An algorithm for approximating subactions

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Abstract

Denote by T the transformation $T(x) = 2\,x \pmod 1$. Given a potential $A : S^1 \to \mathbb{R}$ the main interest in Ergodic Optimization are probabilities μ which maximize $\int A\, d\mu$ (among invariant probabilities) and also calibrated subactions $u : S^1 \to \mathbb{R}$. We will analyze the 1/2-operator $\mathcal{G}$ which acts on Hölder functions $f : S^1 \to \mathbb{R}$. Assuming that the subaction for the Hölder potential A is unique (up to adding constants) it follows from the work of W. Dotson, H. Senter and S. Ishikawa that $\lim_{n\to\infty} \mathcal{G}^n(f_0) = u$ (for any given f_0). $\mathcal{G}$ is not a strong contraction and we analyze here the performance of the algorithm from two points of view: the generic point of view and its action close by the fixed point.

In a companion paper we will consider several examples. The sharp numerical evidence obtained from the algorithm permits to guess explicit expressions for the subaction: among them for $A(x) = \sin^2(2\pi x)$ and $A(x) = \sin(2\pi x)$. There we present a piecewise analytical expression for the calibrated subaction in this case. The algorithm can also be applied to the estimation of the joint spectral radius of matrices.

1 Introduction

Here we analyze some properties of an algorithm designed for approximating subactions. Properties for a general form of such kind of algorithm were considered in [30], [12], [43] and [22] (see also [42] and [1] for more recent results). We analyze here the performance of a **specific** version of the algorithm which is useful in Ergodic Optimization.

In a companion paper [13] we will consider several examples. The sharp numerical evidence obtained from the algorithm permits to guess explicit expressions for the subaction.

One can also consider a similar kind of algorithm for approximating eigenfunctions of the Ruelle operator (see [13]). We will leave for a future paper a more careful analysis of this case. The method works fine when the eigenvalue is not necessarily equal to 1 and the potential is not of the form $-\log f'$, where f is an expanding transformation on the circle.

In recent years several papers considered numerical properties of algorithms that can be used for a better understanding of important questions in Thermodynamic Formalism and Ergodic Theory (see [29], [2], [16], [32] and [14]).

We denote by $T: S^1 \to S^1$ the transformation $T(x) = 2\,x$ (mod 1).

We identify the unitary circle with the interval $[0,1)$. We denote by $\tau_2 : [0,1) \to [0,1/2)$ and $\tau_2 : [0,1) \to [1/2,1)$ the two inverse branches of T ($\tau_2(x) = \frac{1}{2}x$ and $\tau_2(x) = \frac{1}{2}(x+1)$).

We consider here results either for potentials $A : [0,1] \to \mathbb{R}$ or periodic potentials $A : S^1 \to \mathbb{R}$.

Definition 1. *Given a continuous function $A : S^1 \to \mathbb{R}$ (or, $A : [0,1] \to \mathbb{R}$) we denote by*

$$m(A) = \sup_{\rho \, invariant \; for \, T} \int A \, d\rho.$$

Any invariant probability μ attaining such supremum is called a **maximizing probability**.

The properties of the maximizing probabilities μ are the main interest of Ergodic Optimization (see [3], [19], [23], [24], [7] and [31]).

In Statistical Mechanics the limits of equilibrium probabilities when temperature goes to zero (see [3]) are called ground states (they are maximizing probabilities).

A interesting line of reasoning is the following: there is a theory, someone gives a particular example which leads to a problem to solve, then, use the theory to exhibit the solution. Is there a general procedure to find the solution of this kind of problem? Here we will address this kind of query on the present setting.

Definition 2. *Given the Hölder continuous function $A : S^1 \to \mathbb{R}$ the union of the supports of all the maximizing probabilities is called the* **Mather set** *for A.*

We will assume from now on that A is Hölder continuous and that the maximizing probability is unique.

It is known that for a generic Hölder potential A (in the Hölder norm) the maximizing probability is unique and has support on a T-periodic orbit (see [9]). We do not have to assume here that the unique maximizing probability has support on a unique periodic orbit.

Definition 3. *Given the Hölder continuous function $A : S^1 \to \mathbb{R}$, then a continuous function $u : S^1 \to \mathbb{R}$ is called a* **calibrated subaction** *for A, if, for any $x \in S^1$, we have*

$$u(x) = \max_{T(y)=x} [A(y) + u(y) - m(A)]. \tag{1}$$

For Hölder potentials A there exists Hölder calibrated subactions (see [7]). If the **maximizing probability is unique** (our assumption) then the **calibrated subaction is unique** up to add a constant (see [7] or [17]).

One interesting question is the dependence of the calibrated subaction u on the potential A (we will adress this question on [13]).

Calibrated subactions play an important role in Ergodic Optimization (see [3] and [19]). From an explicit calibrated subaction one can guess where is the support of the maximizing probability.

Indeed, given u we have that for all x

$$R(x) \,:=\, u(T(x)) - u(x) - A(x) + m(A) \geq 0, \tag{2}$$

and, for any point x in the Mather set $R(x) = 0$. Moreover, if an invariant probability has support inside the set of points where $R = 0$, then, this probability is maximizing (see [7]).

Example 4. *We show in Figure 1 the graph of a potential A, the graph of the calibrated subaction u and the graph of R. The potential A is zero at the points $1/4, 3/4$ and it is equal to -1 in the points $0, 1/2, 1$. The set $\{1/3, 2/3\}$ is contained on the Mather set (then, it is the support of a maximizing probability) and $m(A) = -1/3$. The calibrated subaction is 0 at the point $1/2$ and equal to $2/3$ at the points $0, 1$. The function R is equal to $2/3$ at the points $0, 1$ and it is equal to zero on the interval $[1/4, 3/4]$. We point out that we easily guessed the explicit expression for the subaction u from the picture obtained from the application of the algorithm on the initial condition $f_0 = 0$.*

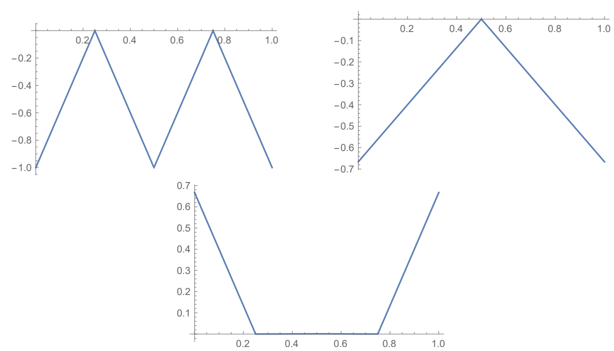

Figure 1: From left to right: the graph of the potential A, the graph of the calibrated subaction u and the graph of R.

Given x, then, $u(x) = A(\tau_j(x)) + u(\tau_j(x)) - m(A)$, for some $j = 1, 2$. We say that $\tau_j(x)$ is a **realizer** for x. There are some points x that eventually get at the same time two realizers.

We are interested in an algorithm for getting a good approximation of the subaction in the case the maximizing probability is unique. As a byproduct we will also get the value $m(A)$. This will help to get R (as above) and eventually to find the support of the maximizing probability.

We will consider a map $\mathcal{G}$ acting on functions such that the subaction u is the unique fixed point. Unfortunately, $\mathcal{G}$ is not a strong contraction but we know that $\lim_{n\to\infty} \mathcal{G}^n(f_0) = u$ (for any given f_0). The performance of the algorithm is quite good for exhibiting good approximations.

We explore here in section 3 the generic point of view and expression (21) (and (22)) in Theorem 15 and also expression (20) in Remark 3 in some way justify the excellent performance one can observe for the algorithm which we will describe here.

A natural question: when the calibrated subaction is unique is there an uniform exponential speed of approximation (or, something numerically good) of the iteration $\mathcal{G}^n(f_0)$ to the subaction? At least close by the subaction? In section 4 we present a very detailed analysis of the action of the map $\mathcal{G}$ close by the fixed point u and we will show that this is not the case. We will consider in Example 29 a case where where $|\mathcal{G}(f_\varepsilon) - \mathcal{G}(u)| = |f_\varepsilon - u|$, $\varepsilon > 0$, for f_ε as close as you want to the calibrated subaction u. One can also show that close by u there are other g_ε, $\varepsilon > 0$, such that, $|\mathcal{G}(g_\varepsilon) - \mathcal{G}(u)| = 1/2\, |g_\varepsilon - u|$ (see Corollary 28).

In section 5 we will describe a method for getting the maximizing probability via a limit procedure in the case we have the explicit expression for the subaction u.

In the Appendix (section 6) we will briefly describe the steps used on the algorithm which was performed on the language C^{++}.

In [13] we will present several examples where the calibrated subaction can be explicitly expressed. When the potential A is analytic the calibrated subaction will be piecewise analytic. The pictures we obtained from the algorithm helped to figure out the explicit solutions we had to look for.

The algorithm we use here can be helpful for estimating the joint spectral radius (see [13]). This question is related to the recent work [28].

2 The 1/2-algorithm

On the set of continuous functions $f : S^1 \to \mathbb{R}$ we consider the sup norm: $|f|_0 = \sup\{|f(x)|,\, x \in S^1\}$.

Definition 5. *For the set of Hölder continuous functions from S^1 to $\mathbb{R}$ we consider the equivalence relation $f \sim g$, if $f - g$ is a constant.*

The set of classes is denoted by $\mathcal{C}$ and, by convention, we will consider in each class a representative which has supremum equal to zero.

In $\mathcal{C}$ we consider the quotient norm (see section 7.2 in [38])

$$|f| = \inf_{\alpha\in\mathbb{R}} |f + \alpha|_0.$$

$(\mathcal{C}, |\cdot|)$ is a Banach space (see [38]). As S^1 is compact we get that: for any given f there exists α, such that, $|f| = |f + \alpha|_0$.

We denote sometimes the constant α associated to f by α_f.

We point out that when we write $|f(x)|$ this means the modulus of an element in $\mathbb{R}$ and $|f|$ means the norm defined above.

Remark 1: Suppose $f, g \in \mathcal{C}$. One can show that

$$|f-g| = \frac{\max(f-g) - \min(f-g)}{2}. \tag{3}$$

Moreover,

$$\alpha_{f-g} = -\frac{\max(f-g) + \min(f-g)}{2}. \tag{4}$$

This also means that given f

$$\alpha_f = -\frac{\max f + \min f}{2}. \tag{5}$$

Indeed, assume that $|(f-g)+d|_0 = |f-g|$ and z_0 are such that $|(f-g)(z_0)+d| = |f-g|$. Without loss of generality we can assume that $(f-g)(z_0)+d > 0$. Note that, if

$$\inf_{v \in S^1} [\,(f-g)(v) + d\,] > -[\,(f-g)(z_0) + d\,],$$

then exists $d_1 < d$, such that, $|\,(f-g) + d_1\,|_0 < |\,(f-g) + d\,|_0$. This is not possible.

It is not possible either that $\inf_{v \in S^1}[(f-g)(z_0) + d] < -[(f-g)(z_0)+d]$, because z_0 maximizes $z \to |(f-g)(z) + d|$.

Then,

$$\inf_{v \in S^1} [\,(f-g)(v) + d\,] = -[\,(f-g)(z_0) + d\,] = -|\,f-g\,|. \tag{6}$$

This show that there exist a point $r \in S^1$, such that,

$$|f-g| \;=\; |\,(f-g)(z_0) + d\,| = \,-\,[\,(f-g)(r) + d\,].$$

It follows that

$$|f-g| = \frac{\max(f-g) - \min(f-g)}{2}. \tag{7}$$

Moreover,

$$d = \alpha_{f-g} = -\frac{\max(f-g) + \min(f-g)}{2}. \tag{8}$$

This also means that given f then equation (5) is true.

$\diamondsuit$

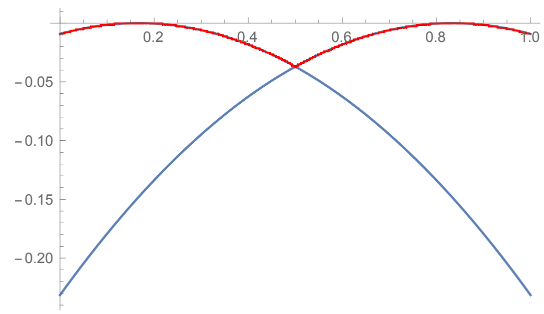

Figure 2: Case $A(x) = -(x - 1/2)^2$ and $T(x) = -2x \pmod 1$ - In this case $m(A) = -1/36$. The red graph describes the values of the approximation (via 1/2-algorithm) to the calibrated subaction u given by $\mathcal{G}^{10}(0)$ (using the language C^{++} and a mesh of points) and the two blue graphs describe, respectively, the graphs of $x \to -1/3x^2 + 1/9x$, and $x \to -1/3x^2 + 5/9x - 2/9$. The supremum of these two functions is the exact analytical expression for the graph of the calibrated subaction u. The red color obliterates the blue color.

Definition 6. *Given a Hölder continuous function $A : S^1 \to \mathbb{R}$ we consider the operator (map) $\hat{\mathcal{L}} = \hat{\mathcal{L}}_A$, such that, for $f : S^1 \to \mathbb{R}$, we have $\hat{\mathcal{L}}_A(f) = g$, if*

$$\hat{\mathcal{L}}_A(f)(x) = g(x) = \max_{T(y)=x} [A(y) + f(y) - m(A)]. \tag{9}$$

for any $x \in S^1$.

Note that u is a fixed point for such operator $f \to \hat{\mathcal{L}}_A(f)$, if and only if, u is a calibrated subaction.

One could hope that a high iterate $\hat{\mathcal{L}}_A^n(f_0)$ (n large) would give an approximation of the calibrated subaction.

This operator will not be very helpful because we have to known in advance the value $m(A)$.

Even if we know the value $m(A)$ the iterations $\hat{\mathcal{L}}_A^n(f_0)$ applied on an initial continuous function f_0 may not converge. This can happen even in the case the calibrated subaction is unique.

Definition 7. *Given a Hölder continuous function $A : S^1 \to \mathbb{R}$ we consider the operator (map) $\mathcal{L} = \mathcal{L}_A : \mathcal{C} \to \mathcal{C}$, such that, for $f : S^1 \to \mathbb{R}$, we have $\mathcal{L}_A(f) = g$, if*

$$\mathcal{L}_A(f)(x) = g(x) = \max_{T(y)=x} [A(y) + f(y)] - \sup_{s \in S^1} \{ \max_{T(r)=s} [A(r) + f(r)] \}. \tag{10}$$

for any $x \in S^1$.

The advantage here is that we do not have to know the value $m(A)$. In the same way as before u is a fixed point for the operator $\mathcal{L}_A(f)$, if and only if, u is a calibrated subaction (see end of the proof of Theorem 11 in [4]).

We call the algorithm (defined below) the 1/2-algorithm. It is a particular case of the algorithm described on [12] and [22]. From these two papers it follows that given any initial function $f_0 \in \mathcal{C}$ we have that $\lim_{n\to\infty} \mathcal{G}^n(f_0)$ exists and it is the subaction u (which belongs to $\mathcal{C}$.)

Remark 2: The iterations $\mathcal{L}_A^n(f_0)$ applied on an initial continuous function f_0 may not converge (see Figure 4). This can happen even in the case the calibrated subaction is unique. On the graphs on the left side in Figure 4 we compare the iteration by $\mathcal{L}$ with the iteration by $\mathcal{G}$ to be introduced next. $\diamondsuit$

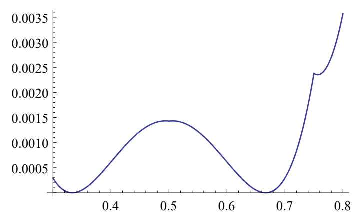

Figure 3: Case $A(x) = -(x-1/3)^2(x-2/3)^2$ and $T(x) = 2\,x$ (mod 1) - This picture shows the graph (plotted on Mathematica) of R (see (12)) obtained from an approximation of the calibrated subaction u after 7 iterations of the 1/2-algorithm. We can infer from this Figure that the maximizing probability has support on the periodic orbit $\{1/3, 2/3\}$ as expected. Therefore, this algorithm has the potential to display the support of the maximizing probability.

In order to show the power of the approximation scheme we consider an example where the subaction u was already known. The dynamics is $T(x) = -2x$ (mod 1) (not $T(x) = 2x$ (mod 1)). The 1/2-algorithm works also fine in this case.

According to example 5 in pages 366-367 in [35] the subaction u (see picture on page 367 in [35]) for the potential $A(x) = -(x-1/2)^2$ is

$$u(x) = \sup\{\,-1/3x^2 + 1/9x,\ -1/3x^2 + 5/9x - 2/9\,\}.$$

More generally, in page 391 in [35] is described a natural procedure to get the subaction u for potentials A which are quadratic polynomials.

The maximizing probability μ in this case has support on the orbit of period two (according to [25], [26] and [27]) and $m(A) = -1/36$.

One can see from Figure 2 a perfect match of the solution obtained from the algorithm described by $\mathcal{G}$ and the graph of the exact calibrated subaction u.

Definition 8. *Given a Hölder continuous function $A : S^1 \to \mathbb{R}$ we consider the operator (map) $\mathcal{G} = \mathcal{G}_A : \mathcal{C} \to \mathcal{C}$, such that, for $f : S^1 \to \mathbb{R}$, we have $\mathcal{G}_A(f) = g$, if*

$$\mathcal{G}_A(f)(x) = g(x) = \frac{\max_{T(y)=x}[A(y) + f(y)] + f(x)}{2} \quad - c_f$$

for any $x \in S^1$, *where*

$$c_f := \sup_{s \in S^1} \frac{\max_{T(r)=s}[A(r) + f(r)] + f(r)}{2}. \tag{11}$$

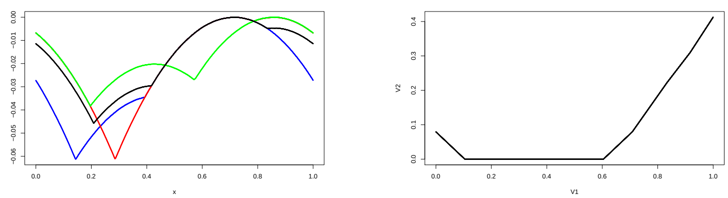

Figure 4: The case of the potential $A(x) = -(x - 1/3)^2$. - On the left side we show in the colors green red and blue the graphs we obtain from high iterates of $\mathcal{L}^n$ applied to the initial function $f_0 = 0$. The iteration of $\mathcal{L}^n$ around $n = 1000$ seems to be in period three where successively we obtain the graphs in the colors blue ($\mathcal{L}^{998}(0)$), red ($\mathcal{L}^{999}(0)$) and green ($\mathcal{L}^{1000}(0)$). In the same picture (on the left) in the color black we show the graph of $\mathcal{G}^{1000}(0)$ (in fact the picture for $\mathcal{G}^{20}(0)$ is almost the same). This iteration procedure using $\mathcal{G}$ stabilizes on this black graph. We use a mesh of 4000 points in $[0, 1]$ and C for all pictures. On the right side we show the picture of the function R (see (12)) we obtain by taking as the function u what we get from the approximation (described by the black graph on the left) via $\mathcal{G}^{1000}(0)$. The graph of R shows that there is a numerical evidence that the support of the maximizing probability for such A is inside an interval of size 0.5. This is just a confirmation of the claim of Corollary 4 in [25]. The maximizing probability in this case is a periodic orbit of period 3 (see [13]). Note that this potential is not periodic. We point out that the period three behavior for the iteration of $\mathcal{L}$ already happens for low iterates like $n = 8, 9, 10$.

We will show later in Theorem 11 that $|\mathcal{G}(f) - \mathcal{G}(g)| \leq |f - g|$, for any $f, g \in \mathcal{C}$. Therefore, $\mathcal{G}$ is Lipschitz continuous.

The operator $\mathcal{G}$ is not linear.

We call the algorithm based on high iterations $\mathcal{G}^n(f_0)$ the 1/2-**algorithm**.

The above Definition 8 was inspired by expressions (5.1) and (5.2) of [8]. This is a particular case of a more general kind of numerical iteration procedure known as the Mann iterative process (see [12], [43], [30], [22] and [39]).

Assuming that the subaction u for the Hölder potential A is unique (up to adding constants) it follows (as particular case) from the general results of W. Dotson, H. Senter and S. Ishikawa (see Corollary 1 in [43], [12] or [22]) that

$$\lim_{n \to \infty} \mathcal{G}^n(f_0) = u,$$

(or any given $f_0 \in \mathcal{C}$.

The special $\mathcal{G}$ presented above was not previously consider in the literature (as far as we know).

Note that $\mathcal{G}_A(f+c) = \mathcal{G}_A(f)$ if c is a constant and for any f the supremum of $\mathcal{G}_A(f)$ is equal to 0.

When running the algorithm on a computer (using the language C^{++}) one fix a mesh of points in $[0,1]$ and perform the operations on each site (see Appendix 6). The pictures we will show here are obtained in this way when we consider a large number of points equally spaced. One can also get as output the realizer for each point x on the lattice. This is very helpful when trying to obtain explicit calibrated subactions in some examples (or, numerical evidence) as we wil show in [13].

Some of our examples in [13] also consider the case when there exists more than one maximizing probability.

One important issue on the companion paper [13] with explicit examples is corroboration. By this we mean: we derive analytically some complicated expressions and we use the algorithm to compare and confirm that our reasoning was correct.

As an example of the kind of result we can get we show in Figure 3 (for the where case $A(x) = -(x-1/3)^2(x-2/3)^2$ and $T(x) = 2\,x \pmod 1$) the graph of R obtained from the calibrated subaction u we can get via the algorithm. Therefore, the algorithm we will consider here can eventually exhibit the support of maximizing probabilities via such function R (see also Figure 4).

The algorithm also performs fine in Mathematica (not using a mesh of discrete points on the interval as in the case where we use the language C^{++}) but not so fast as in C.

In most of the examples where the explicit subaction was previously known when comparing the graph obtained from the algorithm (for the approximated subaction) with the exact one we get that the difference is invisible to the naked eye.

Proposition 9. *If u is such that $\mathcal{G}_A(u) = u$, then, u is a calibrated subaction and*

$$m(A) = \sup_z \max_{T(y)=z} [A(y) + u(y)] + u(z). \tag{12}$$

Proof: If

$$u(x) = \frac{\max_{T(y)=x}[A(y) + u(y)] + u(x)}{2} - c_u, \tag{13}$$

then, for all x, we obtain $u(x) = \frac{\max_{T(y)=x}[A(y)+u(y)]+u(x)}{2} - c$, where $c = c_u = \sup_z \frac{\max_{T(y)=z}[A(y)+u(y)]+u(z)}{2}$ is constant. This means that

$$2u(x) = \max_{T(y)=x} [A(y) + u(y)] + u(x) - 2c,$$

and, finally, we get $u(x) = \max_{T(y)=x}[A(y) + u(y)] - 2c$, for any x.

In the end of the proof of Theorem 11 in [4] it is shown that this implies that $m(A) = 2c$ and it follows that u is a calibrated subaction.

□

Counter example 1: $\mathcal{G}$ may not be a strong contraction (by a factor smaller than 1). We will present an example where $f_0, g_0 \in \mathcal{C}$ but $|\mathcal{G}(f_0) - \mathcal{G}(g_0)| = 1/2 = |f_0 - g_0|$.

Consider the potential A with the graph given by Figure 5. This potential is linear by parts and has the value 0 on the points $1/8, 1/4, 3/4, 7/8$. The value -1 is attained at the points $0, 3/16, 1/2, 13/16, 1$.

Denote $g_0 = 0$ and $f_0 = A$. Then, $|f_0 - g_0| = |f_0 - g_0 + 1/2|_0 = 1/2$. We denote $f_1 = \mathcal{G}(f_0)$ and $g_1 = \mathcal{G}(g_0)$. The graph of the function $x \to |f_1(x) - g_1(x) + 0.5|$ is described by the bottom rigth picture on Figure 5. One can show that $|f_1 - g_1| = |f_1 - g_1 + 1/2|_0 = 1/2$. Therefore, for such potential A the transformation $\mathcal{G}$ is not a strong contraction. Theorem 11 shows that $\mathcal{G}$ is a weak contraction.

$\diamondsuit$

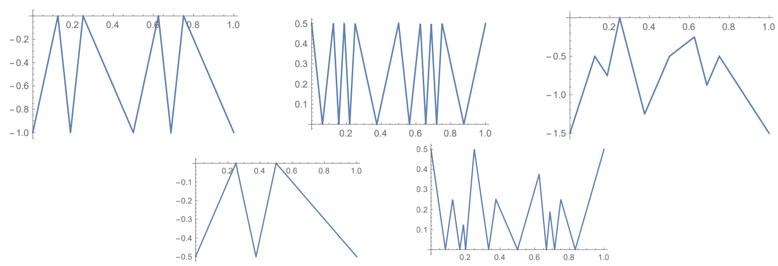

Figure 5: On the top: from left to right the graph of $A = f_0$, the graph of $x \to |(f_0(x) - 0) + 0.5|$, the graph of $f_1 = \mathcal{G}(f_0)$. On the bottom: from left to right the graph of $g_1 = \mathcal{G}(0) = \mathcal{G}(g_0)$ and the graph of $x \to |f_1(x) - g_1(x) + 0.5|$. Therefore, $\mathcal{G}$ is not a strong contraction because $|f_0 - g_0| = 1/2 = |f_1 - g_1| = |\mathcal{G}(f_0) - \mathcal{G}(g_0)|$.

Denote by $\mathcal{C}_K$, the set of Lipschitz (could be also Hölder but in order to simplify the proofs we assume Lipschitz) function $f : S^1 \to \mathbb{R}$, where $f \in \mathcal{C}$, with Lipschitz constant smaller or equal to K.

By Arzela-Ascoli Theorem $\mathcal{C}_K$ is a compact space for the quotient norm on $\mathcal{C}$.

Theorem 10. *Suppose A has Lipschitz constant equal to K. The map $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ takes a function f_0, which has Lipschitz constant smaller or equal K, to the function $f_1 = \mathcal{G}(f_0)$ which has also a Lipschitz constant smaller or equal to K. Therefore, the image of $\mathcal{C}_K$ by $\mathcal{G}$ is compact in the quotient norm.*

Proof: Denote $f_1 = \mathcal{G}(f_0)$.

Given a point y assume without loss of generality that $f_1(x) - f_1(y) \geq 0$.

Then,

$$f_1(x) - f_1(y) \le \left[\frac{A(\tau_{a_0^{x,f_0}}(x))}{2} + \frac{1}{2}(\, f_0(\tau_{a_0^{x,f_0}}(x)) + f_0(x)\,) \right] -$$

$$\left[\frac{A(\tau_{a_0^{x,f_0}}(y))}{2} + \frac{1}{2}(\, f_0(\tau_{a_0^{x,f_0}}(y)) + f_0(y)\,) \right] =$$

$$\frac{1}{2}[A(\tau_{a_0^{x,f_0}}(x)) - A(\tau_{a_0^{x,f_0}}(y))] + \frac{1}{2}[\, f_0(\tau_{a_0^{x,f_0}}(x)) - f_0(\tau_{a_0^{x,f_0}}(y))] + \frac{1}{2}[f_0(x) - f_0(y)] \le$$

$$K\,\frac{1}{2}\,|\tau_{a_0^{x,f_0}}(x)) - \tau_{a_0^{x,f_0}}(y)| + K\,\frac{1}{2}\,|\tau_{a_0^{x,f_0}}(x)) - \tau_{a_0^{x,f_0}}(y)| + \frac{1}{2}K\,|x-y| =$$

$$K\,\frac{1}{2}\,\frac{1}{2}|x-y| + K\,\frac{1}{2}\,\frac{1}{2}|x-y| + \frac{1}{2}K\,|x-y| = K\,|x-y|.$$

□

Theorem 11. *Given the functions $f, g \in \mathcal{C}$ we have*

$$|\mathcal{G}(f) - \mathcal{G}(g)| \le |f - g|.$$

Proof: Let $[f], [g] \in \mathcal{C}$ and $d = \alpha_{f-g} \in \mathbb{R}$ such that

$$|[f] - [g]| = |f - g + d|_0.$$

We denote $k = \alpha_{\mathcal{G}(f)-\mathcal{G}(g)}$ the value such that $|\mathcal{G}([f]) - \mathcal{G}([g])| = |\mathcal{G}([f]) - \mathcal{G}([g]) + k|_0$.

In order to estimate $|\mathcal{G}([f]) - \mathcal{G}([g])|$ consider $\mathcal{G}(f)(x) - \mathcal{G}(g)(x) =$

$$-c_f + \frac{1}{2}f(x) + \frac{1}{2}\max_{i\in\{1,2\}}[(A+f)(\tau_i(x))] + c_g - \frac{1}{2}g(x) - \frac{1}{2}\max_{i\in\{1,2\}}[(A+g)(\tau_i(x))],$$

which means $2\,(\mathcal{G}(f)(x) - \mathcal{G}(g)(x) + c_f - c_g) =$

$$f(x) - g(x) + \max_{i\in\{1,2\}}[(A+f)(\tau_i(x))] - \max_{i\in\{1,2\}}[(A+g)(\tau_i(x))].$$

We add d to both sides obtaining

$$2\,(\mathcal{G}(f)(x) - \mathcal{G}(g)(x) + c_f - c_g + d) =$$

$$f(x) - g(x) + d + \max_{i\in\{1,2\}}[(A+f+d)(\tau_i(x))] - \max_{i\in\{1,2\}}[(A+g)(\tau_i(x))],$$

which can be rewritten as $2\,(\mathcal{G}(f)(x) - \mathcal{G}(g)(x) + c_f - c_g + d) =$

$$(f(x) - g(x) + d) + \max_{i\in\{1,2\}}[(A+g+f-g+d)(\tau_i(x))] - \max_{i\in\{1,2\}}[(A+g)(\tau_i(x))].$$

We notice that $-|[f]-[g]| \leq f(y)-g(y)+d \leq |[f]-[g]|$ for any $y \in X$. By monotonicity of the supremum we get

$$-|[f]-[g]| + \max_{i\in\{1,2\}} [(A+g)(\tau_i(x))] \leq$$

$$\max_{i\in\{1,2\}} [(A+g+f-g+d)(\tau_i(x))] \leq |[f]-[g]| + \max_{i\in\{1,2\}} [(A+g)(\tau_i(x))]\,,$$

which is equivalent to $-|[f]-[g]| \leq$

$$\max_{i\in\{1,2\}} [(A+g+f-g+d)(\tau_i(x))] - \max_{i\in\{1,2\}} [(A+g)(\tau_i(x))] \leq |[f]-[g]|,$$

thus

$$|\max_{i\in\{1,2\}} [(A+g+f-g+d)(\tau_i(x))] - \max_{i\in\{1,2\}} [(A+g)(\tau_i(x))]\,|_0 \leq |[f]-[g]|.$$

We assumed that $|f-g+d|_0 = |[f]-[g]|$. Therefore, using the two last inequalities we get $|2\,(\mathcal{G}(f)-\mathcal{G}(g)+c_f-c_g+d)\,|_0 \leq |[f]-[g]|+|[f]-[g]|$, which is equivalent to

$$|\mathcal{G}(f)-\mathcal{G}(g)+(c_f-c_g+d)|_0 \leq |[f]-[g]|. \tag{14}$$

We recall that $|\mathcal{G}([f])-\mathcal{G}([g])| =$

$$\min_{k\in\mathbb{R}} |\mathcal{G}(f)-\mathcal{G}(g)+k|_0 \leq |\mathcal{G}(f)-\mathcal{G}(g)+(c_f-c_g+d)|_0 \leq |[f]-[g]|,$$

and this finish the proof. $\square$

3 The generic point of view

Definition 12. *Consider the set $\mathfrak{A} \subset \mathcal{C}\times\mathcal{C}$ of pairs of functions (f_0, g_0), such that, if $|f_0-g_0| = |(f_0-g_0)+\alpha_{f_0-g_0)}|_0 = (f_0-g_0)(r)+\alpha_{f_0-g_0}$, for some r, then,*

$$(f_0-g_0)(r) \neq (f_0-g_0)(\tau_2(r)) \text{ and } (f_0-g_0)(r) \neq (f_0-g_0)(\tau_2(r)).$$

Note that the above condition does not depends on the potential A. In the case $|f_0(x)-g_0(x)+\alpha_{f_0-g_0)}|$ attains the supremum in a unique point then $(f_0, g_0) \in \mathfrak{A}$.

We will show in Corollary 18 that the condition $(f, g) \in \mathfrak{A}$ is generic.

Theorem 13. *Given the functions $f_0, g_0 \in \mathcal{C}$, assume $(f_0, g_0) \in \mathfrak{A}$. In this case, if $|\mathcal{G}(f_0)-\mathcal{G}(g_0)| = |f_0-g_0|$, then, $f_0 = g_0$.*

Proof: We denote by $d = \alpha_{f_0 - g_0}$ the value such that $|(f_0 - g_0) + d|_0 = |f_0 - g_0|$.

We denote by z_0 the point such that $|f_0 - g_0| = |f_0(z_0) - g_0(z_0) + d|$. Without loss of generality we assume that $f_0(z_0) - g_0(z_0) + d > 0$.

Note that $|(f_0 - g_0 + d)(z_0)|$ also maximizes

$$x \to |(f_0 - g_0 + d)(x)|. \tag{15}$$

Note that d was determined by the choice $(f_0 - g_0)$ (and, not $(g_0 - f_0)$).

We denote by $k = \alpha_{\mathcal{G}(f_0) - \mathcal{G}(g_0)}$ the value $|\mathcal{G}(f_0) - \mathcal{G}(g_0)| + k|_0 = |\mathcal{G}(f_0) - \mathcal{G}(g_0)|$.

Assuming $|\mathcal{G}(f_0) - \mathcal{G}(g_0)| = |f_0 - g_0|$, then, from (14) we get

$$|\mathcal{G}(f_0) - \mathcal{G}(g_0)| = |\mathcal{G}(f_0) - \mathcal{G}(g_0) + k|_0 \leq |\mathcal{G}(f_0) - \mathcal{G}(g_0) + (c_{f_0} - c_{g_0} + d)|_0 \leq |[f_0] - [g_0]|. \tag{16}$$

Therefore, k can be taken as $k = c_{f_0} - c_{g_0} + d$. Note that k was determined by d and the choice $(f_0 - g_0)$ (and, not $(g_0 - f_0)$).

We denote by z_1 a point such that $|\mathcal{G}(f_0) - \mathcal{G}(g_0)| = |\mathcal{G}(f_0)(z_1) - \mathcal{G}(g_0)(z_1) + k| = |f_1(z_1) - g_1(z_1) + k|$.

In the case $(f_1 - g_1)(z_1) + k \leq 0$, from (6) we know that there exists another point $\tilde{z_1}$, such that, $0 \leq (f_1 - g_1)(\tilde{z_1}) + k = |\mathcal{G}(f_0) - \mathcal{G}(g_0) + k|_0$.

Therefore, without loss of generality, we can always assume that it is true $(f_1 - g_1)(z_1) + k \geq 0$.

Assume that $(f_0, g_0) \in \mathfrak{A}$.

Under the above conditions in f_0, g_0, there exists z_0, z_1, $\bar{z} = \tau_{a_0^{z_1, f_0}}(z_1)$ and $\bar{w} = \tau_{a_0^{z_1, g_0}}(z_1)$ such that

$$\begin{gathered}
(f_0 - g_0)(z_0) + d = |f_0 - g_0| = |\mathcal{G}(f_0) - \mathcal{G}(g_0)| = (f_1 - g_1)(z_1) + k = \\
[\frac{A(\bar{z})}{2} + \frac{1}{2}(f_0(\bar{z}) + f_0(z_1))] - [\frac{A(\bar{w})}{2} + \frac{1}{2}(g_0(\bar{w}) + g_0(z_1))] + k - c_f + c_g \leq \\
[\frac{A(\bar{z})}{2} + \frac{1}{2}(f_0(\bar{z}) + f_0(z_1))] - [\frac{A(\bar{z})}{2} + \frac{1}{2}(g_0(\bar{z}) + g_0(z_1))] + k - c_f + c_g = \\
[\frac{1}{2}(f_0(\bar{z}) + f_0(z_1))] - [\frac{1}{2}(g_0(\bar{z}) + g_0(z_1))] + k - c_f + c_g = \\
\frac{1}{2}(f_0(z_1) - g_0(z_1)) + \frac{1}{2}(f_0(\bar{z}) - g_0(\bar{z}) + k - c_f + c_g = \\
\frac{1}{2}(f_0(z_1) - g_0(z_1)) + \frac{1}{2}(f_0(\bar{z}) - g_0(\bar{z}) + d.
\end{gathered} \tag{17}$$

As $(f_0 - g_0 + d)(z_0) > 0$ is a supremum, it follows from the above that

$$\begin{gathered}
(f_0 - g_0)(z_0) + d \leq \frac{1}{2}[(f_0 - g_0)(z_1) + d] + \frac{1}{2}[(f_0 - g_0)(\bar{z}) + d] \leq \\
\frac{1}{2}[(f_0 - g_0)(z_0) + d] + \frac{1}{2}[(f_0 - g_0)(z_0) + d] = (f_0 - g_0)(z_0) + d.
\end{gathered} \tag{18}$$

$(f_0 - g_0)(z_1) + d$ and $(f_0 - g_0)(\bar{z}) + d$ can not be both negative (because $(f_0 - g_0)(z_0) + d > 0$).

Both $(f_0 - g_0)(z_1) + d$ and $(f_0 - g_0)(\bar{z}) + d$ are positive. Otherwise, from (18) we get $(f_0 - g_0)(z_0) + d < \frac{1}{2}[\,(f_0 - g_0)(z_0) + d\,]$. This implies that $\frac{1}{2}[(f_0 - g_0)(z_1) + d] + \frac{1}{2}[(f_0 - g_0)(\bar{z}) + d] = (f_0 - g_0)(z_0) + d$.

Remember that by (8) we have $d = \alpha_{f_0 - g_0} = -\frac{\max(f_0-g_0)+\min(f_0-g_0)}{2}$. From 18 we get $(f_0 - g_0)(z_1) + d = (f_0 - g_0)(\bar{z}) + d = (f_0 - g_0)(z_0) + d$.

As $(f_0, g_0) \in \mathfrak{A}$ we get by Corollary 18 a contradiction.

□

Remark 3: Given the point z_1 above (supremum of $x \to (f_1(x) - g_1(x)) + k$) we get from (17) that

$$(f_1 - g_1)(z_1) + k \leq \frac{1}{2}(f_0(z_1) - g_0(z_1) + d) + \frac{1}{2}(f_0(\tau_{a_0^{z_1,f_0}}(z_1)) - g_0(\tau_{a_0^{z_1,f_0}}(z_1) + d). \tag{19}$$

Note that if $f_0(z_1) - g_0(z_1) + d$ and $f_0(\tau_{a_0^{z_1,f_0}}(z_1)) - g_0(\tau_{a_0^{z_1,f_0}}(z_1) + d)$ have opposite signals, then we get a better rate

$$|\mathcal{G}(f_0) - \mathcal{G}(g_0)| \;=\; (f_1 - g_1)(z_1) + k \leq \frac{1}{2}|f_0 - g_0|. \tag{20}$$

During the iteration procedure this will happen from time to time for $f_n = \mathcal{G}^n(f_0)$ and $g_n = \mathcal{G}^n(u) = u$. This is a good explanation for the outstanding performance of the algorithm.

♢

Definition 14. *Given a Hölder potential A with a unique subaction $u \in \mathcal{C}$ consider the set $\mathfrak{B} \subset \mathcal{C}$ of functions f_0, such that, if $|f_0 - u| = |(f_0 - u) + \alpha_{f_0-u})|_0 = (f_0 - u)(r) + \alpha_{f_0-u}$, for some r, then,*

$$(f_0 - u)(r) \neq (f_0 - u)(\tau_2(r)) \text{ and } (f_0 - u)(r) \neq (f_0 - u)(\tau_2(r)).$$

The set $\mathfrak{B}$ is generic in $\mathcal{C}$. The proof of this fact is basically the same as the proof that $\mathfrak{A}$ is generic on $\mathcal{C} \times \mathcal{C}$ and will be not presented.

In the same way as before one can show that:

Theorem 15. *Given the function $f_0 \in \mathcal{C}$, assume $f_0 \in \mathfrak{B}$. In this case, if $|\mathcal{G}(f_0) - u| = |f_0 - u|$, then, $f_0 = u$. This implies that if $f_0 \neq u$, then*

$$|\mathcal{G}(f_0) - u| < |f_0 - u|. \tag{21}$$

Therefore, if $\mathcal{G}^n(f_0) \in \mathfrak{B}$ and $\mathcal{G}^n(f_0) \neq u$, then

$$|\mathcal{G}^{n+1}(f_0) - u| < |\mathcal{G}^n(f_0) - u|. \tag{22}$$

Given an initial f_0 from time to time $\mathcal{G}^n(f_0) \in \mathfrak{B}$ for some n, and then the next iterate will experience a better approximation to the calibrated subaction u.

Now we will prove that $\mathfrak{A}$ is generic. We will need first to state some preliminary properties which will be used later. We recall that the norm in $\mathcal{C}$ is given by $|f| = \inf_{d \in R} |f + d|_0$ and the distance in $\mathcal{C} \times \mathcal{C}$ is the max distance $d((f, g), (f', g')) := \max(|f - f'|, |g - g'|)$ which is equivalent to the product topology. We will show now that the set $\mathfrak{A}$ is generic in $\mathcal{C} \times \mathcal{C}$ with respect to this topology.

Consider $X = [0, 1]$ and the maps $\tau_2(x) = \frac{1}{2}x$ and $\tau_2(x) = \frac{1}{2}(x + 1)$. Let $\mathcal{F} = \{(f, g) |\ f, g \text{ are both continuous}\}$. Denote by β the map $\beta : X \times \mathcal{F} \to \mathbb{R}$ given by

$$\beta(x, f, g) = |f - g| - |f(x) - g(x)| + \min_{i \in \{0,1\}} \{|f - g| - |f(\tau_i(x)) - g(\tau_i(x))|\}.$$

We notice that $\beta(x, f, g) \geq 0$, and, moreover

- $\beta(x, f, g) = 0$, if and only if, $|f - g| = |f(x) - g(x)|$, and, $|f - g| = |f(\tau_1(x)) - g(\tau_1(x))|$ or $|f - g| = |f(\tau_2(x)) - g(\tau_2(x))|$;
- $\beta(x, f, g) > 0$, if and only if, one of the two conditions is true

  $|f - g| > |f(x) - g(x)|$, or,
  $|f - g| > |f(\tau_1(x)) - g(\tau_1(x))|$ and $|f - g| > |f(\tau_2(x)) - g(\tau_2(x))|$.

We define the set $\mathcal{O} \subset \mathcal{F}$ as being

$$\mathcal{O}_{\mathcal{F},\delta} = \{(f, g) \in \mathcal{F} | \beta(x, f, g) > 0,\ \forall x \in [\delta, 1 - \delta]\}.$$

By (8), if $d = -\frac{\max(f-g)+\min(f-g)}{2}$, then $|f-g| = |f-g+d|_0 = \frac{\max(f-g)-\min(f-g)}{2}$.
From the previous observation we conclude that for all $(f, g) \in \mathcal{O}_{\mathcal{F},\delta}$, if, x is such that $|f-g+d| = |f(x)-g(x)+d|$, then, $|f-g+d| \neq |f(\tau_1(x))-g(\tau_1(x))+d|$ and $|f - g + d| \neq |f(\tau_2(x)) - g(\tau_2(x)) + d|$.

To motivate our proof we are going to consider an explicit example where we made a perturbation of a pair $(f, g) \in \mathcal{C}$, but $\beta(x, f, g) = 0$, for some x.

Example 16. *Consider* $(f, g) \in \mathcal{C}$ *where*

$$f(x) = \begin{cases} \frac{16}{3}x - 2 & 0 \leq x \text{ and } x < 3/8 \\ 32x^2 - 36x + 9 & 3/8 \leq x \text{ and } x < 3/4 \\ 64x^2 - 104x + 42 & 3/4 \leq x \text{ and } x \leq 7/8 \\ -16x + 14 & 7/8 \leq x \text{ and } x \leq 1. \end{cases}$$

and $g(x) = 0$.

It is easy to see that for $x = 3/4$ *we have* $|f-0| = |f-0+1| = f(3/4)+1 = 1$, $f(\tau_2(3/4))+1 = 1$ *and* $f(\tau_1(3/4))+1 = 1$, *(see Figure 6) thus,* $\beta(3/4, f, 0) = 0$, *meaning that* $(f, 0) \notin \mathcal{O}_{\mathcal{F},\frac{1}{4}}$. *The same is true for* $x = 0$.

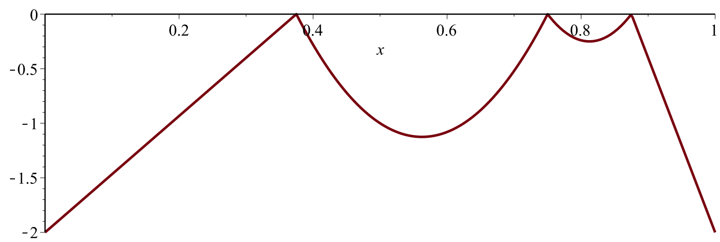

Figure 6: $f(x)$ of example 16

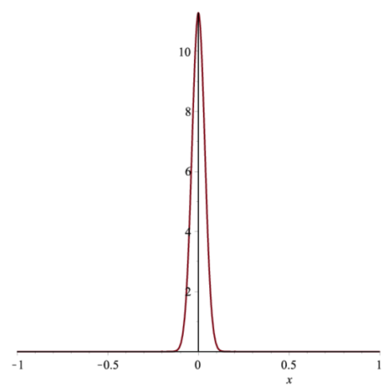

Figure 7: u_ε

In order to obtain the perturbation $(f_\varepsilon, g_\varepsilon)$ we consider an ε-concentrated approximation via Dirac function $u_\varepsilon(x) := \frac{1}{\varepsilon\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{\varepsilon^2}}$ (see Figure 7) and we also we define for $\varepsilon = 0.005$ the modifications (see Figure 8):

$$Q_\varepsilon(x) := \frac{1}{500}u_\varepsilon\left(x - (3/4 - 0.015)\right) \text{ and } W_\varepsilon(x) := -\frac{1}{1000}u_\varepsilon\left(x - (0 + 0.015)\right):$$

We set $f_\varepsilon(x) = f(x) + Q_\varepsilon(x) + W_\varepsilon(x)$ and $g_\varepsilon(x) = g(x)$. In this case $|f_\varepsilon - f| = |-Q_\varepsilon - W_\varepsilon| = (0.113 - (-0.226))/2 = 0.1695$ as we can see by the picture (see Figure 9).

As we can see, after the perturbation the maximum value is attained only for $x_0 = \frac{3}{4} - 0.015$ and for $x_1 = 0 + 0.015$ and neither of them are pre-image one of each other. Therefore, $(f_\varepsilon, g_\varepsilon) \in \mathcal{O}_{\mathcal{F},0}$ (see Figure 16).

Theorem 17. *Let $\Lambda \subset \mathcal{F}$ a compact subset. Then the set $\mathcal{O}_{\Lambda,\delta}$ is an open and dense set. In particular, $\mathcal{O}_\Lambda = \bigcap_{n>2}\mathcal{O}_{\Lambda,\frac{1}{n}}$ is a generic set.*

As a consequence, taking $\mathfrak{A} = \mathcal{O}_\Lambda$, it will follow:

Corollary 18. *The set $\mathfrak{A}$ is generic. In other words, generically, if $|f - g| = |f - g + d| = f(x_0) - g(x_0) + d$, then $f(\tau_i(x_0)) - g(\tau_i(x_0)) + d \neq f(x_0) - g(x_0) + d$, for $i = 1, 2$.*

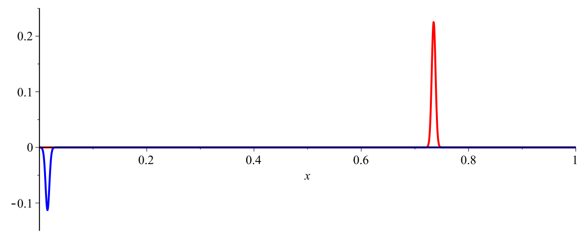

Figure 8: Q_ε (red) and W_ε(blue).

Proof. The first step in the proof of Theorem 17 is the openness of $\mathcal{O}_{\Lambda,\frac{1}{n}}$.

In this direction we observe that β is continuous because the min operation and the sup-norm are continuous. Taking $(f_0, g_0) \in \mathcal{O}_{\Lambda,\frac{1}{n}}$ we obtain $\beta(x, f_0, g_0) > 0,\ \forall x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, as we can see in the Figure 11.

Using the compactness and the continuity we can take $\alpha > 0$, such that, $\beta(x, f_0, g_0) > \alpha,\ \forall x \in [\frac{1}{n}, 1 - \frac{1}{n}]$. Therefore, if $(f, g) \in \mathcal{U}$, where $\mathcal{U}$ is an open neighborhood of (f_0, g_0), we get

$$\beta(x, f, g) - \frac{\alpha}{2} = \beta(x, f, g) - \beta(x, f_0, g_0) + \beta(x, f_0, g_0) - \alpha + \alpha - \frac{\alpha}{2} \geq$$

$$\leq \beta(x, f_0, g_0) - \beta(x, f, g) + \beta(x, f, g) - \alpha + \alpha - \frac{\alpha}{2} > -\varepsilon_x + 0 + \frac{\alpha}{2} > 0,$$

if we choose $\varepsilon_x < \frac{\alpha}{2}$, where ε_x is the continuity constant for the map $(f, g) \to \beta(x, f, g)$, for a fixed $x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$.

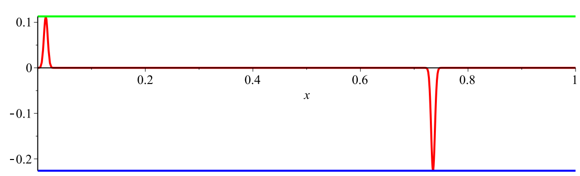

Figure 9: *Calculating* $|-Q_\varepsilon - W_\varepsilon|$.

Since the interval $\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ is compact we can take $0 < \varepsilon \leq \varepsilon_x,\ \forall x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$.

This proves that the set

$$\mathcal{U}_\delta :=$$

$$\left\{(f, g) \mid \text{ if } d((f, g), (f_0, g_0)) < \delta, \text{ then } |\beta(x, f, g) - \beta(x, f_0, g_0)| < \varepsilon,\ \forall x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]\right\}$$

is an open neighborhood of (f_0, g_0) in $\mathcal{O}_{\Lambda,\frac{1}{n}}$.

In order to prove the density of $\mathcal{O}_{\Lambda,\frac{1}{n}}$ we observe that if $x_0 \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, then $\frac{1}{2n} + \frac{i}{2} - x_0 \leq \tau_i(x_0) - x_0 \leq \frac{i}{2} + \frac{1}{2} - \frac{1}{2n} - x_0$. Thus $|\tau_i(x_0) - x_0| \geq \frac{1}{2n}$ for all $x_0 \in [\frac{1}{n}, 1 - \frac{1}{n}]$.

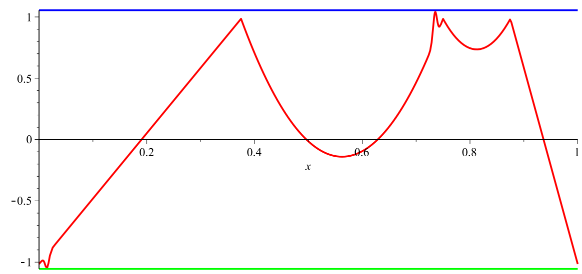

Figure 10: $|f_\varepsilon - g_\varepsilon| = |f_\varepsilon - g_\varepsilon - (-0.985)|_0 = (0.07 - (-2.04))/2 = 1.055$.

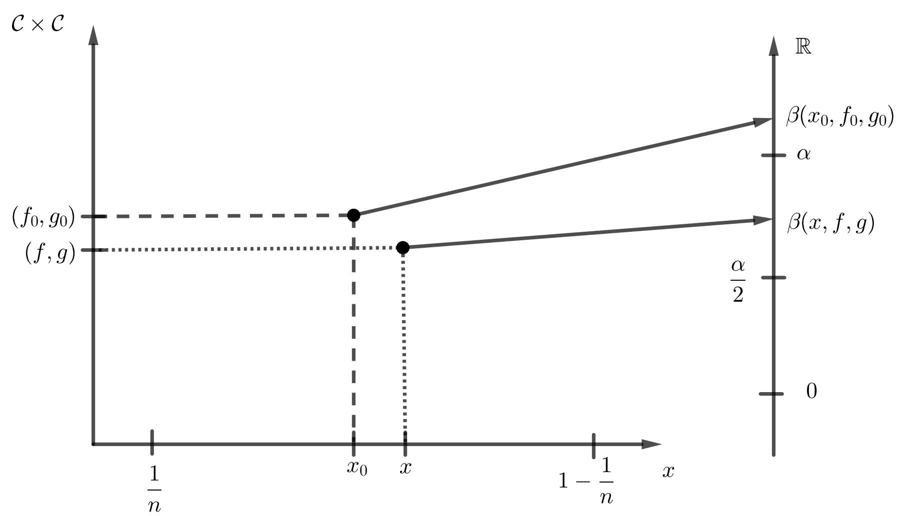

Figure 11: Approximating (x_0, f_0, g_0).

Using this estimate we can apply an ε-concentrated perturbations with $\varepsilon < \frac{1}{2n}$ (see Example 16 for a constructive approach) obtaining a pair $(f_\varepsilon, g_\varepsilon)$, in such way that, $g_\varepsilon = g$, x_0 and x_1 are the only points where $|f_\varepsilon - g_\varepsilon| = |f_\varepsilon(x_0) - g(x_0) + d| = |f_\varepsilon(x_1) - g(x_1) + d|$ and $x_0 \neq \tau_0(x_1), \tau_2(x_1)$, $x_1 \neq \tau_0(x_0), \tau_2(x_0)$.

In particular $\beta(x, f_\varepsilon, g) > 0$, for any $x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, which means that $(f_\varepsilon, g_\varepsilon) \in \mathcal{O}_{\Lambda, \frac{1}{n}}$. $\square$

4 Perturbation theory: close by the fixed point

In this section we analyse the question: when the calibrated subaction is unique is there an uniform exponential speed of approximation of the iteration $\mathcal{G}^n(f_0)$ to the subaction? The question makes sense close by the subaction u. The answer is no.

We will proceed a careful analysis of the action of $\mathcal{G}$ close by the fixed point $u \in \mathcal{C}$.

We denote $C^0(X) = \{f : X \to \mathbb{R} | f \text{ continuous } \}$.

In some examples we may consider a different dynamical system on $X = [0,1]$ given by the maps $\tau_i(x) = \frac{1}{2}(i+1-x)$, for $i = 0, 1$, which are the inverse branches of $T(x) = -2x \mod 1$.

Our main task is to evaluate the effect of a perturbation on the nonlinear operator ψ defined by

$$\psi(f)(x) = \max_{T(y)=x} (A+f)(y) = \max_{i=0,1} (A+f)(\tau_i(x))$$

for a fixed potential $A \in \mathcal{C}_k$.

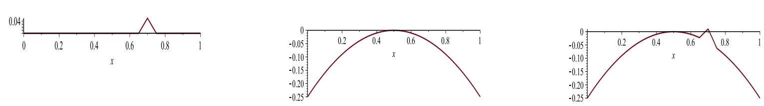

Figure 12: The graph of the function $\alpha_{0.05,0.7}$ in the left side, $f(x) = -(x-1/2)^2$ in the center and $f_{0.05} = f(x) + \alpha_{0.05,0.7}(x)$ in the right side.

The operator $H := H_A$ given by

$$H(f)(x) := \frac{1}{2} f(x) + \frac{1}{2} \psi(f)(x),$$

Note that $\mathcal{G} := \mathcal{G}_A$ is a normalized version of H

$$\mathcal{G}(f)(x) := H(f)(x) - \sup_{x \in X} H(f)(x).$$

It is usual to denote $c_f := \sup_{x\in X} H(f)(x)$ then $H(f)(x) = \mathcal{G}(f)+c_f$ (normalization means that $\sup_{x\in X} \mathcal{G}(f)(x) = 0$). Sometimes it is useful to look at the operator $H - Id$ given by $(H - Id)(f) := \frac{1}{2}\psi(f)(x) - \frac{1}{2}f(x)$.

We assume that there exists **a unique function** $u \in \mathcal{C}_k$ such that $\mathcal{G}(u) = u$ (this is true if the maximizing probability is unique). Thus, $H(u)(x) = \mathcal{G}(u) + c_u = u(x) + c_u$ where $c_u := \sup_{x\in X} H(u)(x)$.

The above equation is equivalent to $u(x) + c_u = \frac{1}{2}u(x) + \frac{1}{2}\psi(f)(x)$ which is equivalent to the sub-action equation

$$u(x) = \max_{i=0,1}(A - 2c_u + u)(\tau_i(x)).$$

We can assume that $m_A = 2c_u = 0$ (by adding a constant to A) and then, $H(u) = u$. It is useful to observe that under this assumption we also get $\psi(u) = u$.

We start with a local perturbation lemma.

Let $\alpha_{\varepsilon,a} : X \to \mathbb{R}$ be a piecewise linear bump function defined by

$$\alpha_{\varepsilon,a}(x) = \begin{cases} 0, & 0 \le x \le a - \varepsilon \\ kx - k(a-\varepsilon), & a-\varepsilon \le x \le a \\ -kx + k(a+\varepsilon), & a \le x \le a+\varepsilon \\ 0, & a+\varepsilon \le x \le 1, \end{cases}$$

where $a \in (0,1)$ and $\varepsilon > 0$ is arbitrary small.

Lemma 19. *If $f \in \mathcal{C}_k$, then $f_\varepsilon = f(x) + \alpha_{\varepsilon,a}(x) \in \mathcal{C}_k$. Moreover, $f_\varepsilon(x) \ge f(x)$ and $f_\varepsilon(x) = f(x)$ outside of the interval $[a-\varepsilon,\ a+\varepsilon]$. Finally, $|f_\varepsilon - f| = \frac{k\varepsilon}{2}$.*

Proof. The proof is straightforward because $|f_\varepsilon - f| = |\alpha_{\varepsilon,a}|$ and $0 \le \alpha_{\varepsilon,a}(x) \le k\varepsilon$. $\square$

We will make the perturbations by choosing a fixed point $x_0 \ne 0, 1, 1/2$ in X and $\varepsilon > 0$, such that, the intervals $I = [x_0 - \varepsilon,\ x_0 + \varepsilon]$ and $T(I) = [T(x_0) - 2\varepsilon,\ T(x_0) + 2\varepsilon]$ are disjoint. Then, we take $f_\varepsilon = f(x) + \alpha_{\varepsilon,a}(x)$ and we will try to estimate $\psi(f_\varepsilon)$.

Lemma 20. *$\psi(f_\varepsilon) = \psi(f)$ outside of $T(I)$.*

Proof. We notice that A remains unchanged and $[T(x_0) - 2\varepsilon,\ T(x_0) + 2\varepsilon] = T([x_0 - \varepsilon,\ x_0 + \varepsilon])$. Therefore, for any y such that $T(y) = x$ we can not have $y \in [x_0 - \varepsilon,\ x_0 + \varepsilon]$. Thus, $f_\varepsilon(y) = f(y)$, proving that $\psi(f_\varepsilon) = \psi(f)$. $\square$

Another question is about what happens in $T(I)$. For any x in this interval one of its pre-images y belongs to I therefore $f_\varepsilon(y) \ge f(y)$. Thus, ψ may change.

We recall that **a turning point** x (see also [34] and [35]) is a point where $(A + f)(\tau_1(x)) = (A + f)(\tau_2(x))$. If x is not a turning point then there exists **a dominant realizer**, that is, $(A + f)(\tau_1(x)) > (A + f)(\tau_2(x))$, or, $(A + f)(\tau_1(x)) < (A + f)(\tau_2(x))$.

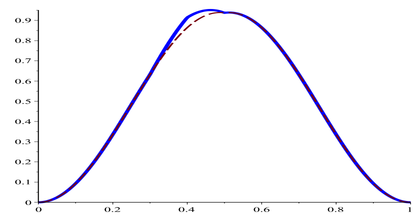

Figure 13: The graph of the functions $\psi(f_\varepsilon)$ (blue line) and $\psi(f)$ (traced line) where, $A(x) = \sin^2(2\pi x)$, $f(x) = -(x-1/2)^2$ and $f_{0.1} = f(x) + \alpha_{0.1,0.7}(x)$. The difference occurs only in the interval $T(I) = [0.2,\ 0.6]$ because $T(0.7) = 0.4$ and $I = [0.6,\ 0.8]$.

Lemma 21. *Suppose that x_0 is such that $T(x_0)$ is not a turning point and j is the dominant symbol. Let $i \in \{0,\ 1\}$ be such that $\tau_i(T(x_0)) = x_0$. We have two possible cases:*

- *If $j \neq i$, then $\psi(f_\varepsilon)(x) = \psi(f)(x)$, for any $x \in T(I)$.*
- *If $j = i$, then $\psi(f_\varepsilon)(x) = \psi(f)(x) + \alpha_{\varepsilon,x_0}(\tau_j(x)) \geq \psi(f)(x)$, for any $x \in T(I)$ and $|\psi(f_\varepsilon)(x) - \psi(f)(x)| = \frac{k\varepsilon}{2}$.*

Proof. In the first case, in order to fix ideas we suppose, without lost of generality, $j = 0$ and $i = 1$, then $\tau_2(T(x_0)) = x_0$ and $(A+f)(\tau_1(T(x_0))) > (A+f)(\tau_2(T(x_0)))$. By the continuity of $A+f$ we can choose $\varepsilon > 0$ small enough in order to have $(A+f_\varepsilon)(\tau_1(x)) > (A+f_\varepsilon)(\tau_2(x))$, for all $x \in T(I)$. Therefore, $\psi(f_\varepsilon)(x) = (A+f_\varepsilon)(\tau_1(x)) = (A+f)(\tau_1(x)) = \psi(f)(x)$, for any $x \in T(I)$.
In the second case, $\tau_j(T(x_0)) = x_0$ and $(A+f)(\tau_j(T(x_0))) > (A+f)(\tau_i(T(x_0)))$. Once more we use the continuity of $A+f$ to choose $\varepsilon > 0$ small enough in order to have $(A+f_\varepsilon)(\tau_j(x)) > (A+f_\varepsilon)(\tau_i(x))$, for all $x \in T(I)$. Therefore, $\psi(f_\varepsilon)(x) = (A+f_\varepsilon)(\tau_j(x)) = (A+f)(\tau_j(x)) + \alpha_{\varepsilon,x_0}(\tau_j(x)) = \psi(f)(x) + \alpha_{\varepsilon,x_0}(\tau_j(x))$, for any $x \in T(I)$. $\square$

Our first task is to compare $H(f)$ and $H(f_\varepsilon)$. We can always assume that $T(I)$ and I are disjoint so the perturbation $f \to f_\varepsilon$ acts separately in each one as described by the previous lemmas.

Lemma 22. *Let f_ε a perturbation of f and x_0 such that is not a pre-image of a turning point (with respect to f). Then, $H(f)(x) \leq H(f_\varepsilon)(x)$, with equality only outside of $[T(x_0)-2\varepsilon,\ T(x_0)+2\varepsilon] \cup [x_0-\varepsilon,\ x_0+\varepsilon]$. Moreover, $H(f_\varepsilon)(x) - H(f)(x) \leq \frac{k\varepsilon}{2}$. (We can prove similar results for $(H - Id)$.)*

The proof is a direct consequence of the previous lemmas.

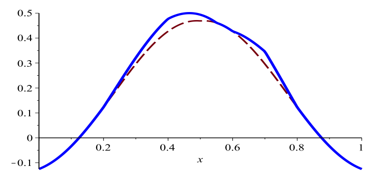

Figure 14: The graph of the functions $H(f_\varepsilon)$ (blue line) and $H(f)$ (traced line) where, $A(x) = \sin^2(2\pi x)$, $f(x) = -(x-1/2)^2$ and $f_{0.1} = f(x)+\alpha_{0.1,0.7}(x)$. The difference occurs only in the interval $[0.2,\ 0.6] \cup [0.6,\ 0.8]$ because $T(0.7) = 0.4$.

We want to study the relation between $|\mathcal{G}(f) - u|$ and $|f - u|$. We also want to see what happens when we make a perturbation $f \to f_\varepsilon$.

We start by choosing $d = \alpha_{f-u}$ such that $\delta = |f - u| = |f - u + d|_0$, then

$$-\delta \leq f(y) - u(y) + d \leq \delta,$$

for all $y \in X$. Multiplying the above by $1/2$ we conclude that

$$-\frac{\delta}{2} \leq \frac{1}{2}(f(y) - u(y)) + \frac{d}{2} \leq \frac{\delta}{2}.$$

Adding $A(y)$ we obtain the inequalities

$$-\delta \leq A(y) + f(y) - (A(y) + u(y)) + d \leq \delta,$$

and

$$-\delta + (A(y) + u(y)) \leq A(y) + f(y) + d \leq \delta + (A(y) + u(y)).$$

Taking the supremum in y, such that, $T(y) = x$, we get $-\delta + \psi(u)(x) \leq \psi(f)(x) + d \leq \delta + \psi(u)(x)$. Multiplying by $1/2$ we conclude that

$$-\frac{\delta}{2} \leq \frac{1}{2}(\psi(f)(x) - \psi(u)(x)) + \frac{d}{2} \leq \frac{\delta}{2}.$$

Note that

$$\mathcal{G}(f)(x) - u(x) + d = \mathcal{G}(f)(x) - \mathcal{G}(u)(x) + d =$$

$$= \frac{1}{2}(f(x) - u(x)) + \frac{1}{2}(\psi(f)(x) - \psi(u)(x)) - c_f + c_u + d =$$

$$\frac{1}{2}(f(x) - u(x) + d) + \frac{1}{2}(\psi(f)(x) - \psi(u)(x) + d) - c_f.$$

Using the inequalities

$$-\frac{\delta}{2} \leq \frac{1}{2}(\psi(f)(x) - \psi(u)(x)) + \frac{d}{2} \leq \frac{\delta}{2},$$

$$-\frac{\delta}{2} \leq \frac{1}{2}(f(y) - u(y)) + \frac{d}{2} \leq \frac{\delta}{2},$$

and, the fact that $c_u = 0$, we finally obtain

$$-\frac{\delta}{2} - \frac{\delta}{2} - c_f \leq \mathcal{G}(f)(x) - u(x) + d \leq \frac{\delta}{2} + \frac{\delta}{2} - c_f,$$

and,

$$-\delta \leq \mathcal{G}(f)(x) - u(x) + (d + c_f) \leq \delta.$$

Therefore,

$$|\mathcal{G}(f)(x) - u(x) + (d + c_f)| \leq \delta = |f - u|,$$

for all $x \in X$.

From this fundamental inequality we get a very important result about the operator $\mathcal{G}$.

We recall that $|\mathcal{G}(f)(x) - u(x)| = \min_\gamma |\mathcal{G}(f) - u + \gamma|_0 \leq |\mathcal{G}(f) - u + (d + c_f)|_0 = \sup_{x \in X} |\mathcal{G}(f)(x) - u(x) + (d + c_f)| \leq |f - u|$.

Theorem 23. *Let $\mathcal{G}$ be the operator associated to A and u the fixed point ($\mathcal{G}(u)(x) = u(x)$), then,*

a) The contraction rate is controlled by $H - Id$;

b) $|H(f) - f|_0 \leq 2|f - u|$;

c) If $|H(f) - f|_0 = \beta$, then $|\mathcal{G}(f)(x) - u(x) + (d + c_f)|_0 \geq |f - u| - \beta$.

Proof. (a) We recall that $\mathcal{G}(f)(x) + c_f = H(f)$, thus,

$$|\mathcal{G}(f)(x) - u(x) + (d + c_f)| \leq |f - u|$$

$$|\mathcal{G}(f)(x) + c_f - f(x) + f(x) - u(x) + d| \leq \sup_{x \in X} |f(x) - u(x) + d|$$

$$|[H(f) - f(x)] + f(x) - u(x) + d| \leq \sup_{x \in X} |f(x) - u(x) + d|$$

$$\sup_{x \in X} |[H(f) - f(x)] + f(x) - u(x) + d| \leq \sup_{x \in X} |f(x) - u(x) + d|.$$

(b) Here we use the triangular inequality

$$|H(f) - f(x)| \leq |[H(f) - f(x)] + f(x) - u(x) + d| + |f(x) - u(x) + d| \leq 2|f - u|.$$

(c) Using the triangular inequality we obtain

$$|f-u| = |f-u+d|_0 \leq |f-u+d+\mathcal{G}(f)(x)+c_f-f(x)-(\mathcal{G}(f)(x)+c_f-f(x))|_0 \leq$$

$$\leq |\mathcal{G}(f)(x)+c_f-f(x)+f-u+d|_0 + |\mathcal{G}(f)(x)+c_f-f(x)|_0 =$$

$$= |\mathcal{G}(f)(x)-u+(d+c_f)|_0 + |H(f)(x)-f(x)|_0 = |\mathcal{G}(f)(x)-u+(d+c_f)|_0 + \beta,$$

or, equivalently,

$$|\mathcal{G}(f)(x)-u+(d+c_f)|_0 \geq |f-u|-\beta.$$

□

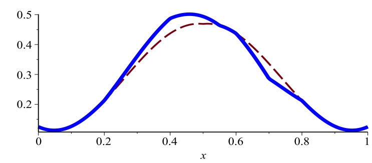

Figure 15: Functions $(H-Id)(f_\varepsilon)$ (blue line) and $(H-Id)(f)$ (traced line) where, $A(x) = \sin^2(2\pi x)$, $f(x) = -(x-1/2)^2$ and $f_{0.1} = f(x) + \alpha_{0.1,0.7}(x)$. The difference occurs only in the interval $[0.2,\ 0.6]$, where the perturbation is bigger, and, the interval $[0.6,\ 0.8]$, where the perturbation is smaller, because $T(0.7) = 0.4$.

We are dealing with a kind of technical problem: $|p(x)+q(x)| \leq |q|_0$, $\forall x \in X$, where $\max q = -\min q$. In our case, $p(x) = H(f) - f(x)$ and $q(x) = f(x) - u(x) + d$ are continuous functions. The first observation is that $|p(x)+q(x)| \leq |q|_0$, $\forall x \in X$, is equivalent to $-|q|_0 - q(x) \leq p(x) \leq |q|_0 - q(x)$. From this we can get interesting examples.

Example 24. *Consider* $p(x) = -4\,(x-1/2)^2$ *and* $q(x) = \cos(2\,\pi\,x)$. *It is easy to see that* $|q|_0 = \max q = -\min q = 1$ *and the inequality* $-1 - q(x) \leq p(x) \leq 1 - q(x)$ *is described in the Figure 16. A simple calculation shows that* $|p+q|_0 = 1 = |q|_0$, *but* $|p+q| = |p+q+0.414|_0 = 0.586$.

The property $\max q = -\min q$ *means that* $|q| = |q+0|_0$, *therefore,* $|p+q| = 0.586 < 1 = |q|$.

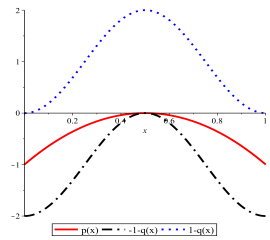

Figure 16: Functions $-1 - q(x)$ and $1 - q(x)$.

Lemma 25. *Consider* $|p(x) + q(x)| \leq |q|_0$, $\forall x \in X$, *with* $\max q = -\min q$. *Then, there exists* $z \in X$, *such that,* $p(z) = 0$. *In particular, taking* $p(x) = H(f) - f(x)$ *and* $q(x) = f(x) - u(x) + d$, *we have*

$$f(z) = \max_{T(y)=z} A(y) + f(y).$$

Proof. We already know that there exists x_0 such that $|q|_0 = q(x_0)$, therefore, $p(x_0) + q(x_0) \leq |q|_0 = q(x_0)$, or, equivalently, $p(x_0) \leq 0$. Analogously, there exists x_1 such that $|q|_0 = -q(x_1)$ and $p(x_1) \geq 0$. Unless $q = cte$ we can always suppose that $x_0 \neq x_1$. If $p(x_0) = 0$ or $p(x_1) = 0$ the problem is solved. Otherwise, if $p(x_0) < 0$ and $p(x_1) > 0$ the intermediate value theorem for continuous functions claims that there exists $z \in [x_0,\ x_1]$, such that, $p(z) = 0$.

Note that for $p(x) = H(f) - f(x)$, the equation $p(z) = 0$ is equivalent to $f(z) = \max_{T(y)=z} A(y) + f(y)$. $\square$

The behaviour of $|\mathcal{G}(f)(x) - u + (d + c_f)|_0$ may be very different from $|\mathcal{G}(f) - u|$. On the one hand $|\mathcal{G}(f) - u| \leq |\mathcal{G}(f) - u + (d + c_f)|_0 \leq |f - u|$ and on the other hand we can find f arbitrarily close to u, such that, $|\mathcal{G}(f) - u| = \frac{1}{4} \leq |f - u|$.

Figure 17: In the left side the graph of u and in the right side the graph of f_ε.

Lemma 26. *Let u be the only sub-action of A ($m_A = 0$). Let $f_\varepsilon = u + \alpha_{\varepsilon,x_0}$ a perturbation of f and take x_0 not a pre-image of a turning point (with respect to f). Then, $|\mathcal{G}(f_\varepsilon) - u| = \frac{1}{2}|f_\varepsilon - u|$ and $|f_\varepsilon - u| = \frac{k\varepsilon}{2}$.*

Proof. First, we observe that $|f_\varepsilon - u| = |\alpha_{\varepsilon,x_0}| = \frac{\max \alpha_{\varepsilon,x_0} - \min \alpha_{\varepsilon,x_0}}{2} = \frac{k\varepsilon - 0}{2} = \frac{k\varepsilon}{2}$.

Rewriting $|\mathcal{G}(f_\varepsilon) - u|$ we obtain

$$|\mathcal{G}(f_\varepsilon) - u| = |H(f_\varepsilon) - c_{f_\varepsilon} - u| = |H(f_\varepsilon) - u| = |\frac{1}{2}f_\varepsilon + \frac{1}{2}\psi(f_\varepsilon) - u| =$$

$$= |\frac{1}{2}(u + \alpha_{\varepsilon,x_0}) + \frac{1}{2}\psi(f_\varepsilon) - \psi(u)| = |\frac{1}{2}\alpha_{\varepsilon,x_0} + \frac{1}{2}(\psi(f_\varepsilon) - \psi(u))|.$$

The function α_{ε,x_0} is zero outside of the set $[x_0 - \varepsilon,\ x_0 + \varepsilon]$, and, $\psi(f_\varepsilon) - \psi(u) = 0$ outside of the set $[T(x_0) - 2\varepsilon,\ T(x_0) + 2\varepsilon]$ by Lemma 20.

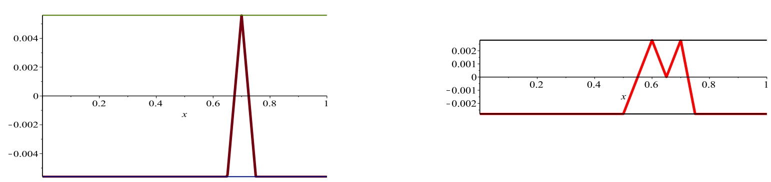

Figure 18: In the left the graph of $f_\varepsilon\,(x) - u\,(x) - \frac{k\varepsilon}{2}$ and in the right the one for $1/2\, f_\varepsilon\,(x) + 1/2\,\psi\,(f_\varepsilon)\,(x) - u\,(x) - \frac{k\varepsilon}{4}$.

Therefore, the min $\frac{1}{2}\alpha_{\varepsilon,x_0} + \frac{1}{2}(\psi(f_\varepsilon) - \psi(u)) = 0$, and, max $\frac{1}{2}\alpha_{\varepsilon,x_0} + \frac{1}{2}(\psi(f_\varepsilon) - \psi(u)) = \frac{k\varepsilon}{2}$. By definition $|\mathcal{G}(f_\varepsilon)(x) - u| = \frac{k\varepsilon}{4}$. $\square$

Example 27. *Consider the dynamics $T(x) = -2x \mod 1$.*

Let $A(x) = -(x - \frac{1}{2})^2 + \frac{1}{36}$ be the potential and u the subaction (see Figures 17, 18 and 19)

$$u(x) = \begin{cases} -1/3\,x^2 + x/9, & 0 \le x \le 1/2 \\ -1/3\,x^2 + 5/9\,x - 2/9, & 1/2 \le x \le 1. \end{cases}$$

From the graph of u we see that $x = \frac{1}{2}$ is the only turning point. Therefore, we can take $x_0 = 0.7$, $\varepsilon = 0.05$ and $f_\varepsilon = u + \alpha_{0.05,0.7}$. We also know that $Lip(A) = 1$ and $Lip(u) = \frac{2}{9}$, thus, we can take $k = \frac{2}{9}$.

As predicted $|\mathcal{G}(f_\varepsilon)(x) - u| = \frac{k\varepsilon}{4} = 0.0028$ and $|f_\varepsilon - u| = \frac{k\varepsilon}{2} = 0.0056$.

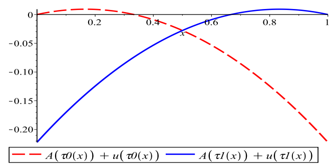

Figure 19: The graph of the functions $(A+u)(\tau_1(x))$ and $(A+u)(\tau_2(x))$.

From Lemma 26 we get

Corollary 28. *For any $\varepsilon > 0$ there exists a function f which is ε-close to u, such that, $\mathcal{G}$ contracts by $1/2$ in f, that is, $|\mathcal{G}(f) - u| = \frac{1}{2}|f - u|$.*

We may ask if there exists some neighborhood of u where $|\mathcal{G}(f_\varepsilon)(x) - u| \leq (1-\delta)|f_\varepsilon - u|$. The answer is no. Actually, it is the opposite of that. We can exhibit a sequence $f_\varepsilon \to u$, and, $|\mathcal{G}(f_\varepsilon)(x) - u| = |f_\varepsilon - u|$.

Example 29. *We will show an example where $|\mathcal{G}(f_\varepsilon) - \mathcal{G}(u)| = |f_\varepsilon - u|$, $\epsilon > 0$, for f_ε as close as you want to the calibrated subaction u.*

Consider again the dynamics $T(x) = -2x \pmod 1$. Let $A(x) = -(x - \frac{1}{2})^2 + \frac{1}{36}$ be the potential and u the subaction

$$u(x) = \begin{cases} -1/3\,x^2 + x/9, & 0 \leq x \leq 1/2 \\ -1/3\,x^2 + 5/9\,x - 2/9, & 1/2 \leq x \leq 1. \end{cases}$$

We fix $x_0 = \frac{2}{3}$. The function α_{ε,x_0} is zero outside of $I = [\frac{2}{3} - \varepsilon,\ \frac{2}{3} + \varepsilon]$ and $\psi(f_\varepsilon) - \psi(u) = 0$ outside of $T(I)$ by Lemma 20.

We know that $T(\frac{1}{3}) = \frac{1}{3}$ and $T(\frac{2}{3}) = \frac{2}{3}$. As we can see in the Figure 19, $\{0,\ \frac{1}{2},\ 1\}$, are the only turning points and the dominant symbol in $x_0 = 2/3$ is $j = 1$. Also, $\tau_2(T(\frac{2}{3})) = \frac{2}{3}$, and thus $i = 1 = j$.

Once more

$$|\mathcal{G}(f_\varepsilon) - u| = \left| \frac{1}{2}\alpha_{\varepsilon,x_0} + \frac{1}{2}(\psi(f_\varepsilon) - \psi(u)) \right|.$$

Since $I \subset T(I)$, we get, by Lemma 21, that α_{ε,x_0} attains the value $k\varepsilon$ and $\psi(f_\varepsilon)(x) - \psi(u)(x) = \alpha_{\varepsilon,x_0}(\tau_2(x))$ attains the value $k\varepsilon$ at least in x_0. Thus, $\frac{1}{2}\alpha_{\varepsilon,x_0} + \frac{1}{2}(\psi(f_\varepsilon) - \psi(u))$ attains the value $\frac{k\varepsilon}{2} = |f_\varepsilon - u|$ (see Figure 20 for $\varepsilon = 0.01$ and $x_0 = \frac{2}{3}$).

Therefore, $|\mathcal{G}(f_\varepsilon) - u| \geq |f_\varepsilon - u|$.

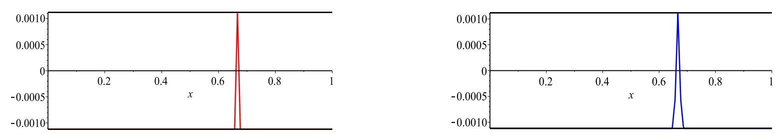

Figure 20: In the left side the graph of $f_\varepsilon\,(x) - u\,(x) - \frac{k\varepsilon}{2}$ and in the right side the graph of $1/2\, f_\varepsilon\,(x) + 1/2\, \psi\,(f_\varepsilon)\,(x) - u\,(x) - \frac{k\varepsilon}{2}$.

5 Approximating the maximizing probability

In this section we will show how one can get the maximizing probability μ from a limit procedure when the calibrated subaction u is explicitly known.

Suppose $u \in \mathcal{C}$ is the subaction for the generic potential A. Given a point x_0 there exist (at least one) x_1, such that,

$$u(x_0) = A(x_1) + u(x_1) - m(A),$$

where $T(x_1) = x_0$.

This means that there exists $a_0 \in \{1, 2\}$, such that, and $\tau_{a_0}(x_0) = x_1$. We say that x_1 is a realizer for x_0. In the same way, for fixed x_0, given the realizer x_1 there exists an $x_2 = \tau_{a_1}(x_1)$, for some $a_1 \in \{1, 2\}$, which is a realizer for x_1. In this way $\tau_{a_1} \circ \tau_{a_0}(x_0) = x_2$.

By induction, for fixed x_0 and for each n we have a sequence of realizers which are described by an element in $(a_0, a_1, .., a_{n-1}) \in \{1, 2\}^n$, such that, $x_n = \tau_{a_{n-1}} \circ .... \circ \tau_{a_1} \circ \tau_{a_0}(x_0)$ is a realizer for x_{n-1}.

We say that x_1 is the first realizer of x_0, and, that x_2 is the second realizer of x_0 and so on.

We denote by $a(x_0) = (a_0, a_1, ..., a_n, ...) \in \{1, 2\}^{\mathbb{N}}$ the string derived by this procedure. We call $a(x) \in \{1, 2\}^{\mathbb{N}}$ the string realizer of x.

The string do not have to be unique but we fix one element $a(x)$ by convention.

Associated to this string there exist a sequence $x_0, x_1, x_2, ..., x_n, ...$ of elements on S^1, such that $T(x_n) = x_{n-1}$ and x_n is a realizer for x_{n-1}.

We say that x_0 is the initial point of the realizer process that gave origin to the sequence $x_0, x_1, x_2, ..., x_n, ...$.

Proposition 30. *Assume the maximizing probability for A is unique. For each x_0 denote by*

$$\mu_n^{x_0} = \mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{x_j},$$

where x_j, $j \in \mathbb{N}$, is the sequence associated to x_0 (as above). Then, for any x_0 the sequence $\mu_n^{x_0}$ converges to the maximizing probability μ.

Proof: For a proof see Proposition 7 in [36] or in [5].

We will present in Proposition 34 a more robust version of the above result.

Proposition 31. *Assume A has a unique maximizing probability.*

Consider a sequence of initial points x_0^k, $k \in \mathbb{N}$, and a sequence of times $N_k \to \infty$, as $k \to \infty$. Denote for each k the realizer sequence with initial point x_0^k by

$$x_0^k, x_1^k, x_2^k, ..., x_{N_k}^k, ...$$

Then, given a open neighborhood Λ of the Mather set, it is not possible that all points x_j^k, $0 \leq j \leq N_k$, $N_k \to \infty$, are not in Λ.

Proof:

For each k denote

$$\mu_k = \frac{1}{N_k} \sum_{j=0}^{N_k - 1} \delta_{x_j^k}.$$

As the set of probabilities on S^1 is sequentially compact there exists a convergent subsequence $\mu_k \to \rho$, when $k \to \infty$.

We will show that ρ is a maximizing probability which is a contradiction because its support is outside K.

Given a continuous function f we have that

$$\int (f \circ T) d\rho = \lim_{k \to \infty} \int f(T(x)) d\mu_k(x) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{j=0}^{N_k - 1} f(T(x_j^k)) =$$

$$\lim_{k \to \infty} \frac{1}{N_k} [\, f(T(x_0^k)) + f(x_0^k) + f(x_1^k) + ... + f(x_{N_k-2}^k) \,] =$$

$$\lim_{k \to \infty} \frac{1}{N_k} [f(x_0^k) + ... + f(x_{N_k-2}^k) + f(x_{N_k-1}^k)] + \frac{1}{N_k} [f(T(x_0^k)) - f(x_{N_k-1})] = \int f \, d\rho.$$

Therefore, ρ is T-invariant

Moreover,

$$\int (A - m(A)) d\rho = \lim_{k \to \infty} \int (A(x) - m(A)) d\mu_k(x) =$$

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} (A(x_j^k) - m(A)) = \lim_{k\to\infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} (u(x_{j+1}^k) - u(x_j^k)) =$$

$$\lim_{k\to\infty} \frac{1}{N_k} (u(x_{n_K}^k) - u(x_0^k)) = 0.$$

Therefore, $\rho = \mu$ is a maximizing probability and this is a contradiction by uniqueness of the maximizing probability for A.

$\square$

Definition 32. *We say that a Hölder potential A is* **good** *if the rate function R associated to the calibrated subaction u satisfies the property: if x is not in the Mather set but $T(x)$ is in the Mather set, then, $R_A(x) > 0$.*

Theorem 33. *Generically the potential A is good.*

For a proof see [18] or Corollary 3 in section 12.3 in [10].

Proposition 34. *Suppose the potential A is good and the maximizing probability is unique. Given $\epsilon > 0$, there exist a compact neighborhood of size ϵ of the Mather set, such that, for any x_0 the associated sequence x_j, $j \in \mathbb{N}$, is such that $x_j \in \Lambda$ for all j large enough.*

There exists an $N > 0$, such that, for any x_0 and any $j > N$, we get that the associated sequence x_j^0, $j \in \mathbb{N}$ satisfies $x_j^0 \in K$, for all $j > N$.

Therefore, uniformly on x_0 we have that $\mu_n^{x_0}$ converges, when $n \to \infty$, to the maximizing probability.

Proof:

As the potential A is good one can get $\epsilon > 0$ and a compact neighborhood $\Lambda$of the Mather set such that $R(x) < \epsilon$, for all $x \in K$. Moreover, we can assume that $r(y) > 2\,\epsilon$ for all y not in the Mather set, such that, $T(y)$ is on the Mather set.

We may assume that $T^{-1}(K) \cap K$ is inside K.

From Proposition 30 given a point x the associated sequence $\mu_n^x(K)$ will be positive for some large n. Therefore, for some large j_0 some element x_{j_0} (obtained at level j_0 on the realizer process beginning on x) will be on K.

The point x_{j_0} has two preimages, let's say $y_1 \in K$ and y_2 not in K

By hypothesis

$$R(y_1) = u(x_{j_0}) - u(y_1) - A(y_1) + m(A) < R(y_2) = u(x_{j_0}) - u(y_2) - A(y_2) + m(A).$$

From this we get

$$u(y_1) + A(y_1) - m(A) > u(y_2) + A(y_2) - m(A).$$

In this way we get that x_{j_0+1} is on K. Therefore, by induction, all $x_j \in K$ for $j \geq j_0$.

If the maximizing probability is unique and is a periodic orbit then the realizer $a(x) \in \{1,2\}^{\mathbb{N}}$ will be a periodic orbit for the shift with the same period of the maximizing periodic orbit. This orbit on the symbolic space is called the dual orbit.

These kind of duality ideas are presented in several papers as [34], [35] and [10].

□

6 Appendix. Implementation of the algorithm via a mesh of points

In this section we will describe the numerical procedure we employed. Assume that the calibrated subation is unique up to add constants.

Given a potential A denote the operator $\mathcal{G}$ (without centering our initial function F by adding a constant).

$$\mathcal{G}(F)(x) = \frac{1}{2} \max_{T(y)=x} [A(y) + F(y)] + F(x)$$

Consider a potential A and two inverse branches of some transformation T which we denote by τ_1, τ_2. We want to approximate numerically the pair of solutions $m(A)$ and V satisfying $V(x) + m(A) = \max_i [A \circ \tau_i(x) + V \circ \tau_i(x)]$ in a given interval I.

We first define $\Omega = \{x_1, x_2, x_3, ..., x_N\}$ a discretization of the interval I. In general, we relate these points by a one-to-one, non-decreasing map $f : \{1, 2, ..., N\} \to \Omega$.

As an example, if $I = [0, 1]$ we could define $f(n) = n/N$ and

$$\Omega = \{f(1), f(2)..., f(N)\},$$

in this case, we would have $f^{-1}(n) = Nn$, where N is the chosen number of points in Ω.

Now, we choose a starting point for our algorithm. In our examples, we chose $F_0(x) = 0$. We want to compute a sequence of functions $F_n(x) = \mathcal{G}^n(F_0)(x) - C_n$, where C_n is a constant such that $\max_{x \in I} F_n(x) = 0$. To do so, we denote F_n by a vector, which approximates the function obtained in the form

$$F_n = (F_n(x_1), F_n(x_2)..., F_n(x_N))$$

(Consider as an example the case where $\Omega = \{x_1, x_2, x_3\}$, with $F = (8, 2, 7)$, which we mean that $F(x_1) = 8$ $F(x_2) = 2$ $F(x_3) = 7$).

Now, given some injective $\tau : I \to I$ we compute

$$F_n \circ \tau = (F_n \circ \tau(x_1), F_n \circ \tau(x_2), ..., F_n \circ \tau(x_N)) = (F_n(x_{j_1}), F_n(x_{j_2}), ..., F_n(x_{j_N})).$$

Where $x_{j_k} \in \Omega$ minimizes $|\tau(x_k) - x_{j_k}|$, and so, x_{j_k} is our best approximation of $\tau(x_k)$ within Ω (for $\tau(x_k)$ need not be in Ω). When given a non-decreasing one-to-one map $f : \{1, 2, ..., N\} \to \Omega$ where the points are equidistant, we mostly computed x_{j_k} simply by making $j_k = \lceil f^{-1}(\tau(x_k)) \rceil$, the closest integer to $f^{-1}(\tau(x_k))$. Now, given the starting point F_0, we define the numerical vector $F_1 = (F_1(x_1), ..., F_1(x_N))$ where

$$F_1(x_k) = G(F_0)(x_k) = \frac{\max_{i\in\{1,2\}}[A(\tau_i(x_k)) + F_0 \circ \tau_i(x_k)] + F(x_k)}{2}.$$

We compute $A(\tau_2(x_k))$ by its definition and $F_0 \circ \tau_i$ by the procedure described above. We then compute the constant $C_1 = \max_k F_1(x_k)$, $k \in \{1, 2, ..., N\}$, and we redefine F_1 as

$$F_1(x_k) = G(F_0)(x_k) - C_1.$$

In this way we obtain $\max_k F_1(x_k) = 0$. Then, we repeat the procedure for $F_2 = (F_2(x_1), ..., F_2(x_N))$ by taking

$$F_2(x_k) = G(F_1)(x_k) - C_2.$$

We check if F_n is an adequate approximation of some sub-action V by the error

$$\epsilon = \max_{k\in\{1,2,...,N\}} \| \max_{i\in\{1,2\}}[A \circ \tau_i(x_k) + F_n \circ \tau_i(x_k) - F_n(x_k) - \hat{m}(A)] \|, \quad (23)$$

where $\hat{m}(A)$ is our numerical approximation for $m(A)$ given by

$$\hat{m}(A) = \max_{k\in\{1,2...,N\}} \max_{i\in\{1,2\}}[A \circ \tau_i(x_k) + F_n \circ \tau_i(x_k)].$$

The reasoning behind such approximation is that, if we had u, a sub-action such that $\max_{x\in[0,1]} u(x) = 0$, then

$$\max_{i\in\{1,2\}}[A \circ \tau_i(x) + u \circ \tau_i(x)] = u(x) + \lambda,$$

which means

$$\max_{x\in[0,1]} \max_{i\in\{1,2\}}[A \circ \tau_i(x) + u \circ \tau_i(x)] = 0 + \lambda.$$

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