An analogy of the charge distribution on Julia sets with the Brownian motion

Artur O. Lopes[a)]
Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

(Received 29 December 1988; accepted for publication 26 April 1989)

A way to compute the entropy of an invariant measure of a hyperbolic rational map from the information given by a Ruelle–Perron–Frobenius operator of a generic Holder-continuous function will be shown. This result was motivated by an analogy of the Brownian motion with the dynamical system given by a rational map and the maximal measure. In the case the rational map is a polynomial, then the maximal measure is the charge distribution in the Julia set. The main theorem of this paper can be seen as a large deviation result. It is a kind of Donsker–Varadhan formula for dynamical systems.

I. INTRODUCTION

We will show an interesting analogy of the Brownian motion on $\mathbb{R}^n$ with the maximal measure of a hyperbolic rational map (the quotient of two polynomials) on the complex plane. In this context, the Ruelle–Perron–Frobenius operator plays the role of the semigroup (at time $t = 1$) associated with the infinitesimal generator of a diffusion process. We will show these results in Sec. IV of this paper. First in Sec. II and Sec. III we will explain carefully the concepts that we want to relate.

We believe it is worthwhile to present all the considerations that motivate the main theorem of this paper.

We refer the reader to Walters,[1] Mañe,[2] and Ruelle[3] for general results about ergodic theory and thermodynamic formalism, and we refer to Varadhan[4] for results about diffusions and large deviation properties of stochastic differential equations. Another source of references for the latter subject is Freidlin–Wentzell,[5] but here we will follow the more concise version of Varadhan.

All analogies presented here are based on some results presented in Ref. 6 about relations of the pressure, entropy, free energy, and large deviation. In Refs. 7–9, results related to the theorems in Ref. 6 are also obtained.

II. DIFFUSION AND BROWNIAN MOTION

Here we will follow the nice presentation of the main ideas about diffusion that appeared in Varadhan.[4]

There are many cases where solutions to problems are expressed as an integral over a space of functions. A simple example below is the (simplified) version of the Feynman–Kac formula that expresses the solution of the equation

$$\frac{\delta u}{\delta t} = \frac{1}{2}\Delta u + v(x)u\,, \quad u(0,x) = 1\,, \tag{2.1}$$

as the function space integral

$$u(t,x) = E_x\left\{\exp \int_0^t v(x(s))\,ds\right\}, \tag{2.2}$$

where E_x refers to the expectation with respect to Brownian motion on $\mathbb{R}^n$, starting from the point x in $\mathbb{R}^n$ at time $t = 0$.

Denote $E_x\{\exp \int_0^t v(x(s))ds\}$ by $\alpha(t), t \in \mathbb{R}$ and consider the limit

$$\lambda = \lim_{t\to\infty} (1/t)\log \alpha(t)\,. \tag{2.3}$$

[a)] Permanent address: Instituto Mat-UFRGS, Porto Alegre, Brazil.

When $v(x)$ is periodic with period 1 in each variable, we can visualize, via the spectral theorem for $\frac{1}{2}\Delta + v$ on the n torus, that the above limit exists and is also the largest eigenvalue of $\frac{1}{2}\Delta + v$.

Now, by the variational principle, we have

$$\lambda = \lim_{t\to\infty} (1/t)\log \alpha(t) = \sup_{\substack{\phi\in L_2(T^n)\\ \|\phi\|^2 = 1}} \left[\int_{T^n} v(x)\phi^2(x)\,dx - \frac{1}{2}\int_{T^n} |\nabla\phi|^2\,dx\right]. \tag{2.4}$$

This last expression can be interpreted in the following way: $\int_{T_n} v(x)\phi^2(x)dx$ is the potential term, that is, the term where the action of the external potential $v(x)$ appears.

The other term is a kind of inertial term. If there is no external potential v, that is $v = 0$, then we just notice the solution given by the regular Brownian motion.

Making analogy with classical mechanics, we can say the first term corresponds to potential energy and the second term to kinetic energy. Hamilton's principle of least action claims that motions of a mechanical system coincide with the extremal of a functional related to the difference of kinetic and potential energy. In the case that we are in a Riemannian manifold, and there is no potential energy, the trajectories are geodesics.

Now, let us return to diffusions. For a Markov process with infinitesimal generator L and domain D, consider the semigroup T_t corresponding to L; then the deviation function (see Ref. 4, Sec. 13) is

$$I(v) = \lim_{t\to\infty} -\frac{1}{t}\inf_{u\in B^+}\left\{\int \log\frac{T_t u(x)}{u(x)}\,dv(x)\right\}. \tag{2.5}$$

Here v is any probability measure on the state space X of the Markov process and B^+ is the space of continuous bounded positive functions.

We also have

$$I(v) = -\inf_{\substack{u\in D\\ \inf_x u(x) > 0}} \int_X \left(\frac{Lu}{u}\right)(x)\,dv(x)\,. \tag{2.6}$$

Note that $L = \log T_1$.

III. THE MAXIMAL MEASURE AND THE PRESSURE

In this section we will explain the main reason to consider the pressure (sometimes called topological pressure) and the Ruelle–Perron–Frobenius operator.

 0022-2488/89/092120-05$02.50

Now we will follow the beautiful and simple motivation of the subject presented in Bowen.[10]

Consider a physical system with possible states $1,2,...,m$ and the energies of these states are $E_1,E_2,...,E_m$, respectively. Suppose our system is put in contact with a much larger "heat source," which is at temperature T. Energy is thereby allowed to pass between the original system and the heat source, and the temperature T of the source remains constant, as it is so much larger than our system. As the energy of our system is not considered fixed, the array of the states can occur. It has been known from statistical mechanics for a long time that the probability P_j that the state j occurs and is given by the Gibbs distribution:

$$P_j = \frac{e^{-BE_j}}{\sum_{i=1}^m e^{-BE_i}}, \quad j \in \{1,2,...,m\}, \tag{3.1}$$

where $B = 1/kT$ and k is a physical constant.

A mathematical formulation of the above consideration in a variational way can be obtained in the following way: consider

$$\widetilde{F}(p_1,p_2,...,p_m) = \sum_{i=1}^m -p_i \log p_i - \sum_{i=1}^m p_i BE_i, \tag{3.2}$$

defined over the simplex in $\mathbb{R}^m$, given by

$$\Big\{(p_1,p_2,...,p_m):\ p_i \geqslant 0, \quad i \in \{1,2,...,m\} \quad \text{and} \quad \sum_{i=1}^m p_i = 1\Big\}.$$

Using Lagrange multipliers, it is easy to show that the maximum of $\widetilde{F}$ in the simplex is obtained for

$$P_j = \frac{e^{-BE_j}}{\sum_{i=1}^m e^{-BE_i}}, \quad j \in \{1,2,...,m\},$$

in accordance with 3.1.

The quantity $H(p_1,p_2,...,p_m) = \sum_{i=1}^m -p_i \log p_i$ is called entropy of the distribution $(p_1,p_2,...,p_m)$. Denote $-\sum_{i=1}^m p_i E_i$ as the average energy $E(p_1,p_2,...,p_m)$.

Then we can say that the Gibbs distribution maximizes

$$H(p_1,p_2,...,p_m) - BE(p_1,p_2,...,p_m). \tag{3.3}$$

The expression $BE - H$ is called, in this context, free energy (in fact, there exist several different concepts in mathematics and physics also called free energy.)

Therefore we can say that nature minimizes free energy.

In the absence of the heat source, that is $E = 0$, nature maximizes entropy. In this case the Gibbs state is the most random probability, namely, $P_j = 1/m$, $j \in \{1,2,...,m\}$. Again, using analogy with classical mechanics, E plays the role of potential energy and H plays the role of kinetic energy.

Now, let us return to Gibbs measures. Generalizing the above considerations, Ruelle proposed the above model: consider the one-dimensional lattice $\mathbb{Z}$. Here one has for each integer a physical system with possible states $1,2,...,m$. A configuration of the system consists of assigning an $x_i \in \{1,2,...,m\}$ for each $i \in \mathbb{Z}$.

Thus a configuration is a point

$$\mathbf{x} = \{x_i\}_{i\in Z} \in \prod_{\mathbf{Z}} \{1,2,...,m\} = \Sigma_m .$$

Considering now the space Σ_m, the shift map

$$\sigma:\ \underset{\{x_i\}_{i\in Z}}{\Sigma_m} \to \underset{\{x_{i+1}\}_{i\in Z}}{\Sigma_m},$$

and $M(\sigma)$, the space of probabilities v, such that for any Borel set A

$$v(A) = v(\sigma^{-1}(A)),$$

we have the well-known Bernoulli shift model.

A continuous function $\phi: \Sigma_m \to \mathbb{R}$, in this setting, plays the role of the energy.

The problem here is to find a way to obtain the Gibbs distribution in the one-dimensional lattice in a similar way as how it was obtained before, in the beginning of Sec. III. Note that it is natural to consider just probabilities $p \in M(\sigma)$, because there is no natural reason to consider a certain distinguished point of the lattice as the origin in $\mathbb{Z}$.

Given a certain continuous function $\phi: \Sigma_m \to \mathbb{R}$ (as we said before will play the role of the energy), consider the following variational problem:

$$\sup_{p \in M(\sigma)} \Big\{ h(p) + \int \phi(z)\, dp(z) \Big\}, \tag{3.4}$$

where $h(p)$ is the entropy of the probability p[1,2].

Denote such supremum by $P(\phi)$, the pressure associated with ϕ. It is natural to ask which properties have a probability p_ϕ that eventually attain such supremum value.

The above setting was proposed by Ruelle. In fact, he was able to find a certain ϕ, such that the above p_ϕ is exactly the Gibbs state for the one-dimensional lattice that with other procedures people in physics already knew a long time ago.[10]

Now, given the above setting, then following Ruelle and Bowen,[3,10–12] consider the below variational problem: given a rational map F of degree d in the complex plane and a Holder-continuous function ϕ on $\mathbb{C}$, consider

$$\sup_{p \in M(F)} \Big\{ h(p) + \int \phi(z)\, dp(z) \Big\} = P(\phi), \tag{3.5}$$

where $h(p)$ is the entropy of the probability p and $M(F)$ is the set of probabilities, such that for any Borel set A,

$$p(A) = p(F^{-1}(A)),$$

$$p(\mathbb{C}) = 1.$$

The support of such measures in $M(F)$ will be contained always in the Julia set.[13,14]

When $\phi = 0$, there always exists a unique measure μ of maximal entropy.[15,16] We will call this measure the maximal measure. The entropy of such measure is $\log d$.

In the case where the rational map is a complex polynomial, the maximal measure is the charge distribution in the Julia set.[13,15] If the rational map is not a polynomial, the maximal measure is not the charge distribution in the Julia set.[14]

The results presented here are for the maximal measure μ. If one considers F a polynomial then, as we said before, our result is, in fact, for the charge distribution in the Julia set. This last measure is also called the harmonic measure seen from ∞.

In any case, given ϕ, the value $P(\phi)$ will be called the pressure of the function ϕ.

There exists a very interesting way, developed by Ruelle, to obtain the above value $P(\phi)$.[10,11,3]

If a measure v satisfies $h(v) + \int \phi(z) dv(z) = P(\phi)$, that is, v attains the supremum of the above-mentioned variational problem, it is natural to call such measure a Gibbs state. In the case where F is hyperbolic and ϕ is Holder continuous, there always exists such a Gibbs state and it is unique.[10,3]

We will denote J the Julia set of F.

Consider $0<\delta<1$ and denote $\mathbf{F}$ the space of δ-Holder-continuous real-valued functions in J with the metric

$$\|g\| = \|g\|_0 + \sup_{x\neq y} \frac{|g(x) - g(y)|}{|x-y|^\delta},$$

where $\| \ \|_0$ is the usual supreme norm and $| \ |$ is the modulus.

Consider now the linear operator on $\mathbf{F}$, L_ψ: $\mathbf{F}\to\mathbf{F}$, given by

$$L_\psi(\Phi(z)) = \sum_{i=1}^{d} e^{\psi(x_i(z))}\Phi(x_i(z)), \tag{3.6}$$

where ψ is considered fixed, $\Phi\in\mathbf{F}$, and $x_i(z)$, $i\in(1,2,...,d)$ are the d solutions (counted with multiplicity) of $F(x) = z$.

In the literature this operator is called the Ruelle–Perron–Frobenius operator associated with $\psi\in\mathbf{F}$.[10,12,13]

The conjugate of L_ψ, denoted by $L_\psi{}^*$, acts on the space of signed measures, and is defined by taking a measure p to the $q = L_\psi{}^*(p)$, the unique one such that for any continuous function Φ,

$$\int \Phi(z)\, dq(z) = \int L_\psi(\Phi)(z) dp(z). \tag{3.7}$$

Theorem 1[11,12]: Let F be a hyperbolic rational map and ψ: $J\to\mathbb{R}$ Holder continuous. Then there exist h: $J\to\mathbb{R}(h\in\mathbf{F})$, a probability v (not necessarily invariant) and $\lambda>0$, such that

(1) $\int h(z)dv(z) = 1$;

(2) $L_\psi(h) = \lambda h$;

(3) $L^*_\psi(v) = \lambda v$;

(4) $\left\| \left| \lambda^{-n} L^n_\psi(\Phi) - h\int \Phi(z)\, dv(z) \right| \right\|_0 \to 0$, $\forall \Phi\in\mathbf{F}$;

(5) h is the unique positive eigenfunction of L_ψ (up to multiplication by scalars);

(6) The probability $u = hv\in M(F)$ satisfies $h(u) + \int \psi(z)du(z) = P(\psi)$ and is the unique solution of the variational problem (3.5). Therefore u is the Gibbs state for ψ;

(7) $P(\psi) = \log\lambda$;

(8) λ is the largest eigenvalue of L_ψ.

The above theorem is proved in Refs. 11 and 12.

Given a probability $p\in M(F)$, consider the limit

$$\lim_{r\to 0} \frac{p(F(B(z,r)))}{p(B(z,r))}.$$

The above limit exists by the Radon–Nykodin theorem for z,p-almost everywhere.

Let us call

$$J(z) = \lim_{r\to 0} \frac{p(F(B(z,r)))}{p(B(z,r))}, \tag{3.8}$$

for z,p-almost everywhere, point $z\in J$, the Jacobian of the measure p. In fact, $h(p) = -\int \log J(z)dp(z)$ (Ref. 17).

In Ref. 6 it is shown that the set of probabilities in $M(F)$, such that the Jacobian is Holder continuous and never zero, is dense in $M(F)$.

Note that $J(z)\leqslant 1$ for $z\in J$, and also $\Sigma_{i=1}^d J(x_i(z)) = 1$ if $p\in M(F)$.

From this fact, it follows easily from Theorem 1 that, if p has Jacobian Holder continuous, then

$$P(\log J) = h(p) + \int \log J(z)dp(z) = 0. \tag{3.9}$$

Using the notation of Theorem 1, we also have $h(z) = 1 \forall z\in J$, $\lambda = 1$, and $u = v = p$.

We will use these results later in this paper.

The Gibbs measure for $\log J$ is sometimes referred to as a "g measure."

Theorem 2[6]: Let F be a hyperbolic rational map and ψ; then

$$P(\psi) = \lim_{n\to\infty} n^{-1} \log \int \exp\left(\sum_{j=0}^{n-1} \psi(F^j(z))\right) d\mu(z) + \log d, \tag{3.10}$$

where μ is the maximal measure.

Denote $\delta(z)$, the Dirac measure, with mass one in the point $z\in J$.

In Ref. 6 the above result was used to prove the following theorem.

Theorem 3[6]: Let F be a hyperbolic rational map of degree d and μ the maximal measure, then for any open convex set G of I continuity[6] G contained in the set of probabilities with support in the Julia set, $G\cap M(F)\neq\phi$, we have the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mu\left\{ z\in J \,\middle|\, \frac{1}{n}\sum_{j=0}^{n-1} \delta(F^j(z))\in G \right\} \tag{3.11}$$

exists and is equal to

$$-\inf_{v\in G\cap M(F)} \{\log d - h(v)\}. \tag{3.12}$$

Therefore $\log d - h(v)$ is a deviation function for the process (F,μ). In fact, this deviation function is the Legendre transform of the pressure minus $\log d$.[6]

IV. BROWNIAN MOTION AND CHARGE DISTRIBUTION: AN ANALOGY

Suppose F is a polynomial. Therefore all considerations made before can be applied for the charge distribution in the Julia set because, in this case, this measure is equal to μ, the maximal measure.

We have that for $\psi\in\mathbf{F}$ and v a continuous function, the expression $\Sigma_{j=0}^{n-1}\psi(F^j(z))$ can be considered a discrete time analogous version of $\int_0^t v(x(s))ds$. In this way

$$P(\psi) - \log d = \lim_{n\to\infty} \frac{1}{n} \log\left(\int \exp\left(\sum_{j=0}^{n-1} \psi(F^j(z))\right) d\mu(z)\right) \tag{4.1}$$

is analogous to

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log E_x \left\{\exp \int_0^t v(x(s))\, ds\right\}, \tag{4.2}$$

where μ plays the role of the Brownian motion.

As we can see, respectively, in Sec. III and Sec. II, ψ and v play the role of a potential energy.

Now from Sec. II the semigroup T_t associated with the diffusion $\frac{1}{2}\Delta + v$ is such that T_1 has the largest eigenvalue e^λ. Observe that it also follows from Theorem 1 that $e^{P(\psi)}$ is the largest eigenvalue of L_ψ. Therefore $(1/d)L_\psi$ is analogous to T_1.

Finally, trying to find some kind of analogy with the last part of Sec. II [remember that L is analogous to $\log(d^{-1}L_\psi)$], then we have the following theorem.

Theorem 4: Consider F a hyperbolic rational map of degree d and $v\in M(F)$ a probability with Jacobian $J(z)$ Holder continuous that never vanishes, then

$$\log d - h(v) - \int \psi(z)dv(z) = -\inf_{u\in B^+}\left\{\int \log \frac{(d^{-1}L_\psi)u(z)}{u(z)}\, dv(z)\right\}. \tag{4.3}$$

We will make some remarks before the proof of this theorem.

Remark 1: The function I of the end of Sec. II is a large deviation function, as can be seen in Sec. 13.[4] By the other way, as we see in Sec. II, in the case $\psi = 0$, then in Theorem 3, the value $\log d - h(v)$ is a deviation function for the process (F,μ). Therefore the above theorem also represents an analogy with a diffusion process. Here we cannot consider a limit as t goes to zero:

$$I(v) = \lim_{t\to 0} -\frac{1}{t} \inf_{u\in B^+}\left\{\int \log \frac{T_t\, u(x)}{u(x)}\, dv(x)\right\}, \tag{4.4}$$

because the time n is discrete. Therefore here in Theorem 4, we consider $n = 1$.

It is usual in large deviation theory to suppose the deviation function is defined in a certain dense set with good properties. In the case of hyperbolic rational maps, this good set is the set of measures with Jacobian Holder continuous (see Ref. 6). Therefore the assumption about v in the setting of large deviation is a mild assumption.

Now we will prove the main theorem of this paper.

Proof of Theorem 4: Consider first $u(z) = e^{-\psi(z)}J(z)$; then

$$\begin{aligned}\log \frac{L_\psi(u(z))}{u(z)} &= \log \frac{\Sigma_{i=1}^d e^{\psi(x_i(z))}u(x_i(z))}{u(z)}\\ &= \log \frac{\Sigma_{i=1}^d (e^{\psi(x_i(z))}e^{-\psi(x_i(z))}J(x_i(z)))}{u(z)}\\ &= \log \sum_{i=1}^d J(x_i(z)) - \log e^{-\psi(z)}J(z)\\ &= \psi(z) - \log J(z).\end{aligned} \tag{4.5}$$

Therefore, for such u,

$$\begin{aligned}\int \log \frac{L_\psi u(z)}{u(z)}\, dv(z) &= \int \psi(z)dv(z) - \int \log J(z)dv(z)\\ &= \int \psi(z)dv(z) + h(v).\end{aligned}$$

Finally

$$\int \log \frac{(d^{-1}L_\psi)u(z)}{u(z)}\, dv(z) = \int \psi(z)dv(z) + h(v) - \log d. \tag{4.6}$$

Now we will show that for any positive continuous function u, we have

$$\int \log \frac{L_\psi u(z)}{u(z)}\, dv(z) \geqslant \int \psi(z)dv(z) + h(v). \tag{4.7}$$

We can , of course, suppose that instead of a general $u\in B^+$, we have $u(z)e^{-\psi(z)}J(z)$, because $e^{-\psi(z)}$ and $J(z)$ are nonzero by hypothesis.

Therefore

$$\begin{aligned}L_\psi(u(z)) &= \sum_{1=i}^d e^{\psi(x_i(z))}e^{-\psi(x_i(z))}J(x_i(z))u(x_i(z))\\ &= \sum_{i=1}^d J(x_i(z))u(x_i(z)) = L_{\log J}u(z).\end{aligned} \tag{4.8}$$

In this case we have

$$\begin{aligned}\log \frac{L_\psi(u(z)e^{-\psi(z)}J(z))}{u(z)e^{-\psi(z)}J(z)} &= \log L_{\log J}u(z)\\ &\quad - \log u(z) + \psi(z) - \log J(z).\end{aligned}$$

From this, it follows that

$$\begin{aligned}\int \log \frac{L_\psi(u(z)e^{-\psi(z)}J(z))}{u(z)e^{-\psi(z)}J(z)}\, dv(z) &= \int \log L_{\log J}u(z)dv(z) - \int \log u(z)dv(z)\\ &\quad + \int \psi(z)dv(z) + h(v).\end{aligned}$$

Therefore all we have to prove is that

$$\int \log L_{\log J}u(z)dv(z) - \int \log u(z)dv(z) \geqslant 0. \tag{4.9}$$

Remember now that from (3) in Theorem 1, $L^*_{\log J}(v) = v$, therefore

$$\int \log u(z)dv(z) = \int L_{\log J} \log u(z)dv(z).$$

If we are able to show that

$$\log L_{\log J}u(z) \geqslant L_{\log J} \log u(z) \tag{4.10}$$

for any $z\in J$, then (4.9) follows.

This last inequality means

$$\log \sum_{i=1}^{d} J(x_i(z))u(x_i(z))$$
$$\geqslant \sum_{i=1}^{d} J(x_i(z))\log u(x_i(z)). \tag{4.11}$$

As $\Sigma_{i=1}^{d} J(x_i(z)) = 1$ for $z \in J$, the last inequality follows from the fact that log is a concave function and u is positive.

This is the end of the proof of Theorem 4.

V. CONCLUSION

Theorem 4 gives a way to compute the entropy of the measure v as an information obtained from the Ruelle–Perron–Frobenius operator of a Holder-continuous function ψ.

ACKNOWLEDGMENTS

We would like to thank F. Ledrapier for a suggestion in the proof of Theorem 4.

This research was partially supported by AFOSR.

[1]P. Walters, *An Introduction to Ergodic Theory* (Springer, Berlin, 1982).

[2]R. Mañe, *Ergodic Theory and Differentiable Dynamics* (Springer, Berlin, 1987).

[3]D. Ruelle, *Thermodynamic Formalism* (Addison–Wesley, Reading, MA, 1978).

[4]S. R. S. Varadhan, "Large deviations and applications," CBMS-NSF Regional Conf. Series in Appl. Math. (1984).

[5]M. I. Friedlin and A. D. Wentzell, *Random Perturbation of Dynamical Systems* (Springer, Berlin, 1984).

[6]A. Lopes, "Entropy and large deviation," University of Maryland preprint.

[7]M. Denker, "Large deviation and the pressure function," University of Göttingen preprint.

[8]S. Orey and S. Pelikan, "Large deviation principles for stationary processes," University of Minnesota preprint.

[9]S. Orey and S. Pelikan, "Deviations of trajectory averages and the defect in Pesin's formula for Anosov diffeomorphisms," University of Minnesota preprint.

[10]R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Difeomorphisma, Lecture Notes in Mathematics* (Springer, Berlin, 1975), Vol. 470.

[11]D. Ruelle, "Repellers for real analytic maps," Erg. Theory Dyn. Syst. **2**, 99 (1982).

[12]M. Pollicott, "A complex Ruelle–Perron–Frobenius operator and two counter examples," Erg. Theory Dyn. Syst. **4**, 135 (1984).

[13]H. Brolin, "Invariant sets under iteration of rational function," Ark. Mat. (Band G) **6**, 103 (1966).

[14]A. Lopes, "Equilibrium measures for rational maps," Erg. Theory Dyn. Syst. **6**, 393 (1986).

[15]A. Freire, A. Lopes, and R. Mañé, "An invariant measure for rational maps," Bol. Soc. Brasil. Mat. **14**, 45 (1983).

[16]V. Lubitsh, "Entropy properties of rational endomorphisms on the Riemann sphere," Erg. Theory Dyn. Syst. **3**, 351 (1983).

[17]R. Mañe, *On the Hausdorff Dimension of Invariant Probabilities of Rational Maps, Lecture Notes in Mathematics* (Springer, Berlin, 1988), Vol. 1331.