(i_1)$$

inductively as

$$\mathscr{H}(i_1, i_2, \ldots, i_n) = \mathscr{H}(i_1, i_2, \ldots, i_{n-1}) \circ \mathscr{H}(i_n)$$

and define

$$\begin{aligned} C(i_1, i_2, \ldots, i_n) &= \mathscr{H}(i_1, i_2, \ldots, i_{n-1})(C(i_n)) \\ &= C(i_1) \cap \mathscr{F}^{-1}(C(i_2)) \cap \mathscr{F}^{-2}(C(i_3)) \cdots \cap \mathscr{F}^{-n+1}(C(i_n)) \end{aligned}$$

$C(i_1, i_2, \ldots, i_n)$ is called the atom of depth n associated to the admissible sequence $i_1, i_2, \ldots, i_n$. The set of these atoms, $\mathscr{P}^n$, is a partition of Ω and it is clear that $\mathscr{P}^{n+1}$ refines $\mathscr{P}^n$.

Let

$$\Delta(i_1, i_2, \ldots, i_n)(x) = \frac{d\mathscr{H}(i_1, i_2, \ldots, i_n)}{d\nu}(x)$$

denote the jacobian of $\mathscr{H}(i_1, i_2, \ldots, i_n)$ with respect to the measure ν at the point $x \in \mathscr{F}(C(i_n))$.

We say that the atom $C(i_1, i_2, \ldots, i_n)$ satisfies Rényi's condition for $K \geq 1$ if

$$\begin{aligned} &\text{ess sup}\{\Delta(i_1, i_2 \ldots, i_n)(x) \mid x \in \mathscr{F}(C(i_n))\} \\ &\leq K \text{ ess inf}\{\Delta(i_1 . i_2, \ldots, i_n)(x) \mid x \in \mathscr{F}(C(i_n))\} \end{aligned} \tag{6.22}$$

Rényi's condition means that the distortion $\mathscr{H}(i_1, i_2, \ldots, i_n)$ produces on the measure of any subset of $\mathscr{F}(C(i_n))$ is essentially the distortion it produces in the measure of $\mathscr{F}(C(i_n))$.

We are ready to state Rényi's result [6] we shall need.

Theorem 6.1. *Let $\mathscr{F}: \Omega \to \Omega$ and $C(i)$ be as above and suppose that ν is finite, $\mathscr{F}(C(i_n)) = \Omega$ for $\forall i$, that there is $K \geq 1$ such that every atom of any depth satisfies Rényi's condition and that $\bigcup_{n=1}^{\infty} \mathscr{P}^n$ generates $\mathscr{B}$. Then $\mathscr{F}$ is ergodic.*

We return now to consider the Gauss map $\mathscr{G}: \mathscr{C} \to \mathscr{C}$. Denote by I the set of pairs $i = (\mathscr{S}, \gamma)$ where $\mathscr{S} \in \{R, L\}$ and $\gamma \in \mathscr{A}$, and define $C(i) = \mathscr{C}_\gamma^{\mathscr{S}}$ and

$$\mathscr{H}(i) = \mathscr{G}^{-1}: \mathscr{G}(C(i)) \to C(i)$$

It is clear that $(C(i))_{i\in I}$ is a finite Markov partition for $\mathscr{G}$. Note that the set $\{\mathscr{G}(C(i)) \mid i \in I\}$ is the set of Farey cells and

$$\mathscr{H}(i): \mathscr{G}(C(i)) \to C(i)$$

which is the inverse to $\mathscr{G}|_{C(i)}$, is a projective isomorphism.

For each $i \in I$ fix $M(i)$ an $\mathbf{n} \times \mathbf{n}$-matrix inducing $\mathscr{H}(i)$ as in the definition of projective maps (3.6).

Since projective maps take straight line segments to straight line segments and therefore convex sets to convex sets it is clear that the atoms are convex.

To show the ergodicity of $\mathscr{G}$ we start by proving that the first return map induced by $\mathscr{G}$ on $C(i_1, \ldots, i_n)$ is ergodic for certain good admissible sequences $i_1, \ldots, i_n$. Observe that there is a first return map since $\mathscr{G}$ is conservative. Then we make use of the identification of $X = \mathbf{T}$ via the stacks associated with the interval exchange maps and prove that if $\mathbf{T}$ satisfies Keane's infinite and distinct orbit condition, i.d.o.c., we can get a good admissible sequence $i_1, \ldots, i_n$ such that $\mathbf{T} \in C(i_1, \ldots, i_n)$. Since the set of i.d.o.c.'s is a set of full measure a well known argument using the transitivity of $\mathscr{T}$ shows the ergodicity of $\mathscr{G}$.

Lemma 6.1. *There is a subset of full Lebesgue measure in $\mathscr{C} = \sum_\gamma \mathscr{C}_\gamma$ such that for every point (L, R) in this set, say $(L, R) \in \mathscr{C}_\gamma$, there is an admissible sequence $i_1, \ldots, i_n$ such that $(L, R) \in C(i_1, \ldots, i_n) \subseteq \operatorname{int}(\mathscr{C}_\gamma)$*

Proof. To prove the lemma recall the interpretation of the Gauss map $\mathscr{G}$ as the change the stacks associated to an interval exchange map $\mathbf{T}$ suffer as we move from one critical iterate to the next one. Given $\gamma \in \mathscr{A}$ we can identify each element (L, R) of this abstract Farey cell with the stacks of an interval exchange map $\mathbf{T}$ in a conveniently fixed convex subset of the simplex of interval exchange maps. This procedure was described in detail in the last section of [7]. Using this identification, the set of full measure we need to establish our lemma is the set of interval exchange maps satisfying the infinite and distinct orbit condition which, as we know, is made of minimal maps. To see the truth of that assertion, fix $\mathbf{T} \in \mathscr{C}_\gamma$ i.d.o.c. and denote by $\beta_1, \ldots, \beta_{m-1}$ its discontinuities. Since for each $i = 1, \ldots, m-1$, $\mathbf{T}^{-k}(\beta_i)$, $k \geq 0$, is dense we can fix k_i such that $\mathbf{T}^{-k}(\beta_i)$ $0 \leq k \leq k_i$ crosses at least twice the interior of each slice of each stack of $\mathbf{T} \in \mathscr{G}_\gamma$; once in the interior of the intervals $L^\sharp$ and the other in the interior of intervals $R^\sharp$. Now denote by s_i the segment of vertical separatrix conecting β_i to $\mathbf{T}^{-k_i}(\beta_i)$ in the vertical foliation of $w(\mathbf{T})$, the quadratic form associated to $\mathbf{T}$, [7]. Each of the segments s_i has possibly several connected components on each stack of $\mathbf{T} \in \mathscr{G}_\gamma$. Now, as we iterate $\mathbf{T}$ under $\mathscr{G}$, the number of these components decrease to one since they start being separated by the $\mathbf{T}$-orbit of 0. Let $n+1$ be the first time each segment s_i is entirely contained in one stack of the corresponding Farey cell. This stack must necessarily be the one with β_i in its top. Take $\mathscr{C}_{\gamma_n}$ the Farey cell containing $\mathscr{G}^n(\mathbf{T})$ with coordinates (L', R'). The itinerary of $\mathscr{G}^k(\mathbf{T})$, $0 \leq k \leq n$, on the atoms $C(i)$ define $C(i_1, \ldots, i_n)$ and it is clear that $C(i_1, \ldots, i_n)$ is contained in the interior of $\mathscr{C}_\gamma$ since each each stack of $\mathscr{C}_{\gamma_n}$ contributes with at least one slice to compose the intervals $L^\sharp$ and $R^\sharp$ of $\mathscr{C}_\gamma$. In fact, if one of the equations defining the boundary of $\mathscr{C}_\gamma$ is satisfied, say $L_1 = 0$, this whould imply that $L' = R' = 0$ which is an absurd. The lemma follows. $\square$

Lemma 6.2. *Let $i_1, \ldots, i_n$ be an admissible sequence satisfying the thesis of the preceding lemma:*

$$C(i_1, \ldots, i_n) \subseteq \operatorname{int}(\mathscr{C}_\gamma)$$

Then the first return map induced by $\mathscr{G}$ on $C(i_1, \ldots, i_n)$ is ergodic.

Proof. Fix $i_1, \ldots, i_n$ an admissible sequence as in the hypothesis and take $J = \{j\}$ the set of admissible sequences $j_1, \ldots, j_\ell$, $\ell > n$, such that

(1) $j_1, \ldots, j_\ell$ starts with the sequence $i_1, \ldots, i_n$, in other words, $i_k = j_k$ for $k = 1, \ldots, n$.
(2) $j_1, \ldots, j_\ell$ ends with the sequence $i_1, \ldots, i_n$.
(3) there are no other ocurrences of the sequence $i_1, \ldots, i_n$ in $j_1, \ldots, j_\ell$ other than the two just considered.

It is clear that $\sum_j C(j_1, \ldots, j_\ell)$ is the domain of the map $\tilde{\mathscr{G}}$ induced by $\mathscr{G}$ on $C(i_1, \ldots, i_n)$ and therefore

$$C(i_1, \ldots, i_n) = \sum_j C(j_1, \ldots, j_\ell)$$

mod $d\mu$ since the first return map is defined a.e.. It is also clear that $\tilde{C}(j) = C(j_1, \ldots, j_\ell)$, $j \in J$, is an irreducible Markov partition for $\tilde{\mathscr{G}}$ and, since $\tilde{\mathscr{G}} = \mathscr{G}^{l-n}$ on $\tilde{C}(j)$, we have that $\tilde{\mathscr{H}}(j)\colon \tilde{\mathscr{G}}(\tilde{C}(j)) \to \tilde{C}(j)$ is given by $\tilde{\mathscr{H}}(j) = \mathscr{H}(j_1, \ldots, j_{l-n})$ on $\tilde{C}(j)$. To prove the lemma we check first Rényi's condition for some $K \geq 1$ that depends only on $i_1, \ldots, i_n$.

Fix $\tilde{C}(j^1, \ldots, j^k)$. We have to bound

$$\frac{\text{ess sup}\{\tilde{\Delta}(j^1, \ldots, j^k)(x)\}}{\text{ess inf}\{\tilde{\Delta}(j^1, \ldots, j^k)(y)\}}$$

for $x, y \in \tilde{\mathscr{G}}(\tilde{C}(j^k))$. Since these set are convex polyhedra we have by lemma 1. that the supremun and infimun are taken at the vertices of the polihedron $\tilde{\mathscr{G}}(\tilde{C}(j^k))$ thus we have to bound the quantity

$$q = \frac{\tilde{\Delta}(j^1, \ldots, j^k)(\tilde{v})}{\tilde{\Delta}(j^1, \ldots, j^k)(\tilde{w})}$$

for $\tilde{v}, \tilde{w}$ vertices of $\tilde{\mathscr{G}}(\tilde{C}(j^k))$. Now,

$$\tilde{\mathscr{H}}(j^1, \ldots, j^k) = \tilde{\mathscr{H}}(j^1) \ldots \tilde{\mathscr{H}}(j^k) =$$

$$\mathcal{H}(j_1^1)\ldots\mathcal{H}(j_{\ell_1-n}^1)\mathcal{H}(j_1^2)\ldots\mathcal{H}(j_{\ell_2-n}^2)\mathcal{H}(j_1^k)\ldots\mathcal{H}(j_{\ell_k-n}^k)$$

and

$$\begin{aligned}\tilde{C}(j^k) &= C(j_1^k,\ldots,j_{\ell_k}^k)\\ \mathcal{G}(\tilde{C}(j^k)) &= C(j_{\ell_k-n+1}^k,\ldots,j_{\ell_k}^k) = C(i_1,\ldots,i_n)\\ &\quad \mathcal{H}(i_1)\ldots\mathcal{H}(i_n)(\mathcal{G}(C(i_n)))\end{aligned}$$

therefore $\tilde{v} = \mathcal{H}(i_1)\ldots\mathcal{H}(i_n)(v)$ and $\tilde{w} = \mathcal{H}(i_1)\ldots\mathcal{H}(i_n)(w)$ where v and w are vertices of $\mathcal{G}(C(i_n))$. Using the chain rule we can write

$$q = \frac{\mathrm{Jac}(\mathcal{H}(j_1^1)\ldots\mathcal{H}(j_{\ell_k-n}^k)\mathcal{H}(i_1)\ldots\mathcal{H}(i_n))(v).\mathrm{Jac}(\mathcal{H}(i_1)\ldots\mathcal{H}(i_n))(w)}{\mathrm{Jac}(\mathcal{H}(j_1^1)\ldots\mathcal{H}(j_{\ell_k-n}^k)\mathcal{H}(i_1)\ldots\mathcal{H}(i_n))(w).\mathrm{Jac}(\mathcal{H}(i_1)\ldots\mathcal{H}(i_n))(v)}$$

where Jac denotes the Jacobian with respect to the Lebesgue measure. We have then to get a bound for

$$q = \frac{\mathrm{Jac}(\mathcal{H}(j_1^1)\ldots\mathcal{H}(j_{\ell_k-n}^k)\mathcal{H}(i_1)\ldots\mathcal{H}(i_n))(v)}{\mathrm{Jac}(\mathcal{H}(j_1^1)\ldots\mathcal{H}(j_{\ell_k-n}^k)\mathcal{H}(i_1)\ldots\mathcal{H}(i_n))(w)}$$

using Lemma 1. we see that we have to get a bound for

$$q = \frac{\mathbf{u}M(j_1^1)\ldots M(j_{\ell_k-n}^k)M(i_1)\ldots M(i_n)w}{\mathbf{u}M(j_1^1)\ldots M(j_{\ell_k-n}^k)M(i_1)\ldots M(i_n)v}$$

where $\mathbf{u}$ is the $\mathbf{n}$-columns row matrix with all entries 1. Since the vertices of $C(i_1,i_2,\ldots,i_n)$ are in the interior of $\mathcal{C}_\gamma$ we can fix a matrix $A = A(i)$ all of whose entries are positive such that $WA = M(i_1)\cdots M(i_n)V$ where V is the matrix with columns the vertices of $\mathcal{C}_\gamma$ and W is the matrix with columns the vertices of the Farey cell containing $C(i_1)$.

Setting $X = \mathbf{u}.M(j_1^1)\ldots M(j_{\ell_k-n}^k)W$ we have

$$q = \left(\frac{Xa}{Xa'}\right)^m$$

where a and a' are columns of A. But then, for $X_{k_0} = \max\{X_k \mid 0 \le k \le \mathbf{n}\}$, we have.

$$q = \left(\frac{\sum_{k=1}^{\mathbf{n}} X_k a_k}{\sum_{k=1}^{\mathbf{n}} X_k a'_k}\right)^m = \left(\frac{\sum_{k=1}^{\mathbf{n}} \frac{X_k}{X_{k_0}} a_k}{\sum_{k=1}^{\mathbf{n}} \frac{X_k}{X_{k_0}} a'_k}\right)^m$$

$$\begin{aligned}&\le \left(\frac{\sum_{k=1}^{\mathbf{n}} a_k}{a'_{k_0}}\right)^m \le \left(\frac{\sum_{k=1}^{\mathbf{n}} a_k}{\min\{A(i)\}}\right)^m\\ &\le \left(\frac{\mathbf{n}.\max\{A(i)\}}{\min\{A(i)\}}\right)^m\end{aligned}$$

where $\max\{A(i)\}$ and $\min\{A(i)\}$ are, respectively, the maximun and minimum of the entries of $A(i)$ We have then shown that Rényi's condition holds for

$$K = \max\left\{\left(\frac{\mathbf{n}.\max\{A(i)\}}{\min\{A(i)\}}\right)^m \mid i \in I\right\}$$

To finish the proof of the lemma using Theorem 1. we have to exhibit a subset $\mathcal{T}$ of full measure of $C(i_1,i_2,\ldots,i_n)$ such that the diameter of the atom of $\mathcal{P}^n$ around $x \in \mathcal{T}$, $A_n(x)$, goes to 0 as $n \to \infty$. Now, the set of points in $C(i_1,i_2,\ldots,i_n)$ which, under the action of $\mathcal{G}$, recur infinitely often to this set has this property. This follows from Lemma 3.28, p. 240 of [8] on account of the infinitely repeated matrix product $M(i_1).M(i_2).\cdots.M(i_n)$ occurring in the definition of $A_n(x)$. This product, as we know from lemma 1. , has all entries positive. This finishes the proof of the lemma. □

Theorem 6.2. *Given π an irreducible and discontinuous permutation, the set of interval exchange maps $\mathbf{T} = \mathbf{T}(\pi,\alpha)$, $\alpha \in \mathcal{S}_m$, which are uniquely ergodic is a set of full Lebesgue measure on $\mathcal{S}_m$.*

Proof. Using the notation and results of the last section of [7] we have to show that the set of uniquely ergodic interval exchange maps of an arbitrary but fixed integral type $\gamma \in \mathcal{A}$ form a set of full measure. But, as remarked above, the set of these interval exchange maps can be identified with the points in the Farey cell $\mathcal{C}_\gamma$, $\mathbf{T}$ being uniquely ergodic iff, in our present notation, □

(6.23) $$\delta(C(i_1,i_2,\ldots,i_n)) \to 0$$

where δ denotes diameter and $C(i_1,i_2,\ldots,i_n)$ is the depth n atom containing $\mathbf{T}$. Now, we just saw in the proof of the preceding lemma a set of full measure with this property. The theorem follows.

Theorem 6.3. *$\mathcal{G}:(\mathcal{C},\mu) \to (\mathcal{C},\mu)$ is ergodic.*

Proof. Let E be a measurable $\mathcal{G}$-invariant set with $\mu(E) > 0$. It is enough to show that for any admissible sequence $i_1,\ldots,i_n$ such that $C(i_1,\ldots,i_n)$

satisfies the condition of lemma 1. we have

$$\mu(E\cap C(i_1,i_2,\ldots,i_n))=\mu(C(i_1,i_2,\ldots,i_n))$$

As $E\cap C(i_1,\ldots,i_n)$ is invariant by $\tilde{\mathscr{G}}$, the map induced by $\mathscr{G}$ on $C(i_1,\ldots,i_n)$ all we have to do is show that $\mu(E\cap C(i_1,i_2,\ldots,i_n))>0$ since by lemma 2. $\tilde{\mathscr{G}}$ is ergodic. Now, by lemma 1. , as $\mu(E)>0$, there is $i'_1,\ldots,i'_\ell$ an admissible sequence such that $\mu(E\cap C(i'_1,\ldots,i'_\ell))>0$ and since $\mathscr{T}$ is irreducible there is an admissible sequence $j_1,j_2,\ldots,j_k$ which starts with $i'_1,\ldots,i'_\ell$ and ends with $i_1,i_2,\ldots,i_n$. But the maps $\mathscr{H}$ are non-singular and as E is $\mathscr{G}$-invariant it follows that $\mu(E\cap C(i_1,i_2,\ldots,i_n))>0$ thus proving the theorem. □

7. Appendix. In this appendix we establish necessary conditions for an integral of the type we dealt with in Section 5 to be finite.

Let $\mathbf{s}$ be the n dimensional simplex with vertices $e_0=0$ and $e_1,\ldots,e_n$ the canonical basis of $\mathbf{R}^n$, i.e.,

$$\begin{aligned}\mathbf{s}\ &=\ \Big\{\sum_{i=0}^{n}x_ie_i\mid\sum_{i=0}^{n}x_i=1\ ,\ 0\le x_i\Big\}\\ &=\ \Big\{\sum_{i=1}^{n}x_ie_i\mid\sum_{i=1}^{n}x_i\le 1\ ,\ 0\le x_i\Big\}\\ &=\ \Big\{(x_1,\ldots,x_n)\mid\sum_{i=1}^{n}x_i\le 1\ ,\ 0\le x_i\Big\}\end{aligned}$$

and $L(x)=c_1x_1+\cdots+c_nx_n+b$ an affine functional. Suppose $L(x)>0$ for $x\in\mathbf{s}^\circ$, the interior of $\mathbf{s}$. Then $L(x)\ge 0$ for $x\in\mathbf{s}$ and, taking $x=e_0,e_1,\ldots,e_n$, we get $c_0+b\ge 0,\ c_1+b\ge 0,\ldots,c_n+b\ge 0$, where $c_0=0$.

If $\{L=0\}\cap\mathbf{s}\neq\varnothing$ there are $x_0,x_1,\ldots,x_n$ such that $\sum_{i=0}^n x_i=1,\ 0\le x_i$ with $c_1x_1+\cdots+c_nx_n+b=0$, or $(c_0+b)x_0+\cdots+(c_n+b)x_n=0$. This shows that there are indices i such that $c_i+b=0$. Let $0\le i_1<i_2<\ldots<i_k\le n$, $1\le k\le n$, be this set of indices. It is easy to see that $\{L=0\}\cap\mathbf{s}$ is the simplex generated by $e_{i_1},\ldots,e_{i_k}$. In other words, $\{L=0\}$ cuts $\mathbf{s}$ in a subsimplex.

A simple consequence of these remarks is that if L vanishes in a point in the interior of a face $\mathbf{f}$ of $\mathbf{s}$, then it vanishes in the entire face $\mathbf{f}$.

Given $P=\prod_{i=1}^N L_i$, where $L_i(x)=c_{i1}x_1+\cdots+c_{in}x_n+b_i$ for $i=1,\ldots,N$, and $\mathbf{s}$ a simplex as above, define the degree of a face $\mathbf{f}$ of $\mathbf{s}$, degree($\mathbf{f}$) as the number of factors of P, counting multiplicities, which vanish on the entire face $\mathbf{f}$.

Theorem 7.1. *Let P and $\mathbf{s}$ be as above satisfying $L_i(x)>0$ for $i=1,\ldots,n$ and $x\in\mathbf{s}^\circ$. If*

$$\text{dimension}(\mathbf{f})+\text{degree}(\mathbf{f})<n \tag{7.24}$$

for every face $\mathbf{f}$ of $\mathbf{s}$ we have

$$\int_{\mathbf{s}}\frac{dx}{P}<\infty$$

where dx is the Lebesgue measure on $\mathbf{R}^n$.

Proof. Take $\mathscr{B}=\{\mathbf{t}\}$ the baricentric subdivision of $\mathbf{s}$. We have to prove that

$$\int_{\mathbf{t}}\frac{dx}{P}<\infty$$

for each $\mathbf{t}\in\mathscr{B}$. Fix $\mathbf{t}\in\mathscr{B}$ and let $v_0,v_1,\ldots,v_n$ be its vertices ordered in such a way that v_j is the baricenter of a j-th dimensional face of $\mathbf{s}$, $j=0,1,\ldots,n$. Take $\mathbf{f}$ a face of $\mathbf{t}$ with vertices $v_{j_0},\ldots,v_{j_k}$, $0\le j_0<\ldots<j_k\le n$, and L_i a factor of P such that $L_i(\mathbf{f})=0$. Using the remark just preceding the statement of this theorem we conclude that $L_i(\mathbf{f}_{j_\ell})=0$ where $\mathbf{f}_{j_\ell}$ is the face of $\mathbf{s}$ with baricenter v_{j_ℓ}, $\ell=0,\ldots,k$. This shows that our hypothesis (7.24) holds for $\mathbf{t}$ (since $\{L_i\mid L_i(\mathbf{f})=0\}\subseteq\{L_i\mid L_i(\mathbf{f}_{j_k})=0\}$ and this set has cardinality $<n-j_k\le n-k$). After an affine change of coordinates we can suppose that $v_0=0$ and $v_1,\ldots,v_n$ is the canonical basis of $\mathbf{R}^n$. Using the same remark again we see that every factor of P that vanishes at a point of $\mathbf{t}$ must vanish at a vertex of $\mathbf{t}$ and therefore at all previous vertices of this simplex. In particular this factor must be homogeneous. Thus, since $L_i(x)>0$ for $i=1,\ldots,n-1$ and $x\in\mathbf{s}^\circ$ we can write $L_i=c_{i1}x_1+\cdots+c_{in}x_n$ for $i=1,\ldots,n-1$ and non-negative c_{ij}'s such that if $c_{ij}=0$ for some j, $c_{ik}=0$ for $k<j$. Since (7.24) hold for $\mathbf{f}=\mathbf{t}$, at most $n-1$ factors of P vanish at a point of $\mathbf{t}$. Factors which are finite on $\mathbf{t}$ won't matter for our thesis so we will ignore them and suppose we have at most $n-1$ factors. In fact, to simplify the notation, we suppose that P has exactly $n-1$ factors by multiplying P by a convenient number of factors equal to $x_1+\cdots+x_n$. Reordering the L_i's if necessary we can assume that the number of vanishing c_{ij} does not decrease with i. We claim that the j-th column of the matrix c_{ij} has at least j positive entries. In fact if $n-j$ entries of this column are zero $n-j$ factors of P vanish at e_j and therefore at the face generated by $e_0,e_1,\ldots,e_j$ contradicting our hypothesis. Thus $c_{ij}>0$ at least for $1\le i\le j$ and then

$$P=\prod_{i=1}^{n-1}\sum_{j=1}^{n}c_{ij}x_i\ge\prod_{i=1}^{n-1}c_{ii}x_i+c_{in}x_n\ge c\prod_{i=1}^{n-1}x_i+x_n$$

where c is the minimum of the positive c_{ij}. Denoting by $\mathbf{c}$ the cube $[0,1]^n \supseteq \mathbf{t}$ we have

$$\begin{aligned}
\int_{\mathbf{t}} \frac{dx}{P} &\leq \frac{1}{c} \int_{\mathbf{t}} \frac{dx}{\prod_{i=1}^{n-1} x_i + x_n} \\
&\leq \frac{1}{c} \int_{\mathbf{c}} \frac{dx}{\prod_{i=1}^{n-1} x_i + x_n} \\
&= \frac{1}{c} \int_0^1 \left[\ln\left(\frac{1+x_n}{x_n}\right)\right]^{n-1} dx_n < \infty
\end{aligned}$$

which finishes the proof of the theorem. □

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