Pentadiagonal Matrices and an Application to the Centered MA(1) Stationary Gaussian Process

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Abstract

In this work, we study the properties of a pentadiagonal symmetric matrix with perturbed corners. More specifically, we present explicit expressions for characterizing when this matrix is non-negative and positive definite in two special and important cases. We also give a closed expression for the determinant of such matrices. Previous works present the determinant in a recurrence form but not in an explicit one. As an application of these results, we also study the limiting cumulant generating function associated to the bivariate sequence of random vectors $\left(n^{-1}(\sum_{k=1}^{n} X_k^2\,,\ \sum_{k=2}^{n} X_k X_{k-1})\right)_{n\in\mathbb{N}}$, when $(X_n)_{n\in\mathbb{N}}$ is the centered stationary moving average process of first order with Gaussian innovations. We exhibit the explicit expression of this limiting cumulant generating function. Finally, we present three examples illustrating the techniques studied here.

Keywords: Pentadiagonal symmetric matrices, Determinant, Eigenvalues, Non-negative and Positive definite matrices, Moving average process, Limiting cumulant generating function, Time series.

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1 Introduction

Pentadiagonal matrices have been explored in many possible ways in recent decades, most of them for the symmetric case (sometimes, assuming that the symmetric matrix is Toeplitz). Some results address the analysis of its eigenvalues (see Elouafi [6] and Fasino [9]), others focus on explicit formulas for its determinant (see Elouafi [7, 8], Jia *et al.* [15], Marr and Vineyard [17] and Solary [22]). Other authors examine faster algorithms for computing the determinant of such matrices (see Cinkir [5] and Sogabe [21]), its use in solving systems of linear equations (see Jia *et al.* [14], McNally [18] and Nemani [19]), and in the search of explicit formulas for the inverse matrix (see Wang *et al.* [25] and Zhao and Huang [26]).

However, there are not many works dedicated to the case of pentadiagonal matrices with perturbed corners; to be defined below.

A pentadiagonal matrix is described in the literature as having zeros everywhere except in its five principal diagonals. In the present work, we shall consider the following pentadiagonal matrix with

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perturbed corners

$$D_n = \begin{bmatrix} r & q & s & 0 & \cdots & 0 \\ q & p & q & s & \ddots & \vdots \\ s & q & p & \ddots & \ddots & 0 \\ 0 & s & \ddots & \ddots & q & s \\ \vdots & \ddots & \ddots & q & p & q \\ 0 & \cdots & 0 & s & q & r \end{bmatrix}. \tag{1.1}$$

Our purpose with this study is to present few properties of the matrix D_n, with relation to its determinant and positive and non-negative definiteness. Working around with the matrix D_n is non-trivial. The pentadiagonal matrices found in Cinkir [5], Elouafi [7], Wang *et al.* [25], or Jia *et al.* [15] serve as particular cases from the matrix presented in (1.1). A more advanced study is given in Solary [22], where the author presents computational properties for a pentadiagonal band matrix with perturbed corners, similar to ours, but the elements are disposed in $N \times N$ blocks of $m \times m$ matrices in its five main diagonals, with $m, N \in \mathbb{N}$.

As we will show here, a particular case of the pentadiagonal matrix in (1.1) appears in a problem relating to the centered stationary moving average process of first order (MA(1)) with Gaussian innovations, defined by the equation

$$X_n = \varepsilon_n + \phi\, \varepsilon_{n-1}, \quad \text{with } |\phi| < 1 \text{ and } n \in \mathbb{N},$$

where $(\varepsilon_n)_{n \geqslant 0}$ is a sequence of independent and identically distributed (i.i.d.) random variables following a Gaussian distribution with zero mean and unitary variance ($\varepsilon_n \sim \mathcal{N}(0,1)$, for all $n \geqslant 0$). We are interested in the asymptotics of the bivariate *normalized cumulant generating function*

$$L_n(\boldsymbol{\lambda}) = \frac{1}{n} \log \mathbb{E}(\exp(n\langle(\lambda_1, \lambda_2), \mathcal{W}_n\rangle)) = \frac{1}{n} \log\left(\mathbb{E} \exp\left[\lambda_1 U_n + \lambda_2 V_n\right]\right), \quad \text{for } \boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2,$$

associated to the random vectors sequence $(\mathcal{W}_n)_{n \geqslant 2}$, where

$$\mathcal{W}_n = n^{-1}(U_n, V_n) = n^{-1} \left(\sum_{k=1}^{n} X_k^2,\ \sum_{k=2}^{n} X_k X_{k-1} \right). \tag{1.2}$$

The results we obtain for pentadiagonal matrices will help us in this direction. The main result in this part of the paper is to give an explicit expression for the limit $\mathcal{L}(\boldsymbol{\lambda}) := \lim_{n\to\infty} L_n(\boldsymbol{\lambda})$, when it is well defined. A similar discussion appeared in Karling *et al.* [16], where the authors analyzed the bivariate normalized cumulant generating function associated with the sequence $(\mathcal{W}_n)_{n \geqslant 2}$, when $(X_n)_{n\in\mathbb{N}}$ is a centered stationary autoregressive process of first order with Gaussian innovations. In that work, the treatment of the positive definiteness of a tridiagonal matrix was required.

The normalized cumulant generating function is of great help for obtaining the moments of a given random vector. We point out that for the practical use of this property it is required to have an explicit expression for it. The analytic expression we obtain for $\mathcal{L}(\cdot)$ is quite complex (see Proposition 4.1) but its partial derivatives can be calculated using the Wolfram Mathematica software.

The present work is organized as follows. Section 2 is dedicated to obtaining a closed expression for the domain when D_n is non-negative definite in the presence of the restriction $r \geqslant p-s$. Furthermore, we analyze the special case $r = p-s$ to give the explicit domain for which D_n is a positive definite matrix. In Section 3 we compute the determinant of the matrix D_n by using a recurrence relation proposed in Sweet [23]. An application to the MA(1) process is presented in Section 4, where we analyze the asymptotic behavior of the bivariate normalized cumulant generating function associated to the sequence $(\mathcal{W}_n)_{n\geqslant 2}$, given in (1.2), and we provide its limiting function. A few examples to illustrate the theory in practice are exhibited in Section 5. In Section 6 some conclusions are presented.

2 Non-negative and positive definiteness of D_n

We scrutinize in the following subsections when the matrix D_n in (1.1) is non-negative definite if the restriction $r \geqslant p - s$ is considered. In addition to this, a sharper result can be provided for the positive definiteness of D_n in the special case when $r = p - s$. Both reasonings rely on the results proved in Fasino [9] and Solary [22]. Despite being well known, we recall two equivalent definitions of non-negative (positive) definite matrices in the real symmetric case.

Definition 2.1. A real symmetric matrix $M = [m_{i,j}]_{n\times n}$ of order $n \times n$ is said to be non-negative (positive) definite if (see Gilbert [10] and Horn [13]):

1. the scalar $\boldsymbol{x}^T M \boldsymbol{x}$ is non-negative (positive) for every non-zero column vector $\boldsymbol{x} \in \mathbb{R}^n$;
2. the eigenvalues of M are all non-negative (positive).

2.1 Case $r \geqslant p - s$

The approach presented in Fasino [9] yields a nice criterion based on a second-order polynomial to determine when D_n in (1.1) is a non-negative definite matrix. We use this criterion to provide an explicit expression for the domain which characterizes when D_n is non-negative definite. It is although necessary to require a priori that $r \geqslant p - s$.

Lemma 2.1. *Let D_n be the pentadiagonal matrix defined in* (1.1) *with $p \geqslant 0$. Consider the sets*

$$\begin{aligned}
\mathcal{D}_1 &= \left\{-\frac{p}{2} \leqslant s < 0\,,\ -\frac{1}{2}(p+2s) \leqslant q \leqslant \frac{1}{2}(p+2s)\right\},\\
\mathcal{D}_2 &= \{s = 0\,,\ p \geqslant 2|q|\}\,,\\
\mathcal{D}_3 &= \left\{0 < s \leqslant \frac{p}{2}\,,\ -\sqrt{4s(p-2s)} \leqslant q \leqslant \sqrt{4s(p-2s)}\right\} \ \text{and}\\
\mathcal{D}_4 &= \left\{0 < s < \frac{p}{6}\,,\ -\frac{1}{2}(p+2s) \leqslant q < -\sqrt{4s(p-2s)}\ \vee\ \sqrt{4s(p-2s)} < q \leqslant \frac{1}{2}(p+2s)\right\}.
\end{aligned} \tag{2.1}$$

If $r \geqslant p - s$ and p, q, s lie inside $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$, then D_n is non-negative definite for all $n \in \mathbb{N}$.

Proof. First we observe that if $p = 0$, then, the only possible case where D_n might be non-negative definite is the trivial one, when $p = q = s = 0$. Thus, we can assume hereafter that $p > 0$. The remaining of the proof stands on proposition 5 in Fasino [9], which states that, given

$$g(x) = sx^2 + qx + (p - 2s), \quad \text{for } x \in \mathbb{R}, \tag{2.2}$$

the matrix D_n is non-negative definite, for all $n \in \mathbb{N}$, if and only if $g(x) \geqslant 0$, for all $x \in [-2, 2]$.

We separate our analysis in three cases:

- *Case* $s < 0$*:* by hypothesis $p > 0$, hence, it follows that $q^2 - 4s(p - 2s) > 0$ and the equation $g(x) = 0$ has two real roots, given by

$$x_1 = \frac{-q - \sqrt{q^2 - 4s(p - 2s)}}{2s} \quad \text{and} \quad x_2 = \frac{-q + \sqrt{q^2 - 4s(p - 2s)}}{2s}. \tag{2.3}$$

  For the condition $g(x) \geqslant 0$ to be true for all $x \in [-2, 2]$, we must have simultaneously $x_2 \leqslant -2$ and $x_1 \geqslant 2$. The latter relations are verified if and only if p, q, s lie inside $\mathcal{D}_1$.

- *Case* $s = 0$*:* in this case, notice that D_n is a tridiagonal matrix and that $g(x) = qx+p$. Therefore, if $p \geqslant 2|q|$, then $g(x) \geqslant 0$ for all $x \in [-2, 2]$. Hence, p, q, s must lie inside $\mathcal{D}_2$ for D_n to be non-negative definite.

- *Case* $s > 0$*:* here we observe that there are two possibilities. Either $q^2 - 4s(p-2s) \leqslant 0$ and $g(x) \geqslant 0$, for all $x \in \mathbb{R}$, or either $q^2 - 4s(p - 2s) > 0$ and $g(x) = 0$ has two real distinct roots, namely, x_1 and x_2 given in (2.3). In the former case, p, q, s must lie inside $\mathcal{D}_3$. In the later case, $g(x) \geqslant 0$, for all $x \in [-2, 2]$, if and only if $x_2 \leqslant -2$ or $x_1 \geqslant 2$, which gives us the domain $\mathcal{D}_4$ in (2.1).

□

Remark 1. Note that, if p, q, s belong to $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ and $p \geqslant 0$, then $r \geqslant p - s$ implies that $r \geqslant 0$.

Remark 2. When considering proposition 5 in Fasino [9], the term positive definite should be read as non-negative definite. Additionally, the same proposition cannot be proved for positive definite matrices in the strict positive sense, i.e., by just replacing the condition $g(x) \geqslant 0$, for all $x \in [-2, 2]$, by $g(x) > 0$, for all $x \in [-2, 2]$.

An illustration of the domain $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ is given in Figure 1. We note that outside this set it may happen that D_n is non-negative definite for some $n \in \mathbb{N}$, but this does not generate a contradiction to the result of Lemma 2.1. In fact, the statement of this lemma considers the non-negative definiteness of the matrices D_n for all $n \in \mathbb{N}$.

2.2 Special case $r = p - s$

It may happen that $r = p - s$ and as a consequence we obtain the following.

Lemma 2.2. *If the elements of the matrix* D_n *in* (1.1) *satisfy the relation* $r = p - s$*, then its eigenvalues are given by*

$$\alpha_{n,k} = 4s\cos^2\left(\frac{k\pi}{n+1}\right) + 2q\cos\left(\frac{k\pi}{n+1}\right) + p - 2s, \quad \textit{for} \;\; 1 \leqslant k \leqslant n.$$

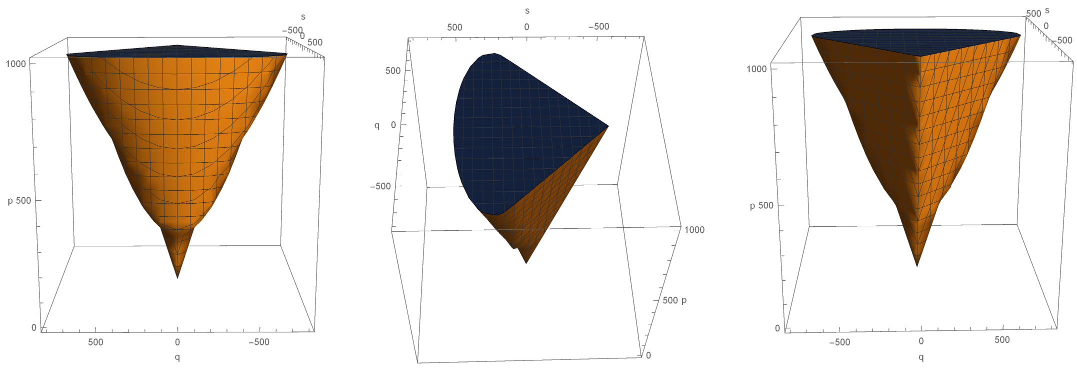

Figure 1: Domain $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ illustrated, for the cases when $s, q \in [-800, 800]$ and $p \in [0, 1000]$.

Proof. See theorem 4 in Solary [22]. □

Since we have explicitly the general representation for the eigenvalues of D_n in the special case when $r = p - s$, it is now easy to obtain the determinant of such matrix. As a consequence from Lemmas 2.1 and 2.2, the following corollary is of extreme importance.

Corollary 2.1. *Let D_n be the matrix in* (1.1) *with $r = p - s$. Then, it follows that*

1. *D_n has a null eigenvalue if and only if*

$$4s\cos^2\left(\frac{k\pi}{n+1}\right) + 2q\cos\left(\frac{k\pi}{n+1}\right) + p - 2s = 0,$$

for some k such that $1 \leqslant k \leqslant n$.

2. *A closed expression for the determinant of D_n is given by*

$$\det(D_n) = \prod_{k=1}^{n}\left(4s\cos^2\left(\frac{k\pi}{n+1}\right) + 2q\cos\left(\frac{k\pi}{n+1}\right) + p - 2s\right).$$

3. *Consider*

$$\mathcal{D}_0 = \bigcup_{n\in\mathbb{N}}\left\{p, q, s \;\middle|\; 0 < s\,,\; p = 2s\left(1 + 2\cos^2\left(\frac{k\pi}{n+1}\right)\right)\,,\; q = -4s\cos\left(\frac{k\pi}{n+1}\right),\; \text{for } k \in \mathbb{N}\right\}.$$

If p, q, s lie inside $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \setminus \mathcal{D}_0$, then D_n is positive definite, for all $n \in \mathbb{N}$.

Proof. By Lemma 2.2, the eigenvalues of D_n are given as $\alpha_{n,k} = 4s\cos^2\left(\frac{k\pi}{n+1}\right) + 2q\cos\left(\frac{k\pi}{n+1}\right) + p - 2s$, for $1 \leqslant k \leqslant n$. Hence, statement 1 is evident. For the proof of statement 2, we note that the determinant of D_n is equal to the product of its eigenvalues.

Statement 3 is the only one that requires more caution. In the proof of Lemma 2.1, we note that inside $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_4$ we have $\alpha_{n,k} > 0$ for all $k, n \in \mathbb{N}$. Indeed, if $p, q, s \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_4$, the polynomial

$g(\cdot)$, defined in (2.2), is non-negative for all $x \in [-2, 2]$ and, in the worst scenario, it has a real root at $x = -2$ or $x = 2$. Since $\alpha_{n,k} = g\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)$ and $\left|\cos\left(\frac{k\pi}{n+1}\right)\right| < 1$, for all $k, n \in \mathbb{N}$, it follows that $\alpha_{n,k} > 0$, for all $k, n \in \mathbb{N}$. The only section that D_n can actually have a null eigenvalue is inside the domain $\mathcal{D}_3$ with $q^2 = 4s(p - 2s)$. In this case, when $q^2 = 4s(p - 2s)$, it follows that $-q/2s$ is the only root of the polynomial $g(\cdot)$ and, therefore,

$$\alpha_{n,k} = g\left(2\cos\left(\frac{k\pi}{n+1}\right)\right) = 0 \;\Leftrightarrow\; q = -4s\cos\left(\frac{k\pi}{n+1}\right).$$

As a solution to the equation $s\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)^2 + q\left(2\cos\left(\frac{k\pi}{n+1}\right)\right) + (p - 2s) = 0$, we obtain

$$p = 2s\left(1 + 2\cos^2\left(\frac{k\pi}{n+1}\right)\right).$$

Therefore, the matrix D_n, with $r = p - s$, has an eigenvalue equal to zero if and only if $s > 0$, $q = -4s\cos\left(\frac{k\pi}{n+1}\right)$ and $p = 2s\left(1 + 2\cos^2\left(\frac{k\pi}{n+1}\right)\right)$. □

3 An explicit formula for the determinant of the matrix D_n

It is possible to find in the literature explicit formulas for the determinant of pentadiagonal symmetric Toeplitz matrices (see e.g. Andelić and da Fonseca [1], Elouafi [6, 7], and Jia *et al.* [15]). However, little has been done concerning pentadiagonal symmetric matrices with perturbed corners. Recently, Solary [22] proposed a closed expression for the determinant and computational properties for a pentadiagonal matrix disposed by blocks, where the corners in the main diagonal are perturbed. This matrix by blocks serves as a generalization of the matrix D_n in (1.1) and its determinant can be computed from equation (22) in Solary [22]. The formula of the determinant was given with the help of the Sherman-Morrison-Woodbury formula.

In the present section, we show a closed expression for the determinant of the matrices D_n and E_n, defined in (3.1), by considering a recursive relation proposed in Sweet [23]. We also show the explicit expressions for some cases not covered by this author (see Lemmas 3.2 for matrices D_n and E_n and Lemma 3.3-3.5 for the matrix E_n). In Theorem 3.1 we exhibit a closed expression for the determinant of the matrix D_n, based on the results of Lemmas 3.1-3.5. As far as we know, this explicit expression is totally new and it provides a quicker and efficient way to compute the determinant of D_n. To achieve such aim, we shall consider the sub-matrix

$$E_n = \begin{bmatrix} r & q & s & 0 & \cdots & 0 \\ q & p & q & s & \ddots & \vdots \\ s & q & p & \ddots & \ddots & 0 \\ 0 & s & \ddots & \ddots & q & s \\ \vdots & \ddots & \ddots & q & p & q \\ 0 & \cdots & 0 & s & q & p \end{bmatrix}. \tag{3.1}$$

Let us denote the determinants of D_n and E_n by d_n and e_n, respectively. The recursive relation presented in Sweet [23] gives us the following lemma.

Lemma 3.1. *For $n \geqslant 6$ and $q \neq 0$, the following recursive relations hold*

$$d_n = (r-s)\, e_{n-1} + (p\, s - q^2)\, (e_{n-2} - s\, e_{n-3}) + s^3\, (s-p)\, e_{n-4} + s^5\, e_{n-5}, \tag{3.2}$$

$$e_n = (p-s)\, e_{n-1} + (p\, s - q^2)\, (e_{n-2} - s\, e_{n-3}) + s^3\, (s-p)\, e_{n-4} + s^5\, e_{n-5}, \tag{3.3}$$

with the initial conditions

$$\begin{aligned} e_1 &= r, \\ e_2 &= p\, r - q^2, \\ e_3 &= p^2 r - q^2(r-2s) - p\,(q^2+s^2), \\ e_4 &= p^3 r - p^2\left(q^2+s^2\right) - p\left(2q^2(r-s) + rs^2\right) + q^4 + 2q^2 s(r-s) + s^4, \\ e_5 &= p^4 r + q^4(r-4s) + rs^4 + 2q^2 s^2(-r+s) - p^3(q^2+s^2) + p(2q^4 + 4q^2 rs + s^4) \\ &\quad + p^2(-2rs^2 + q^2(-3r+2s)). \end{aligned} \tag{3.4}$$

Proof. Immediate from equations (1), (5) and (11) in Sweet [23]. □

Remark 3. The initial conditions e_1, e_2, e_3, e_4, e_5 in (3.4) are defined as the first, second, third, fourth and fifth principal minor of E_n, respectively.

The case when $q = 0$ is not covered by Sweet's [23] recurrence relations, but it is not difficult to prove the following.

Lemma 3.2. *For $n \geqslant 5$ and $q = 0$, the following recursive relations hold*

$$d_n = r\, e_{n-1} - p\, s^2\, e_{n-3} + s^4\, e_{n-4}, \tag{3.5}$$

$$e_n = p\, e_{n-1} - p\, s^2\, e_{n-3} + s^4\, e_{n-4}, \tag{3.6}$$

with the initial conditions

$$e_1 = r, \qquad e_2 = p\, r, \qquad e_3 = p\,(p\, r - s^2), \qquad e_4 = (p^2 - s^2)(p\, r - s^2).$$

Proof. The proof follows by the induction principle. □

From (3.3), we obtain the following lemma.

Lemma 3.3. *If $q \neq 0$ and $s \neq 0$, then $e_n = \det(E_n)$ may be given by*

$$e_n = \sum_{j=1}^{5} \kappa_j\, \mu_j^n, \tag{3.7}$$

where $\mu_1, \cdots, \mu_5$ are given in (3.8). The coefficients $\kappa_1, \cdots, \kappa_5$ are described in the following way:

1. *if* $q^2 \notin \{4s(p-2s),\,(p+2s)^2/4\}$, *then it holds* (3.9);
2. *if* $q^2 = 4s(p-2s)$ *and* $p > 6s$, *then it holds* (3.11);
3. *if* $q^2 = 4s(p-2s)$ *and* $p < 6s$, *then it holds* (3.14);
4. *if* $q^2 = (p+2s)^2/4$ *and* $p \neq 6s$, *then it holds* (3.15);
5. *if* $q^2 \in \{4s(p-2s),\,(p+2s)^2/4\}$ *and* $p = 6s$, *then it holds* (3.16).

Proof. The result follows by applying the characteristic roots technique to the associated auxiliary polynomial

$$\rho(z) = z^5 - (p-s)\,z^4 - (p\,s - q^2)\,(z^3 - s\,z^2) - s^3\,(s-p)\,z - s^5, \qquad \text{for } z \in \mathbb{C}.$$

The roots of $\rho(\cdot)$ are given by

$$\begin{gathered}\mu_1 = \frac{p-2s-\alpha-\beta_1}{4}, \qquad \mu_2 = \frac{p-2s-\alpha+\beta_1}{4}, \qquad \mu_3 = \frac{p-2s+\alpha-\beta_2}{4},\\ \mu_4 = \frac{p-2s+\alpha+\beta_2}{4} \qquad \text{and} \qquad \mu_5 = s,\end{gathered} \tag{3.8}$$

with

$$\alpha = \sqrt{(p+2s)^2 - 4q^2}, \quad \beta_1 = \sqrt{2(p-2s)(p+2s-\alpha) - 4q^2} \ \text{ and } \ \beta_2 = \sqrt{2(p-2s)(p+2s+\alpha) - 4q^2}.$$

Let us separate the proof in four cases.

Case 1: if $q^2 \notin \{4s\,(p-2s),\,(p+2s)^2/4\}$, then α, β_1 and β_2 are non-zero, and as a consequence, $\mu_1, \cdots, \mu_5$ are distinct roots of the polynomial $\rho(\cdot)$. Thus, each solution to the recurrence in (3.3) is of the form (3.7), where the coefficients κ_j, for $j = 1, \cdots, 5$, are the solution to the 5-by-5 Vandermonde linear system

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 & \mu_4^2 & \mu_5^2 \\ \mu_1^3 & \mu_2^3 & \mu_3^3 & \mu_4^3 & \mu_5^3 \\ \mu_1^4 & \mu_2^4 & \mu_3^4 & \mu_4^4 & \mu_5^4 \end{bmatrix} \begin{bmatrix} \kappa_1' \\ \kappa_2' \\ \kappa_3' \\ \kappa_4' \\ \kappa_5' \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}$$

with $\kappa_j' = \kappa_j\,\mu_j$ and e_j representing the initial conditions given in (3.4), for $j = 1, \cdots, 5$. We used the Wolfram Mathematica software (version 11.2) to find these coefficients, obtaining the expressions:

$$\begin{gathered}\kappa_1 = \mathcal{K}(-\alpha, \beta_2, -\beta_1), \qquad \kappa_2 = \mathcal{K}(-\alpha, \beta_2, \beta_1), \qquad \kappa_3 = \mathcal{K}(\alpha, \beta_1, -\beta_2),\\ \kappa_4 = \mathcal{K}(\alpha, \beta_1, \beta_2) \qquad \text{and} \qquad \kappa_5 = \frac{2s\,(r+s-p)}{q^2 - 4s(p-2s)},\end{gathered} \tag{3.9}$$

where

$$\mathcal{K}(x,y,z) = \frac{64\begin{pmatrix} 2s^4(2s+3p+x+z)+ps^2(4q^2-2s(p-x)-(p+x)(2p+z)) \\ -2sq^2(4q^2-2p^2-(p-2s)(2x+z)-x\,z) \\ +r\begin{pmatrix} 4q^4-q^2(p+x)(2p+z)+2sq^2(p-2s+3x+2z) \\ -2s^2p\,(p+x+z)-2s^3(p-2s-x+z)\end{pmatrix} \\ +(q^2-pr)\begin{pmatrix} 2s^2(2s+3p+x)+2q^2(3p-4s+x+z) \\ -(s\,(p-x)+p\,(p+x))(2p+z)\end{pmatrix}\end{pmatrix}}{z\,(p-2s+x+z)\,(p-6s+x+z)\,((2x+z)^2-y^2)}, \quad (3.10)$$

for $x,y,z\in\mathbb{C}$. We note that the coefficients $\kappa_1,\cdots,\kappa_5$ in (3.9)-(3.10) are not well defined when $q^2\in\big\{4p\,(p-2s),\,(p+2s)^2/4\big\}$. In these cases, some of the roots $\mu_1,\cdots,\mu_5$ have multiplicity greater than 1. Thus, the solution to the recurrence in (3.3) takes another form and the coefficients might depend on n.

Case 2: if $q^2=4s\,(p-2s)$, then $\alpha=|p-6s|$. Let us consider $\gamma=\sqrt{(p-6s)(p-2s)}$. On the one hand, if $p>6s$, we get $\beta_1=0$ and $\beta_2=2\gamma$, implying that $\mu_1=\mu_2=\mu_5=s$, $\mu_3=(p-4s-\gamma)/2$ and $\mu_4=(p-4s+\gamma)/2$. It follows that (3.7) is a solution to the recurrence in (3.3), with

$$\kappa_1=\mathcal{K}_1,\quad \kappa_2=2\,n\,\mathcal{K}_2(2),\quad \kappa_3=\mathcal{K}_3(\gamma),\quad \kappa_4=\mathcal{K}_3(-\gamma)\quad\text{and}\quad \kappa_5=n^2\,\mathcal{K}_2(1), \quad (3.11)$$

where

$$\mathcal{K}_1=\frac{p^2-p\,(r+8s)+2s\,(2r+11s)}{(p-6s)^2},\qquad \mathcal{K}_2(j)=\frac{p-r-j\,s}{p-6s},\quad\text{for } j=1,2, \quad (3.12)$$

and

$$\mathcal{K}_3(z)=\frac{2\,s^2\,(p-2s)^2\,(p-4s+z)\,\big((r-p)(p-4s+z)+2s^2\big)}{z\,(4s\,(3s-z)+p\,(p-8s+z))^3},\quad\text{for } z\in\mathbb{C}. \quad (3.13)$$

Note that κ_2 and κ_5 are dependent on n and n^2, respectively. On the other hand, if $p<6s$, we get $\beta_1=2\gamma$ and $\beta_2=0$, implying that $\mu_1=(p-4s-\gamma)/2$, $\mu_2=(p-4s+\gamma)/2$ and $\mu_3=\mu_4=\mu_5=s$. Then, it follows that (3.7) is a solution to the recurrence in (3.3), with

$$\kappa_1=\mathcal{K}_3(\gamma),\quad \kappa_2=\mathcal{K}_3(-\gamma),\quad \kappa_3=\mathcal{K}_1,\quad \kappa_4=2\,n\,\mathcal{K}_2(2)\quad\text{and}\quad \kappa_5=n^2\,\mathcal{K}_2(1), \quad (3.14)$$

for $\mathcal{K}_1$ and $\mathcal{K}_2(\cdot)$ defined in (3.12) and $\mathcal{K}_3(\cdot)$ defined in (3.13). Note that in this case, κ_4 and κ_5 are dependent on n and n^2, respectively.

Case 3: if $q^2=(p+2s)^2/4$ and $p\neq 6s$, let us denote $\delta=\sqrt{(p-6s)(p+2s)}$. Then $\alpha=0$ and $\beta_1=\beta_2=\delta$, implying that $\mu_1=\mu_3=(p-2s-\delta)/4$ and $\mu_2=\mu_4=(p-2s+\delta)/4$, with $\mu_5=s\neq\mu_j$, for $j=1,2,3,4$. The solution of the recurrence in (3.3) is given in this case by (3.7) with

$$\kappa_1=\mathcal{K}_4(\delta),\quad \kappa_2=\mathcal{K}_4(-\delta),\quad \kappa_3=n\,\mathcal{K}_5(\delta),\quad \kappa_4=n\,\mathcal{K}_5(-\delta)\quad\text{and}\quad \kappa_5=\frac{8\,s\,(r+s-p)}{(p-6\,s)^2}, \quad (3.15)$$

where

$$\mathcal{K}_4(z)=\frac{8\begin{pmatrix} p^5\,(12s-p+z)-2\,p^4s\,(24s-4r+5z)-4\,p^3s\,(2r(10s+z)-s(16s+9z)) \\ +8\,p^2s^2\,(4r(5s+2z)+s(20s-7z))+16\,ps^3\,(2r(8s-3z)-5s(8s+z)) \\ -32\,s^4\,(2\,r\,(6s+z)+s\,(6s-7z))\end{pmatrix}}{z\,(p-6s)\,(p-2s-z)^2\,(p-6s-z)^2}$$

and

$$\mathcal{K}_5(z) = \frac{4\begin{pmatrix} p^4\,(2r+4s-p+z) - 2\,p^3\,(r(6s+z)-s(6s-z)) - 8\,p^2 s\,(r(s-z)+s(s+z)) \\ +\,8\,p\,s^2\,(r(8s+z)-s(6s+z)) - 16\,s^3\,\left(4s^2-r(2s-z)\right)\end{pmatrix}}{z^2\,(p-6s-z)\,(p-2s-z)^2},$$

for $z \in \mathbb{C}$. Note that κ_3 and κ_4 are both dependent on n.

Case 4: if $q \in \left\{4s\,(p-2s),\ (p+2s)^2/4\right\}$ and $p = 6s$, then $\mu_1 = \cdots = \mu_5 = s$ and the solution of the recurrence in (3.3) is given by (3.7) with

$$\kappa_1 = 1,\ \kappa_2 = n\left(\frac{r+8s}{6s}\right),\ \kappa_3 = n^2\left(\frac{5r-7s}{12s}\right),\ \kappa_4 = n^3\left(\frac{r-4s}{3s}\right) \text{ and } \kappa_5 = n^4\left(\frac{r-5s}{12s}\right). \tag{3.16}$$

Note that κ_j depends on n^{j-1}, for $j = 2, 3, 4, 5$. □

An analogous result follows when $q = 0$.

Lemma 3.4. *If $q = 0$ and $s \neq 0$, then $e_n = \det(E_n)$ may be given by*

$$e_n = \sum_{j=1}^{4} \kappa_j\,\nu_j^n, \tag{3.17}$$

where

$$\nu_1 = -s, \quad \nu_2 = s, \quad \nu_3 = \frac{1}{2}\left(p - \sqrt{p^2-4s^2}\right) \quad \text{and} \quad \nu_4 = \frac{1}{2}\left(p + \sqrt{p^2-4s^2}\right).$$

The coefficients $\kappa_1, \cdots, \kappa_4$ are described in the following way:

1. *if $p^2 \neq 4s^2$, then $\kappa_1 = \mathcal{K}_6(1)$, $\kappa_2 = \mathcal{K}_6(-1)$, $\kappa_3 = \mathcal{K}_7(1)$ and $\kappa_4 = \mathcal{K}_7(-1)$, where*

$$\mathcal{K}_6(j) = \frac{p-r+j\,s}{2(p+2\,j\,s)} \quad \text{and} \quad \mathcal{K}_7(j) = \frac{p^3 r - p^2 s^2 - 3\,p\,r s^2 + 2s^4 + j\left(ps^2+rs^2-p^2 r\right)\sqrt{p^2-4s^2}}{(p^2-4s^2)\left(p^2-2s^2-j\,p\,\sqrt{p^2-4s^2}\right)},$$

   for $j = -1, 1$;

2. *if $p = 2s$, then*

$$\kappa_1 = \frac{3s-r}{8s}, \quad \kappa_2 = \frac{r+5s}{8s}, \quad \kappa_3 = n\left(\frac{r}{2s}\right) \quad \text{and} \quad \kappa_4 = n^2\left(\frac{r-s}{4s}\right);$$

3. *if $p = -2s$, then*

$$\kappa_1 = \frac{5s-r}{8s}, \quad \kappa_2 = \frac{r+3s}{8s}, \quad \kappa_3 = -n\left(\frac{r}{2s}\right) \quad \text{and} \quad \kappa_4 = -n^2\left(\frac{r+s}{4s}\right).$$

Note that, if $p = \pm 2s$, then κ_3 and κ_4 depend on n and n^2, respectively.

Proof. The proof is similar to the one of Lemma 3.3. □

In the case when $s = 0$ we get the following lemma.

Lemma 3.5. *If $s = 0$ then E_n in* (3.1) *is a tridiagonal matrix with determinant equal to*

$$e_n = \begin{cases} \kappa_1 \xi_1^n + \kappa_2 \xi_2^n, & \text{if } q \neq 0, \\ r\, p^{n-1}, & \text{if } q = 0, \end{cases} \tag{3.18}$$

where ξ_1 and ξ_2 are given by (3.20)*. The coefficients κ_1, κ_2 are described in the following way:*

1. *if $p^2 \neq 4q^2$, then it holds* (3.21)*;*
2. *if $p^2 = 4q^2$, then it holds* (3.22)*.*

Proof. If $s = 0$ then (3.3) simplifies to

$$e_n = p\, e_{n-1} - q^2\, e_{n-2}. \tag{3.19}$$

In the case when $q = 0$, it follows that E_n is a diagonal matrix with determinant equal to $\det(E_n) = r\, p^{n-1}$. Whereas if $s \neq 0$, consider

$$\xi_1 = \frac{p - \sqrt{p^2 - 4q^2}}{2} \quad \text{and} \quad \xi_2 = \frac{p + \sqrt{p^2 - 4q^2}}{2}. \tag{3.20}$$

The solutions to the recurrence relation in (3.19) are thus given by

$$e_n = \kappa_1 \xi_1^n + \kappa_2 \xi_2^n,$$

with

$$\kappa_1 = \frac{p - 2r + \sqrt{p^2 - 4q^2}}{2\sqrt{p^2 - 4q^2}} \quad \text{and} \quad \kappa_2 = \frac{2r - p + \sqrt{p^2 - 4q^2}}{2\sqrt{p^2 - 4q^2}}, \quad \text{if } p^2 \neq 4q^2, \tag{3.21}$$

and

$$\kappa_1 = 1 \quad \text{and} \quad \kappa_2 = n\left(\frac{2r - p}{p}\right), \quad \text{if } p^2 = 4q^2. \tag{3.22}$$

□

By inserting the formulas in (3.7) and (3.18) into the recurrence relation (3.2) and the formula (3.17) into the recurrence relation (3.5), we obtain an explicit formula for the determinant of D_n.

Theorem 3.1. *The determinant of the matrix D_n in* (1.1) *is given by*

$$\det D_n = \begin{cases} \sum_{j=1}^{5} \kappa_j\, f(\mu_j)\, \mu_j^{n-5}, & \text{if } q \neq 0,\ s \neq 0 \text{ and } n \geqslant 6, \\ \sum_{j=1}^{4} \kappa_j\, f(\nu_j)\, \nu_j^{n-4}, & \text{if } q = 0,\ s \neq 0 \text{ and } n \geqslant 5, \\ \sum_{j=1}^{2} \kappa_j\, f(\xi_j)\, \xi_j^{n-2}, & \text{if } q \neq 0,\ s = 0 \text{ and } n \geqslant 3, \\ r^2\, p^{n-2}, & \text{if } q = s = 0 \text{ and } n \geqslant 3. \end{cases} \tag{3.23}$$

where $f(\cdot)$ is the polynomial function defined by

$$f(z) = \begin{cases} (r - s)\, z^4 + (ps - q^2)(z^3 - sz^2) + s^3(s - p)\, z + s^5, & \text{if } q \neq 0 \text{ and } s \neq 0, \\ r\, z^3 - p\, s^2\, z + s^4, & \text{if } q = 0 \text{ and } s \neq 0, \\ r\, z - q^2, & \text{if } s = 0. \end{cases}$$

Proof. The result in (3.23) is a consequence of Lemmas 3.1–3.5. □

Remark 4. If we consider D_n defined for $n = 3$ and $n = 4$, respectively, as

$$D_3 = \begin{bmatrix} r & q & s \\ q & p & q \\ s & q & r \end{bmatrix} \qquad \text{and} \qquad D_4 = \begin{bmatrix} r & q & s & 0 \\ q & p & q & s \\ s & q & p & q \\ 0 & s & q & r \end{bmatrix}, \tag{3.24}$$

then the expression of the determinant in (3.23) is true for all cases when $n \geqslant 3$.

4 Application to the centered MA(1) stationary Gaussian process

Consider the stochastic process $(X_n)_{n\in\mathbb{N}}$ defined by the equation

$$X_n = \varepsilon_n + \phi\,\varepsilon_{n-1}, \quad \text{with } |\phi| < 1 \text{ and } n \in \mathbb{N}, \tag{4.1}$$

where $(\varepsilon_n)_{n\geqslant 0}$ is a sequence of i.i.d. random variables, with $\varepsilon_n \sim \mathcal{N}(0,1)$, for each $n \geqslant 0$. The spectral density function associated to $(X_n)_{n\in\mathbb{N}}$ is given by

$$h_\phi(\omega) = 1 + \phi^2 + 2\,\phi\,\cos(\omega), \quad \text{for } \omega \in \mathbb{T} = [-\pi, \pi).$$

Since $(X_n)_{n\in\mathbb{N}}$ is stationary (see definition 3.4 in Shumway and Stoffer [20]), we have by (4.1) that $X_n \sim \mathcal{N}(0, 1+\phi^2)$. Moreover, the hypothesis $|\phi| < 1$ guarantees that this process is also invertible (see theorem 3.1.2 in Brockwell and Davis [4]).

It is common in natural sources to appear data sets that may be modeled by a process as the one given in equation (4.1). The job of a statistician is to identify the pattern of these data sets and associate it with such a model. The process given in (4.1) is called a moving average process of first order (MA(1) process). The book by Brockwell and Davis [4] gives a full treatment in the subject of MA(1) processes, of which we recall the most important properties related to it:

- if X is a random variable defined on a probability space $(\Omega, \Sigma, \mathbb{P})$, the expected value is defined by the Lebesgue integral
$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\,\mathbb{P}(\omega)$$
and the variance of X is given by $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$;
- the spectral density function of the process $(X_n)_{n\in\mathbb{N}}$ in (4.1) satisfies $h_\phi(\omega) = h_\phi(-\omega) > 0$, for all $\omega \in \mathbb{T}$, and $\int_{-\pi}^{\pi} h_\phi(\omega) d\omega < \infty$;
- the autocovariance function $\gamma_X(k) = \mathbb{E}(X_{n+k}X_n) - \mathbb{E}(X_{n+k})\,\mathbb{E}(X_n)$ of $(X_n)_{n\in\mathbb{N}}$ depends on $h_\phi(\cdot)$ in the sense that
$$\gamma_X(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\omega} h_\phi(\omega) d\omega;$$

- the Toeplitz matrix $T_n(h_\phi)$ associated with $h_\phi(\cdot)$ coincides with the autocovariance matrix of the process $(X_n)_{n\in\mathbb{N}}$ and it is given by

$$T_n(h_\phi) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(j-k)\omega}\, h_\phi(\omega)\, d\omega\right)_{1\leqslant j,k\leqslant n}; \tag{4.2}$$

- the matrix $T_n(h_\phi)$ is symmetric and positive definite.

Here we tackle the following problem: let us assume that there is a set of observations $X_1, \cdots, X_n$ from the process given in (4.1). For $\boldsymbol{X}_n = (X_1, \ldots, X_n)$ and $\boldsymbol{X}_n^T$ denoting the transpose of $\boldsymbol{X}_n$, consider the random vector

$$\mathcal{W}_n = \frac{1}{n}\left(U_n, V_n\right),$$

where

$$U_n = \boldsymbol{X}_n^T\, T_n(\varphi_1)\, \boldsymbol{X}_n = \sum_{k=1}^{n} X_k^2, \qquad V_n = \boldsymbol{X}_n^T\, T_n(\varphi_2)\, \boldsymbol{X}_n = \sum_{k=2}^{n} X_k X_{k-1},$$

and $T_n(\varphi_j)$ being, respectively, the Toeplitz matrices associated with $\varphi_j(\cdot): \mathbb{T} \to \mathbb{R}$, for $j = 1, 2$, defined by the functions

$$\varphi_1(\omega) = 1, \qquad \varphi_2(\omega) = \cos(\omega).$$

We are interested in the asymptotic behavior of the normalized cumulant generating function associated to $\mathcal{W}_n$, defined by

$$\begin{aligned} L_n(\boldsymbol{\lambda}) &= \frac{1}{n}\log \mathbb{E}(\exp(n\langle(\lambda_1,\lambda_2), \mathcal{W}_n\rangle)) = \frac{1}{n}\log\left(\mathbb{E}\exp\left[\lambda_1 U_n + \lambda_2 V_n\right]\right) \\ &= \frac{1}{n}\log\left(\mathbb{E}\exp\left[\boldsymbol{X}_n^T\left(\lambda_1 T_n(\varphi_1) + \lambda_2 T_n(\varphi_2)\right)\boldsymbol{X}_n\right]\right), \end{aligned}$$

for each $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. From the definition given in (4.2), it is easy to show that linearity holds on Toeplitz matrices. If we set $\varphi_{\boldsymbol{\lambda}}(\cdot) = \lambda_1\varphi_1(\cdot) + \lambda_2\varphi_2(\cdot)$, we note that

$$L_n(\boldsymbol{\lambda}) = \frac{1}{n}\log\left(\mathbb{E}\exp\left[\boldsymbol{X}_n^T T_n(\varphi_{\boldsymbol{\lambda}})\boldsymbol{X}_n\right]\right),$$

with

$$T_n(\varphi_{\boldsymbol{\lambda}}) = \frac{1}{2}\begin{pmatrix} 2\lambda_1 & \lambda_2 & 0 & \cdots & 0 \\ \lambda_2 & 2\lambda_1 & \lambda_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \lambda_2 & 2\lambda_1 & \lambda_2 \\ 0 & \cdots & 0 & \lambda_2 & 2\lambda_1 \end{pmatrix}.$$

Since the random vector $\boldsymbol{X}_n$ follows a n-variate Gaussian distribution and the matrix $T_n(\varphi_{\boldsymbol{\lambda}})$ is symmetric, as observed in Bercu *et al.* [3], we may rewrite ${\boldsymbol{X}_n}^T T_n(\varphi_{\boldsymbol{\lambda}})\boldsymbol{X}_n$ as

$${\boldsymbol{X}_n}^T\, T_n(\varphi_{\boldsymbol{\lambda}})\, \boldsymbol{X}_n = \sum_{k=1}^{n} \alpha_{n,k}^{\boldsymbol{\lambda}} Z_{n,k},$$

where $\{\alpha_{n,k}^{\boldsymbol{\lambda}}\}_{k=1}^{n}$ are the eigenvalues of $T_n(\varphi_{\boldsymbol{\lambda}})\,T_n(h_\phi)$ and $\{Z_{n,k}\}_{k=1}^{n}$ is a sequence of i.i.d. random variables, each one having a chi-squared distribution with one degree of freedom. A simple algebraic proof shows that $\{\alpha_{n,k}^{\boldsymbol{\lambda}}\}_{k=1}^{n}$ and $\{1-2\,\alpha_{n,k}^{\boldsymbol{\lambda}}\}_{k=1}^{n}$ are also the eigenvalues of $T_n(h_\phi)^{1/2}\,T_n(\varphi_{\boldsymbol{\lambda}})\,T_n(h_\phi)^{1/2}$ and $I_n-2\,T_n(\varphi_{\boldsymbol{\lambda}})\,T_n(h_\phi)$, respectively. Hence, from the independence of the random variables $\{Z_{n,k}\}_{k=1}^{n}$, it turns out that $L_n(\cdot)$ can bee expressed as (see Karling *et al.* [16]):

$$L_n(\boldsymbol{\lambda})=\frac{1}{n}\log\left(\mathbb{E}\exp\left[\boldsymbol{X}_n{}^T\,T_n(\varphi_{\boldsymbol{\lambda}})\,\boldsymbol{X}_n\right]\right)=\begin{cases}-\frac{1}{2n}\sum_{k=1}^{n}\log(1-2\alpha_{n,k}^{\boldsymbol{\lambda}}), & \text{if } \alpha_{n,k}^{\boldsymbol{\lambda}}<\frac{1}{2},\ \forall\, 1\leqslant k\leqslant n,\\ +\infty, & \text{otherwise.}\end{cases}\tag{4.3}$$

From (4.3) we note that the condition $\alpha_{n,k}^{\boldsymbol{\lambda}}<\frac{1}{2}$, for all k such that $1\leqslant k\leqslant n$, is the equivalent of requiring that $D_{n,\boldsymbol{\lambda}}=I_n-2\,T_n(\varphi_{\boldsymbol{\lambda}})\,T_n(h_\phi)$ must be a positive definite matrix, where

$$D_{n,\boldsymbol{\lambda}}=\begin{bmatrix} r & q & s & 0 & \cdots & 0\\ q & p & q & s & \ddots & \vdots\\ s & q & p & \ddots & \ddots & 0\\ 0 & s & \ddots & \ddots & q & s\\ \vdots & \ddots & \ddots & q & p & q\\ 0 & \cdots & 0 & s & q & r\end{bmatrix},\quad\text{with}\quad \begin{cases} r &= 1-2\lambda_1(1+\phi^2)-\lambda_2\phi,\\ p &= 1-2\lambda_1(1+\phi^2)-2\lambda_2\phi,\\ q &= -2\lambda_1\phi-\lambda_2(1+\phi^2),\\ s &= -\lambda_2\phi.\end{cases}\tag{4.4}$$

To avoid confusion, we shall adopt the notation $D_{n,\boldsymbol{\lambda}}$ to distinguish the particular case in (4.4) from the general one in (1.1), and in the sequel, we say that $D_{n,\boldsymbol{\lambda}}$ is the pentadiagonal matrix associated to the MA(1) process. Thus, it follows that

$$L_n(\boldsymbol{\lambda})=\begin{cases}-\frac{1}{2n}\log(\det(D_{n,\boldsymbol{\lambda}})), & \text{if } D_{n,\boldsymbol{\lambda}} \text{ is positive definite,}\\ +\infty, & \text{otherwise.}\end{cases}\tag{4.5}$$

It remains to check for the convergence of $-(1/2n)\log(\det(D_{n,\boldsymbol{\lambda}}))$, which is given by the next proposition.

Proposition 4.1. *Let* $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)\in\mathbb{R}^2$ *and* $\mathcal{D}_{\boldsymbol{\lambda}}=\mathcal{D}_{\boldsymbol{\lambda}}^1\cup\mathcal{D}_{\boldsymbol{\lambda}}^2$, *with* $\mathcal{D}_{\boldsymbol{\lambda}}^1$ *and* $\mathcal{D}_{\boldsymbol{\lambda}}^2$ *given in* (4.8) *and* (4.9), *respectively. Then,* $\mathcal{L}(\boldsymbol{\lambda}):=\lim_{n\to\infty}L_n(\boldsymbol{\lambda})=\lim_{n\to\infty}-(1/2n)\log(\det(D_{n,\boldsymbol{\lambda}}))$, *where*

$$\mathcal{L}(\boldsymbol{\lambda})=\begin{cases}-\dfrac{1}{2}\log\left[\dfrac{(p-2s)(1+\sqrt{1-A^2})(1+\sqrt{1-B^2})}{4}\right], & \textit{for } \boldsymbol{\lambda}\in\mathcal{D}_{\boldsymbol{\lambda}}\setminus\overline{\mathcal{D}_{\boldsymbol{\lambda}}^0},\\ +\infty, & \textit{otherwise,}\end{cases}\tag{4.6}$$

with

$$A=\frac{q-\sqrt{q^2-4s(p-2s)}}{p-2s}\quad\textit{and}\quad B=\frac{q+\sqrt{q^2-4s(p-2s)}}{p-2s},\tag{4.7}$$

p,q *and* s *defined as in* (4.4), *and* $\overline{\mathcal{D}_{\boldsymbol{\lambda}}^0}$ *denotes the closure of* $\mathcal{D}_{\boldsymbol{\lambda}}^0$, *given in* (4.10).

Proof. As $D_{n,\boldsymbol{\lambda}}$ is a matrix that satisfies the relation $r=p-s$, Lemma 2.1 and Corollary 2.1 are applicable. The domains $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3,\mathcal{D}_4$ in (2.1) can be rewritten in terms of λ_1,λ_2 and ϕ, as the union of the two following sets

$$\mathcal{D}^1_{\boldsymbol{\lambda}} = \left\{ \frac{1+4\lambda_2\phi}{2(1+\phi^2)} \leqslant \lambda_1 \leqslant \frac{1}{2(1+\phi^2)} \,,\; (2\lambda_1\phi + \lambda_2(1+\phi^2))^2 \leqslant -4\lambda_2\phi(1-2\lambda_2(1+\phi^2)) \right\} \tag{4.8}$$

and

$$\mathcal{D}^2_{\boldsymbol{\lambda}} = \left\{ \frac{-1+2\lambda_1(1+\phi^2)}{4} < \lambda_2\phi \leqslant \frac{1-2\lambda_1(1+\phi^2)}{4} \,,\; \lambda_1 - \frac{1}{2(1-\phi)^2} \leqslant \lambda_2 \leqslant \frac{1}{2(1+\phi)^2} - \lambda_1 \right\}. \tag{4.9}$$

From Corollary 2.1 we conclude that $D_{n,\boldsymbol{\lambda}}$ has at least one null eigenvalue inside $\mathcal{D}_{\boldsymbol{\lambda}}$ if $\boldsymbol{\lambda}$ belongs to

$$\mathcal{D}^0_{\boldsymbol{\lambda}} = \bigcup_{\substack{1 \leqslant k \leqslant n \\ n \in \mathbb{N}}} \left\{ \lambda_1 = \frac{1+\phi^2+4\phi\cos\left(\frac{k\pi}{n+1}\right)}{2\left(1+\phi^2+2\phi\cos\left(\frac{k\pi}{n+1}\right)\right)^2} \,,\; \lambda_2 = \frac{-\phi}{\left(1+\phi^2+2\phi\cos\left(\frac{k\pi}{n+1}\right)\right)^2} \right\}. \tag{4.10}$$

As a result of that, $D_{n,\boldsymbol{\lambda}}$ is positive definite, for all $n \in \mathbb{N}$, if (λ_1, λ_2) is considered inside $\mathcal{D}_{\boldsymbol{\lambda}} \setminus \mathcal{D}^0_{\boldsymbol{\lambda}}$, implying that $-\frac{1}{2n}\log(\det(D_{n,\boldsymbol{\lambda}}))$ is finite, for all $n \in \mathbb{N}$. However, we need to be careful when taking the limit as $n \to \infty$. Although $D_{n,\boldsymbol{\lambda}}$ is positive definite in $\overline{\mathcal{D}^0_{\boldsymbol{\lambda}}} \setminus \mathcal{D}^0_{\boldsymbol{\lambda}}$, asymptotically speaking, the limit $\lim_{n\to\infty} L_n(\boldsymbol{\lambda})$ does not exist over this set. Consequently, we may define $\mathcal{L}(\boldsymbol{\lambda}) = +\infty$, if $\boldsymbol{\lambda} \notin \mathcal{D}_{\boldsymbol{\lambda}} \setminus \overline{\mathcal{D}^0_{\boldsymbol{\lambda}}}$. Henceforth, we shall restrict our analysis to the set $\mathcal{D}_{\boldsymbol{\lambda}} \setminus \overline{\mathcal{D}^0_{\boldsymbol{\lambda}}}$.

Consider in what follows the measure space $L^\infty(\mathbb{T}) := L^\infty(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathbb{L})$, were $\mathbb{L}(\cdot)$ is the Lebesgue measure acting on $\mathcal{B}(\mathbb{T})$, the Borel σ-algebra over $\mathbb{T} = [-\pi, \pi)$. Since $\varphi_{\boldsymbol{\lambda}}$, $h_\phi \in L^\infty(\mathbb{T})$, it is straightforward to show that

$$|\alpha^{\boldsymbol{\lambda}}_{n,k}| \leqslant ||\varphi_{\boldsymbol{\lambda}}||_\infty ||h_\phi||_\infty, \quad \text{for all } 1 \leqslant k \leqslant n \text{ and } n \in \mathbb{N}, \tag{4.11}$$

where $||\cdot||_\infty$ denotes the usual norm in $L^\infty(\mathbb{T})$ (see definition 6.15 in Bartle [2]). The function $\varphi_{\boldsymbol{\lambda}}\, h_\phi : \mathbb{T} \to \mathbb{R}$, defined by

$$(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\omega) = \varphi_{\boldsymbol{\lambda}}(\omega)\, h_\phi(\omega) = (\lambda_1 + \lambda_2\cos(\omega))(1+\phi^2+2\phi\cos(\omega)),$$

is continuous and bounded in $\mathbb{T}$, hence it attains a maximum and a minimum in that interval. Let $m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}$ and $M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}$ denote, respectively, the *minimum* and the *maximum* of $(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\cdot)$, i.e.,

$$m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi} = \min_{\omega\in\mathbb{T}}\{(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\omega)\} \quad \text{and} \quad M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi} = \max_{\omega\in\mathbb{T}}\{(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\omega)\}.$$

It follows that $m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}$ and $M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}$ coincide, respectively, with the *essential lower and upper bounds* of $(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\cdot)$ (see page 65 in Grenander and Szegö [12]). Moreover, one can verify that

$$m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}, M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi} \in \left\{ (\lambda_1+\lambda_2)(1+\phi)^2,\; (\lambda_1-\lambda_2)(1-\phi)^2,\; -\frac{\left(\lambda_2(1+\phi^2) - 2\lambda_1\phi\right)^2}{8\lambda_2\phi} \right\}.$$

Note that

$$\begin{aligned} 1 - 2\,(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\omega) &= 1 - 2\,(\lambda_1 + \lambda_2\cos(\omega))(1+\phi^2+2\phi\cos(\omega)) \\ &= 1 - 2\lambda_1(1+\phi^2) - 2\,(2\lambda_1\phi + \lambda_2(1+\phi^2))\cos(\omega) - 4\lambda_2\phi\cos^2(\omega) \\ &= (p-2s) + q(2\cos(\omega)) + s(2\cos(\omega))^2 = g(2\cos(\omega)), \end{aligned} \tag{4.12}$$

where $g(\cdot)$ is the second-order polynomial given in (2.2), but for the particular case when p, q and s are given by (4.4). If $\boldsymbol{\lambda} \in \mathcal{D}_{\boldsymbol{\lambda}} \setminus \overline{\mathcal{D}^0_{\boldsymbol{\lambda}}}$, we have $g(x) \geqslant 0$, for all $x \in [-2, 2]$, and from (4.12) it follows that

$$1 - 2\,(\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\omega) = g(2\cos(\omega)) \geqslant 0, \forall\, \omega \in \mathbb{T} \;\Rightarrow\; (\varphi_{\boldsymbol{\lambda}}\, h_\phi)(\omega) \leqslant 1/2, \forall\, \omega \in \mathbb{T}.$$

Thus, $M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi} \leqslant 1/2$ for all $(\lambda_1, \lambda_2) \in \mathcal{D}_{\boldsymbol{\lambda}} \setminus \overline{\mathcal{D}^0_{\boldsymbol{\lambda}}}$. On the other hand, from

$$||\varphi_{\boldsymbol{\lambda}}||_\infty ||h_\phi||_\infty \geqslant ||\varphi_{\boldsymbol{\lambda}}\, h_\phi||_\infty = \max\{|M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}|, |m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}|\} \geqslant -m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi},$$

we obtain $m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi} \geqslant -||\varphi_{\boldsymbol{\lambda}}||_\infty ||h_\phi||_\infty$. Therefore, if $\boldsymbol{\lambda} \in \mathcal{D}_{\boldsymbol{\lambda}} \setminus \overline{\mathcal{D}^0_{\boldsymbol{\lambda}}}$, then

$$[m_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}, M_{\varphi_{\boldsymbol{\lambda}}\, h_\phi}] \subseteq [-||\varphi_{\boldsymbol{\lambda}}||_\infty ||h_\phi||_\infty, 1/2]. \tag{4.13}$$

Let us consider the continuous extended function $F : [-||\varphi_{\boldsymbol{\lambda}}||_\infty ||h_\phi||_\infty, 1/2] \to \mathbb{R} \cup \{\infty\}$, defined by

$$F(x) = -\frac{\log(1-2x)}{2}.$$

Note that $F(\cdot)$ has a bounded support (i.e., the set of those $x \in \mathbb{R}$ for which $F(x) \neq 0$ is bounded) and, as a consequence from (4.11) and (4.13), if $\boldsymbol{\lambda} \in \mathcal{D}_{\boldsymbol{\lambda}} \setminus \overline{\mathcal{D}^0_{\boldsymbol{\lambda}}}$, we infer that $F(\alpha^{\boldsymbol{\lambda}}_{n,k})$ are finite for every $1 \leqslant k \leqslant n$ and $n \in \mathbb{N}$. Then, it follows from theorem 5.1 in Tyrtyshnikov [24] that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(\alpha^{\boldsymbol{\lambda}}_{n,k}) = \frac{1}{2\pi} \int_{\mathbb{T}} (F \circ (\varphi_{\boldsymbol{\lambda}}\, h_\phi))(\omega)\, d\omega.$$

In particular, we have

$$\begin{aligned}
\lim_{n\to\infty} L_n(\lambda_1, \lambda_2) \;&=\; \lim_{n\to\infty} -\frac{1}{2n} \sum_{k=1}^{n} \log(1 - 2\alpha_{n,k}) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(\alpha_{n,k}) \;=\; \frac{1}{2\pi} \int_{\mathbb{T}} (F \circ (\varphi_{\boldsymbol{\lambda}}\, h_\phi))(\omega)\, d\omega \\
&= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\Big(1 - 2\, h_\phi(\omega)\, \varphi_{\boldsymbol{\lambda}}(\omega)\Big) d\omega = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\left[1 - 2\,(1+\phi^2 + 2\phi\cos(\omega))\,(\lambda_1 + \lambda_2\cos(\omega))\right] d\omega \\
&= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\left[1 - 2\lambda_1(1+\phi^2) - 2\left(2\lambda_1\phi + \lambda_2(1+\phi^2)\right)\cos(\omega) - 4\lambda_2\phi\cos^2(\omega)\right] d\omega \\
&= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\left[p - 2s + 2q\cos(\omega) + 4s\cos^2(\omega)\right] d\omega,
\end{aligned} \tag{4.14}$$

where p, q and s are given by (4.4). Considering A and B as in (4.7), from Lemma A.1 (see Appendix A) it follows that

$$\int_{-\pi}^{\pi} \log\left[p - 2s + 2q\cos(\omega) + 4s\cos^2(\omega)\right] d\omega = 2\pi \log\left[\frac{(p-2s)(1+\sqrt{1-A^2})(1+\sqrt{1-B^2})}{4}\right]. \tag{4.15}$$

In conclusion, (4.6) now follows from (4.14) and (4.15). □

In Figure 2, we ploted the domain $\mathcal{D}_{\boldsymbol{\lambda}} = \mathcal{D}^1_{\boldsymbol{\lambda}} \cup \mathcal{D}^2_{\boldsymbol{\lambda}}$, for $\mathcal{D}^1_{\boldsymbol{\lambda}}$ and $\mathcal{D}^2_{\boldsymbol{\lambda}}$ given, respectively, in (4.8) and (4.9) for $\boldsymbol{\lambda} \in [-2, 0.5] \times [-3, 2]$ and $\phi = 1/3$. In this figure, we also ploted some of the points (λ_1, λ_2) that belong to $\mathcal{D}^0_{\boldsymbol{\lambda}}$, given in (4.10). Notice how they scatter just over one side of the boundary of $\mathcal{D}_{\boldsymbol{\lambda}}$.

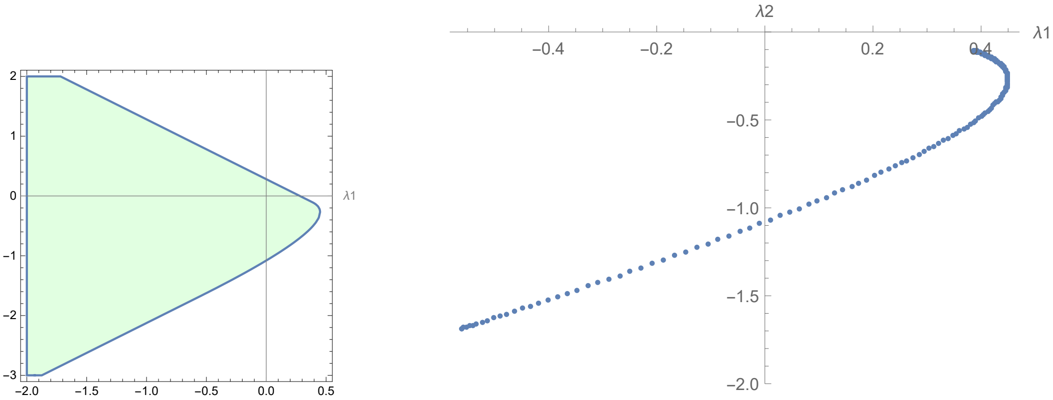

Figure 2: Domain $\mathcal{D}_{\boldsymbol{\lambda}} = \mathcal{D}^1_{\boldsymbol{\lambda}} \cup \mathcal{D}^2_{\boldsymbol{\lambda}}$, for $\mathcal{D}^1_{\boldsymbol{\lambda}}$ and $\mathcal{D}^2_{\boldsymbol{\lambda}}$ given, respectively, in (4.8) and (4.9), for the case when $\phi = 1/3$, with $(\lambda_1, \lambda_2) \in [-2, 0.5] \times [-3, 2]$. On the left is the union $\mathcal{D}_{\boldsymbol{\lambda}} = \mathcal{D}^1_{\boldsymbol{\lambda}} \cup \mathcal{D}^2_{\boldsymbol{\lambda}}$. On the right, the points (λ_1, λ_2) with $\lambda_1 = \frac{1+\phi^2+4\phi\cos\left(\frac{\pi k}{n+1}\right)}{2\left(1+\phi^2+2\phi\cos\left(\frac{\pi k}{n+1}\right)\right)^2}$ and $\lambda_2 = \frac{-\phi}{\left(1+\phi^2+2\phi\cos\left(\frac{\pi k}{n+1}\right)\right)^2}$, for $n = 200$ and $1 \leqslant k \leqslant 200$, are plotted.

5 Examples

Here we introduce three examples that illustrate the theory presented in the preceding sections. The first one gives a counterexample to show that proposition 5 in Fasino [9] (and by consequence Lemma 2.1) is not true if the condition $r \geqslant p - s$ is not verified.

Example 5.1. Consider $p = 5$, $q = -1$, $r = 1$ and $s = 2$, so that D_n, defined in (1.1), is given by

$$D_n = \begin{bmatrix} 1 & -1 & 2 & 0 & \cdots & 0 \\ -1 & 5 & -1 & 2 & \ddots & \vdots \\ 2 & -1 & 5 & \ddots & \ddots & 0 \\ 0 & 2 & \ddots & \ddots & -1 & 2 \\ \vdots & \ddots & \ddots & -1 & 5 & -1 \\ 0 & \cdots & 0 & 2 & -1 & 1 \end{bmatrix}. \tag{5.1}$$

Note that $1 = r < p - s = 2$. Since $q^2 - 4s(p-2s) = -7$, the polynomial function $g(x) = sx^2 + qx + p - 2s = 2x^2 - x + 1$ has no real roots. We observe that, even though $g(x) > 0$ for all $x \in \mathbb{R}$, the matrix D_n in (5.1) cannot be non-negative definite for all $n \in \mathbb{N}$. In fact, when computing its eigenvalues, we observe that D_n has one negative eigenvalue if $5 \leqslant n \leqslant 8$, and two negative eigenvalues if $9 \leqslant n \leqslant 100$, suggesting that proposition 5 in Fasino [9] does not hold in the absence of the condition $r \geqslant p - s$. Nevertheless, it is still possible to compute the determinant of D_n by using the result of Theorem 3.1. The coefficients required for this computation are (in approximated form)

$$\kappa_1 = 0.163717 + 0.05368i,\ \kappa_2 = 0.163717 - 0.05368i,\ \kappa_3 = -0.395173,\ \kappa_4 = -0.075118,\ \kappa_5 = 1.14286,$$

and

$$\mu_1 = -1.94374 - 0.471031i, \quad \mu_2 = -1.94374 + 0.471031i, \quad \mu_3 = 1.03951, \quad \mu_4 = 3.84797, \quad \mu_5 = 2.$$

Although $\kappa_1, \kappa_2, \mu_1$ and μ_2 are complex numbers, the determinant is real and it is given by

$$\det(D_n) = \sum_{j=1}^{5} \kappa_j \, f(\mu_j)\, \mu_j^{n-5},$$

where $f(z) = -z^4 + 9z^3 - 18z^2 - 24z + 32$.

Table 5.1 presents the values of $\det(D_n)$ for four different values of n, obtained with the help of the Wolfram Mathematica software, operating in an Intel Core i7-8565U processor. For comparison reasons, when using the determinant function available in this software, the computational time registered for $n = 2000$ was 0.828125 seconds. As Table 5.1 shows, the formula presented in Theorem 3.1 allows to compute the determinant of the matrix in (5.1) much faster than the usual algorithms do.

Table 5.1: Approximated values of $\det(D_n)$, when $n \in \{5,\, 5\times 10^6,\, 5\times 10^7,\, 5\times 10^8\}$.

n	5	5×10^6	5×10^7	5×10^8
$\det(D_n)$	-40	$1.65193\times 10^{2926158}$	$8.47348\times 10^{29261604}$	$1.06844\times 10^{292616072}$
Time (in seconds)	≈ 0	0.015625	0.046875	0.453125

◇

The next example clarifies the theory presented in Section 4.

Example 5.2. Consider $\phi = 1/3$ fixed and let $(X_n)_{n\in\mathbb{N}}$ denote the MA(1) process defined in Section 4. We demonstrated that the normalized cumulant generating function associated to the random sequence $\left(n^{-1}\left(\sum_{k=1}^{n} X_k^2, \sum_{k=2}^{n} X_k X_{k-1}\right)\right)_{n\geqslant 2}$ can be written as in (4.5). For instance, if $\boldsymbol{\lambda} = (-1,-1)$, then $D_{n,\boldsymbol{\lambda}}$ is the pentadiagonal matrix given by

$$D_{n,(-1,-1)} = \begin{bmatrix} \frac{32}{9} & \frac{16}{9} & \frac{1}{3} & 0 & \cdots & 0 \\ \frac{16}{9} & \frac{35}{9} & \frac{16}{9} & \frac{1}{3} & \ddots & \vdots \\ \frac{1}{3} & \frac{16}{9} & \frac{35}{9} & \ddots & \ddots & 0 \\ 0 & \frac{1}{3} & \ddots & \ddots & \frac{16}{9} & \frac{1}{3} \\ \vdots & \ddots & \ddots & \frac{16}{9} & \frac{35}{9} & \frac{16}{9} \\ 0 & \cdots & 0 & \frac{1}{3} & \frac{16}{9} & \frac{32}{9} \end{bmatrix}.$$

The vector $(-1,-1)$ belongs to the interior of $\mathcal{D}^2_{\boldsymbol{\lambda}}$, defined in (4.9). Hence, from Proposition 4.1 we conclude that

$$\lim_{n\to\infty} L_n(-1,-1) = \lim_{n\to\infty} -(1/2n)\log[\det(D_{n,(-1,-1)})] = \mathcal{L}(-1,-1)$$

where

$$\mathcal{L}(-1,-1) = -\frac{1}{2}\log\left[\frac{(p-2s)(1+\sqrt{1-A^2})(1+\sqrt{1-B^2})}{4}\right] \approx -0.548981, \tag{5.2}$$

with $p = 35/9$, $q = 16/9$, $s = 1/3$, and

$$A = \frac{q - \sqrt{q^2 - 4s(p-2s)}}{p-2s} = \frac{16 - 2i\sqrt{23}}{29} \quad \text{and} \quad B = \frac{q + \sqrt{q^2 - 4s(p-2s)}}{p-2s} = \frac{16 + 2i\sqrt{23}}{29}.$$

Table 5.2 presents the values of $L_n(-1,-1) = -(1/2n)\log[\det(D_{n,(-1,-1)})]$ for $n \in \{5, 10, 50, 100, 500\}$. Notice that, even for a small value of $n = 5$, the term $L_n(-1,-1)$ is relatively close to the asymptotic value in (5.2).

Table 5.2: Approximated values of $L_n(-1,-1)$, for $n \in \{5, 10, 50, 100, 500\}$.

n	5	10	50	100	500
$L_n(-1,-1)$	−0.554116	−0.551548	−0.549495	−0.549238	−0.549032

◊

In the following example, we show that in the case when $r = p - s$, the eigenvalues of the matrix D_n in (1.1) feature a periodic behavior. This is due to the result of Lemma 2.2.

Example 5.3. Consider once more the MA(1) process with $\phi = 1/3$. As shown in Lemma 2.2, since the matrix $D_{n,\boldsymbol{\lambda}}$ in (4.4) satisfies the relation $r = p - s$, for any pair $\boldsymbol{\lambda} \in \mathbb{R}^2$, the eigenvalues of this matrix are given by $\alpha_{n,k} = 4s\cos^2\left(\frac{k\pi}{n+1}\right) + 2q\cos\left(\frac{k\pi}{n+1}\right) + p - 2s$, for p, q, r, s defined in (4.4) and $1 \leqslant k \leqslant n$. If we take a point $\boldsymbol{\lambda}$ outside the range of $\mathcal{D}_{\boldsymbol{\lambda}}$, we shall have an enumerable set of negative eigenvalues of $D_{n,\boldsymbol{\lambda}}$. For instance, if $\boldsymbol{\lambda} = (0,1)$, then $p = 1/3$, $q = -10/3$, $r = 2/3$ and $s = -1/3$. The matrix $D_{n,(0,1)}$ is therefore given by

$$D_{n,(0,1)} = \begin{bmatrix} \frac{2}{3} & -\frac{10}{3} & -\frac{1}{3} & 0 & \cdots & 0 \\ -\frac{10}{3} & \frac{1}{3} & -\frac{10}{3} & -\frac{1}{3} & \ddots & \vdots \\ -\frac{1}{3} & -\frac{10}{3} & \frac{1}{3} & \ddots & \ddots & 0 \\ 0 & -\frac{1}{3} & \ddots & \ddots & -\frac{10}{3} & -\frac{1}{3} \\ \vdots & \ddots & \ddots & -\frac{10}{3} & \frac{1}{3} & -\frac{10}{3} \\ 0 & \cdots & 0 & -\frac{1}{3} & -\frac{10}{3} & \frac{2}{3} \end{bmatrix}.$$

If $n = 5$, the eigenvalues of $D_{n,(0,1)}$ are

$$\alpha_{5,1} = -\frac{10}{3\sqrt{3}}, \quad \alpha_{5,2} = -\frac{4}{9}, \quad \alpha_{5,3} = 1, \quad \alpha_{5,4} = \frac{16}{9}, \quad \alpha_{5,5} = \frac{10}{3\sqrt{3}}. \tag{5.3}$$

If we take $n = 11$, these same eigenvalues will appear as

$$\alpha_{11,2} = -\frac{10}{3\sqrt{3}}, \quad \alpha_{11,4} = -\frac{4}{9}, \quad \alpha_{11,6} = 1, \quad \alpha_{11,8} = \frac{16}{9}, \quad \alpha_{11,10} = \frac{10}{3\sqrt{3}}.$$

In fact, if $k \equiv 5 \pmod 6$, then the values in (5.3) will be eigenvalues of $D_{k,(0,1)}$. Consequently, since $\alpha_{5,1}$ and $\alpha_{5,2}$ are already negative, $D_{k,(0,1)}$ cannot be non-negative definite.

This reasoning is not restricted to the pentadiagonal matrix associated to the MA(1) process. If D_n in (1.1) has arbitrary values for p, q, s, and r is such that $r = p - s$, then its eigenvalues also share this periodic property, due to Lemma 2.2. The point here is that the presence of periodic eigenvalues does not allow the existence of some $n_0 \in \mathbb{N}$ such that D_n is positive or non-negative definite for $n \geqslant n_0$.

$\diamondsuit$

6 Conclusions

In this work, we have examined some determinantal properties of the general matrix D_n in (1.1). We gave explicit expressions for the determinant via recurrence relations, providing an alternative to the existing expressions given in the literature. Theorem 3.1, with the help of five lemmas, presents the explicit expression for the determinant of D_n, showing its dependence on $n \geqslant 3$. We also analyzed when the matrix D_n is non-negative definite in the presence of the restriction $r \geqslant p - s$. This is achieved through the use of proposition 5 in Fasino [9], that helped us to provide the precise expressions for the domains in (2.1). Furthermore, when $r = p - s$, we showed the explicit domain in which D_n is actually positive definite (in the strict sense). Example 5.1 is important to show that the condition $r \geqslant p - s$ is essential for proposition 5 in Fasino [9].

We have indicated the importance of the linear algebra analysis, self-contained in the present work, by applying these results to the stationary centered moving average process of first order with Gaussian innovations. An explicit expression for the normalized cumulant generating function $L_n(\cdot)$, associated to $\mathcal{W}_n$, described in expression (1.2), was exhibited. Proposition 4.1 presents the limit of this function $L_n(\cdot)$, when n goes to infinity, in a closed expression, whenever it is well defined. In Example 5.2, with the help of Proposition 4.1, we expressed the value of the limit $\mathcal{L}(\cdot)$ in a particular case. Whereas in Example 5.3 we exhibit the case $r = p - s$, where the eigenvalues of the matrix D_n in (1.1) feature a periodic behavior, due to Lemma 2.2. Finally, we mention that one can obtain the partial derivatives of $\mathcal{L}(\boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$ by using the Wolfram Mathematica software. From this, one can access an explicit form of the moments for the underline random process.

A A Useful Lemma

In this section, we show a useful lemma that makes it possible to compute the integral in (4.15) and which extends the result given in equation 4.224(9) in Gradshteyn and Ryzhik [11].

Lemma A.1. *Consider $a, b, c \in \mathbb{R}$ such that $a + bx + cx^2 \geqslant 0$, for $|x| \leqslant 1$. Let*

$$\gamma_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} \quad \textit{and} \quad \gamma_2 = \frac{b + \sqrt{b^2 - 4ac}}{2a}.$$

Then, it follows that

$$\int_0^{\pi} \log\left[a + b\cos(\omega) + c\cos^2(\omega)\right] d\omega = \begin{cases} \pi \log\left(\frac{c}{4}\right) & \textit{if } \; a = b = 0, \; c > 0, \\ \pi \log\left[\frac{a\,(1+\sqrt{1-\gamma_1^2})(1+\sqrt{1-\gamma_2^2})}{4}\right], & \textit{otherwise}. \end{cases} \tag{A.1}$$

Proof. Note that, the assumption $a+bx+cx^2 \geqslant 0$ for $|x| \leqslant 1$, guarantees that $a+b\cos(\omega)+c\cos^2(\omega) \geqslant 0$ for all $\omega \in [0, \pi]$, since $|\cos(\omega)| \leqslant 1$. In particular, it follows that $a \geqslant 0$, and for this reason, two cases are considered for the proof.

- *Case 1:* If $a = 0$, then we must have, necessarily, $b = 0$ and $c > 0$. Hence

$$\int_0^{\pi} \log\left[a + b\cos(\omega) + c\cos^2(\omega)\right] d\omega = \int_0^{\pi} \log\left[c\cos^2(\omega)\right] d\omega = \int_0^{\pi} \log(c)\, d\omega + \int_0^{\pi} \log\left[\cos^2(\omega)\right] d\omega$$
$$= \pi \log(c) + 4 \int_0^{\pi/2} \log\left[\cos(\omega)\right] d\omega.$$

From equation 4.224(6) in Gradshteyn and Ryzhik [11], we know that $\int_0^{\pi/2} \log\left[\cos(\omega)\right] d\omega = -\frac{\pi}{2}\log(2)$, hence

$$\int_0^{\pi} \log\left[c\cos^2(\omega)\right] d\omega = \pi\log(c) - \pi\log(4) = \pi\log\left(\frac{c}{4}\right).$$

- *Case 2:* If $a > 0$, note that

$$a + b\cos(\omega) + c\cos^2(\omega) = a\,(1 + \gamma_1\cos(\omega))(1 + \gamma_2\cos(\omega)). \tag{A.2}$$

The proof of (A.1) then follows from the identify

$$\int_0^{\pi} \log\left[1 + z\cos(\omega)\right] d\omega = \pi\log\left[\frac{1+\sqrt{1-z^2}}{2}\right], \quad \text{for} \quad z \in \mathbb{C}.$$

Indeed, we have

$$\int_0^{\pi} \log\left[a + b\cos(\omega) + c\cos^2(\omega)\right] d\omega = \int_0^{\pi} \log\left[a\,(1+\gamma_1\cos(\omega))(1+\gamma_2\cos(\omega))\right] d\omega$$
$$= \pi\log(a) + \int_0^{\pi} \log\left[1+\gamma_1\cos(\omega)\right] d\omega + \int_0^{\pi} \log\left[1+\gamma_2\cos(\omega)\right] d\omega$$
$$= \pi\log(a) + \pi\log\left[\frac{1+\sqrt{1-\gamma_1^2}}{2}\right] + \pi\log\left[\frac{1+\sqrt{1-\gamma_2^2}}{2}\right]$$
$$= \pi\log\left[\frac{a\,(1+\sqrt{1-\gamma_1^2})(1+\sqrt{1-\gamma_2^2})}{4}\right].$$

□

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Declaration of competing interest

None declared.

References

[1] Andelić, M. and da Fonseca, C. M. (2020). Some determinantal considerations for pentadiagonal matrices. *Linear and Multilinear Algebra*. Available at https://doi.org/10.1080/03081087.2019.1708845.

[2] Bartle, R. G. (1995). *The elements of integration and Lebesgue measure*. John Wiley & Sons, New York.

[3] Bercu, B., Gamboa, F. and Rouault, A. (1997). Large deviations for quadratic forms of stationary Gaussian processes. *Stochastic Processes and their Applications* **71(1),** 75–90.

[4] Brockwell, P. J. and Davis, R. A. (1991). *Time Series: Theory and methods*, 2nd edn. Springer, New York.

[5] Cinkir, Z. (2012). An elementary algorithm for computing the determinant of pentadiagonal Toeplitz matrices. *Journal of Computational and Applied Mathematics* **236(9),** 2298–2305.

[6] Elouafi, M. (2011). An eigenvalue localization theorem for pentadiagonal symmetric Toeplitz matrices. *Linear Algebra ans its Applications* **435(11),** 2986–2998.

[7] Elouafi, M. (2013). A note for an explicit formula for the determinant of pentadiagonal and heptadiagonal symmetric Toeplitz matrices. *Applied Mathematics and Computation* **219(9),** 4789–4791.

[8] Elouafi, M. (2018). On formulae for the determinant of symmetric pentadiagonal Toeplitz matrices. *Arabian Journal of Mathematics* **7,** 91–99.

[9] Fasino, D. (1988). Spectral and structural properties of some pentadiagonal symmetric matrices. *Calcolo* **25,** 301–310.

[10] Gilbert, G. T. (1991). Positive Definite Matrices and Sylvester's Criterion. *The American Mathematical Monthly* **98(1),** 44-46.

[11] Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*, 7th edn. Academic Press, San Diego.

[12] Grenander, U. and Szegö, G. (1958). *Toeplitz Forms and Their Applications*, 2nd edn. Cambridge University Press, Cambridge.

[13] Horn, R. A. (2013). *Matrix Analysis*, 2nd edn. Cambridge University Press, New York.

[14] Jia, J., Kong, Q. and Sogabe, T. (2012). A new algorithm for solving nearly penta-diagonal Toeplitz linear systems. *Computers and Mathematics with Applications* **63(7),** 1238-1243.

[15] Jia, J., Yang, B. and Li, S. (2016). On a homogeneous recurrence relation for the determinants of general pentadiagonal Toeplitz matrices. *Computers and Mathematics with Applications* **71(4),** 1036–1044.

[16] Karling, M. J., Lopes, A. O. and Lopes, S. R. C. (2020). Explicit bivariate rate functions for large deviations in AR(1) and MA(1) processes with Gaussian innovations. Available at http://arxiv.org/abs/2102.09637.

[17] Marr, R. B. and Vineyard, G. H. (1988). Five-Diagonal Toeplitz Determinants an Their Relation to Chebyshev Polynomials. *SIAM Journal on Matrix Analysis and Applications* **9(4),** 579–586.

[18] McNally, J. M. (2010). A fast algorithm for solving diagonally dominant symmetric pentadiagonal Toeplitz systems. *Journal of Computational and Applied Mathematics* **234(4),** 995–1005.

[19] Nemani, S. S. (2010). A fast algorithm for solving Toeplitz penta-diagonal systems. *Applied Mathematics and Computation* **215(11),** 3830–3838.

[20] Shumway, R. H. and Stoffer, D. S. (2016). *Time Series Analysis and its Applications: With R Examples*, 4th edn. Springer, New York.

[21] Sogabe, T. (2008). A fast numerical algorithm for the determinant of a pentadiagonal matrix. *Applied Mathematics and Computation* **196(2),** 835–841.

[22] Solary, M. S. (2020). Computational properties of pentadiagonal and anti-pentadiagonal block bandmatrices with perturbed corners. *Soft Computing* **24,** 301–309.

[23] Sweet, R. A. (1969). A recursive relation for the determinant of a pentadiagonal matrix. *Communications of the ACM* **12(6),** 330–332.

[24] Tyrtyshnikov, E. E. (1994). Influence of matrix operations on the distribution of eigenvalues and singular values of Toeplitz matrices. *Linear Algebra and its Applications* **207,** 225–249.

[25] Wang, C., Li, H. and Zhao, D. (2015). An explicit formula for the inverse of a pentadiagonal Toeplitz matrix. *Journal of Computational and Applied Mathematics* **278,** 12–18.

[26] Zhao, X -L. and Huang, T -Z. (2008). On the inverse of a general pentadiagonal matrix. *Applied Mathematics and Computation* **202,** 639–646.