Gibbs States and Gibbsian Specifications on the space $\mathbb{R}^{\mathbb{N}}$

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Abstract

We are interested in the study of Gibbs and equilibrium probabilities on the space state $\mathbb{R}^{\mathbb{N}}$. We consider the unilateral full-shift σ defined on the non-compact set $\mathbb{R}^{\mathbb{N}}$, that is $\sigma(x_1, x_2, .., x_n, ..) = (x_2, x_3, .., x_n, ..)$, and a Hölder continuous potential $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. From a suitable class of a priori probability measures ν we define the Ruelle operator associated to A and we show the existence of eigenfunctions, conformal probability measures and equilibrium states associated to A. Moreover, we prove the existence of an involution kernel for A, we build a Gibbsian specification for the Borelian sets on $\mathbb{R}^{\mathbb{N}}$ and we show that this family of probabilities satisfies a FKG-inequality.

Keywords: Equilibrium States, Gibbs States, Gibbsian Specifications, Ruelle-Perron-Frobenious Theorem, Thermodynamic Formalism, XY-Model.

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1 Introduction

Thermodynamic Formalism is a useful branch of mathematics, mainly in the rigorous study of some interesting problems of Statistical Mechanics and Ergodic Theory which arise on the analysis of physical systems of particles, like molecules of water, noble gases and other type of fluids. These systems usually consist of a large number of elements, commonly of the order of 10^{27} elements. In some cases it is required to study these type of problems when

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the lattice is such that in each site the set of possible spins is unbounded. When the set of spins is countable several results are already known (see [12], [4], [13], [14], [15], [11] and Section 5 in [17])

Given a potential A (which describes a certain interaction among spins) one is interested in the equilibrium probability (or DLR probability) associated to such potential.

One of the principal tools used in this area is the Ruelle operator (also called transfer operator) associated to the potential A. This operator was introduced initially by David Ruelle in [23] as an instrument for the study of one-dimensional lattice gas. This has been generalized in several directions, for example, Mauldin, Urbański and Sarig in [19] and [24] consider a non-compact setting on $\mathbb{N}^{\mathbb{N}}$.

[1], [17], [18] and [21] analyze the case of a potential $A : M^{\mathbb{N}} \to \mathbb{R}$, where M is a compact metric space (maybe not countable) and they consider the shift σ acting on $M^{\mathbb{N}}$, that is $\sigma(x_1, x_2, .., x_n, ..) = (x_2, x_3, .., x_n, ..)$. When M is not countable it is necessary to introduce an *a priori* probability ν on M. Then, a Ruelle operator can be defined and the classical results of Thermodynamic Formalism can be obtained. In all the above cases some regularity of the potential A (like Holder) is assumed on the potential A.

We emphasize the fact that from the point of view of Physics (Statistical Mechanics) the important thing is the existence of equilibrium measures (or, DLR) for a given potential and the Ruelle operator can be considered as a device to obtain them (or, to be able to find special properties of them, or, to be able to approximate them, etc..).

[8] consider the case where the potential $A : M^{\mathbb{N}} \to \mathbb{R}$ is just continuous and the metric space M is compact. We point out that a continuous potential may not have continuous positive eigenfunctions. Even in the case of existence of a continuous positive eigenfunction the uniqueness of the equilibrium probability can not be guaranteed. Examples (of potentials not of Hölder class) with phase transitions (more than one equilibrium state) are well known even in the compact case. In [8] it is considered bounded extensions of the Ruelle operator to the Lebesgue space of integrable functions with respect to the eigenmeasure and it is studied the problem of existence of maximal positive eigenfunctions.

[9] consider the case where the metric space M is not compact and the potential $A : M^{\mathbb{N}} \to \mathbb{R}$ is continuous with bounded support. It is shown the existence of finite additive probabilities which are equilibrium probabilities. Uniqueness of the equilibrium probability can not be guaranteed.

We will study here the case where the state space is $\mathbb{R}^{\mathbb{N}}$. In this case the metric space $M = \mathbb{R}$ is not countable and not compact.The assumptions (on the shift space or in the potential) which are usually considered in most of the cases which analyze the lattice $\mathbb{N}^{\mathbb{N}}$ (or, $\mathbb{Z}^{\mathbb{N}}$) are not natural on the present setting.

We will analyze here potentials $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ which are Holder with respect to some natural metric. We point out that in order to study continuous potentials on lattices (with a countable set of sites), such that the fiber of spins is an unbounded set, some kind of constrictive assumption is needed (for instance, in order that some part of the initial probability mass do no go to infinity under the dynamical evolution). Here the only technical assumption is in the potential (not related to conditions on the symbolic space) and is just the Hölder assumption on the potential A in its action on $\mathbb{R}^{\mathbb{N}}$.

Our analysis will be based on the Ruelle operator and we will get existence and uniqueness of the equilibrium probability (see Theorem 2.4).

In order to define the Ruelle operator we need an *a priori* probability measure ν with **support equal** to the all fiber $\mathbb{R}$. In our approach compactness is not a necessary condition to define the transfer operator. We will consider the following class of *a priori* probabilities: given a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is strictly positive and also satisfying the condition $\int_{\mathbb{R}} f(a)da = 1$, we take $d\nu = f(x)\, dx$ as the *a priori* probability measure

The Ruelle operator allows the construction of Gibbs states, conformal probability measures and DLR-Gibbs probability measures through the study of the behavior of its eigenvalues and eigenfunctions. The foregoing provides a good instrument for studying variational problems related with the existence and properties of equilibrium states through properties of linear operators. Nevertheless, the utilities of this operator permeate another important areas of mathematics; for example, W. Parry and M. Pollicott in [22] used this operator in order to exhibit important results on thermodynamic formalism of topological Markov chains with some important applications to complex analysis, geometry and number theory.

One important issue here is the fact that as the fiber $\mathbb{R}$ is a vector space one can consider questions related to differentiability of the potential $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ (see sections 3 and 4).

Our main interest here is to study the Thermodynamic Formalism for Hölder continuous potentials defined on both, the set of sequences of real numbers and the set of bi-sequences of real numbers. This will help on the study of ergodic properties of Gibbs measures for a certain class of potentials using an adaptation of results considered in [1] and [17]. Besides that, using some of the features of the Ruelle operator, which is defined from an adequate a priori probability measure, we obtain certain tools that will help to show the existence of calibrated sub-actions, A-maximizing probability measures and ground states. Moreover, from properties of this operator, it is obtained a family of probability measures that satisfies the conditions of Gibbsian specification and an *FKG*-inequality, which is an extension of results presented in [1, 7, 6] which are now extended (once more) to the continuous unbounded space of spins. Due to the natural differentiable nature of $\mathbb{R}$ (and, also - naturally - the one for the product $\mathbb{R}^{\mathbb{N}}$) we are able to study some differentiable properties of the eigenfunctions of the Ruelle operator using an involution kernel - we will also consider an example for the case of Markov chains.

This paper is organized as follows.

In Section 2 we show the existence of Gibbs states and conformal probability measures through properties of the spectrum of the Ruelle operator and its corresponding dual. Furthermore, using the above we build a calibrated sub-action and we present a definition of entropy that extend definitions presented in [17], [18] and [20]. We point out that the entropy depends of the *a priori* probability (see expression (5)).

Moreover, we show that the given definition of entropy is suitable for the study of the variational principle of pressure. We use this fact later in order to show the existence of A-maximizing measures through a construction related to the existence of ground states associated to A. At the end of this Section, under mild assumptions, we show that there exists accumulation points in the zero temperature limit in our non-compact setting.

The papers [10] and [1] analyze the limit of equilibrium probabilities when temperature goes to zero for the compact XY model.

In Section 3 we present the concept of involution kernel in our setting and we show its existence. Besides that, we build an extension of the Gibbs states to the bilateral case in terms of the involution kernel and the conformal probability measure associated to the potential A and its respective adjunct A^*. We also construct an example - for the case of stationary Markov probability measures - where we show some properties of differentiability of the eigenfunctions associated to the Ruelle operator in the case of locally constant potentials.

In Section 4 we build a family of probability **kernels** which defines a Gibbsian specification on the Borel sets of $\mathbb{R}^{\mathbb{N}}$ and we demonstrate that this family of probability measures satisfies an *FKG*-inequality under certain assumptions for the potential A. Besides that, from the classical approach of thermodynamic limit, we show that any thermodynamic limit defined by the Gibbsian specification considered here is actually a DLR-Gibbs measure (see [7] for definitions) and we also show that the Gibbs state and the conformal probability measure constructed in Section 2 are *DLR*-Gibbs probability measures.

2 Ruelle operator

The Ruelle operator is one of the main tools used in Thermodynamical Formalism. This operator allows the construction of equilibrium states through an algebraic approach, which helps a lot the study of variational principles for the topological pressure (among other important features). With this aim in this Section we will introduce a Ruelle operator from a suitable *a priori* probability measure ν.

Let $a, b \in \mathbb{R}$, note that the map $(a, b) \mapsto \frac{1}{\pi}|\arctan(a) - \arctan(b)|$ defines a metric on $\mathbb{R}$.

Points x in $\mathbb{R}^{\mathbb{N}}$ are denoted by $x = (x_1, x_2, .., x_n, ..)$. We consider on $\mathbb{R}^{\mathbb{N}}$ the metric

$$d(x, y) = \sum_{n\in\mathbb{N}} \frac{1}{\pi 2^n}|\arctan(x_n) - \arctan(y_n)|\,.$$

We will analyze here potentials $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ which are Hölder with respect to such metric.

By defining $\arctan(\pm\infty) = \lim_{x\to\pm\infty} \arctan(x)$ one can get an extension of this metric to the set of extended real numbers defined as the two-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Moreover, the set $\overline{\mathbb{R}}$ equipped with this metric is a compact metric space. Therefore, follows from the Tychonoff's Theorem that $\overline{\mathbb{R}}^{\mathbb{N}} = \{(x_n)_{n\in\mathbb{N}} : x_n \in \overline{\mathbb{R}}, \forall n\}$ is a compact metric space equipped with the metric

$$\widehat{d}(x, y) = \sum_{n\in\mathbb{N}} \frac{1}{\pi 2^n}|\arctan(x_n) - \arctan(y_n)|\,.$$

Indeed, note that for any basic open set in the product topology, which is of the form

$$[U_1 \ldots U_k] = \prod_{n\in\mathbb{N}} U_n\,,$$

with $U_n = \mathbb{R}$ for any $n > k$ and $U_1, \ldots, U_k$ open sets in $\mathbb{R}$, we get that for each $x \in [U_1 \ldots U_k]$, there is $\epsilon_0 > 0$ such that

$$B_{\widehat{d}}(x; \epsilon_0) \subset [U_1 \ldots U_k] \subset \bigcup_{x\in[U_1\ldots U_k]} B_{\widehat{d}}(x; \epsilon_0)\,.$$

Hereafter, we will denote by $\mathcal{C}_b(X)$ the set of bounded continuous functions from X into $\mathbb{R}$. Consider the Lebesgue measure dx on the Borel sigma-algebra of $\mathbb{R}$.

Fixing $f : \mathbb{R} \to \mathbb{R}$ an strictly positive continuous function satisfying $\int_{\mathbb{R}} f(a)da = 1$, choosing $\nu = f dx$ as a priori probability measure, and using the notation $ax = (a, x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, we define the Ruelle operator $\mathcal{L}_A$ associated to a Hölder continuous potential $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ as the map assigning to each $\varphi \in \mathcal{C}_b(\mathbb{R}^{\mathbb{N}})$ the function

$$\mathcal{L}_A(\varphi)(x) = \int_{\mathbb{R}} e^{A(ax)} \varphi(ax) d\nu(a) = \int_{\mathbb{R}} e^{A(ax)} \varphi(ax) f(a) da\,.$$

By the above definition, follows that for with $a^n = (a_n, \ldots, a_1)$ and $S_n A(x) = \sum_{k=0}^{n-1} A(\sigma^n(x))$, the n-th iterative of the Ruelle operator is given by

$$\mathcal{L}_A^n(\varphi)(x) = \int_{\mathbb{R}^n} e^{S_n A(a^n x)} \varphi(a^n x) f(a_n) \ldots f(a_1) da_1 \ldots da_n\,.$$

Furthermore, using that $\mathcal{L}_A$ is an operator from $\mathcal{C}_b(\mathbb{R}^{\mathbb{N}})$ into $\mathcal{C}_b(\mathbb{R}^{\mathbb{N}})$, it is possible to define the dual of the Ruelle operator, as the map from the set of regular additive finite Borel measures into itself, satisfying for any $\varphi \in \mathcal{C}_b(\mathbb{R}^{\mathbb{N}})$ the following equation

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi d\left(\mathcal{L}_A^* \mu\right) = \int_{\mathbb{R}^{\mathbb{N}}} \mathcal{L}_A(\varphi) d\mu\,.$$

Given a metric space (X, d) we denote by $\mathcal{H}_\alpha(X)$ the set of Hölder continuous functions from X into $\mathbb{R}$. As usual, we will denote the Hölder constant of $A \in \mathcal{H}_\alpha(X)$ as $\mathrm{Hol}_A = \sup_{x \neq y} \frac{|A(x) - A(y)|}{d(x,y)^\alpha}$. It is widely known that $\mathcal{H}_\alpha(X)$ equipped with the norm $\|A\|_\alpha = \|A\|_\infty + \mathrm{Hol}_A$ is a Banach space. Besides that, given a continuous map $T : X \to X$ we are going to denote by $\mathcal{M}_T(X)$ the set of T-invariant Borel probability measures on X.

Following a similar procedure that in Proposition 4 of [1], it is possible showing that the image of any Hölder continuous function by the Ruelle operator it is also Hölder continuous, in other words, the restriction of $\mathcal{L}_A$ to $\mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ is a map into $\mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$.

The following Lemma provides a useful tool which will help us in getting most of properties in the case that we are studying in this work.

Lemma 2.1. *Suppose that $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, then A can be extended to a unique function $A' \in \mathcal{H}_\alpha(\overline{\mathbb{R}}^{\mathbb{N}})$.*

Proof. For any $x \in \overline{\mathbb{R}}^{\mathbb{N}}$ we define $A'(x) = \lim_{y \to x} A(y)$. Note that this limit exists because $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, moreover, it is finite since A is a bounded potential. Then, for $x, y \in \overline{\mathbb{R}}^{\mathbb{N}}$ there exist sequences $(x^n)_{n \in \mathbb{N}}$ and $(y^m)_{m \in \mathbb{N}}$ taking values in $\mathbb{R}^{\mathbb{N}}$ such that $\lim_{n \in \mathbb{N}} x^n = x$ and $\lim_{m \in \mathbb{N}} z^m = z$, which implies

$$|A'(x) - A'(y)| = \lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} |A(x^n) - A(x^m)| \leq \mathrm{Hol}_A \widehat{d}(x, y)\,.$$

The above implies that $A' \in \mathcal{H}_\alpha(\overline{\mathbb{R}^{\mathbb{N}}})$. The uniqueness of A' follows of the properties of limits using that $\overline{\mathbb{R}^{\mathbb{N}}}$ is a perfect set. $\square$

Corollary 2.2. *$\mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ is isometrically isomorphic to $\mathcal{H}_\alpha(\overline{\mathbb{R}^{\mathbb{N}}})$.*

Proof. We first observe that for each $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ we have that $\|A\|_\infty = \|A'\|_\infty$ and $\mathrm{Hol}_A = \mathrm{Hol}_{A'}$. Therefore, the identity map $I : \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}}) \to \mathcal{H}_\alpha(\overline{\mathbb{R}^{\mathbb{N}}})$ is a linear isometry and therefore an injective map. From the uniqueness and continuity of the extension provided by Lemma 2.1 it follows that I is onto. Indeed, if $A \in \mathcal{H}_\alpha(\overline{\mathbb{R}^{\mathbb{N}}})$ then $A = (A|_{\mathbb{R}^{\mathbb{N}}})' = I(A|_{\mathbb{R}^{\mathbb{N}}})$. $\square$

Using the Lemma 2.1 and taking $A' \in \mathcal{H}_\alpha(\overline{\mathbb{R}}^{\mathbb{N}})$, we can extend the definition of Ruelle operator to $\mathcal{H}_\alpha(\overline{\mathbb{R}}^{\mathbb{N}})$ in a natural way as the map assigning to each $\varphi' \in \mathcal{H}_\alpha(\overline{\mathbb{R}}^{\mathbb{N}})$ the function

$$\mathcal{L}_{A'}(\varphi)(x) = \int_{\mathbb{R}} e^{A'(ax)}\varphi'(ax)d\nu(a) = \int_{\mathbb{R}} e^{A'(ax)}\varphi'(ax)f(a)da\,.$$

Moreover in the following Lemma we will show that $\mathcal{L}_A$ is topologically conjugated to $\mathcal{L}_{A'}$ through the isometry defined in the Corollary 2.2.

Lemma 2.3. *Consider $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$. Then $\mathcal{L}_{A'} \circ I = I \circ \mathcal{L}_A$, where $I : \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}}) \to \mathcal{H}_\alpha(\overline{\mathbb{R}^{\mathbb{N}}})$ is the isometry provided by Corollary 2.2.*

Proof. For any $\varphi \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, and each $x \in \mathbb{R}^{\mathbb{N}}$ we have we have

$$\begin{aligned}(I^{-1} \circ \mathcal{L}_{A'} \circ I)(\varphi)(x) &= I^{-1}(\mathcal{L}_{A'} \circ I)(\varphi)(x)\\ &= I^{-1}(\mathcal{L}_{A'}(I(\varphi)))(x)\\ &= I^{-1}(\mathcal{L}_{A'}(\varphi'))(x)\\ &= I^{-1}(\mathcal{L}_{A}(\varphi))(x)\\ &= \mathcal{L}_A(\varphi)(x)\,.\end{aligned}$$

Note that the second last equality is a consequence of the assumption $\lim_{a\to\pm\infty} f(a)da = 0$ and the boundedness of A, which implies that

$$\begin{aligned}0 &\le \left|\lim_{R\to+\infty}\int_{(-R,R)^c} e^{A'(ax)}\varphi'(ax)f(a)da\right|\\ &\le e^{\|A\|_\infty}\|\varphi\|_\infty \lim_{R\to+\infty}\int_{(-R,R)^c} f(a)da = 0\,.\end{aligned}$$

$\square$

Observe that compactness of $\overline{\mathbb{R}}^{\mathbb{N}}$ with the metric $\widehat{d}$, implies that results demonstrated in [17] are valid for $A' \in \mathcal{H}_\alpha(\overline{\mathbb{R}}^{\mathbb{N}})$.

Theorem 2.4. *Consider $A \in \mathcal{H}_\alpha(\mathbb{R}^\mathbb{N})$, then:*

a) There is $\lambda_A > 0$, and an strictly positive Hölder continuous function $\psi_A : \mathbb{R}^\mathbb{N} \to \mathbb{R}$, such that, $\mathcal{L}_A(\psi_A)(x) = \lambda_A \psi_A(x)$, for all $x \in \mathbb{R}^\mathbb{N}$. Moreover, the main eigenvalue is simple.

b) Defining $\bar{A} = A + \log(\psi_A) - \log(\psi_A \circ \sigma) - \log(\lambda_A)$, there is a unique fixed point μ_A of $\mathcal{L}^*_{\bar{A}}$ which belongs to $\mathcal{M}_\sigma(\mathbb{R}^\mathbb{N})$. This probability measure is called Gibbs state associated for A.

c) Choosing adequately the eigenfunction ψ_A, we have that $d\rho_A = \frac{1}{\psi_A} d\mu_A$ is a probability measure satisfying $\mathcal{L}^*_A \rho_A = \lambda_A \rho_A$. This measure is called conformal probability measure for A (also called eigenprobability).

d) For any $w \in \mathcal{H}_\alpha(\mathbb{R}^\mathbb{N})$ there exist the following uniform limits:

$$\lim_{n\in\mathbb{N}} \mathcal{L}^n_{\bar{A}} w = \int_{\mathbb{R}^\mathbb{N}} w d\mu_A \, .$$

and

$$\lim_{n\in\mathbb{N}} \frac{\mathcal{L}^n_A w}{\lambda^n_A} = \psi_A \int_{\mathbb{R}^\mathbb{N}} w d\rho_A \, .$$

Proof. By Lemma 2.1, there is a Hölder continuous extension A' of A defined on the set of extended real numbers $\overline{\mathbb{R}}^\mathbb{N}$. Moreover, by Theorem 2 in [17] the items *a*), *b*), *c*) and *d*) of this Theorem are valid for A'.

Now, taking $\lambda_A = \lambda_{A'}$ and $\psi_A = \psi_{A'}|_{\mathbb{R}^\mathbb{N}}$, we obtain item *a*) of this Theorem. Indeed, by Lemma 2.3 we have $\mathcal{L}_{A'} \circ I = I \circ \mathcal{L}_A$, which implies that

$$\begin{aligned} \mathcal{L}_A(\psi_A) &= (I^{-1} \circ \mathcal{L}_{A'} \circ I)(\psi_A) \\ &= (I^{-1} \circ \mathcal{L}_{A'} \circ I)(\psi_{A'}|_{\mathbb{R}^\mathbb{N}}) \\ &= (I^{-1} \circ \mathcal{L}_{A'})(\psi_{A'}) \\ &= \lambda_{A'} I^{-1}(\psi_{A'}) \\ &= \lambda_{A'} \psi_A \, . \end{aligned}$$

Therefore, in the process of showing the remaining items, it is enough to show that

$$\mu_{A'}(\overline{\mathbb{R}}^\mathbb{N} \setminus \mathbb{R}^\mathbb{N}) = 0 \, . \tag{1}$$

Indeed, under the assumption that (1) is valid, we can define $\mu_A = \mu_{A'}|_{\mathbb{R}^\mathbb{N}}$, in other words, the probability measure assigning to any Borelian set $E' \subset \overline{\mathbb{R}^\mathbb{N}}$, the value

$$\mu_{A'}|_{\mathbb{R}^\mathbb{N}}(E') = \frac{\mu_{A'}(E' \cap \mathbb{R}^\mathbb{N})}{\mu_{A'}(\mathbb{R}^\mathbb{N})} = \mu_{A'}(E' \cap \mathbb{R}^\mathbb{N}) \, .$$

Therefore $\mu_A(E) = \mu_{A'}(E \cap \mathbb{R}^{\mathbb{N}}) = \mu_{A'}(E)$ for any Borelian set $E \subset \mathbb{R}^{\mathbb{N}}$, which implies that μ_A satisfies the items b), c) and d) of this Theorem.

Now, observe that $\mu_{A'}(\{x \in \overline{\mathbb{R}}^{\mathbb{N}} : x_1 = \pm\infty\}) = 0$ implies (1). Now it follows from invariance of $\mu_{A'}$ regarding the map σ and the above condition that $\mu_{A'}(\{x \in \overline{\mathbb{R}}^{\mathbb{N}} : x_n = \pm\infty\}) = 0$, for all $n \in \mathbb{N}$. Therefore,

$$\mu_{A'}(\overline{\mathbb{R}}^{\mathbb{N}} \setminus \mathbb{R}^{\mathbb{N}}) \leq \sum_{n\in\mathbb{N}} \mu_{A'}(\{x \in \overline{\mathbb{R}}^{\mathbb{N}} : x_n = \pm\infty\}) = 0\,.$$

In order to demonstrate that $\mu_{A'}(\{x \in \overline{\mathbb{R}}^{\mathbb{N}} : x_1 = \pm\infty\}) = 0$ we will use the following procedure.

Fixing $R > 0$, by Urysohn's Lemma (and the the fact that the Hölder functions are dense in the set continuous functions) we can choose a Hölder continuous function $w_R : \overline{\mathbb{R}}^{\mathbb{N}} \to [0, 1]$ that satisfies the following conditions:

i) $\chi_{\{x\in\overline{\mathbb{R}}^{\mathbb{N}}:x_1=\pm\infty\}}(x) \leq w_R(x) \leq 1$.

ii) $w_R(x) = 0$, if $x \in (-R, R)$.

Then, we have

$$\begin{aligned}
&\mu_{A'}(\{x \in \overline{\mathbb{R}}^{\mathbb{N}} : x_1 = \pm\infty\})\\
&= \int_{\overline{\mathbb{R}}^{\mathbb{N}}} \chi_{\{x\in\overline{\mathbb{R}}^{\mathbb{N}}:x_1=\pm\infty\}} d\mu_{A'}\\
&\leq \int_{\overline{\mathbb{R}}^{\mathbb{N}}} w_R d\mu_{A'}\\
&= \lim_{n\in\mathbb{N}} \mathcal{L}^n_{\overline{A'}}(w_R)(x)\\
&= \lim_{n\in\mathbb{N}} \int_{\mathbb{R}^n} e^{S_n\overline{A'}(a^n x)} w_R(a^n x) f(a_n)\dots f(a_1) da_1 \dots da_n\\
&= \lim_{n\in\mathbb{N}} \int_{(-R,R)^c} e^{\overline{A'}(a_n x)}\\
&\quad \left(\int_{\mathbb{R}^{n-1}} e^{S_{n-1}\overline{A'}(a^{n-1}x)} f(a_{n-1})\dots f(a_1) da_1 \dots da_{n-1}\right) f(a_n) da_n\\
&= \lim_{n\in\mathbb{N}} \int_{(-R,R)^c} e^{\overline{A'}(a_n x)} f(a_n) da_n \leq e^{\|\overline{A'}\|} \int_{(-R,R)^c} f(a) da\,.
\end{aligned}$$

Taking the limit when $R \to +\infty$ we get our claim. □

Note that under the hypothesis of the above Theorem, for any $\beta > 0$ we have that $\lambda_{\beta A}$, $\psi_{\beta A}$, $\mu_{\beta A}$, and $\rho_{\beta A}$ are well defined, and satisfy the same conclusions of the Theorem. The following Lemma shows a bound for the family of logarithms of the eigenvalues associated to the family $(\beta A)_{\beta>0}$. This bound will be useful in the proofs of the results that appear below.

Lemma 2.5. *If $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, then for any $\beta > 0$ we have*

$$-\|A\| \leq \frac{1}{\beta}\log(\lambda_{\beta A}) \leq \|A\| \,.$$

Proof. Let $A' : \overline{\mathbb{R}}^{\mathbb{N}} \to \mathbb{R}$ be the Hölder continuous extension of A, then $\inf(\psi_{\beta A}) = \min(\psi_{\beta A'})$, and $\sup(\psi_{\beta A}) = \max(\psi_{\beta A'})$. The foregoing implies that for any $n \in \mathbb{N}$ there exists $\tilde{x}^n \in \mathbb{R}^{\mathbb{N}}$, such that $\psi_{\beta A}(\tilde{x}^n) < \inf(\psi_{\beta A}) + 1/n$. Therefore, using that $\mathcal{L}_{\beta A}(\psi_{\beta A})(\tilde{x}^n) = \lambda_{\beta A}\psi_{\beta A}(\tilde{x}^n)$, follows

$$\begin{aligned}
\lambda_{\beta A} = \frac{1}{\psi_{\beta A}(\tilde{x}^n)}\mathcal{L}_{\beta A}(\psi_{\beta A})(\tilde{x}^n) &= \frac{1}{\psi_{\beta A}(\tilde{x}^n)}\int_{\mathbb{R}} e^{\beta A(a\tilde{x}^n)}\psi_{\beta A}(a\tilde{x}^n)f(a)da \\
&> \frac{1}{\inf(\psi_{\beta A}) + 1/n}\int_{\mathbb{R}} e^{\beta A(a\tilde{x}^n)}\psi_{\beta A}(a\tilde{x}^n)f(a)da \\
&> \frac{1}{\inf(\psi_{\beta A})}\int_{\mathbb{R}} e^{\beta A(a\tilde{x}^n)}\psi_{\beta A}(a\tilde{x}^n)f(a)da \,.
\end{aligned}$$

Therefore, we have

$$\lambda_{\beta A} \geq \int_{\mathbb{R}} e^{\beta A(a\tilde{x}^n)}f(a)da \geq e^{-\beta\|A\|} \,.$$

Using a similar argument, it is demonstrated that $\lambda_{\beta A} \leq e^{\beta\|A\|}$, which concludes the proof. □

One of the main problems in non-compact setting is existence or not of maximizing measures. Nevertheless, the next Theorem provides conditions in which there exists a maximizing measure for a Hölder continuous potential.

Theorem 2.6. *Set $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$. If there exists $z_0 \in \mathbb{R}$ such that the extension $A' \in \mathcal{H}_\alpha(\overline{\mathbb{R}}^{\mathbb{N}})$ satisfies*

$$A'(x_1, \ldots, x_{n-1}, \pm\infty, x_{n+1}, \ldots) < A'(x_1, \ldots, x_{n-1}, z_0, x_{n+1}, \ldots) \,, \tag{2}$$

for all $x \in \overline{\mathbb{R}}^{\mathbb{N}}$ and all $n \in \mathbb{N}$, then:

a) Any maximizing measure μ_∞ of A' has support contained in $\mathbb{R}^{\mathbb{N}}$, therefore μ_∞ is a maximizing measure of A.

b) A has a calibrated sub-action V defined on $\mathbb{R}^{\mathbb{N}}$*, in other words, V is a continuous function that satisfies*

$$m(A) = \max_{a \in \mathbb{R}} \{A(ax) + V(ax) - V(x)\} \,.$$

With $m(A) = \sup\Big\{\int_{\mathbb{R}^{\mathbb{N}}} A d\mu : \mu \in \mathcal{M}_\sigma(\mathbb{R}^{\mathbb{N}})\Big\}$.

c) $\lim_{\beta \to \infty} \frac{1}{\beta} \log(\lambda_{\beta A}) = m(A)$.

Proof. Consider A' the Hölder continuous extension of A to the set $\overline{\mathbb{R}}^{\mathbb{N}}$, using that $\frac{1}{\beta}\log(\psi_{\beta A'})$ is Hölder continuous with constant $\frac{2^\alpha}{2^\alpha - 1}\mathrm{Hol}_A$ for all $\beta > 0$, follows that the family $(\frac{1}{\beta}\log(\psi_{\beta A'}))_{\beta>0}$ is equi-continuous and uniformly bounded, therefore, by Arzela-Ascoli's Theorem, there exists a convergent sub-sequence $(\frac{1}{\beta_n}\log(\psi_{\beta_n A'}))_{n\in\mathbb{N}}$ and, by Proposition 10 in [17], $V' = \lim_{n\in\mathbb{N}} \frac{1}{\beta_n}\log(\psi_{\beta_n A'})$ is a calibrated sub-action for A', in other words, for all $x \in \overline{\mathbb{R}}^{\mathbb{N}}$ is satisfied

$$m(A') = \max_{a \in \overline{\mathbb{R}}} \{A'(ax) + V'(ax) - V'(x)\} \,.$$

Moreover, a similar result in [17], guarantees that $\lim_{\beta\to\infty} \frac{1}{\beta}\log(\lambda_{\beta A'}) = m(A')$, therefore $\lim_{\beta\to\infty}\frac{1}{\beta}\log(\lambda_{\beta A}) = m(A')$.

Now, using (2), we obtain the following inequalities for each $x \in \overline{\mathbb{R}}^{\mathbb{N}}$

i) $\psi_{\beta A'}(\pm\infty, x_1, x_2, \ldots) \leq \psi_{\beta A'}(z_0, x_1, x_2, \ldots)$.

ii) $V'(\pm\infty, x_1, x_2, \ldots) \leq V'(z_0, x_1, x_2, \ldots)$

Indeed, since for all $x \in \overline{\mathbb{R}}^{\mathbb{N}}$

$$\rho_{\beta A'}(\overline{\mathbb{R}}^{\mathbb{N}})\psi_{\beta A'}(x) = \lim_{n\in\mathbb{N}} \int_{\overline{\mathbb{R}}^n} e^{S_n \beta A'(a^n x) - n\log(\lambda_{\beta A'})} f(a_n)\ldots f(a_1) da_1 \ldots da_n \,. \tag{3}$$

and the points $y = (\pm\infty x)$ and $z = (z_0 x)$ satisfy for each $n \in \mathbb{N}$ the condition

$$\int_{\overline{\mathbb{R}}^n} e^{S_n\beta A'(a^n y)} f(a_n)\ldots f(a_1) da_1\ldots da_n < \int_{\overline{\mathbb{R}}^n} e^{S_n\beta A'(a^n z)} f(a_n)\ldots f(a_1) da_1 \ldots da_n \,.$$

Then, it follows from (3), that $\psi_{\beta A'}(y) \leq \psi_{\beta A'}(z)$, therefore using that $V' = \lim_{n\in\mathbb{N}} \frac{1}{\beta_n}\log(\psi_{\beta_n A'})$, we obtain $V'(y) \leq V'(z)$. The last inequality joint with (2) implies that for all $x \in \overline{\mathbb{R}}^{\mathbb{N}}$

$$A'(\pm\infty x) + V'(\pm\infty x) - V'(x) < A'(z_0 x) + V'(z_0 x) - V'(x) \,.$$

Therefore, we have

$$\max_{a\in\overline{\mathbb{R}}}\{A'(ax)+V'(ax)-V'(x)\} = \max_{a\in\mathbb{R}}\{A'(ax)+V'(ax)-V'(x)\}\,.$$

In other words, using that $A = A'|_{\mathbb{R}^{\mathbb{N}}}$, and taking $V = V'|_{\mathbb{R}^{\mathbb{N}}}$, it follows from the properties of the calibrated sub-action of V', that for all $x \in \mathbb{R}^{\mathbb{N}}$

$$m(A') = \max_{a\in\mathbb{R}}\{A(ax)+V(ax)-V(x)\}\,.$$

Thus, in the process of finalizing the proof, we only need to demonstrate that $m(A) = m(A')$, and for that, it is enough showing that any A'-maximizing measure it is supported in $\mathbb{R}^{\mathbb{N}}$.

Indeed, observe that any A'-maximizing probability measure μ_∞ satisfies

$$\int_{\overline{\mathbb{R}}^{\mathbb{N}}}(A'+V'-V'\circ\sigma-m(A'))d\mu_\infty = 0\,,$$

then, taking in account that $A'+V'-V'\circ\sigma-m(A')$ is less than or equal to zero and continuous, it follows that this function vanishes at the support of μ_∞.

Now observe that, for $x \in \overline{\mathbb{R}}^{\mathbb{N}} \setminus \mathbb{R}^{\mathbb{N}}$ there is $k = k(x) \in \mathbb{N}$, such that, $x_k = \pm\infty$, and $x_l \neq \pm\infty$ for $1 \leq l < k$, then taking y defined by $y_i = x_i$ for all $i \neq k$ and $y_k = z_0$, it follows that

$$\begin{aligned}&A'(\sigma^{k-1}(x))+V'(\sigma^{k-1}(x))-V'(\sigma^k(x))-m(A')\\&< A'(\sigma^{k-1}(y))+V'(\sigma^{k-1}(y))-V'(\sigma^k(y))-m(A') \leq 0\,,\end{aligned}$$

which implies that x does not belongs to the support of any maximizing measure. □

From now on, we will denote the set of Gibbs states associated to the a priori probability measure $\nu = fdx$ as $\mathcal{G}$, in other words, the set of $\mu \in \mathcal{M}_\sigma(\mathbb{R}^{\mathbb{N}})$, such that, $\mathcal{L}_B^*(\mu) = \mu$ for some Hölder normalized potential B. In this case we define the entropy of $\mu \in \mathcal{G}$ as

$$h(\mu) = -\int_{\mathbb{R}^{\mathbb{N}}} Bd\mu\,. \tag{4}$$

In particular, when $B = \bar{A}$ for some Hölder continuous potential A, we have

$$\begin{aligned}h(\mu_A) &= -\int_{\mathbb{R}^{\mathbb{N}}} \bar{A}d\mu_A\\&= -\int_{\mathbb{R}^{\mathbb{N}}} A+\log(\psi_A)-\log(\psi_A\circ\sigma)-\log(\lambda_A)d\mu_A\\&= -\int_{\mathbb{R}^{\mathbb{N}}} Ad\mu_A+\log(\lambda_A)\,.\end{aligned}$$

We can extend (4) for the set $\mathcal{M}_\sigma(\mathbb{R}^\mathbb{N})$ in the following way

$$h(\mu) = \inf_{u \in \mathcal{C}_b^+(\mathbb{R}^\mathbb{N})} \left\{ \int_{\mathbb{R}^\mathbb{N}} \log\left(\frac{\mathcal{L}_0(u)}{u}\right) d\mu \right\} . \tag{5}$$

With $\mathcal{C}_b^+(\mathbb{R}^\mathbb{N})$ the set of strictly positive bounded continuous functions from $\mathbb{R}^\mathbb{N}$ into $\mathbb{R}$.

Henceforth we are going to show that the (5) coincides with (4) in the case that $\mu \in \mathcal{G}$, furthermore, we are going to show that this definition in fact satisfies a variational principle.

Lemma 2.7. *Let $\mu \in \mathcal{G}$ a Gibbs state associated to a normalized potential B. Then,*

$$h(\mu) = \inf_{u \in \mathcal{C}_b^+(\mathbb{R}^\mathbb{N})} \left\{ \int_{\mathbb{R}^\mathbb{N}} \log\left(\frac{\mathcal{L}_0(u)}{u}\right) d\mu \right\} .$$

Proof. Set $u_0 = e^B$, then u_0 belongs to $\mathcal{C}_b^+(\mathbb{R}^\mathbb{N})$, moreover, since B is a normalized potential, it follows that

$$\log\left(\frac{\mathcal{L}_0(u_0)}{u_0}\right) = -B \, .$$

Therefore, integrating on both sides of the equation with respect to the measure μ we get

$$\int_{\mathbb{R}^\mathbb{N}} \log\left(\frac{\mathcal{L}_0(u_0)}{u_0}\right) d\mu = h(\mu) \, .$$

The rest of the proof follows similar arguments as in [17]. □

Note that the above Lemma shows that (5) extends (4) to the set of $\mathcal{M}_\sigma(\mathbb{R}^\mathbb{N})$. Moreover for any $\mu \in \mathcal{M}_\sigma(\mathbb{R}^\mathbb{N})$ we have $h(\mu) \leq 0$. Indeed, it follows by (5) that

$$h(\mu) \leq \int_{\mathbb{R}^\mathbb{N}} \log\left(\mathcal{L}_0(1)(x)\right) d\mu(x) = \int_{\mathbb{R}^\mathbb{N}} \log\left(\int_\mathbb{R} f(a) da\right) d\mu(x) = 0 \, .$$

Besides that, for any $A \in \mathcal{H}_\alpha(\mathbb{R}^\mathbb{N})$ and each $\mu \in \mathcal{M}_\sigma(\mathbb{R}^\mathbb{N})$, we have

$$
\begin{aligned}
h(\mu) + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu &= \inf_{u \in \mathcal{C}_b^+(\mathbb{R}^{\mathbb{N}})} \left\{ \int_{\mathbb{R}^{\mathbb{N}}} \log\left(\frac{\mathcal{L}_0(u)}{u}\right) d\mu \right\} + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu \\
&\leq \int_{\mathbb{R}^{\mathbb{N}}} \log\left(\frac{\mathcal{L}_0(e^A\psi_A)}{e^A\psi_A}\right) d\mu + \int_{\mathbb{R}^{\mathbb{N}}} \log(e^A) d\mu \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \log\left(\frac{\mathcal{L}_0(e^A\psi_A)}{\psi_A}\right) d\mu \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \log\left(\frac{\mathcal{L}_A(\psi_A)}{\psi_A}\right) d\mu \\
&= \log(\lambda_A)\,.
\end{aligned}
$$

The next theorem shows that (5) satisfies a variational principle.

Theorem 2.8. *Consider $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ and λ_A be the maximal eigenvalue of $\mathcal{L}_A$ obtained in Theorem 2.4. Then*

$$
\log(\lambda_A) = \sup_{\mu \in \mathcal{M}_\sigma(\mathbb{R}^{\mathbb{N}})} \left\{ h(\mu) + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu \right\}.
$$

Moreover, the supremum in the above expression is attained at μ_A.

Proof. Since $A \in \mathcal{H}_\alpha^+(\mathbb{R}^{\mathbb{N}})$, follows that

$$
\begin{aligned}
&\sup_{\mu \in \mathcal{M}_\sigma(\mathbb{R}^{\mathbb{N}})} \left\{ h(\mu) + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu \right\} \\
&\leq \sup_{\mu \in \mathcal{M}_\sigma(\mathbb{R}^{\mathbb{N}})} \left\{ -\int_{\mathbb{R}^{\mathbb{N}}} A d\mu + \log(\lambda_A) + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu \right\} \\
&= \log(\lambda_A)\,.
\end{aligned}
$$

Conversely, we have that

$$
\log(\lambda_A) = h(\mu_A) + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu_A \leq \sup_{\mu \in \mathcal{M}_\sigma(\mathbb{R}^{\mathbb{N}})} \left\{ h(\mu) + \int_{\mathbb{R}^{\mathbb{N}}} A d\mu \right\}.
$$

□

The following lemma provides conditions to guarantee existence of maximizing probability measures through the existence of ground states.

Lemma 2.9. *Let $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$. If the family $(\mu_{\beta A})_{\beta > 0}$ has an accumulation point μ_∞ at infinity, then this point is an A-maximizing probability measure.*

Proof. By hypothesis, there is a sequence $(\beta_n)_{n\in\mathbb{N}}$, with $\beta_n \to +\infty$, such that $\lim_{n\in\mathbb{N}} \mu_{\beta_n A} = \mu_\infty$. Using that $h(\mu) \leq 0$, follows that

$$\begin{aligned} m(A) = \lim_{n\in\mathbb{N}} \frac{1}{\beta_n} \log(\lambda_{\beta_n A}) &= \lim_{n\in\mathbb{N}} \left(\frac{1}{\beta_n} h(\mu_{\beta_n A}) + \int_{\mathbb{R}^\mathbb{N}} A d\mu_{\beta_n A} \right) \\ &\leq \int_{\mathbb{R}^\mathbb{N}} A d\mu_{\beta_n A} \\ &\leq \int_{\mathbb{R}^\mathbb{N}} A d\mu_\infty \\ &= m(A)\,. \end{aligned}$$

Therefore, $\int_{\mathbb{R}^\mathbb{N}} A d\mu_\infty = m(A)$. □

The next proposition shows the existence of ground states. Some interesting results in this direction can be found in [3, 5, 12, 13, 15, 16].

Proposition 2.10. *Consider $A \in \mathcal{H}_\alpha(\mathbb{R}^\mathbb{N})$. If there exists $z_0 \in \mathbb{R}$ such that the extension $A' \in \mathcal{H}_\alpha(\overline{\mathbb{R}}^\mathbb{N})$ satisfies (2) for all $x \in \overline{\mathbb{R}}^\mathbb{N}$ and all $n \in \mathbb{N}$, then the family $(\mu_{\beta A})_{\beta>0}$ has an accumulation point μ_∞ at infinity.*

Proof. Let $\beta > 0$, and $\beta A' : \overline{\mathbb{R}}^\mathbb{N} \to \mathbb{R}$ the Hölder continuous extension of βA, by Theorem 2.4 we have $\mu_{\beta A'}(\overline{\mathbb{R}}^\mathbb{N} \setminus \mathbb{R}^\mathbb{N}) = 0$ and $\mu_{\beta A} = \mu_{\beta A'}|_{\mathbb{R}^\mathbb{N}}$. Therefore, follows of compactness of $\overline{\mathbb{R}}^\mathbb{N}$, that $(\mu_{\beta A'})_{\beta>0}$ has an accumulation point μ'_∞ at infinity, in other words, there is a sequence $(\beta_n)_{n\in\mathbb{N}}$ with $\beta_n \to \infty$ such that $\lim_{n\in\mathbb{N}} \mu_{\beta_n A'} = \mu'_\infty$. By Theorem 5 in [18] we have that μ'_∞ is an A'-maximizing measure, then using part a) of Theorem 2.6 we obtain that this probability measure is supported in $\mathbb{R}^\mathbb{N}$.

Let $g : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ a Lipschitz continuous function, observe that this implies that g is bounded. Defining $g'(x) = \lim_{y\to x} g(y)$ for each $x \in \overline{\mathbb{R}}^\mathbb{N}$, we obtain that $g' : \overline{\mathbb{R}}^\mathbb{N} \to \mathbb{R}$ is a bounded Lipschitz continuous function, which implies

$$\lim_{n\in\mathbb{N}} \int_{\mathbb{R}^\mathbb{N}} g d\mu_{\beta_n A} = \lim_{n\in\mathbb{N}} \int_{\overline{\mathbb{R}}^\mathbb{N}} g' d\mu_{\beta_n A} = \int_{\overline{\mathbb{R}}^\mathbb{N}} g' d\mu'_\infty = \int_{\mathbb{R}^\mathbb{N}} g d\mu_\infty\,.$$

In other words, $(\mu_{\beta A})_{\beta>0}$ has an accumulation point μ_∞ at infinity, and by Lemma 2.9 this probability measure is A-maximizing. □

3 Involution kernel

It is widely known that Livsic's Theorem guarantees that a potential with some regularity defined from $\mathbb{R}^\mathbb{Z}$ into $\mathbb{R}$ is co-homologous (via the bilateral shift) to a potential defined from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}$. Conversely, also there exists a tool, known as involution kernel, that provides the desired co-homology between a potential defined from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}$ and a potential defined from $\mathbb{R}^{\mathbb{Z}}$ into $\mathbb{R}$. The case of the shift acting on $\{1,2,...,d\}^{\mathbb{N}}$ was considered in [2] .

In this Section we construct an involution kernel for the non-compact case studied in Section 2 and we show some properties that provides an extension of the Gibbs states defined in the last Section for the bilateral case joint with an interesting example for the case of stationary Markov probability measures.

We define $(\mathbb{R}^{\mathbb{N}})^* = \{(\ldots, y_2, y_1) \in \mathbb{R}^{\mathbb{N}}\}$ and the map $\sigma^* : (\mathbb{R}^{\mathbb{N}})^* \to (\mathbb{R}^{\mathbb{N}})^*$ as $\sigma^*(\ldots, y_2, y_1) = (\ldots, y_3, y_2)$. For each pair of points $(y, x) \in (\mathbb{R}^{\mathbb{N}})^* \times \mathbb{R}^{\mathbb{N}}$ we will denote by $(y|x)$ the element $(\ldots, y_2, y_1|x_1, x_2, \ldots)$. The set of ordered pairs $(y|x)$ with $x \in \mathbb{R}^{\mathbb{N}}$ and $y \in (\mathbb{R}^{\mathbb{N}})^*$ will be called $\widehat{\mathbb{R}^{\mathbb{N}}}$, which is isomorphic to $\mathbb{R}^{\mathbb{Z}}$. In this case we can define a bilateral sub-shift $\widehat{\sigma} : \widehat{\mathbb{R}^{\mathbb{N}}} \to \widehat{\mathbb{R}^{\mathbb{N}}}$ as the map $\widehat{\sigma}(y|x) = (\tau^*_x(y)|\sigma(x))$, with $\tau^*_x(y) = (\ldots, y_2, y_1, x_1)$.

Now, fixing $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, we say that $W : \widehat{\mathbb{R}^{\mathbb{N}}} \to \mathbb{R}$ is an involution kernel for A, if the adjunct potential A^* defined by

$$A^* = A \circ \widehat{\sigma}^{-1} + W \circ \widehat{\sigma}^{-1} - W \,,$$

depends only of the variable y. In [2] was shown that in fact A has a Hölder continuous involution kernel defined by

$$W(y|x) = \sum_{n \in \mathbb{N}} A(\tau_{y,n}(x)) - A(\tau_{y,n}(x')) \,, \tag{6}$$

with $\tau_{y,n}(x) = (y_n, \ldots, y_1, x_1, x_2, \ldots)$. Note that this involution kernel is Holder, because $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ and $\tau_{y,n}(x)$ is a contraction. Furthermore, the above implies that A^* is also a Hölder potential.

Now, we are going to define a natural extension of the Gibbs state associated to A in the bilateral sub-shift $\widehat{\mathbb{R}^{\mathbb{N}}}$. Taking a constant $c \in \mathbb{R}$ satisfying

$$\int_{\widehat{\mathbb{R}^{\mathbb{N}}}} e^{W(y|x)} d(\rho_{A^*} \times \rho_A)(y, x) = e^c \,,$$

which is possible because $e^{W(y|x)}$ is an strictly positive function, we define $K(y|x) = e^{W(y|x) - c}$ and, using the above function, $\widehat{\mu}_A$ is defined by

$$d\widehat{\mu}_A(y, x) = K(y|x) d(\rho_{A^*} \times \rho_A)(y, x) \,.$$

Using a similar procedure to the one which was used in [2] it is possible to show that $\widehat{\mu} \in \mathcal{M}_{\widehat{\sigma}}(\widehat{\mathbb{R}^{\mathbb{N}}})$ and extends the Gibbs states μ_A and μ_{A^*} in the following way:

if $\varphi : \widehat{\mathbb{R}^{\mathbb{N}}} \to \mathbb{R}$ is an $\widehat{\mu}_A$-integrable function, such that, $\varphi(y|x) = \varphi(z|x)$, for all $y, z \in (\mathbb{R}^{\mathbb{N}})^*$, then using the notation $\varphi(y|x) = \varphi(x)$, we have

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi(x) d\mu_A(x) = \int_{\widehat{\mathbb{R}^{\mathbb{N}}}} \varphi(x) d\widehat{\mu}_A(y, x) .$$

Analogously, if $\varphi : \widehat{\mathbb{R}^{\mathbb{N}}} \to \mathbb{R}$ satisfies $\varphi(y|x) = \varphi(y|z)$ for all $x, z \in \mathbb{R}^{\mathbb{N}}$, then using the notation $\varphi(y|x) = \varphi(y)$, we obtain that

$$\int_{(\mathbb{R}^{\mathbb{N}})^*} \varphi(y) d\mu_{A^*}(y) = \int_{\widehat{\mathbb{R}^{\mathbb{N}}}} \varphi(y) d\widehat{\mu}_A(y, x) .$$

The proofs of the above claims are a consequence of ones for the compact case that appears in [17]. Here we use the fact that in our case the Gibbs states $\mu_{A'}$ and $\mu_{(A')^*}$ are supported in $\mathbb{R}^{\mathbb{N}}$. Moreover, using the kernel K, we can find an explicit form for the eigenfunctions of $\mathcal{L}_A$ and $\mathcal{L}_{A^*}$ associated to $\lambda_A = \lambda_{A^*}$, which are given by $\psi_A(x) = \int_{(\mathbb{R}^{\mathbb{N}})^*} K(y|x) d\rho_{A^*}(y)$ and $\psi_{A^*}(y) = \int_{\mathbb{R}^{\mathbb{N}}} K(y|x) d\rho_A(x)$ respectively.

Besides that, the j-th partial derivative of this involution kernel is well defined for all $j \in \mathbb{N}$, when A satisfies the following conditions:

i) There exists the partial derivative of A regarding the j-th coordinate at the point x, for all $x \in \mathbb{R}^{\mathbb{N}}$.

ii) Given $\epsilon > 0$, there exists $H_\epsilon > 0$, such that, for all $x \in \mathbb{R}^{\mathbb{N}}$, if $h < H_\epsilon$, then for all $j \in \mathbb{N}$ it is satisfied the expression

$$\left| \frac{A(x + he_j) - A(x)}{h} - D_j A(x) \right| < \frac{\epsilon}{2^j} .$$

From the definition of partial derivative and ii), it is possible to demonstrate, in a similar way as in [17], that the j-th partial derivative of the involution kernel W satisfies the following equation

$$D_j W(y|x) = \sum_{n \in \mathbb{N}} D_{n+j} A(\tau_{y,n}(x)) . \tag{7}$$

Therefore, using the explicit form of the eigenfunctions (associated to $\mathcal{L}_A$ and $\mathcal{L}_{A^*}$) and also using the fact that $K(\cdot|x)$ is integrable (regarding ρ_{A^*}) and $K(y|\cdot)$ is integrable (with respect to ρ_A), we obtain the following expressions for the partial derivatives of the eigenfunctions associated to $\mathcal{L}_A$ and $\mathcal{L}_{A^*}$.

$$D_j \psi_A(x) = \int_{(\mathbb{R}^{\mathbb{N}})^*} K(y|x) \sum_{n \in \mathbb{N}} D_{n+j} A(\tau_{y,n}(x)) d\rho_{A^*}(y) , \tag{8}$$

and

$$D_j\psi_{A^*}(x)=\int_{\mathbb{R}^{\mathbb{N}}}K(y|x)\sum_{n\in\mathbb{N}}D_{n+j}A(\tau_{y,n}(x))d\rho_A(x)\,.$$

Henceforth, we will interested in describing the differentiable property of the eigenfunction of the Ruelle operator for a differentiable potential A (in the case of Markov chains). Some results are known in the compact setting for the XY model (see [17]). We are interested in study partial derivatives of the involution kernel and the entropy for induced stationary Markov measures associated to potentials that depends only of two coordinates, in other words, potentials $A:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ such that $A(x)=A(x_1,x_2)$, which implies that in this case when can consider that $A:\mathbb{R}^2\to\mathbb{R}$.

Stationary Markov measures are defined using a transition kernel P, and an stationary vector for P, which we will denote by θ. The transition kernel is a strictly positive function $P:\mathbb{R}^2\to\mathbb{R}$ satisfying

$$\int_{\mathbb{R}}P(x_1,x_2)f(x_2)dx_2=1\,. \tag{9}$$

The stationary map for P is an strictly positive function $\theta:\mathbb{R}\to\mathbb{R}$ that satisfies

$$\int_{\mathbb{R}}\theta(x_1)P(x_1,x_2)f(x_1)dx_1=\theta(x_2)\,. \tag{10}$$

Using the foregoing equations, we can define the stationary Markov measure induced by P and θ, as the probability measure

$$\mu([A_1\ldots A_n])=$$

$$\int_{A_1\times\ldots\times A_n}\theta(x_1)P(x_1,x_2)\ldots P(x_{n-1},x_n)f(x_n)\ldots f(x_1)dx_1\ldots dx_n\,.$$

Besides that, it is possible to show, using (6), that any Hölder potential $A:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ that depends of two coordinates, has a Hölder involution kernel defined by $W(y|x)=A(y_1,x_1)$, and in this case $A^*(y)=A(y_2,y_1)$.

The following Theorem shows that Gibbs states associated to potentials that depends of two coordinates are stationary Markov measures and conversely.

Theorem 3.1. *a) Consider $A\in\mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ such that $A(x)=A(x_1,x_2)$, then there exists an stationary Markov measure μ, that is Gibbs state associated to A.*

b) Given an stationary Markov measure μ induced by P and θ, there exists $A\in\mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, with $A(x)=A(x_1,x_2)$, such that this measure is a Gibbs state for A.

Proof. a) Let ψ_A and $\overline{\psi}_A$ be the eigenfunctions associated to $\mathcal{L}_A$ and $\mathcal{L}_{A^*}$, respectively. Note that $\psi_A(x) = \psi_A(x_1)$ and $\overline{\psi}_A(x) = \overline{\psi}_A(x_1)$, because the potential A depends only on its first two coordinates.

Now we are going to demonstrate that $P_A(x_1, x_2) = \frac{e^{A(x_1,x_2)}\overline{\psi}_A(x_2)}{\lambda_A\overline{\psi}_A(x_1)}$ is a transition kernel, and $\theta(x_1) = \frac{\psi_A(x_1)\overline{\psi}_A(x_1)}{\pi_A}$ is its respectively stationary map, with $\pi_A = \int_{\mathbb{R}} \psi_A(x_1)\overline{\psi}_A(x_1)f(x_1)dx_1$.

Using a similar procedure to the one which was used in Theorem 16 of [1] it is possible to demonstrate that for any $g_n \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, satisfying $g_n(x) = g_n(x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$, it is true the equality

$$\int_{\mathbb{R}^{\mathbb{N}}} \mathcal{L}_{\overline{A}}(g_n)d\mu = \int_{\mathbb{R}^{\mathbb{N}}} g_n d\mu\,.$$

The foregoing equation guarantees part *a*) of this Theorem, because in the general case, i.e. when $g \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ depends on an arbitrary number of coordinates, defining $g_n(x) = g(x^n)$, with $x^n = (x_1, \ldots, x_n, 1^\infty)$ we have

$$|g(x) - g_n(x)| \leq K\widehat{d}(x, x^n)^\alpha \leq \frac{K}{2^{n\alpha}}\,.$$

This implies that $(g_n)_{n\in\mathbb{N}}$ converges pointwise to g. Then, by Dominated Convergence Theorem it follows that

$$\int_{\mathbb{R}^{\mathbb{N}}} \mathcal{L}_{\overline{A}} g d\mu = \lim_{n\in\mathbb{N}} \int_{\mathbb{R}^{\mathbb{N}}} \mathcal{L}_{\overline{A}} g_n d\mu = \lim_{n\in\mathbb{N}} \int_{\mathbb{R}^{\mathbb{N}}} g_n d\mu = \int_{\mathbb{R}^{\mathbb{N}}} g d\mu\,.$$

b) If P and θ satisfy (9) and (10), respectively, taking $A = \log(P)$, we obtain that

$$\mathcal{L}_{A^*}(1)(x_1) = \int_{\mathbb{R}} e^{A(x_1,x_2)} f(x_2)dx_2 = \int_{\mathbb{R}} P(x_1, x_2)f(x_2)dx_2 = 1\,,$$

which implies that $\lambda_A = 1$ and $\overline{\psi}_A \equiv 1$. Therefore, defining $\theta_A = \frac{\psi_A}{\pi_A}$, it follows that

$$\begin{aligned}\int_{\mathbb{R}} \theta_A(x_1)P(x_1, x_2)f(x_1)dx_1 &= \int_{\mathbb{R}} \frac{\psi_A(x_1)}{\pi_A} e^{A(x_1,x_2)} f(x_1)dx_1 \\ &= \frac{1}{\pi_A}\mathcal{L}_A(\psi_A)(x_2) \\ &= \frac{\psi_A(x_2)}{\pi_A} \\ &= \theta_A(x_2)\,.\end{aligned}$$

This implies that the measure induced by P and θ is a Gibbs state for A.

$\square$

Proposition 3.2. *Let μ be an stationary Markov measure defined by P and θ, then the entropy of this measure, when given by*

$$S(\theta P) = -\int_{\mathbb{R}^2} \theta(x_1)P(x_1,x_2)\log(P(x_1,x_2))f(x_2)f(x_1)dx_1dx_2\,,$$

coincides with the entropy given by the usual definition for Gibbs states.

Proof. Set μ an stationary Markov measure induced by P and θ, then, it follows of part $b)$ of the claim of above Theorem that μ is a Gibbs measure associated to the normalized potential $A = \log(P)$, therefore the entropy of this measure is given by

$$\begin{aligned} h(\mu) &= -\int_{\mathbb{R}^2} \log(P(x_1,x_2))d\mu(x_1,x_2) \\ &= -\int_{\mathbb{R}^2} \log(P(x_1,x_2))\theta(x_1)P(x_1,x_2)f(x_1)f(x_2)dx_2dx_1\,. \end{aligned}$$

$\square$

Now we will make some observations about differentiability of eigenfunction of the Ruelle operator associated to a differentiable potential A that depends of two coordinates.

Observe that defining $G(x_1) = \int_{\mathbb{R}} e^{A(y_1,x_1)}\varphi(y_1)f(y_1)dy_1$, with $\varphi : \mathbb{R} \to \mathbb{R}$ a Hölder continuous function, and using the fact that $e^{A(y_1,\cdot)}\varphi(y_1)f(y_1)$ is integrable for all $x_1 \in \mathbb{R}$ and differentiable for each $y_1 \in \mathbb{R}$, it follows from (7) that

$$\frac{\partial G}{\partial x_1}(x_1) = \int_{\mathbb{R}} e^{A(y_1,x_1)}\frac{\partial A}{\partial x_1}(y_1,x_1)\varphi(y_1)f(y_1)dy_1\,.$$

Furthermore, in particular when $\varphi = \psi_A$, we obtain

$$\begin{aligned} \frac{\partial \psi_A}{\partial x_1}(x_1) &= \frac{1}{\lambda_A}\int_{\mathbb{R}} e^{A(y_1,x_1)}\frac{\partial A}{\partial x_1}(y_1,x_1)\psi_A(y_1)f(y_1)dy_1 \\ &= \frac{1}{\lambda_A}\int_{\mathbb{R}} e^{A(y_1,x_1)}\frac{\partial A}{\partial x_1}(y_1,x_1)d\rho_{A^*}(y_1)\,. \end{aligned}$$

Which coincides with the partial derivative obtained in the general case using the involution kernel approach, such as appears in (8).

4 *FKG*-Inequality

DLR-Gibbs probability measures are an interesting topic in Statistical Mechanics. In this Section we are going to show existence of this type of measures in a similar fashion as in Sections 2 and 3. Moreover, we are going to show that these probability measures satisfy a *FKG*-inequality and we will show the connection between *DLR*-Gibbs probability measures, Thermodynamic limit Gibbs probability measures and Gibbs states such as was defined in Section 2.

If $x, y \in \mathbb{R}^{\mathbb{N}}$, we say that $x \preceq y$ if, and only if $x_i \leq y_i$ for each $i \in \mathbb{N}$. Using the above definition we say that a function $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is an increasing function, if for any pair $x, y \in \mathbb{R}^{\mathbb{N}}$ such that $x \preceq y$, we have $\varphi(x) \leq \varphi(y)$.

For each $n \in \mathbb{N}$, $t \in \mathbb{R}$, and $x, y, z \in \mathbb{R}^{\mathbb{N}}$ we are going to use the following notation

$$\begin{aligned}
[x|y]_n &= (x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots) \in \mathbb{R}^{\mathbb{N}}\,, \\
[x|t|y]_n &= (x_1, \ldots, x_n, t, y_{n+2}, y_{n+3}, \ldots) \in \mathbb{R}^{\mathbb{N}}\,, \\
[x|y|z]_{n,n+r} &= (x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+r}, z_{n+r+1}, z_{n+r+2}, \ldots) \in \mathbb{R}^{\mathbb{N}}\,, \\
[t|y]_1 &= (t, y_2, y_3, \ldots) \in \mathbb{R}^{\mathbb{N}}\,.
\end{aligned}$$

Note that under the above notation $[t|\sigma^n(y)]_1 = (t, y_{n+2}, y_{n+3}, \ldots) \in \mathbb{R}^{\mathbb{N}}$.

Fixing $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$, $y \in \mathbb{R}^{\mathbb{N}}$, and $n \in \mathbb{N}$, we define the probability measure μ_n^y on all the Borelian sets in $\mathbb{R}^{\mathbb{N}}$, assigning to each $E \subset \mathbb{R}^{\mathbb{N}}$ the value

$$\mu_n^y(E) = \frac{1}{Z_n^y} \int_{\mathbb{R}^n} e^{S_n A([x|y]_n)} \chi_E([x|y]_n) f(x_1) \ldots f(x_n) dx_1 \ldots dx_n\,, \tag{11}$$

with $Z_n^y = \int_{\mathbb{R}^n} e^{S_n A([x|y]_n)} f(x_1) \ldots f(x_n) dx_1 \ldots dx_n$. In this case, the integral of any Hölder continuous function $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^y &= \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_n) d\mu_n^{[x|y]_n}(x) \\
&= \frac{1}{Z_n^y} \int_{\mathbb{R}^n} e^{S_n A([x|y]_n)} \varphi([x|y]_n) f(x_1) \ldots f(x_n) dx_1 \ldots dx_n\,. \qquad (12)
\end{aligned}$$

In the next Lemma we are going to show that the family of probability measures defined in (11) with $n \in \mathbb{N}$ and $y \in \mathbb{R}^{\mathbb{N}}$, is in fact a Gibbsian specification, with kernel $K_n(E, y) = \mu_n^y(E)$.

Lemma 4.1. *The family* $K_n : (\mathcal{B}, \mathbb{R}^{\mathbb{N}}) \to [0, 1]$ *with* $n \in \mathbb{N}$*, defined by*

$$K_n(E, y) = \mu_n^y(E)\,,$$

is a Gibbsian specification, in other words, $(K_n)_{n\in\mathbb{N}}$ satisfies the following properties

a) The map $y \mapsto K_n(E, y)$ is $\sigma^n\mathcal{B}$-measurable for any $E \in \mathcal{B}$.

b) The map $E \mapsto K_n(E, y)$ is a probability measure for each $y \in \mathbb{R}^{\mathbb{N}}$.

c) For any $n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, and any Hölder continuous function $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ we have the compatibility condition

$$K_{n+r}(\varphi, z) = K_{n+r}(K_n(\varphi, y), z)\,,$$

with $K_n(\varphi, y) = \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^y$.

Proof. Note that for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^{\mathbb{N}}$ we have that the maps assigning $y \mapsto S_nA([x|y]_n)$ and $y \mapsto \chi_E([x|y]_n)$ are measurable, which implies that the map $y \mapsto K_n(E, y)$ is $\sigma^n\mathcal{B}$-measurable for any $E \in \mathcal{B}$, therefore part $a)$ of this Lemma is obtained.

Part $b)$ is obvious by the definition of K_n. In order to demonstrate part $c)$ of this Lemma, we observe that the equation $K_{n+r}(\varphi, z) = K_{n+r}(K_n(\varphi, y), z)$ is equivalent the following equality

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi([y|z]_{n+r}) d\mu_{n+r}^{[y|z]_{n+r}}(y) = \int_{\mathbb{R}^{\mathbb{N}}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y|z]_{n,n+r}) d\mu_n^{[x|y|z]_{n,n+r}}(x) \right) d\mu_{n+r}^{[y|z]_{n+r}}(y)\,.$$

Indeed, defining

$$\psi([y|z]_{n+r}) = \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y|z]_{n,n+r}) d\mu_n^{[x|y|z]_{n,n+r}}(x)\,,$$

it will be enough to show that

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi([y|z]_{n+r}) d\mu_{n+r}^{[y|z]_{n+r}}(y) = \int_{\mathbb{R}^{\mathbb{N}}} \psi([y|z]_{n+r}) d\mu_{n+r}^{[y|z]_{n+r}}(y)\,,$$

which is equivalent to show that

$$\int_{\mathbb{R}^{n+r}} e^{S_{n+r}A([y|z]_{n+r})} \varphi([y|z]_{n+r}) f(y_1) \ldots f(y_{n+r}) dy_1 \ldots dy_{n+r}$$
$$= \int_{\mathbb{R}^{n+r}} e^{S_{n+r}A([y|z]_{n+r})} \psi([y|z]_{n+r}) f(y_1) \ldots f(y_{n+r}) dy_1 \ldots dy_{n+r}\,.$$

Indeed, observe that

$$\int_{\mathbb{R}^{n+r}} e^{S_{n+r}A([y|z]_{n+r})}\psi([y|z]_{n+r})f(y_1)\dots f(y_{n+r})dy_1\dots dy_{n+r}$$
$$= \int_{\mathbb{R}^{n+r}} \frac{1}{Z^{[y|z]_{n+r}}}$$
$$\left(\int_{\mathbb{R}^n} e^{S_{n+r}A([y|z]_{n+r})+S_nA([x|y|z]_{n,n+r})}\varphi([x|y|z]_{n,n+r})f(x_1)\dots f(x_n)dx_1\dots dx_n\right)$$
$$f(y_1)\dots f(y_{n+r})dy_1\dots dy_{n+r}\,.$$

Now, using the fact that any ergodic sum satisfies

$$S_{n+r}A([y|z]_{n+r}) + S_nA([x|y|z]_{n,n+r}) = S_{n+r}A([x|y|z]_{n,n+r}) + S_nA([y|z]_{n+r})\,,$$

it follows that the above integral is equal to

$$\int_{\mathbb{R}^{n+r}} \frac{1}{Z^{[y|z]_{n+r}}}$$
$$\left(\int_{\mathbb{R}^n} e^{S_{n+r}A([x|y|z]_{n,n+r})+S_nA([y|z]_{n+r})}\varphi([x|y|z]_{n,n+r})f(x_1)\dots f(x_n)dx_1\dots dx_n\right)$$
$$f(y_1)\dots f(y_{n+r})dy_1\dots dy_{n+r}$$
$$= \int_{\mathbb{R}^r} (\int_{\mathbb{R}^n} \frac{1}{Z^{[y|z]_{n+r}}}$$
$$\left(\int_{\mathbb{R}^n} e^{S_{n+r}A([x|y|z]_{n,n+r})+S_nA([y|z]_{n+r})}\varphi([x|y|z]_{n,n+r})f(x_1)\dots f(x_n)dx_1\dots dx_n\right)$$
$$f(y_1)\dots f(y_n)dy_1\dots dy_n\Big)f(y_{n+1})\dots f(y_{n+r})dy_{n+1}\dots dy_{n+r}$$
$$= \int_{\mathbb{R}^r} \left(\int_{\mathbb{R}^n} \frac{e^{S_nA([y|z]_{n+r})}}{Z^{[y|z]_{n+r}}} f(y_1)\dots f(y_n)dy_1\dots dy_n\right)$$
$$\left(\int_{\mathbb{R}^n} e^{S_{n+r}A([x|y|z]_{n,n+r})}\varphi([x|y|z]_{n,n+r})f(x_1)\dots f(x_n)dx_1\dots dx_n\right)$$
$$f(y_{n+1})\dots f(y_{n+r})dy_{n+1}\dots dy_{n+r}$$

$$= \int_{\mathbb{R}^r} \left(\int_{\mathbb{R}^n} e^{S_{n+r}A([x|y|z]_{n,n+r})}\varphi([x|y|z]_{n,n+r})f(x_1)\dots f(x_n)dx_1\dots dx_n\right)$$
$$f(y_{n+1})\dots f(y_{n+r})dy_{n+1}\dots dy_{n+r}$$
$$= \int_{\mathbb{R}^{n+r}} e^{S_{n+r}A([y|z]_{n+r})}\varphi([y|z]_{n+r})f(y_1)\dots f(y_{n+r})dy_1\dots dy_{n+r}\,.$$

This concludes our proof. ∎

Fixing a Gibbsian specification $(K_n)_{n\in\mathbb{N}}$ determined by a Hölder continuous potential A, we say that a probability measure μ is a *DLR*-Gibbs probability measure associated to A, if $E_\mu(\varphi|\sigma^n\mathcal{B})(y) = K_n(\varphi, y)$ for almost every point $y \in \mathbb{R}^{\mathbb{N}}$, any Hölder continuous function φ and each $n \in \mathbb{N}$. The set of all such μ will be denoted from now on by $\mathcal{G}^{DLR}(A)$.

On the other hand, the set of Thermodynamic limit Gibbs probability measures, denoted by $\mathcal{G}^{TL}(A)$, is defined as the closure of the convex hull of the set of cluster points of $\{K_n(\cdot, y_n) : n \in \mathbb{N}, y_n \in \mathbb{R}^{\mathbb{N}}\}$.

The next Lemma shows that $\mathcal{G}^{TL}(A) \subset \mathcal{G}^{DLR}(A)$ using a classical approach in statistical mechanics known as *DLR*-equations.

Lemma 4.2. *Consider $(K_n)_{n\in\mathbb{N}}$ be the Gibbsian specification determined by kernels of the form $K_n(\varphi, z) = \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^z$. If there exists a sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$, such that $\lim_{j\in\mathbb{N}} K_{n_j}(\cdot, z) = \lim_{j\in\mathbb{N}} \mu_{n_j}^z = \mu^z$ in the weak* topology. Then, for any Hölder continuous function $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{N}$ we have*

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi(y) d\mu^z(y) = \int_{\mathbb{R}^{\mathbb{N}}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_n) d\mu_n^{[x|y]_n}(x) \right) d\mu^z(y)\,.$$

Proof. By the Lemma (4.1) we have

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi([y|z]_{n+r}) d\mu_{n+r}^{[y|z]_{n+r}}(y) = \int_{\mathbb{R}^{\mathbb{N}}} \psi([y|z]_{n+r}) d\mu_{n+r}^{[y|z]_{n+r}}(y)\,,$$

with $\psi([y|z]_{n+r}) = \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y|z]_{n,n+r}) d\mu_n^{[x|y|z]_{n,n+r}}(x)$. Therefore, taking the limit when r goes to infinity in both of the sides of the equation, we obtain that

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi(y) d\mu^z(y) = \int_{\mathbb{R}^{\mathbb{N}}} \psi(y) d\mu^z(y)\,.$$

In other words

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi(y) d\mu^z(y) = \int_{\mathbb{R}^{\mathbb{N}}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_n) d\mu_n^{[x|y]_n}(x) \right) d\mu^z(y)\,.$$

□

In the next Theorem we will show the relation between the conformal probability measure and the Gibbs state associated to the Ruelle operator $\mathcal{L}_A$, with the set of *DLR*-Gibbs probability measures associated to A.

Theorem 4.3. *Suppose that $A \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$ and $(K_n)_{n\in\mathbb{N}}$ a Gibbsian specification such as defined in Lemma 4.1. Then, the eigenmeasure ρ_A and the Gibbs state μ_A, such as were defined in Theorem 2.4, belongs to $\mathcal{G}^{DLR}(A)$.*

Proof. In the first case it is enough to show that $E_{\rho_A}(\varphi|\sigma^n\mathcal{B})(z) = K_n(\varphi, z)$ for almost every point $z \in \mathbb{R}^{\mathbb{N}}$, any Hölder continuous function φ and each $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{N}}} K_n(\varphi, z) d\rho_A(z) &= \int_{\mathbb{R}^{\mathbb{N}}} \frac{\mathcal{L}_A^n(\varphi)(\sigma^n(z))}{\mathcal{L}_A^n(1)(\sigma^n(z))} d\rho_A(z) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \frac{\mathcal{L}_A^n(\varphi)(\sigma^n(z))}{\mathcal{L}_A^n(1)(\sigma^n(z))} \frac{1}{\psi_A(z)} d\mu_A(z) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \frac{1}{\lambda_A^n} \mathcal{L}_A^n \left(\frac{\mathcal{L}_A^n(\varphi)(\sigma^n(y))}{\mathcal{L}_A^n(1)(\sigma^n(y))} \right) (z) \frac{1}{\psi_A(z)} d\mu_A(z) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \frac{1}{\lambda_A^n} \mathcal{L}_A^n \left(\frac{\mathcal{L}_A^n(\varphi)(\sigma^n(y))}{\mathcal{L}_A^n(1)(\sigma^n(y))} \right) (\sigma^n(z)) \frac{1}{\psi_A(\sigma^n(z))} d\mu_A(z) \, .
\end{aligned}$$

Now, using the fact that for any $n \in \mathbb{N}$ it is satisfied the equation $K_n(K_n(\varphi, y), z) = K_n(\varphi, z)$ (proved in Lemma 4.1), it follows that the above integral is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^{\mathbb{N}}} \frac{1}{\lambda_A^n} \mathcal{L}_A^n(\varphi)(\sigma^n(z)) \frac{1}{\psi_A(\sigma^n(z))} d\mu_A(z) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \frac{1}{\lambda_A^n} \mathcal{L}_A^n(\varphi)(z) \frac{1}{\psi_A(z)} d\mu_A(z) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \frac{1}{\lambda_A^n} \mathcal{L}_A^n(\varphi)(z) d\rho_A(z) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \varphi(z) d\rho_A(z) \, .
\end{aligned}$$

The foregoing implies that $\rho_A \in \mathcal{G}^{DLR}(A)$. The proof for μ_A follows a procedure similar to the above using the fact that $\mathcal{L}_{\overline{A}}\mu_A = \mu_A$ and also that $\overline{A}$ is a Hölder continuous potential. □

The main result of this Section is to show that probability measures associated to kernels of the Gibbsian specification (such as was defined above) have positive correlation, which is usually knowing as *FKG*-inequality. The following Lemma provide necessary tools to prove this result.

Lemma 4.4. *Let η a Borel probability measure on $\mathbb{R}$. If $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing integrable functions, then*

$$\int_{\mathbb{R}} \varphi\psi d\eta \geq \int_{\mathbb{R}} \varphi d\eta \cdot \int_{\mathbb{R}} \psi d\eta \, .$$

Proof. The proof is similar to the one in [6]. □

The following Lemma shows a decomposition of μ_{n+1}^{y} in terms of μ_n^y and a suitable Borel measure defined on the set of real numbers.

Lemma 4.5. *Let $y \in \mathbb{R}^{\mathbb{N}}$ and $\varphi \in \mathcal{H}_\alpha(\mathbb{R}^{\mathbb{N}})$. Then the following equation is valid for each $n \in \mathbb{N}$.*

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n) d\mu_n^{[x|t|y]_n}(x) \right) d\eta(t) = \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x) \,,$$

with $\eta(E) = \frac{Z_n^{[y|t|y]_n}}{Z_{n+1}^y} \int_{\mathbb{R}} e^{A([t|\sigma^n(y)]_1)} \chi_E(t) f(t) dt$, for each mensurable set $E \subset \mathbb{R}$.

Proof. It follows from the definitions of η and μ_n^y that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n) d\mu_n^{[x|t|y]_n}(x) \right) d\eta(t) \\
&= \int_{\mathbb{R}} \frac{Z_n^{[y|t|y]_n}}{Z_{n+1}^y} e^{A([t|\sigma^n(y)]_1)} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n) d\mu_n^{[x|t|y]_n}(x) \right) f(t) dt \\
&= \frac{1}{Z_{n+1}^y} \int_{\mathbb{R}} e^{A([t|\sigma^n(y)]_1)} \\
&\quad \left(\int_{\mathbb{R}^n} e^{S_n A([x|t|y]_n)} \varphi([x|t|y]_n) f(x_1) \dots f(x_n) dx_1 \dots dx_n \right) f(t) dt \\
&= \frac{1}{Z_{n+1}^y} \int_{\mathbb{R}^{n+1}} e^{S_{n+1} A([x|y]_{n+1})} \varphi([x|y]_{n+1}) f(x_1) \dots f(x_{n+1}) dx_1 \dots dx_{n+1} \\
&= \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x) \,.
\end{aligned}$$

We want to study the behavior of the map $t \mapsto \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^{[y|t|y]_n}$. This is an important tool to show that the probability measures defined in (11) have the FKG property. Note that in the non-countable non-compact setting positive correlations is stronger than FKG property. On this way, it is necessary to define an especial class of functions. We say that a Hölder continuous potential $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ belongs to the class $\mathcal{E}$, if it is continuously differentiable in each coordinate, and the derivative of the n-th ergodic sum regarding each coordinate defined by the map

$$(x_1, x_2, \dots) \mapsto \frac{d}{dt} S_n A([x|t|y]_n) \,,$$

is increasing. Note that the function defined above depends only of its first n coordinates.

On the other hand, fixing $n \in \mathbb{N}$, $y \in \mathbb{R}^{\mathbb{N}}$, and a potential A in the class $\mathcal{E}$, we say that the probability measure μ_n^y, such as was defined in (11), satisfies the *FKG*-inequality if

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi\psi d\mu_n^y \geq \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^y \cdot \int_{\mathbb{R}^{\mathbb{N}}} \psi d\mu_n^y \,. \tag{13}$$

for any pair of increasing Hölder continuous functions φ, ψ from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}$ that depends only of its first n coordinates. □

The following Lemma shows that under suitable conditions the map $t \mapsto \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^{[y|t|y]_n}$ is increasing.

Lemma 4.6. *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}^{\mathbb{N}}$ fixed. If the potential $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ belongs to the class $\mathcal{E}$ and the probability measure μ_n^y satisfies (13). Then, for any Hölder continuous increasing function $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ that depends only of its first n coordinates, the map*

$$t \mapsto \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^{[y|t|y]_n} \,,$$

is a real increasing function.

Proof. We fix $n \geq 1$ and consider the function

$$t \to \mathbb{E}_t[\varphi] = \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^{[y|t|y]_n} \,,$$

Let $t < s \in \mathbb{R}$, fixed parameters and consider the auxiliar function

$$\log \psi(x) = S_n A([x|t|y]_n) - S_n A([x|s|y]_n).$$

Since $A \in \mathcal{E}$ it follows from the mean value theorem that there exist $\varepsilon \in (t, s)$, such that,

$$x \to \log \psi(x) = S_n A([x|t|y]_n) - S_n A([x|s|y]_n) = (t-s) \frac{d}{dt} S_n A([x|t|y]_n)|_{t=\varepsilon}.$$

is a decreasing function in the FKG sense. A straightforward computation shows that

$$\mathbb{E}_t[\varphi] = \frac{Z_n^{[y|s|y]}}{Z_n^{[y|t|y]}} \mathbb{E}_s[\varphi\, \psi].$$

Note that by taking $\varphi = 1$, in the above identity, we get

$$\mathbb{E}_s[\psi] = \frac{Z_n^{[y|t|y]}}{Z_n^{[y|s|y]}}.$$

Since we are assuming μ_n^z has the FKG property for any boundary condition $z \in \mathbb{R}^{\mathbb{N}}$ it follows that if φ is a non-decreasing local function (depending only on the first n coordinates), then,

$$\mathbb{E}_t[\varphi] = \frac{\mathbb{E}_s[\varphi\,\psi]}{\mathbb{E}_s[\psi]} \leq \mathbb{E}_s[\psi]$$

finishing the proof of the lemma.

□

The following Theorem, which is the main result of this Section, provides some conditions in order to guarantee that each probability measure μ_n^y such as was defined in (11) satisfies the *FKG*-inequality. In other words, each one of the members of the Gibbsian specification $(K_n)_{n\in\mathbb{N}}$ satisfies the *FKG*-inequality.

Theorem 4.7. *Let $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a potential that belongs to the class $\mathcal{E}$. Then, for any $n \in \mathbb{N}$ and each $y \in \mathbb{R}^{\mathbb{N}}$, the probability measure μ_n^y, which was defined in (11) satisfies (13).*

Proof. We are going to demonstrate this theorem by induction. The case $n = 1$ follows directly of Lemma (4.4), since the probability measure μ_1^y is defined on the Borelian sets in $\mathbb{R}$.

We assume the claim of this Theorem for $n \geq 1$, and we will demonstrate that this implies the same result for $n+1$. Let φ and ψ increasing Hölder continuous functions that depends only of its first $n+1$ coordinates, then it follows from the definition of μ_{n+1}^y that

$$\begin{aligned}
&\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1})\psi([x|y]_{n+1})d\mu_{n+1}^{[x|y]_{n+1}}(x)\\
&= \frac{1}{Z_{n+1}^y}\int_{\mathbb{R}^{n+1}} e^{S_{n+1}A([x|y]_{n+1})}\varphi([x|y]_{n+1})\psi([x|y]_{n+1})f(x_1)\ldots f(x_{n+1})dx_1\ldots dx_{n+1}\\
&= \frac{1}{Z_{n+1}^y}\int_{\mathbb{R}} e^{A([t|\sigma^n(y)]_1)}\\
&\qquad\left(\int_{\mathbb{R}^n} e^{S_nA([x|t|y]_n)}\varphi([x|t|y]_n)\psi([x|t|y]_n)f(x_1)\ldots f(x_n)dx_1\ldots dx_n\right)f(t)dt\\
&= \int_{\mathbb{R}} \frac{Z_n^{[y|t|y]}}{Z_{n+1}^y} e^{A([t|\sigma^n(y)]_1)}\left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n)\psi([x|t|y]_n)d\mu_n^{[x|t|y]_n}(x)\right)f(t)dt\\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n)\psi([x|t|y]_n)d\mu_n^{[x|t|y]_n}(x)\right)d\eta(t)\\
&\geq \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n)d\mu_n^{[x|t|y]_n}(x)\right)\cdot\left(\int_{\mathbb{R}^{\mathbb{N}}} \psi([x|t|y]_n)d\mu_n^{[x|t|y]_n}(x)\right)d\eta(t)\,,
\end{aligned}$$

where the last inequality is obtained by the induction hypothesis. On other hand, by Lemma (4.6) we have that the maps $t \mapsto \int_{\mathbb{R}^{\mathbb{N}}} \varphi d\mu_n^{[y|t|y]_n}$ and $t \mapsto \int_{\mathbb{R}^{\mathbb{N}}} \psi d\mu_n^{[y|t|y]_n}$ are real increasing functions. Therefore, it follows from Lemma (4.4) that

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1})\psi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x)$$
$$\geq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n) d\mu_n^{[x|t|y]_n}(x) \right) d\eta(t) \cdot \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \psi([x|t|y]_n) d\mu_n^{[x|t|y]_n}(x) \right) d\eta(t)\,.$$

Furthermore, observe that the same procedure used above guarantees that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|t|y]_n) d\mu_n^{[x|t|y]_n}(x) \right) d\eta(t) = \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x)\,,$$

and the same conclusion is valid for ψ. Therefore,

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1})\psi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x)$$
$$\geq \int_{\mathbb{R}^{\mathbb{N}}} \varphi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x) \cdot \int_{\mathbb{R}^{\mathbb{N}}} \psi([x|y]_{n+1}) d\mu_{n+1}^{[x|y]_{n+1}}(x)\,.$$

□

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