SPECTRAL ANALYSIS OF CHAOTIC TRANSFORMATIONS

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ABSTRACT

The purpose of this paper is to show explicitly the spectral distribution function of some stationary stochastic processes as

$$Z_t = X_t + \xi_t = \phi(F_\theta(X_{t-1})) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z},$$

where ϕ is a given continuous function, F_θ is a deterministic invertible map with parameter $\theta \in \Theta \subseteq \mathbf{R}^n$ and $\{\xi_t\}_{t\in\mathbf{Z}}$ is a noise process.

We present several examples of transformations F_θ and ϕ and for each one we analyze spectral properties for the above process. One of the examples considered here generalizes the classical harmonic model

$$Z_t = A\cos(\omega_0\, t + \psi) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z}.$$

The harmonic model is the motivation for this work.

1. INTRODUCTION

We will consider the parametric analysis of several examples of time series determined by deterministic systems given by chaotic bijective transformations.

When $F : [0,1] \to [0,1]$ is given by $F(\psi) = \omega_0 + \psi \pmod{2\pi}$, then the classical harmonic model

$$Z_t = A\cos(\omega_0\, t + \psi) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z},$$

can alternatively be given by

$$Z_t = A\cos(F^t(\psi)) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z},$$

where $\{\xi_t\}_{t\in\mathbf{Z}}$ is a white noise process.

We want to analyze time series obtained from stochastic processes as

$$Z_t = (\phi \circ F)(X_{t-1}) + \xi_t = \phi(F^t(X_0)) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z},$$

where $\{\xi_t\}_{t\in\mathbf{Z}}$ is a white noise process, ϕ is a random variable and F is an invertible transformation on $\mathbf{R^n}$.

We will consider the noise process $\{\xi_t\}_{t\in\mathbf{Z}}$ independent of the signal process $\{\phi(F^t(X_0))\}_{t\in\mathbf{Z}}$. Therefore, for practical purposes we can omit it. One can obtain the spectral density function of $Z_t = \phi(F^t(X_0)) + \xi_t$ from the spectral density function of $X_t = \phi(F^t(X_0))$.

We shall show that of the periodogram is a good estimator for a large class of transformations (see Section 4) and the explicit expression of the spectral density function in several examples.

The parameter θ can be estimated by the method of moments and this is analyzed in Lopes and Lopes (1995).

2. STATIONARY STOCHASTIC PROCESSES

The general setting of chaotic time series we shall analyze is the following. Consider K a compact subset of $\mathbf{R}^n$ with a given Borel σ-algebra $\mathcal{F}$, a bijective continuous transformation $F : K \to K$ (or F_θ), an invariant probability $\mathcal{P}$ on K (that is, $\mathcal{P}(F^{-1}(A)) = \mathcal{P}(A)$, for any set $A \in \mathcal{F}$) and $\phi : K \to \mathbf{R}$ a continuous function. We will analyze the stationary stochastic process $\{Z_t\}_{t\in\mathbf{Z}}$ given by

$$Z_t = X_t + \xi_t = (\phi \circ F)(X_{t-1}) + \xi_t, \quad \text{for } t \in \mathbf{Z}. \tag{2.1}$$

The natural measure on $K^{\mathbf{Z}}$ is the product measure on $K^{\mathbf{Z}}$ and it is invariant for the stationary process $\{X_t\}_{t\in\mathbf{Z}}$ or $\{Z_t\}_{t\in\mathbf{Z}}$. The process $\{\xi_t\}_{t\in\mathbf{Z}}$ is considered to be a Gaussian white noise process (see Brockwell and Davis (1987)) independent of $\{(\phi\circ F)(X_t)\}_{t\in\mathbf{Z}}$, with zero mean and variance σ_ξ^2. One observes that in the model (2.1) the random variables X_t (or Z_t) and X_{t+1} (or Z_{t+1}) are generally not independent.

We shall denote the above system by $(K, F, \mathcal{P}, \phi, \mathcal{F}, \sigma_\xi^2)$. Following the terminology in Tong (1990) we may call the system (2.1), when $\sigma_\xi^2 = 0$, the *skeleton* of the system.

Given a certain measurable function $\phi : K \to \mathbf{R}$ *the autocovariance function at lag* $h \in \mathbf{Z}$ (see Brockwell and Davis (1987)) *of the process* $\{X_t\}_{t\in\mathbf{Z}}$ as in (2.1) is given by

$$R_{XX}(h) = E(X_t X_{t+h}) - [E(X_t)]^2 = \int \phi(x)\phi(F^h(x))d\mathcal{P}(x) - \left[\int \phi(x)d\mathcal{P}(x)\right]^2. \tag{2.2}$$

The autocovariance function $R_{XX}(h)$ in (2.2) measures the covariance between two values of the process $\{X_t\}_{t\in\mathbf{Z}}$ separated by lag h. The *autocorrelation function at lag h of the process* $\{X_t\}_{t\in\mathbf{Z}}$ (see Brockwell and Davis (1987)) is given by

$$\rho_X(h) = \frac{R_{XX}(h)}{R_{XX}(0)}, \quad \text{for} \quad h \in \mathbf{Z}, \tag{2.3}$$

where $R_{XX}(0) = E(X_t^2) - [E(X_t)]^2 = \mathrm{Var}(X_t)$ is the variance of the process.

The reason to consider F a bijective map and not just a map is for defining $R_{XX}(h)$ also for negative values of $h \in \mathbf{Z}$.

From the Herglotz's theorem (see Brockwell and Davis (1987)) a function $\rho_X(h)$ is non-negative definite if and only if

$$\rho_X(h) = \int_{-\pi}^{\pi} e^{i\lambda h} dF_X(\lambda), \quad \text{for any} \quad h \in \mathbf{Z}, \tag{2.4}$$

where $F_X(\cdot)$ is a right-continuous, non-decreasing, bounded function on $[-\pi, \pi]$ with $F_X(-\pi) = 0$. The function $F_X(\cdot)$ is called *the spectral distribution function* of $\{X_t\}_{t\in\mathbf{Z}}$ and if

$$F_X(\lambda) = \int_{-\pi}^{\lambda} f_X(\omega)\, d\omega, \quad \text{for } -\pi \leq \lambda \leq \pi, \tag{2.5}$$

then $f_X(\cdot)$ is called *the spectral density function* of the process $\{X_t\}_{t\in\mathbf{Z}}$. When

$$\sum_{h=-\infty}^{\infty} |\rho_X(h)| < \infty,$$

then $\rho_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda)\, d\lambda$, for $h \in \mathbf{Z}$, where $f_X(\cdot)$ is given by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \rho_X(h). \tag{2.6}$$

This function has real values if $\rho_X(h) = \rho_X(-h)$, for all $h \in \mathbf{N}$.

Each particular invertible transformation F will require a different technique in order to obtain explicitly the spectral distribution function.

Example: When the compact subset K is equal to $[-\pi, \pi]$, the transformation F is given by $F(x) = \omega_0 + x \pmod{2\pi}$, with $\omega_0 \in (0, \pi)$, and $\phi(x) = \cos(x)$ (this is the classical harmonic model), the spectral distribution function of the process $\{X_t\}_{t\in\mathbf{Z}} = \{(\phi \circ F)(X_{t-1})\}_{t\in\mathbf{Z}}$ as in (2.1) is not a function but a *generalized spectral distribution function* exists and it is given by

$$dF_X(\lambda) = \frac{1}{2}(\delta_{\omega_0} + \delta_{-\omega_0}), \tag{2.7}$$

where δ_{ω_0} is the Dirac delta function concentrated at ω_0.

Remark: Expanding maps (see Section 3 for the definition) always have an exponential decay of autocorrelations, for any ϕ Holder continuous function (see Parry and Pollicott (1990)). Therefore, in this case (see Examples 1, 3 and 4), the spectral density function always exists and it is an analytic function. The function F of Example 1 in Section 5 is an expanding map but the one of Example 2 in Section 6 is not.

3. THE NATURAL EXTENSION F_θ OF T_θ

It is well known that in general larger the dimension of the set K, more difficult is to analyze the dynamics of the map F_θ.

When K is one-dimensional, that is, when K is a segment, the diffeomorphism $F_\theta : K \to K$ has a simple dynamics. When F_θ is linear (mod 1) then one obtains the harmonic model by taking $\phi(x) = \cos(x)$.

In general the dynamics of an one-dimensional diffeomorphism is too simple (see Section 6 for a more difficult case).

The simplest example in two dimensions, that is, when K is a square $[0,1] \times [0,1]$, is obtained when F_θ is the natural extension of an one dimensional map T_θ. The map T_θ is not an one-to-one map, but F_θ is.

When the transformation T_θ is an *expanding map* (see Examples 1, 3 and 4), that is, there exists $\lambda > 1$ such that $|T'_\theta(x)| > \lambda$, for all $x \in [0,1]$, then there exists (see Lasota and Yorke (1973)) a density $g(x)$ such that $d\mu(x) = g(x)\,dx$ is invariant for T_θ (that is, $\mu(T_\theta^{-1}(A)) = \mu(A)$, for any Borel set A). The probability μ is ergodic (see Parry and Pollicott (1990) for the definition) for such map T_θ. There exists a natural way to obtain from such T_θ a bijective map F_θ, called *the natural extension of* T_θ. Denote by (x,y) a vector in the domain K and by $(x',y') = F_\theta(x,y)$ its image by the map F_θ. Then, (see Bogomolny and Carioli (1995))

$$T_\theta(x) = x' \quad \text{and} \quad T_\theta(y') = y$$

defines F_θ.

The invariant probability μ for T_θ on $[0,1]$ has a natural extension to a probability ν on $K = [0,1] \times [0,1]$ invariant for F_θ.

When T is an expanding map, the transformation F is Axiom A (see Robinson (1995) for definitions).

Consider now the random variable $\phi : K \to \mathbf{R}$ of the form $\phi(x,y) = \phi(x)$. Then, the time series

$$X_t = \phi(F_\theta^t(x,y)) = \phi(T_\theta^t(x)), \quad \text{for} \quad 1 \le t \le N,$$

and the probability ν define the simplest example of a chaotic time series.

The dynamics comes basically from an one-dimensional map even if the setting is for a two-dimensional bijective map. As we mentioned before the reason to consider bijective maps is to obtain $R_{XX}(h)$, for $h \in \mathbf{Z}$.

For a certain class of such maps (see Examples 1,3 and 4) we shall be able to show explicitly the spectral density function. We call a stochastic process obtained from the system (F_θ, ϕ) as above *a standard stochastic process* obtained from (T_θ, ϕ).

The spectral density functions of maps T are important for the spectral analysis of chaotic time series and also because the zeta function associated with the potential $-\log T'(x)$ has poles on the same values of the poles of the spectral density function (see Ruelle (1987) and Rugh (1992)).

4. THE PERIODOGRAM IS A GOOD ESTIMATOR

We analyze in this section the periodogram for (ϕ, T_θ) (or for (ϕ, F_θ)) when T_θ defines a standard time series. Our purpose here is to show how to obtain an approximation of the spectral density $f_X(\lambda)$ from a time series data $X_t = \phi(T_\theta^t(X_0))$, for $1 \le t \le N$, (that is, when $\phi(x,y) = \phi(x)$), where X_0 is chosen at random according to the measure μ (or according to the Lebesgue measure).

One can say from the reasoning below that in this case the periodogram is a good estimator in the sense of generalized functions (see Rudin (1986)). Suppose, for the sake of simplicity, that $E(X_t) = 0$. We can alternatively estimate

$$f_X(\lambda)\, 2\pi\, Var(X_0) = \sum_{h=-\infty}^{\infty} E(X_0 X_h)\, \exp(-ih\lambda),$$

with $X_h = \phi(F_\theta^h(x,y))$ and from this result estimate the spectral density $f_X(\lambda)$. By abuse of the notation we shall also call the above expression as the spectral density function.

We will not present a rigorous proof of the facts we consider in this section, we just want to explain the procedure to obtain the periodogram. The formal proof of the reasoning in this section will appear in a forthcoming paper.

Notice that as the random variable $\phi(x,y)$ depends only on x (for positive t, $\phi(F_\theta^t(x,y)) = \phi(T_\theta^t(x))$ independently of y) we shall consider the periodogram for T_θ instead of F_θ.

Consider the transformation $T_\theta : [0,1] \to [0,1]$, where $\theta \in \Theta \subseteq \mathbf{R}^n$, an expanding map.

We shall assume that ϕ is the random variable $\phi(x,y) = \phi(x)$ and $d\mu(x) = g(x)dx$ is the unique ergodic and absolutely continuous invariant probability for T_θ.

The goal here is to sketch the proof of the smoothed periodogram's consistency (in the sense of generalized functions) for the above setting. One denotes X_t by $(\phi \circ T_\theta^t)(X_0) = \phi(F_\theta^t(X_0, Y_0))$, and $\{X_t\}_{t=1}^N$ is a time series of N observations where (X_0, Y_0) is an initial point chosen randomly according to μ. From the Birkhoff's Ergodic Theorem (μ is ergodic for T_θ), for each subinterval $\Delta_j = (a_j, b_j) \subset [0,1]$ and for μ-almost every $x_0 \in [0,1]$

$$\mu(\Delta_j) = \int_{\Delta_j} g(x)dx = \lim_{N\to\infty} \frac{1}{N}(\#\{t \mid 1 \le t \le N, T_\theta^t(x_0) \in \Delta_j\}).$$

If $|b_j - a_j| = \epsilon$ is small and N is large enough, then

$$A_N(\epsilon) = \frac{1}{N}(\#\{t \mid 1 \le t \le N, T_\theta^t(X_0) \in \Delta_j\}) \approx g(c_j)\Delta_j = B_N(\epsilon), \qquad (4.1)$$

for some $c_j = c_j(N) \in \Delta_j$.

The expression $A_N(\epsilon) \approx B_N(\epsilon)$ means that the quotient $A_N(\epsilon)/B_N(\epsilon)$ goes to one when N goes to infinity and ϵ goes to zero.

Consider the discrete Fourier transform of the spatial position of the data obtained as the sampled time series $X_t = \phi(T_\theta^t(X_0))$, for $1 \leq t \leq N$,

$$f(k) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} X_t \exp(-i\omega_k t),$$

where $\omega_k = 2\pi k N^{-1}, k = 1, 2, \cdots, N$, are the so-called *Fourier frequencies* of the time series X_t, $1 \leq t \leq N$. The periodogram value $I(\omega_k)$ at the frequency ω_k, for

$$k \in \left\{ j \in \mathbf{Z}; 0 < \omega_j = \frac{2\pi j}{N} \leq 2\pi \right\},$$

is defined in terms of the discrete Fourier transform $f(k)$ of a sample X_t, for $1 \leq t \leq N$, by

$$I(\omega_k) = f(k)\overline{f(k)} = \frac{1}{N} \sum_{t=1}^{N} X_t \exp(-i\omega_k t) \sum_{s=1}^{N} X_s \exp(i\omega_k s) =$$
$$= \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} X_t X_s \exp(-i(t-s)\omega_k),$$

where $\overline{z}$ denotes the complex conjugate of z.

For each $h \in \mathbf{Z}$ consider t and s such that $t - s = h$. Then,

$$I(\omega_k) = \frac{1}{N} \left(\sum_{h=0}^{N-1} \sum_{s=1}^{N-h} X_s X_{s+h} \exp(-ih\omega_k) + \sum_{h=-1}^{1-N} \sum_{s=-h}^{N} X_s X_{s+h} \exp(-ih\omega_k) \right) =$$

$$= \frac{1}{N} \sum_{h=0}^{N-1} \sum_{s=1}^{N-h} X_s \phi(F_\theta^h(X_s, Y_s)) \exp(-ih\omega_k) +$$
$$+ \frac{1}{N} \sum_{h=-1}^{1-N} \sum_{s=-h}^{N} X_s \phi(F_\theta^h(X_s, Y_s)) \exp(-ih\omega_k). \tag{4.2}$$

Now if we take Δ_j, $1 \leq j \leq v$, as a partition by intervals (of the same size) of the interval [0,1], with $|\Delta_j| = \epsilon = 1/v$ small, one observes from (4.1) that

$$\frac{\#[X_j \in \Delta_j]}{N} \approx \Delta_j \, g(c_j),$$

where $c_j \in \Delta_j$, $1 \leq j \leq v$.

We shall sum up $X_s = \phi(T_\theta^s(X_0)) = \phi(F_\theta^s(X_0, Y_0))$ according to its position in each Δ_j. Hence,

$$\Delta_j \, g(c_j) N \approx \#\{s \; |1 \leq s \leq N, X_s \in \Delta_j\}.$$

Then, from (4.2)

$$I(\omega_k) \approx \frac{1}{N} \sum_{|h|<N} \sum_{j=1}^{v} \phi(c_j, y_j)\, \phi(F_\theta^h(c_j, y_j))(\Delta_j\, g(c_j)N) \exp(-ih\omega_k) =$$

$$= \sum_{|h|<N} \sum_{j=1}^{v} \phi(c_j, y_j)\, \phi(F_\theta^h(c_j, y_j))\, g(c_j)\, \Delta_j\, \exp(-ih\omega_k). \tag{4.3}$$

We shall show that for any X_0 chosen at random, then $\sum_{k=1}^{N} I(\omega_k) \frac{1}{N} \delta_{\omega_k}$ converges in the distribution sense to the spectral density function

$$\frac{1}{2\pi} \sum_{h\in\mathbf{Z}} E(X_0 X_h)\, \exp(-ih\lambda),$$

where δ_{ω_k} is the Dirac delta function concentrated at the frequency ω_k, $1 \leq k \leq N$. Hence, we will show that for any test function $z(\lambda)$, $\lambda \in [0, 2\pi)$,

$$\int_0^{2\pi} z(\lambda) d \left(\sum_{k=1}^{N} I(\omega_k) \frac{1}{N} \delta_{\omega_k} \right)$$

converges to

$$\int_0^{2\pi} z(\lambda) \left(\sum_{h\in\mathbf{Z}} E(X_0 X_h)\, \exp(-ih\lambda) \right) d\lambda$$

when N goes to infinity.

By integrating the smoothed periodogram against a test function $z(\lambda)$, $\lambda \in [0, 2\pi)$, and by using (4.3)

$$\lim_{v\to\infty} \lim_{N\to\infty} \sum_{k=1}^{N} I(\omega_k)\, \frac{1}{N}\, z\left(\frac{2\pi k}{N}\right) =$$

$$= \lim_{v\to\infty} \lim_{N\to\infty} \sum_{k=1}^{N} \left(\sum_{|h|<N} \sum_{j=1}^{v} \phi(c_j, y_j) \phi(F_\theta^h(c_j, y_j)) g(c_j) \Delta_j\, \exp(-ih\frac{2\pi k}{N}) \right) \frac{1}{N}\, z\left(\frac{2\pi k}{N}\right) =$$

$$= \int_0^{2\pi} \left[\sum_{h\in\mathbf{Z}} \left(\int_0^{2\pi} \phi(x, y)\, \phi(F_\theta^h(x, y))\, g(x) dx \right) \exp(-ih\lambda) \right] z(\lambda) d\lambda =$$

$$= \int_0^{2\pi} \left(\sum_{h\in\mathbf{Z}} E(X_0 X_h) \exp(-ih\lambda) \right) z(\lambda) d\lambda. \tag{4.4}$$

Therefore, the smoothed periodogram converges in distribution sense to the spectral density function.

The property considered above in (4.4) describes a method for obtaining a good approximation to the spectral density function. This method will be explained below.

Consider $z(\lambda) = I_{[x-\epsilon,x+\epsilon]}(\lambda)$ for a fixed x and a small fixed ϵ.

From the reasoning described before, for such $z(\lambda)$, $(2\epsilon)^{-1} \sum_{k=1}^{N} I(\omega_k) \frac{1}{N} z(2\pi k/N)$ is approximately equal to

$$\frac{1}{2\pi} \sum_{h=1-N}^{N-1} E(X_0 X_h) \exp(-ih\lambda),$$

if N is large and ϵ small enough.

There will approximately exist $2\epsilon N/2\pi$ elements of the form $2\pi k/N$ in the interval $[x-\epsilon, x+\epsilon] \subset [0, 2\pi)$ if N is large. Therefore,

$$(2\epsilon)^{-1} \sum_{k=1}^{N} \frac{1}{N} I(\omega_k) z(2\pi k/N)$$

is approximately the mean value of $(2\pi)^{-1} I(w_k)$ in the interval $[x-\epsilon, x+\epsilon]$.

One can alternatively obtain the approximated value of $\sum_{h=1-N}^{N-1} E(X_0 X_h) \exp(-ihx)$ by taking directly the mean value of $I(w_k)$ in a small interval around x.

Considering now several $z_i(\lambda) = I_{[x_i-\epsilon,x_i+\epsilon]}(\lambda)$, where x_i are equally spaced,

$$[x_1-\epsilon, x_1+\epsilon) \cup [x_2-\epsilon, x_2+\epsilon) \cup .... \cup [x_n-\epsilon, x_n+\epsilon]$$

is a partition of $[0, 2\pi)$ and applying the same reasoning to each $z_i(\lambda)$, we obtain the approximated shape of the graph of

$$\sum_{h=1-N}^{N-1} E(X_0 X_h) \exp(-i\, h\lambda) \quad , \quad \lambda \in [0, 2\pi),$$

as a function of λ.

From the above expression, one can derive (see the expression (7.6)) the approximated graph of the spectral density $f_X(\lambda)$ or $f_Z(\lambda)$.

The proceeding just described above is called *smoothing the periodogram* (see Brockwell and Davis (1987)). For instance, if one takes a large sample $T_\theta^t(x_0)$, for $1 \leq t \leq 10,000$, the periodogram is given by

$$I(\omega_k) = N^{-1} \sum_{t=1}^{N} X_t \exp(-i\omega_k t) \sum_{s=1}^{N} X_s \exp(i\omega_k s) =$$

$$= N^{-1} \sum_{t=1}^{N} \sum_{s=1}^{N} X_s X_t \exp(-i(t-s)\omega_k)$$

and one can plot this real function in the interval $[0, 2\pi)$ as a function of ω_k. This graph will show a sparse amount of data, but if one takes a partition of the interval in small intervals and takes means of this data in each small interval (also called *smoothing the periodogram*), then the graph of a well defined spectral density function

$$\sum_{h=-\infty}^{\infty} E(X_0 X_h) \exp(-ih\lambda),$$

as described in this section, will be obtained.

5. EXAMPLE 1

Sakai and Tokumaru (1980) (see also Grossmann and Thomae (1977)) introduce the following model of chaotic time series. For a given constant $a \in (0,1)$, consider the transformation $T_a : [0,1] \to [0,1]$ given by

$$T_a(x) = \begin{cases} \dfrac{x}{a}, & \text{if} \quad 0 \leq x < a \\ \dfrac{1-x}{1-a}, & \text{if} \quad a \leq x \leq 1. \end{cases} \tag{5.1}$$

The Lebesgue measure dx is invariant and ergodic for the transformation T_a (see Lasota and Yorke (1973)). In the notation of Section 2, $\mathcal{P}(A)$ is the length of A, for any interval A.

We now consider the stochastic process

$$Z_t = X_t + \xi_t = T_a(X_{t-1}) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z}, \tag{5.2}$$

where $\phi(x) = x$ according to the notation of Section 2.

The autocovariance function at lag h of the process $\{X_t\}_{t\in\mathbf{Z}}$ in (5.2) (see Sakai and Tokumaru (1980)) is given by

$$R_{XX}(h) = \int_0^1 x T_a^h(x) dx - [E(X_t)]^2 = \frac{1}{12}(2a-1)^h, \quad \text{for} \quad h > 0, \tag{5.3}$$

where $E(X_t) = \frac{1}{2}$ and $R_{XX}(0) = \mathrm{Var}(X_t) = \frac{1}{12}$.

The main obstacle to proceed in the spectral analysis of Example 1 is that the map T_a is not invertible. Therefore, the autocovariance function $R_{XX}(h)$ of the process $\{X_t\}_{t\in\mathbf{Z}}$, given by expression (5.2), for negative lag h does not have a precise meaning.

We shall analyze the natural extension F_a of T_a, instead of T_a itself.

As a particular example, we mention that the Baker map is the natural extension of the tent map (with inclination 2).

In Example 1, the natural extension of T_a is the map $F_a : [0,1]\times[0,1] \to [0,1]\times[0,1]$ defined by

$$F_a(x,y) = (T_a(x), G_a(x,y)), \quad \text{for any} \quad (x,y) \in [0,1]\times[0,1], \tag{5.4}$$

where

$$G_a(x,y) = \begin{cases} ya, & \text{if} \quad 0 \le x < a \\ (a-1)y+1, & \text{if} \quad a \le x \le 1. \end{cases}$$

The map F_a is invertible and it is easy to see that the Lebesgue measure $dxdy$ is invariant and ergodic for F_a.

Therefore, we shall consider the dynamical system $(K, F_a, \mathcal{P})$ where $K = [0,1] \times [0,1]$ and $\mathcal{P}$ is the Lebesgue measure $dxdy$ on $[0,1] \times [0,1]$. Instead of $\phi(x) = x$, one can consider $\phi(x,y) = \Pi(x,y) = x$ for any $(x,y) \in [0,1] \times [0,1]$ as a random variable. In the setting of Section 2, we shall analyze in this section the system $(K, F_a, \mathcal{P}, \Pi, \mathcal{F}, \sigma_\xi^2)$. Now, if $h \ge 0$ then

$$\int_0^1 x T_a^h(x) dx = \int_0^1 \int_0^1 x \Pi(F_a^h(x,y)) dxdy = \int_0^1 \int_0^1 \Pi(x,y) \Pi(F_a^h(x,y)) dxdy$$

and we obtain, from the expression (5.3), $R_{XX}(h)$ for positive h when $X_t = \Pi \circ F_a^t$. As the map F_a is invertible, it makes sense to estimate, for $h > 0$, the integral

$$\int_0^1 \int_0^1 \Pi(x,y) \Pi(F_a^{-h}(x,y)) dxdy.$$

Now, as $dxdy$ is invariant for F_a^h, one obtains the following

$$\int_0^1 \int_0^1 \Pi(x,y) \Pi(F_a^{-h}(x,y)) dxdy = \int_0^1 \int_0^1 \Pi(F_a^h(x,y)) \Pi(x,y) dxdy = \int_0^1 T_a^h(x)\, x\, dx.$$

The example considered above defines a standard time series. After these results one can have the spectral density function associated with the stochastic process $\{X_t\}_{t \in \mathbf{Z}}$. The last term in the above equalities has already been calculated (see (5.3)).

Theorem 5.1: *The spectral density function of the stochastic process*

$$Z_t = X_t + \xi_t = (\Pi \circ F_a)(X_{t-1}) + \xi_t, \quad \textit{for } t \in \mathbf{Z},$$

where F_a is defined by the expression (5.4), is given by

$$f_Z(\lambda) = \frac{2a(1-a)}{\pi[1 - 2(2a-1)\cos(\lambda) + (2a-1)^2]} + \frac{\sigma_\xi^2}{2\pi}, \quad \textit{for } \lambda \in [0, 2\pi). \tag{5.5}$$

Proof: Since $R_{XX}(h)$ is given by the expression (5.3) and goes to zero exponentially when

$h \to +\infty$, the spectral density function (see (2.6)) does exist and it is given by

$$
\begin{aligned}
f_X(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \rho_X(h) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} (2a-1)^{|h|} = \\
&= \frac{1}{2\pi} \left[\sum_{h \geq 0} ((2a-1)e^{-i\lambda})^h + \sum_{h=-\infty}^{-1} ((2a-1)e^{i\lambda})^{-h} \right] = \\
&= \frac{1}{2\pi} \left[\frac{1}{1-(2a-1)e^{-i\lambda}} + \frac{(2a-1)e^{i\lambda}}{1-(2a-1)e^{i\lambda}} \right] = \\
&= \frac{2a(1-a)}{\pi[1-2(2a-1)\cos(\lambda)+(2a-1)^2]},
\end{aligned}
$$

for all $\lambda \in [0, 2\pi)$, since $|(2a-1)e^{\pm i\lambda}| < 1$ when $a \in (0,1)$. The spectral density function of the process $\{Z_t\}_{t \in \mathbf{Z}}$ follows from this.

The spectral density function of the signal process $\{X_t\}_{t \in \mathbf{Z}}$ is continuous and its graph is shown in Figure 1 (a), (b) and (c). Notice that if a is small then the function $f_X(\lambda)$ has a maximum on π and if a is large it has a maximum on zero.

We refer the reader to Lopes and Lopes (1995) for more details in the example considered in this section.

6. EXAMPLE 2

Consider the two parameters mapping family $\{F_{a,b} : [0,1] \to [0,1]; \quad a, b \in \mathbf{R}\}$ where $F_{a,b}$ is given by

$$
F_{a,b}(x) = \begin{cases} a + \dfrac{1-a}{b} x, & \text{if} \quad 0 \leq x < b \\ \dfrac{a}{1-b}(x-b), & \text{if} \quad b \leq x \leq 1, \end{cases} \tag{6.1}
$$

with a and b constants. This map is not an expanding one. Let α be the derivative of $F_{a,b}$ on $[0, b)$ and β its derivative on $[b, 1]$. Then,

$$
\alpha = F'_{a,b}(x) = \frac{1-a}{b}, \quad \text{if} \quad 0 \leq x < b \quad \text{and} \quad \beta = F'_{a,b}(x) = \frac{a}{1-b}, \quad \text{if} \quad b \leq x \leq 1. \tag{6.2}
$$

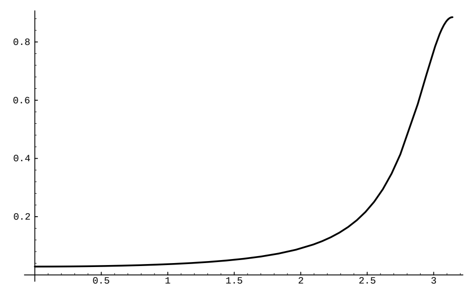
The ergodic properties of the family $\{F_{a,b} : [0,1] \to [0,1]; \quad a, b \in \mathbf{R}\}$ are analyzed in Coelho et al. (1995). This map does not define a standard time series.

In Example 2 we want to consider the spectral analysis of the process $\{X_t\}_{t \in \mathbf{Z}}$ defined in (6.3) below.

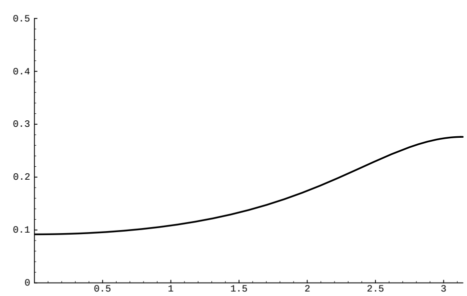
Notice that when $b = 1 - a$, the transformation $F_{a,b}$ of Example 2 is $F(x) = a + x$ (mod 1), which corresponds to the harmonic model analyzed by Lopes and Kedem (1994).

Figure 1: The spectral density function $f_X(\lambda)$, $0 \le \lambda \le \pi$, for Example 1 as in (5.4) when $\sigma_\xi^2 = 0$ and

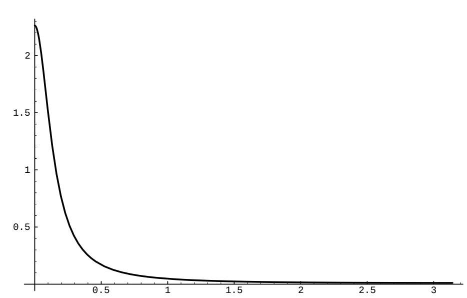
(a) $a = 0.15240$; **(b)** $a = 0.36570$; **(c)** $a = 0.93459$.



(a)



(b)



(c)

Therefore, the presented analysis of Example 2 is a generalization of that work when there exists only one frequency.

By using the notation introduced in Section 2, for a given transformation $F_{a,b}$ and $\phi(x) = x$ one considers the signal process $\{X_t\}_{t \in \mathbf{Z}}$ given by

$$X_t = F_{a,b}(X_{t-1}), \quad \text{for} \quad t \in \mathbf{Z}. \tag{6.3}$$

In the present example the spectral density function does not exist and the spectral density distribution has a quite different behavior compared to the Examples 1 and 3.

To consider the constants a and b is the same as to consider α and β, since one has the following identities

$$\alpha = \frac{1-a}{b} \quad \text{and} \quad \beta = \frac{a}{1-b} \Longleftrightarrow a = \frac{\beta(\alpha-1)}{\alpha-\beta} \quad \text{and} \quad b = \frac{1-\beta}{\alpha-\beta}. \tag{6.4}$$

Therefore, for the sake of simplicity, we shall consider the parameters α and β in our analysis.

The invariant measure $\mathcal{P}_{\alpha,\beta} = \mathcal{P}$ (see Coelho et al. (1995)) for the process $\{X_t\}_{t\in\mathbf{Z}}$, in terms of α and β, is given by the density

$$\varphi_{\alpha,\beta}(x) = \varphi(x) = \frac{1}{c}\frac{1}{x+\frac{\beta}{\alpha}(1-x)} = \frac{1}{c}\frac{\alpha}{(\alpha-\beta)x+\beta}, \tag{6.5}$$

where

$$c = \frac{1}{\beta-\alpha}\log\left(\frac{\beta}{\alpha}\right) = \frac{\alpha}{\frac{\beta}{\alpha}-1}\log\left(\frac{\beta}{\alpha}\right). \tag{6.6}$$

For a set $A \subset [0,1]\times[0,1]$, with Lebesgue measure equal to 1, for all $(\alpha,\beta)\in A$, the map $T_{\alpha,\beta}$ is ergodic for $\mathcal{P}_{\alpha,\beta} = \mathcal{P}$. We will assume $(\alpha,\beta)\in A$ in the sequel.

In other words, in this case $\mathcal{P}$ given by

$$\mathcal{P}(A) = \int_A \varphi(x)dx, \quad \text{for all } A \in \mathcal{F},$$

where now $\mathcal{F}$ is the Borel σ-algebra in $[0,1]$, defines an invariant ergodic probability measure for $F_{\alpha,\beta}$.

From the expressions (6.1) and (6.4) the transformation $F_{\alpha,\beta}$ is given by

$$F_{\alpha,\beta}(x) = \begin{cases} \dfrac{\beta(\alpha-1)}{\alpha-\beta} + \alpha x, & \text{if} \quad 0 \le x < \dfrac{1-\beta}{\alpha-\beta} \\ \beta\left(x - \dfrac{1-\beta}{\alpha-\beta}\right), & \text{if} \quad \dfrac{1-\beta}{\alpha-\beta} \le x \le 1. \end{cases} \tag{6.7}$$

The list of integrals below are useful to understand the spectral analysis that we shall present in the sequel.

1. $\displaystyle\int_0^y \varphi(x)dx = \frac{\log(\frac{(\alpha-\beta)y+\beta}{\beta})}{\log(\frac{\alpha}{\beta})}.$

2. $\displaystyle E(Z_t) = E(X_t) = \int_0^1 x\varphi(x)dx = \frac{1}{\log(\frac{\alpha}{\beta})} - \frac{\beta}{\alpha-\beta}.$

3. $\displaystyle E(Z_t^2) = E(X_t^2) + \sigma_\xi^2 = \int_0^1 x^2\varphi(x)dx + \sigma_\xi^2 = \left(\frac{\beta}{\alpha-\beta}\right)^2 + \frac{\alpha-3\beta}{2(\alpha-\beta)\log(\frac{\alpha}{\beta})} + \sigma_\xi^2.$

4. $\displaystyle E(Z_tZ_{t+1}) = E(X_tX_{t+1}) = \int_0^1 xF_{\alpha,\beta}(x)\varphi(x)dx = \left(\frac{\beta}{\alpha-\beta}\right)^2 + \frac{1+\alpha\beta-4\beta}{2(\alpha-\beta)\log(\frac{\alpha}{\beta})}.$

(6.8)

Some of these integrals are obtain after long calculations.

For a given $F = F_{\alpha,\beta}$ and the corresponding invariant density $\varphi = \varphi_{\alpha,\beta}$ we consider the signal process $\{X_t\}_{t\in\mathbf{Z}} = \{(\phi \circ F_{\alpha,\beta})(X_{t-1})\}_{t\in\mathbf{Z}}$.

From the expressions (6.5) and (6.6) one observes that the density function $\varphi_{\alpha,\beta}(x)$ depends only on the quotient $\Delta = \frac{\alpha}{\beta}$. Consider now the transformation F^h, for any $h \in \mathbf{Z}$, where $F = F_{\alpha,\beta}$ is given by the expression (6.7). From Coelho et al. (1995) it is known that

$$F^h(x) = F_{\alpha_h,\beta_h}(x) \quad \text{where} \quad \alpha_h = \frac{b_h}{1-a_h} \quad \text{and} \quad \beta_h = \frac{a_h}{1-b_h}$$

with $a_h = F^h(0)$ and $b_h = F^{-h}(0)$. From Coelho et al. (1995) it is also known that

$$\frac{\alpha_h}{\beta_h} = \frac{\alpha}{\beta}, \quad \text{for any} \quad h \in \mathbf{N},$$

and hence

$$\varphi_{\alpha_h,\beta_h} = \varphi_{\alpha,\beta}, \quad \text{for any} \quad h \in \mathbf{N}.$$

The conclusion is that, for any continuous function ϕ and $h \in \mathbf{N}$,

$$E(X_tX_{t+h}) = \int \phi(x)\phi(F^h(x))\varphi(x)dx = \int \phi(x)\phi(F_{\alpha_h,\beta_h}(x))\varphi_{\alpha,\beta}(x)dx =$$
$$= \int \phi(x)\phi(F_{\alpha_h,\beta_h}(x))\varphi_{\alpha_h,\beta_h}(x)dx. \qquad (6.9)$$

As we know $\int \phi(x)\phi(F_{\alpha,\beta}(x))\varphi_{\alpha,\beta}(x)dx$ (see integral 4. in (6.8)), for any α and β, one can calculate $\int \phi(x)\phi(F_{\alpha_h,\beta_h}(x))\varphi_{\alpha_h,\beta_h}(x)dx$, for any $h \in \mathbf{N}$.

Notice that $E(X_tX_{t+h}) = E(X_tX_{t-h})$, for all $h \in \mathbf{N}$.

Therefore, we are able to obtain the exact values of $R_{XX}(h)$, for all $h \in \mathbf{Z}$, from the positive and negative orbit of zero by F (since α_h and β_h depend only on a_h and b_h).

We now consider $\phi(x) = x$. It is known (see Lopes and Lopes (1995)) that, for fixed α and β, there exists Δ such that $\alpha_h = \Delta\beta_h$, for all $h \in \mathbf{Z}$. From integral 4. in (6.8), a simple calculation shows that (see Lopes and Lopes (1995)) there exist $c_1(\Delta)$ and $c_2(\Delta)$ such that

$$\int_0^1 xF^h(x)\varphi(x)dx = c_1(\Delta) + c_2(\Delta)\left(\frac{1}{\beta_h} + \alpha_h\right).$$

As α_h and $\frac{1}{\beta_h}$ wander around the interval $[0,1]$, then the above integral does not converge to zero as $h \to \infty$. Therefore, the spectral density function is not a function, but there exists the spectral distribution function also called the *generalized spectral density function*.

First one observes that the process $\{X_t\}_{t\in\mathbf{Z}} = \{F_{\alpha,\beta}(X_{t-1})\}_{t\in\mathbf{Z}}$ has mathematical expectation given by the integral 2. in expression (6.8), that is,

$$E(X_t) = \frac{1}{\log\left(\frac{\alpha}{\beta}\right)} - \frac{\beta}{\alpha-\beta}, \quad \text{for all} \quad t \in \mathbf{Z}.$$

We want to derive the spectral distribution function of the process $\{Z_t\}_{t\in\mathbf{Z}}$. We first consider the autocorrelation $\rho_X(h)$ at lag h of the process $\{X_t\}_{t\in\mathbf{Z}} = \{F_{\alpha,\beta}(X_{t-1})\}_{t\in\mathbf{Z}}$ and then use the Herglotz's theorem (see (2.4)) for the process $\{X_t\}_{t\in\mathbf{Z}}$.

Remark: The Fourier coefficients of the spectral distribution function in the case where $F(x) = \omega_0 + x$ are given by $\rho_X(h) = \cos(h\omega_0) = \cos(F^h(0))$, for $h \in \mathbf{Z}$, that is, they are determined by the iterates F^h of zero. The next theorem claims a similar property for the transformation $F_{\alpha,\beta}$ and $\phi(x) = x$.

Theorem 6.1: *The spectral distribution function of the process*

$$Z_t = F^t_{\alpha,\beta}(\cdot) + \xi_t = F_{\alpha,\beta}(X_{t-1}) + \xi_t, \quad \textit{for } t \in \mathbf{Z},$$

where $F_{\alpha,\beta}$ is defined by the expression (6.7), is given by

$$dF_Z(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\rho_X(h) + \frac{\sigma_\xi^2}{2\pi}, \quad \textit{for } \lambda \in [0, 2\pi), \tag{6.10}$$

where $\rho_X(h)$ is given by $\frac{R_{XX}(h)}{R_{XX}(0)}$ (see the expression (2.3)) with

$$R_{XX}(h) = \frac{1+\alpha_h\beta_h}{2(\alpha_h-\beta_h)\log(\frac{\alpha_h}{\beta_h})} - \frac{1}{\left[\log(\frac{\alpha_h}{\beta_h})\right]^2} \tag{6.11}$$

and

$$R_{XX}(0) = \frac{\alpha+\beta}{2(\alpha-\beta)\log(\frac{\alpha}{\beta})} - \frac{1}{\left[\log(\frac{\alpha}{\beta})\right]^2}, \tag{6.12}$$

where α and β are given by the expression (6.2) and

$$\alpha_h = \frac{1-a_h}{b_h}, \quad \beta_h = \frac{a_h}{1-b_h}, \quad a_h = F^h(0) \quad \textit{and} \quad b_h = F^{-h}(0).$$

Now we consider $\phi(x) = \cos(2\pi x)$. One wants to calculate the spectral distribution of the process

$$Z_t = X_t + \xi_t = \cos(2\pi F_{\alpha,\beta}(X_{t-1})) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z}.$$

For this purpose we need the following integral:

$$E(X_t X_{t+1}) = \int_0^1 \cos(2\pi x)\cos(2\pi F(x))\varphi(x)dx = \frac{1}{2\log\left(\frac{\alpha}{\beta}\right)} \times k, \tag{6.13}$$

where

$$\begin{aligned} k = {} & \cos(2d\beta)[ci(d(\alpha+1)) + ci(d\alpha(\beta+1)) - ci(d\beta(\alpha+1)) - ci(d(\beta+1))] + \\ & + \sin(2d\beta)[si(d(\alpha+1)) + si(d\alpha(\beta+1)) - si(d\beta(\alpha+1)) - si(d(\beta+1))] + \\ & + ci(d(\alpha-1)) + ci(d\alpha(\beta-1)) - ci(d(\beta-1)) - ci(d\beta(\alpha-1)), \end{aligned}$$

with $d = \frac{2\pi}{\alpha-\beta}$, $ci(x)$ is the cosine integral and $si(x)$ is the sine integral (see Gradshteyn and Ryzhik (1965), page 928). The integral (6.13) comes after a long calculation.

In order to calculate the spectral distribution function, one should obtain the Fourier coefficients of such distribution by substituting in (6.13) the values of α and β by α_h and β_h (see expression (6.9)).

Theorem 6.2: *The spectral distribution function of the process*

$$Z_t = F^t_{\alpha,\beta}(\cdot) + \xi_t = \cos(2\pi\, F_{\alpha,\beta}(X_{t-1})) + \xi_t, \quad \textit{for } t \in \mathbf{Z},$$

where $F_{\alpha,\beta}$ is defined by the expression (6.7), is given by

$$dF_Z(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\rho_X(h) + \frac{\sigma_\xi^2}{2\pi}, \quad \textit{for } \lambda \in [0, 2\pi), \tag{6.14}$$

where $\rho_X(h)$ is given by $\frac{R_{XX}(h)}{R_{XX}(0)}$ (see the expression (2.3)) with

$$R_{XX}(h) = \frac{1}{2\log(\frac{\alpha_h}{\beta_h})} \times k_h - \frac{1}{[\log(\frac{\alpha_h}{\beta_h})]^2} \times l_h$$

where

$$\begin{aligned} k_h = {} & \cos(2d_h\beta_h)[ci(d_h(\alpha_h+1)) + ci(d_h\alpha_h(\beta_h+1)) - ci(d_h\beta_h(\alpha_h+1)) - ci(d_h(\beta_h+1))] \\ & + \sin(2d_h\beta_h)[si(d_h(\alpha_h+1)) + si(d_h\alpha_h(\beta_h+1)) - si(d_h\beta_h(\alpha_h+1)) - si(d_h(\beta_h+1))] \\ & + ci(d_h(\alpha_h-1)) + ci(d_h\alpha_h(\beta_h-1)) - ci(d_h(\beta_h-1)) - ci(d_h\beta_h(\alpha_h-1)), \end{aligned}$$

and

$$l_h = \{\cos(d_h\beta_h)[ci(d_h\alpha_h) - ci(d_h\beta_h)] + \sin(d_h\beta_h)[si(d_h\alpha_h) - si(d_h\beta_h)]\}^2$$

with

$$d_h = \frac{2\pi}{\alpha_h - \beta_h}, \quad \alpha_h = \frac{1-a_h}{b_h}, \quad \beta_h = \frac{a_h}{1-b_h},$$

$a_h = F^h(0)$ *and* $b_h = F^{-h}(0)$. *The variance of* X_t *is given by*

$$R_{XX}(0) = \frac{1}{2\log(\frac{\alpha}{\beta})}\{\cos(2d\beta)[ci(2d\alpha) - ci(2d\beta)] + \sin(2d\beta)[si(2d\alpha) - si(2d\beta)]\} + \frac{1}{2} - \\ - \frac{1}{[\log(\frac{\alpha}{\beta})]^2} \times l,$$

where

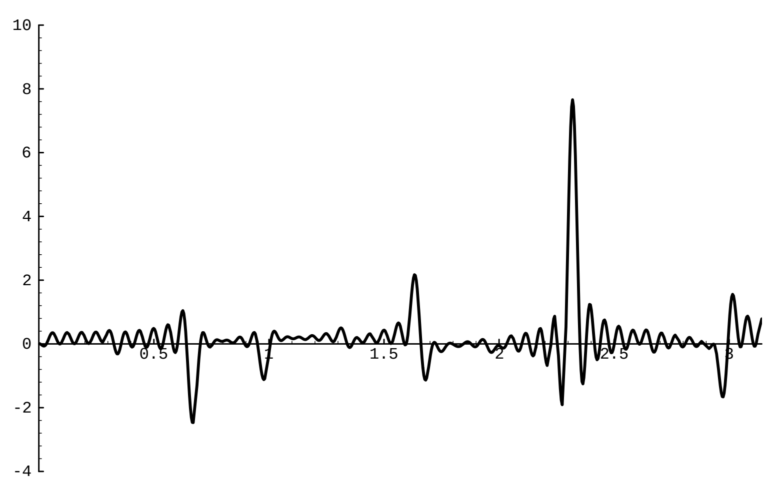
$$l = \{\cos(d\beta)[ci(d\alpha) - ci(d\beta)] + \sin(d\beta)[si(d\alpha) - si(d\beta)]\}^2$$

with

$$d = \frac{2\pi}{\alpha - \beta}, \quad \alpha = \frac{1-a}{b} \quad \textit{and} \quad \beta = \frac{a}{1-b}.$$

In Figure 2 we plot the graph of the Fourier series $\frac{1}{2\pi}\sum_{h=-100}^{100} e^{-i\lambda h}\rho_X(h)$ when $\alpha = 2.41809$ and $\beta = 0.22052$. Therefore, we are considering here an approximation of the generalized spectral density function $f_X(\lambda)$ up to an order of 100.

Figure 2: The generalized spectral density function $f_X(\lambda)$, $0 \le \lambda \le \pi$, for Example 2 as in (6.14) when $\sigma_\xi^2 = 0$, $\alpha = 2.41809$ and $\beta = 0.22052$.



Remark: The rotation number (see Devaney (1989)) of $F_{\alpha,\beta}$ is

$$\theta_1 = \frac{\log(\alpha)}{\log(\frac{\alpha}{\beta})}$$

and the rotation number of $F_{\tilde{\alpha},\tilde{\beta}} = F_{\alpha,\beta}^{-1}$ is

$$\theta_2 = \frac{\log(\beta)}{\log(\frac{\beta}{\alpha})}.$$

One observes that $\theta_1 + \theta_2 = 1$. We denote by ζ the smallest value between θ_1 and θ_2. Therefore, $\zeta \leq 0.5$. We call ζ *the rotation number of the stochastic process.*

It is extremely interesting the fact that, for any α and β, the spectral measure is not a Dirac delta function concentrated on the rotation number of $F_{\alpha,\beta}$ (we checked the coefficients $\rho_X(h)$) but it has a very strong peak on the value $2\pi\zeta$ where ζ is the rotation number of the process. In other words, the spectral distribution is very close to

$$\frac{1}{2}(\delta_{2\pi\,\zeta} + \delta_{-2\pi\,\zeta}) = \frac{1}{2}(\delta_{2\pi\,\theta_1} + \delta_{-2\pi\,\theta_1}),$$

where $\theta_1 \leq 0.5 \leq \theta_2$ were defined above.

In conclusion, if one applies the Fourier transform to the data it will appear a strong peak in the rotation number.

This property requires, in the future, a deeper analysis in order to understand the spectral distribution function given by (6.14). Notice in Figure 2 the strong peak in the value $2\pi\,\zeta = 2.31671$, where ζ is the rotation number of the process when $\alpha = 2.41809$ and $\beta = 0.22052$ (corresponding to the values of $a = 0.1423$ and $b = 0.3547$).

We remind the reader that if $a = 1 - b$ then the rotation number of $F_{\alpha,\beta}$ is equal to a and, in fact, in this case, the spectral distribution function is a Dirac delta function $\frac{1}{2}(\delta_{\pi a} + \delta_{-\pi a})$, when $\phi(x) = \cos(2\pi\, x)$.

Notice that for $F_{\alpha,\beta}(x) = a + x \pmod 1$, the inverse map $F_{\alpha,\beta}^{-1} = F_{\tilde{\alpha},\tilde{\beta}}$ is such that $F_{\tilde{\alpha},\tilde{\beta}}(x) = x - a \pmod 1$. In this case, $\zeta = \pi|a|$.

We refer the reader to Lopes and Lopes (1995) for more details about the example considered in this section.

7. EXAMPLE 3

We shall present a complete spectral analysis of the stationary stochastic process

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0, Y_0)) + \xi_t = \phi(F_\alpha(X_{t-1}, Y_{t-1}) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z}, \tag{7.1}$$

where $\phi(x,y) = x$ is a random variable, $\{\xi_t\}_{t\in\mathbf{Z}}$ is a Gaussian white noise process, F_α is a transformation defined below and (X_0, Y_0) is an initial point chosen at random according to the measure ν also defined below.

The map F_α is defined from $K = ([0,1] \times (0,\alpha)) \cup ([0,\alpha] \times [\alpha, 1])$ to itself and it is given by $F_\alpha(x,y) = (T_\alpha(x), G_\alpha(x,y))$ where the transformation $T_\alpha : [0,1] \to [0,1]$ has definition

$$T_\alpha(x) = \begin{cases} \dfrac{x}{\alpha}, & \text{if} \quad 0 \leq x < \alpha \\ \dfrac{\alpha(x-\alpha)}{1-\alpha}, & \text{if} \quad \alpha \leq x \leq 1, \end{cases} \tag{7.2}$$

with $\alpha \in (0,1)$ as a constant, and

$$G_\alpha(x,y) = \begin{cases} \alpha\, y, & \text{if} \quad 0 \le x < \alpha \\ \alpha + \left(\dfrac{1-\alpha}{\alpha}\right) y, & \text{if} \quad \alpha \le x \le 1. \end{cases} \tag{7.3}$$

The graph of the map T_α is shown in Figure 3. The action of the piecewise diffeomorphism F_α is presented in Figure 4. The transformation F_α is a modification of the well known Baker transformation. It defines a standard time series.

The map T_α describes a model for a particle that moves around in the interval [0,1]. If the particle is at position x, then after a unit of time it jumps to $T_\alpha(x)$ and so on. According to the model considered here suppose the spatial position of the particle is $T_\alpha^t(x) = X_t$, $t \in \mathbf{N}$, in the interval [0,1]. If the particle X_t is in the interval $[0,\alpha)$, it has a uniformly spread possibility to jump to any point X_{t+1} in [0,1]. However, if it is in the interval $[\alpha,1)$ it has a uniformly spread possibility to jump to any point X_{t+1} in the interval $[0,\alpha)$.

We are primarily interested in the expanding map T_α, but for defining the spectral density we need a bijective map. Therefore, we have to consider F_α, *the natural extension of* T_α (as mentioned in Section 3).

The piecewise diffeomorphism F_α leaves invariant (see Lasota and Yorke (1973)) an ergodic probability ν on $K \subset \mathbf{R}^2$, absolutely continuous with respect to the Lebesgue measure, that will be described later.

Choosing a point (x_0, y_0) at random, according to the Lebesgue measure (or according to ν), the spectral properties of the process Z_t will be analyzed.

One observes that F_α is a piecewise homeomorphism of K and F_α^n is of the form

$$F_\alpha^n(x,y) = (T_\alpha^n(x), G_{\alpha,n}(x,y)),$$

that is, the action of F_α in the first variable is just the action of T_α.

Now we shall define the F_α-invariant measure ν on K, absolutely continuous with respect to the Lebesgue measure $dxdy$.

From Lopes, Lopes and Souza (1996) the transformation T_α has an invariant absolutely continuous measure $d\mu = g(x)\,dx$ where

$$g(x) = \begin{cases} \dfrac{1}{\alpha(2-\alpha)}, & \text{if} \quad 0 \le x < \alpha \\ \dfrac{1}{2-\alpha}, & \text{if} \quad \alpha \le x \le 1. \end{cases} \tag{7.4}$$

Consider in the sequel the following notation

$$c = \frac{1}{\alpha(2-\alpha)} \quad \text{and} \quad d = \frac{1}{2-\alpha}. \tag{7.5}$$

Now we shall define ν on subsets of K by using the μ above.

For sets of the form $A_1 \times A_2$, where $A_1 \subset (0,\alpha)$ and $A_2 \subset (\alpha,1)$ or $A_1 \subset (\alpha,1)$ and $A_2 \subset (0,\alpha)$, we define $\nu(A_1 \times A_2) = (2-\alpha)\,\mu(A_1)\,\mu(A_2)$.

For sets of the form $A_1 \times A_2$, where $A_1 \subset (0,\alpha)$ and $A_2 \subset (0,\alpha)$, we define $\nu(A_1 \times A_2) = (2-\alpha)\,\alpha\,\mu(A_1)\,\mu(A_2)$.

It is not difficult to see that ν is invariant for F_α and is absolutely continuous with respect to the Lebesgue measure. The measure ν satisfies $\nu(A \times (0,1)) = \mu(A)$, when $A \subset (0,\alpha)$ and $\nu(A \times (0,\alpha)) = \mu(A)$, when $A \subset (\alpha,1)$.

The next theorem gives the spectral density function for the process (7.1) and the proof can be found in Lopes, Lopes and Souza (1995).

Theorem 7.1: *The spectral density function of the process*

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0,Y_0)) + \xi_t, \quad \textit{for} \quad t \in \mathbf{Z},$$

where F_α is defined by the expressions (7.2) and (7.3) and the point (X_0,Y_0) is chosen randomly according to the measure ν or according to the Lebesgue measure $dxdy$, is given by

$$f_Z(\lambda) = \frac{1}{2\pi\, Var(X_t)}\left[\gamma(e^{i\lambda}) + \gamma(e^{-i\lambda}) - \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)}\right] + \frac{\sigma_\xi^2}{2\pi}, \quad \textit{for all} \quad \lambda \in [0,2\pi), \quad (7.6)$$

where $Var(X_t) = \frac{(\alpha^2-\alpha+1)(\alpha^2-5\alpha+5)}{12(2-\alpha)^2}$ and $\gamma(z)$ is given by

$$\gamma(z) = \frac{2\alpha^2(1-\alpha) + 2 + \alpha z(2-\alpha-\alpha^2)}{6(2-\alpha)} + \\ + \left[\frac{\alpha z + (1-\alpha)^2 z^2}{2-\alpha}\right]\psi(z) + \frac{\alpha(1-\alpha)z^2}{2-\alpha}\varphi(z),$$

with

$$\varphi(z) = \frac{1+\alpha z(1-\alpha)}{2[(1-\alpha)z+1](1-z)} \qquad \textit{and} \qquad \psi(z) = \frac{2-\alpha z(\alpha^2+\alpha-2) + 6\alpha(1-\alpha)z^2\varphi(z)}{6[1-\alpha^2 z - (1-\alpha)^2 z^2]}.$$

Remark: The power series $\gamma(z)$ is an analytic function on the disc $\{z \in \mathbf{C} \mid \| z \| < 1\}$ and the expression (7.6) has the meaning of the radial limit

$$\lim_{r \to 1} r e^{i\lambda} = e^{i\lambda} = z.$$

In this sense, the series

$$\sum_{n \in \mathbf{Z}} e^{in\lambda} = 2\mathcal{R}\mathrm{e}\left(\frac{1}{1-e^{i\lambda}}\right) - 1 = 0, \quad \text{for} \quad \lambda \neq 0,$$

even though the series $\sum_{n \in \mathbf{Z}} e^{in\lambda}$ does not converge. We are using this fact in the expression (7.6).

8. EXAMPLE 4

The example in this section generalizes the results by Grossmann and Thomae (1977).

Let $a_1, a_2, \cdots, a_n$ be any positive real numbers such that $\sum_{i=1}^{n} a_i = 1$ and, for each $1 \le i \le n$, let $b_{i1}, b_{i2}, \cdots, b_{ij}$ be any positive real numbers such that $\sum_{j=1}^{n} b_{ij} = a_i$.

For each $i \in \{1, 2, \cdots, n\}$ one defines

$$B_i = \left[\sum_{l=1}^{i-1} a_l, \sum_{l=1}^{i} a_l \right],$$

where $\sum_{l=1}^{0} a_l = 0$.

For each fixed $i \in \{1, 2, \cdots, n\}$, one defines for any $1 \le j \le n$

$$B_{ij} = \left[\sum_{l=1}^{i-1} a_l + \sum_{m=1}^{j-1} b_{im}, \sum_{l=1}^{i-1} a_l + \sum_{m=1}^{j} b_{im} \right],$$

where $\sum_{m=1}^{0} b_{im} = 0$, for all $1 \le i \le n$. Note that length$(B_i) = a_i$ and length$(B_{ij}) = b_{ij}$. Consider now the following function $T : [0,1] \to [0,1]$ given by

$$T(x) = \sum_{l=1}^{j-1} a_l + \left(x - \sum_{l=1}^{i-1} a_l - \sum_{m=1}^{j-1} b_{im} \right) \frac{a_j}{b_{ij}}, \quad \text{for all} \quad x \in B_{ij}. \tag{8.1}$$

In Figure 5 we show the graph of T when $n = 4$. Consider F the natural extension of such function T.

One is interested in the first order autocorrelation function of the stochastic process

$$X_t = T(X_{t-1}) = F(X_{t-1}, Y_{t-1}), \quad \text{for all} \quad t \in \mathbf{Z},$$

when $\sigma_\xi = 0$ and $\phi(x, y) = x$.

First we want to prove that the invariant measure associated with the function T is of the form $\mu(A) = \int_A \sum_{i=1}^{n} p_i I_{B_i}(x)\, dx$, that is, the density of μ is given by

$$g(x) = \sum_{i=1}^{n} p_i I_{B_i}(x).$$

From the definition of the function T in expression (8.1), if $x \in B_{ij}$ then

$$T(x) \in \left[\sum_{l=1}^{j-1} a_l, \sum_{l=1}^{j} a_l \right] \subset B_j.$$

It is easy to see that $B_{ij} = \{x \in [0,1] \mid x \in B_i, \ \ T(x) \in B_j\}$.

Suppose $\mu(A) = \int_A \sum_{i=1}^n p_i I_{B_i}(x)\, dx$, where $p_i \geq 0$, dx is the Lebesgue measure and it is an invariant measure for T.

Let ω_i be $\mu([x \in B_i])$. Then $\sum_{i=1}^n \omega_i = 1$. From the invariance of μ, one obtains $\omega_i = \mu([T(x) \in B_i])$. Since $[T(x) \in B_j] = \cup_{i=1}^n [x \in B_i] \cap [T(x) \in B_j] = \cup_{i=1}^n B_{ij}$, hence

$$\omega_j = \mu([T(x) \in B_j]) = \sum_{i=1}^n \mu([x \in B_i] \cap [T(x) \in B_j]) = \sum_{i=1}^n \mu(B_{ij}) = \sum_{i=1}^n \frac{\mu(B_{ij})}{\mu(B_i)} \mu(B_i) =$$

$$= \sum_{i=1}^n \frac{p_i \mathrm{length}(B_{ij})}{p_i \mathrm{length}(B_i)} \omega_i = \sum_{i=1}^n \frac{b_{ij}}{a_i} \omega_i.$$

Therefore, for all $j \in \{1, 2, \cdots, n\}$, $\omega_j = \sum_{i=1}^n \frac{b_{ij}}{a_i} \omega_i$ or, in matrix form,

$$\omega = B\omega$$

where $\omega = (\omega_1, \cdots, \omega_n)$ and $B = \left(\frac{b_{ij}}{a_i}\right)_{i,j}$. It is easy to see that the matrix B is a stochastic matrix.

In this way one can obtain, from the Perron- Frobenius Theorem, the invariant density $\sum_{i=1}^n p_i I_{B_i}(x)$, by taking $\frac{\omega_i}{\mathrm{length} B_i} = p_i$, for $1 \leq i \leq n$, where $\omega = (\omega_1, \cdots, \omega_n)$ satisfies $\omega = B\omega$.

This shows that the values of p_i, $1 \leq i \leq n$, can be explicitly obtained by solving an eigenvalue equation in w_i.

The values w_i can be alternatively obtained by iteration of the stochastic matrix. This follows from the contraction fixed point theorem.

It will be necessary to obtain the value of the first and second moments of the random variable X_t. These moments are given as folows.

1. $$E(X_t) = \int_0^1 z\, d\mu(z) = \sum_{i=1}^n \frac{p_i}{2} \left[\left(\sum_{j=1}^i a_j \right)^2 - \left(\sum_{j=1}^{i-1} a_j \right)^2 \right].$$

2. $$E(X_t^2) = \int_0^1 z^2\, d\mu(z) = \sum_{i=1}^n \frac{p_i}{3} \left[\left(\sum_{j=1}^i a_j \right)^3 - \left(\sum_{j=1}^{i-1} a_j \right)^3 \right].$$

Denote $A(k)$, $B(k, i)$ and $V(k, i)$ by

$$A(k) = \int_0^1 x\, T^k(x) g(x)\, dx, \quad B(k, i) = \int_{B_i} T^k(x)\, dx \quad \text{and} \quad V(k, i) = \int_{B_i} x\, T^k(x)\, dx. \tag{8.2}$$

The values of $B(k, i)$ and $V(k, i)$ can be obtained from the recurrence formula

$$
\begin{aligned}
&a. \quad B(k+1,i) = \sum_{j=1}^{n} \frac{b_{ij}}{a_j} B(k,j). \\
&b. \quad V(k+1,i) = \sum_{j=1}^{n} \frac{b_{ij}}{a_j} \left[\frac{b_{ij}}{a_j} V(k,j) + \left(\sum_{l=1}^{i-1} a_l + \sum_{m=1}^{j-1} b_{im} - \frac{b_{ij}}{a_j} \sum_{l=1}^{j-1} a_l \right) B(k,j) \right].
\end{aligned} \tag{8.3}
$$

One can describe the quantities $A(k)$, $B(k,i)$ and $V(k,i)$ by the following power series

$$
\varphi(z) = \sum_{k \geq 0} A(k) z^k, \quad \Psi_i(z) = \sum_{k \geq 0} B(k,i) z^k \quad \text{and} \quad \gamma_i(z) = \sum_{k \geq 0} V(k,i) z^k. \tag{8.4}
$$

From (8.3 a.), the second power series in expression (8.4) is given by

$$
\Psi_i(z) = B(0,i) + z \sum_{j=1}^{n} \frac{b_{ij}}{a_j} \Psi_j(z), \quad \text{for all} \quad 1 \leq i \leq n, \tag{8.5}
$$

where the values $B(0,i)$ can be calculated by

$$
B(0,i) = \int_{B_i} x \, dx = \frac{1}{2} \left[(\sum_{l=1}^{i} a_l)^2 - (\sum_{l=1}^{i-1} a_l)^2 \right], \quad \text{for all} \quad 1 \leq i \leq n.
$$

Consider the vector $v = (B(0,1), B(0,2), \cdots, B(0,n))$ and A the $n \times n$ matrix $A = (\frac{b_{ij}}{a_j})$. Then one can easily find the vector $\Psi(z) = (\Psi_1(z), \Psi_2(z), \cdots, \Psi_n(z))$ by solving the linear system (8.5) $\Psi = v + A(\Psi)z$. In this way we obtain the values $\Psi_i(z)$, $1 \leq i \leq n$.

From (8.3 b.), the third power series in expression (8.4) is given by

$$
\gamma_i(z) = V(0,i) + z \sum_{j=1}^{n} \left(\frac{b_{ij}}{a_j} \right)^2 \gamma_j(z) + z \sum_{j=1}^{n} \frac{b_{ij}}{a_j} \left(\sum_{l=1}^{i-1} a_l + \sum_{m=1}^{j-1} b_{im} - \frac{b_{ij}}{a_j} \sum_{l=1}^{j-1} a_l \right) \Psi_j(z). \tag{8.6}
$$

The value $V(0,i)$ can be calculated as

$$
V(0,i) = \int_{B_i} x^2 \, dx = \frac{1}{3} \left[(\sum_{l=1}^{i} a_l)^3 - (\sum_{l=1}^{i-1} a_l)^3 \right], \quad \text{for all} \quad 1 \leq i \leq n.
$$

As we also know $\Psi_j(z)$, one can solve the linear system (8.6) and finally find $\gamma_i(z)$, for $1 \leq i \leq n$.

From the first power series in expression (8.4) one obtains

$$
\varphi(z) = \sum_{k \geq 0} A(k) z^k = \sum_{i=1}^{n} p_i \gamma_i(z). \tag{8.7}
$$

It is easy to see (Lopes, Lopes and Souza (1995)), by taking $z = e^{i\theta}$ and $z = e^{-i\theta}$ that, from the expression for $\varphi(z)$ in (8.7), the explicit expression of the spectral density function associated with T can be obtained by

$$f_Z(\lambda) = \frac{1}{2\pi\, Var(X_t)}[\varphi(e^{i\lambda}) + \varphi(e^{-i\lambda}) - E(X_t^2)] + \frac{\sigma_\xi^2}{2\pi}, \quad \text{for all} \quad \lambda \in [0, 2\pi).$$

where $Var(X_t) = E(X_t^2) - [E(X_t)]^2 = \int x^2 g(x)dx - \left(\int xg(x)dx\right)^2$.

A more general result for piecewise expanding linear maps is given in Lopes, Lopes and Souza (1996).

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