SPECTRAL PROPERTIES OF THE RUELLE OPERATOR FOR PRODUCT TYPE POTENTIALS ON SHIFT SPACES

L. CIOLETTI, M. DENKER, A. O. LOPES, AND M. STADLBAUER

Abstract. We study a class of potentials f on one sided full shift spaces over finite or countable alphabets, called potentials of product type. We obtain explicit formulae for the leading eigenvalue, the eigenfunction (which may be discontinuous) and the eigenmeasure of the Ruelle operator. The uniqueness property of these quantities is also discussed and it is shown that there always exists a Bernoulli equilibrium state even if f does not satisfy Bowen's condition.

We apply these results to potentials $f : \{-1,1\}^{\mathbb{N}} \to \mathbb{R}$ of the form

$$f(x_1, x_2, \ldots) = x_1 + 2^{-\gamma} x_2 + 3^{-\gamma} x_3 + \ldots + n^{-\gamma} x_n + \ldots$$

with $\gamma > 1$. For $3/2 < \gamma \leq 2$, we obtain the existence of two different eigenfunctions. Both functions are (locally) unbounded and exist a.s. (but not everywhere) with respect to the eigenmeasure and the measure of maximal entropy, respectively.

1. Introduction

The theory of Gibbs states in physics and mathematics led to the notion of the pressure function and its variational formula for dynamical systems (Walters 1975, [14]). Since then a variety of results has been published to clarify existence and uniqueness of equilibriums states maximizing the pressure, and this note is in the same spirit.

The classical condition for uniquess of the equilibrium state requires summable variations and was relaxed by Bowen ([3]) using a condition which is named after him. This has been further investigated by Walters in 2005 ([17]) who introduced a slighly stronger condition, which is referred to as Walters' condition. Yuri in 1998 ([19]) coined the term weak bounded variation and also showed uniqueness. For many classes of maps on compact spaces uniqueness has been proved as well, as a recent examples for this, Climenhaga and Thompson in 2013 ([6]) used a restricted Bowen condition, and Iommi and Todd ([8]) studied the existence of phase transitions for grid potentials (see [11]) on full shift spaces. We finally mention Sarig's work in 2001 ([12]) which opened the a new field of studying this question on countable subshifts (the non-compact case) using Gurevic' pressure, or, for a more general approach to pressure, the notion introduced by Stratmann and Urbański in 2007 ([13]).

In expansive dynamical systems an equilibrium state always exists, leading to the problem of uniqueness and continuity properties of the density of the equilibrium state with respect to *canonical* measures. These canonical measures may be defined as conformal measures (in many cases the eigen-measure of the Ruelle operator

---

L. Cioletti, O.A. Lopes and M. Stadlbauer are partially supported by CNPq. M. Denker was partially funded by CAPES Grant 158/2012–Pesquisador Visitante Especial.

and then called Gibbs measures on shift spaces) or - as we show below - product measures (for example a Bernoulli measure on shift spaces).

In this note we deal with a dynamical system $T : X \to X$ where T denotes the shift transformation on $X = \mathcal{A}^{\mathbb{N}}$, where $\mathcal{A}$ is a finite or countable set, called the alphabet of the dynamics, and where X is equipped with the product topology of pointwise convergence and the associated Borel σ-field. We consider potential functions

$$f : X \to \mathbb{R}$$

which can be written in form of

$$f(x) = \sum_{n=1}^{\infty} f_n(x_n), \qquad x = (x_n)_{n\in\mathbb{N}} \in X$$

and call these functions of product type (see Section 3), where $f_n : \mathcal{A} \to \mathbb{R}$ are fixed functions so that the sum converges. Given a function $f : X \to \mathbb{R}$ the Ruelle operator

$$\mathcal{L}_f \phi(x) = \sum_{T(y)=x} \phi(y) \exp f(y) \tag{1}$$

acts on bounded measurable functions if $\mathcal{L}_f(1)(x) < \infty$ for all $x \in X$.

The initial motivation for the present note was to show the existence of positive measurable eigenfunctions of $\mathcal{L}_f$ and obtain criteria for its continuity (see [16] for details). In Section 6.2, we show that, for a continuous potential f less regular than a Bowen potential, the eigenfunction might oscillate between 0 and ∞ on any open set (see Theorem 6.1).

If f is of product type, the function $g = e^f$ appearing in the Ruelle operator (1) has indeed a product structure. It is not hard to see that $\mathcal{L}_f$ and its dual act on functions with a product structure and product measures, respectively. These basic observations permit explicit representations of eigenfunctions, conformal measures and equilibrium states (which are of possible interest in connection with computer experiments or applications in mathematical physics). There are examples of potentials of product type which belong to Bowen's and Walters' class (see [16, 17]), but also examples having less regularity properties than potentials in these two classes.

We consider the following classes of potentials of product type. We say that $g = e^f$ is *ℓ_1-bounded* if $(\|f_k\|_\infty)_{k\geq 2} \in \ell_1$, i.e. $\sum_{k=2}^{\infty} \|f_k\|_\infty < \infty$ and is *summable* if $\sum_{a\in\mathcal{A}} \exp(f_1(a)) < \infty$. Moreover, g is a *balanced potential*, if $\sum_{a\in\mathcal{A}} f_k(a) = 0$ for all $k \geq 1$. Note that the first condition is equivalent to the condition that $g(x_1, x_2 \ldots)/\exp(f_1(x_1))$ is uniformly bounded. Combined with summability, this implies that $\|\mathcal{L}_{\log g}(1)\|_\infty < \infty$, independently of $\mathcal{A}$ being finite or not. A balanced potential may be considered as a kind of normal form for potentials of product type.

These conditions on potentials of product type can be used to describe the properties of the corresponding Ruelle operator. We obtain the following results for the existence of conformal and equilibrium measures under rather weak assumptions. If $\|g\|_\infty < \infty$, then there is an explicitly given product measure which is $1/g$-conformal (Theorem 3.1). Furthermore, if g is summable and ℓ_1-bounded, then there exists an explicitly given Bernoulli measure which is an equilibrium state.

In order to obtain uniqueness of these measures, we have to impose Bowen's condition. We say that $g = e^f$ is in *Bowen's class* if g is of locally bounded distortion (see Proposition 2.1). That is, there exists $k \in \mathbb{N}$, referred to as index, such that

$$\sum_{m=k}^{\infty} \sum_{n=m}^{\infty} \sup\{|f_n(x) - f_n(y)| : x, y \in \mathcal{A}\} < \infty.$$

If Bowen's condition holds for $k = 2$, observe that a summable and balanced potential automatically is locally bounded. Under these assumptions, we show that that there exist an explicitly given continuous eigenfunction of $\mathcal{L}_{\log g}$ (Theorem 4.1) and, if $\mathcal{A}$ is finite, that the conformal measure and the equilibrium state are unique (Theorems 3.3, 4.2).

Beyond Bowen's condition, the situation is very different. If $\mathcal{A}$ is finite and for some k,

$$\sum_{i=k}^{\infty} \max_{a \in \mathcal{A}} \left(\sum_{j=i}^{\infty} \log g_j(a) \right)^2 < \infty, \tag{2}$$

there are three canonical measures, first the conformal measure μ for $1/g$, secondly the equilibrium measure $\tilde{\mu}$ and last the measure of maximal entropy ρ. All three measures are Bernoulli (i.e. the coordinate process is independent) and μ and $\tilde{\mu}$ are absolutely continuous with respect to each other, whereas μ and ρ have disjoint support. Moreover, there exist functions $h_\mu \in L^1(X, \mu)$ and $h_\rho \in L^1(X, \rho)$ who may exist only almost surely. Furthermore, these functions are eigenfunctions for the action of the operator on $L^1(X, \mu)$ and $L^p(X, \rho)$ (for $1 \leq p < \infty$), respectively. The relationship between h and the equilibrium measure is explained by ergodicity of a natural operator on $L^1(X, \rho)$ defined by $\tilde{\mu}$.

In order to illustrate the results we will study an explicit example in section 6. In there, we consider the potential $f : \{-1, 1\}^{\mathbb{N}} \to \mathbb{R}$ of the form

$$f(x_1, x_2, \ldots) = x_1 + 2^{-\gamma} x_2 + 3^{-\gamma} x_3 + \ldots + n^{-\gamma} x_n + \ldots$$

which is a summable, locally bounded and balanced potential for $\gamma > 1$. If $\gamma > 2$, then e^f is in Bowen's class, and for $3/2 < \gamma \leq 2$, condition (2) is satisfied. For the latter case, we obtain that h_μ and h_ρ are locally unbounded and therefore discontinuous. Furthermore, for $1 < \gamma \leq 3/2$, these eigenfunctions do not exist and the measures μ, $\tilde{\mu}$ and ρ are pairwise singular to each other.

The paper is structured as follows. In Section 2, we recall the regularity classes of Bowen, Walters and Yuri adapted to the setting of potentials of product type. In our setting, the classes of Bowen and Walters coincide, and in particular, the existence of conformal measures and continuous eigenfunctions for finite $\mathcal{A}$ could also be obtained by results in [16]. For the sake of completeness, we also include the notion of weak bounded variation by Yuri, even though the results for this type of potentials are not applicable in here. This is due to the fact that the results by Yuri rely on a tower construction whose associated potential is of bounded variation - or, from a more abstract viewpoint, on the existence of an isolated critical set or isolated indifferent fixed points.

In Section 3, we then provide a very general condition for the existence of a conformal measure and a condition for uniqueness. These results essentially rely on the observation that the Radon-Nikodym derivative of a measure of product type is a function of product type, and an ergodicity argument, respectively. In

Section 4, we explicitly construct eigenfunctions and equilibrium states, including the existence only ρ-almost everywhere when Bowen's condition is not satisfied and where ρ denotes the measure of maximal entropy. This is extended in the following section to the action of the Ruelle operator on $L^1(X,\rho')$ for certain product measures and a condition for the uniqueness of h is given. Section 6 is then dedicated to the analysis of the above mentioned example.

2. Regularity classes of potentials

In order to adapt the conditions by Bowen, Walters and Yuri to functions of product type we begin specifying a metric on $X = \mathcal{A}^{\mathbb{N}}$. For $(x_n),(y_n) \in X$, let

$$d(x,y) = 2^{-\max\{n : x_k = y_k \forall k \leq n\}}.$$

As it is well known, d generates the product topology of pointwise convergence and (X,d) is a complete metric space which is compact if and only if $\mathcal{A}$ is finite. The cylinder sets form a a basis of this topology, where, for a k-word $(x_1,\ldots,x_k) \in \mathcal{A}^k$, the associated cylinder set is defined by $[x_1,\ldots,x_k] := \{(y_n)_{n\geq 1} \in X : y_i = x_i \forall i = 1,\ldots,k\}$.

The shift on X is defined by $T : X \to X, (x_1,x_2,\ldots) \mapsto (x_2,\ldots)$ and, as it is well known, is a continuous transformation which expands distances by 2. In order to put emphasis on the underlying topology and Borel σ-algebra, we will refer to (X,T) as a *topological Bernoulli* shift over the alphabet $\mathcal{A}$.

Using a slighly different notation as in [16], for a function $\phi : X \to \mathbb{R}$ we let

$$\mathrm{var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : d(x,y) \leq 2^{-n}\}$$

denote the variation of ϕ over cylinders of length n. Then ϕ has summable variations ([15]) if

$$\sum_{n=1}^{\infty} \mathrm{var}_n(\phi) < \infty.$$

Denoting by $S_n(\phi) = \phi + \ldots + \phi \circ T^{n-1}$ the decay of $\mathrm{var}_n(\cdot)$ now allows to define the following regularity classes. We say that a function $\phi : X \to \mathbb{R}$ belongs to

1. Walters' class ([16]) if $\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \mathrm{var}_{n+k}(S_n(\phi)) = 0$,
2. Bowen's class ([3]) if $\exists\, k \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \mathrm{var}_{n+k}(S_n(\phi)) < \infty$,[1]
3. Yuri's class ([19]) if $\lim_{n\to\infty} \frac{1}{n}\mathrm{var}_n(S_n(\phi)) = 0$.

Observe that for shift spaces, Walters' condition is equivalent to equicontinuity of the family $\{S_n(\phi)) : n \geq 1\}$, whereas Bowen's condition provides a uniform local bound on the local distortion of $(S_n(\phi))_{n\geq 1}$. Yuri's condition is also known as weak bounded variation ([19]). We now deduce necessary conditions for functions of product type to belong to these classes. Assume that $f : X \to \mathbb{R}$ is of the form

$$f((x_n)_{n\geq 1}) = \sum_{n=1}^{\infty} f_n(x_n),$$

where $(f_n : \mathcal{A} \to \mathbb{R})_{n\geq 1}$ is a sequence such that $\sum_n f_n(x_n)$ converges for all $x = (x_n)_{n\geq 1} \in X$ and set

$$v_n(f) := \sup\{|f_n(x) - f_n(y)| : x,y \in \mathcal{A}\}, \quad s_n(f) := \sum_{k>n} v_k(f).$$

---

[1] It has been remarked in [16] that the definition here is equivalent to Bowen's original definition.

Proposition 2.1. *For f of product type as above, the following holds.*

1. *If $\sum_{n=1}^{\infty} s_n(f) < \infty$ then ϕ has summable variation.*
2. *If $\sum_{n=k}^{\infty} s_n(f) < \infty$ for some $k \in \mathbb{N}$, then f belongs to Bowen's and Walters' class.*
3. *If $\lim_{m\to\infty} \frac{1}{m}\sum_{n=1}^{m} s_n(f) = 0$, then f belongs to Yuri's class.*

Proof. For $x = (x_n)_{n\geq 1}$, $y = (y_n)_{n\geq 1}$ with $x_j = y_j$ for all $j \leq m+k$, it follows that

$$\begin{aligned}|S_m(f)(x) - S_m(f)(y)| &= \left|\sum_{j=0}^{m-1}\sum_{n=1}^{\infty} f_n(x_{n+j}) - f_n(y_{n+j})\right| \\ &= \left|\sum_{j=0}^{m-1}\sum_{n=m-j+k+1}^{\infty} f_n(x_{n+j}) - f_n(y_{n+j})\right| \\ &\leq \sum_{l=1}^{m}\sum_{n=l+k+1}^{\infty} v_n(f) = \sum_{l=1}^{m} s_{l+k}(f).\end{aligned}$$

Hence, $\mathrm{var}_{m+k}(S_m(f)) \leq \sum_{l=1}^{m} s_{l+k}(f) \leq \sum_{l>k} s_l(f)$. Assertions 2. and 3. easily follow from this estimate. The assertion 1. is shown similarly. ☐

Example 1. Assume that $\|f_n\|_\infty \ll n^{-\gamma}$ for some $\gamma > 1$, where $a_n \ll b_n$ stands for the existence of $C > 0$ with $a_n \leq C b_n$ for all $n \in \mathbb{N}$. As $\gamma > 1$, it follows that $\sum_n \|f_n\|_\infty < \infty$. Moreover, the estimate $v_n(f) \leq 2\|f_n\|_\infty \leq 2n^{-\gamma}$ implies that $s_n(f) \ll n^{1-\gamma}$. In particular, $\sum_{n=1}^{m} s_n(f) \ll n^{2-\gamma}$. Hence, if $\gamma > 2$, then f has summable variation and is in Bowen's and Walters' class, and if $\gamma > 1$, then f is in Yuri's class.

Example 2. In order to see that this classification through γ is sharp, we consider the specific example $f : \{-1,1\}^{\mathbb{N}} \to \mathbb{R}$ of the form $f(x) = \sum_n x_n n^{-\gamma}$. Then, for $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ with $x_j = y_j$ for all $j \leq m+k$ and $x_j = 1$ and $y_j = -1$ for all $j > m+k$, one obtains as in the proof of Proposition 2.1 that, for $\gamma \neq 2$,

$$\begin{aligned}S_m(f)(x) - S_m(f)(y) &= \sum_{l=1}^{m}\sum_{n>l+k} f_n(1) - f_n(-1) = 2\sum_{l=1}^{m}\sum_{n>l+k} n^{-\gamma} \\ &\gg \sum_{l=1}^{m}(l+k+1)^{1-\gamma} \gg \left|(k+2)^{2-\gamma} - (m+k+2)^{2-\gamma}\right|.\end{aligned}$$

By the same argument, it follows that $S_m(f)(x) - S_m(f)(y) \gg \log(m+k+2) - \log(k+2)$ for $\gamma = 2$. Hence, for this particular choice of f, it follows that f is in Bowen's or Walters' class if and only if $\gamma > 2$. Furthermore, f is in Yuri's class if and only if $\gamma > 1$.

3. CONFORMAL MEASURES OF PRODUCT TYPE

3.1. **Existence.** Conformal measures are used to denote the existence of probability measures μ with a prescribed Jacobian $J = d\mu \circ T / d\mu$. In this section we study their existence and uniqueness for a given potential f of product type, where the Jacobian is given by $J = e^{-f}$. Hence if $g : X \to \mathbb{R}_+$ is a given positive function (also called a potential), $f = \log g$ is the potential for the associated Ruelle operator $\mathcal{L}_f$ (see below), and g is said to be of product type if the associated f is, in particular, g can be written in the form $g(x) = \prod_{n=1}^{\infty} g_n(x_n)$ $(x = (x_n)_{n\geq 1})$ where the g_n are uniquely determined up to non-zero constants. In analogy to product type functions, we also call a product measure $\mu = \otimes_{i=1}^{\infty}\mu_i$ on $X = \mathcal{A}^{\mathbb{N}}$ a measure of product type, where μ_i are probability measures on $\mathcal{A}$. These product measures are uniquely defined by their values on cylinders:

$$\mu([a_1,\dots,a_n]) = \prod_{i=1}^{n} \mu_i(a_i) \qquad a_1,...,a_n \in \mathcal{A}.$$

Recall from [7] that a Borel probability measure μ on $(X,\mathcal{B})$ is ϕ-conformal if there exists $\lambda > 0$,

$$\mu(T(A)) = \lambda \int_A \phi\, d\mu$$

for all measurable sets A such that the shift map $T : X \to X$ restricted to A is injective. If the Ruelle operator $\mathcal{L}_{-\log\phi}$ acts on continuous functions its dual operator also acts on finite signed measures, and it is well known that a measure μ is ϕ-conformal if and only if $\mathcal{L}^*_{-\log\phi}(\mu) = \lambda\mu$, for some $\lambda > 0$. Also note that λ usually is equal to the spectral radius of $\mathcal{L}_{-\log\phi}$.

Theorem 3.1. *Let (X,T) be a topological Bernoulli shift over a finite or countable alphabet $\mathcal{A}$ and let $g = \prod_{n=1}^{\infty} g_n$ be a potential of product type.*

1. *There exists at most one conformal measure μ of product type for g which is positive on open sets. This measure μ is given by*

$$\mu_n(a) = \left(\sum_{b\in\mathcal{A}} \prod_{i=1}^{n} \frac{g_i(a)}{g_i(b)}\right)^{-1} \quad \textit{for all } n\in\mathbb{N},\ a\in\mathcal{A}. \tag{3}$$

2. *If $\inf_{x\in X} g(x) > 0$, then a conformal measure of product type for g exists and is positive on open sets.*

Proof. We begin with the proof of the first assertion. Let $\mu = \otimes_{i=1}^{\infty}\mu_i$ be a product measure which is positive on open sets, in particular on each cylinder set. In order that it is conformal for g, that is

$$\mu(T[a_1,\dots,a_n]) = \mu([a_2,\dots,a_n]) = \lambda \int_{[a_1,\dots,a_n]} g(x)\mu(dx)$$

for every cylinder set $[a_1,\dots,a_n]$, it is necessary and sufficient that

$$1 = \mu(T[a]) = \lambda \int_{[a]} g_1(x_1)\mu_1(dx_1) \prod_{i=2}^{\infty} g_i(y)\mu_i(dy) \tag{4}$$

for some $\lambda > 0$ and

$$\begin{aligned} \mu_1(a_2)\dots\mu_{n-1}(a_n) &= \mu(T([a_1,\dots,a_n])) \\ &= \lambda \prod_{i=1}^{n} g_i(a_i)\mu_i(a_i) \prod_{i=n+1}^{\infty} \int g_i(y)\mu_i(dy) \end{aligned} \tag{5}$$

for any $a_1,\dots,a_n \in \mathcal{A}$. Varying $a \in \mathcal{A}$ in equation (4) yields

$$g_1(a)\mu_1(a) = g_1(b)\mu_1(b)$$

and hence

$$\mu_1(a) = \left(g_1(a) \sum_{b\in\mathcal{A}} \frac{1}{g_1(b)} \right)^{-1}. \tag{6}$$

The similarly equations (5) yield

$$\mu_n(a) = \frac{\mu_{n-1}(a)}{g_n(a)} \left(\sum_{b\in\mathcal{A}} \frac{\mu_{n-1}(b)}{g_n(b)} \right)^{-1}. \tag{7}$$

It follows that the conformality equalities (4) amd (5) uniquely determine the conformal measure (which is positive on open sets and a product measure), hence the uniqueness of μ. Moreover, by (5),

$$\frac{\mu_n(a)}{\mu_n(b)} = \frac{g_n(b)}{g_n(a)} \cdot \frac{\mu_{n-1}(a)}{\mu_{n-1}(b)}.$$

Hence, the first part of the theorem follows by induction.

For the proof of the second part, note that the uniform lower bound on g is equivalent to

$$\sum_{i=1}^{\infty} \log \|g_i^{-1}\|_\infty^{-1} > -\infty.$$

Hence, for any sequence of measures μ_i on $\mathcal{A}$,

$$\int_X g(x) \prod_{i=1}^{\infty} \mu_i(dx) = \prod_{i=1}^{\infty} \int_{\mathcal{A}} g_i(u)\mu_i(du) \geq \prod_{i=1}^{\infty} \|g_i^{-1}\|_\infty^{-1} > 0.$$

Hence the equations (4) and (5) show that the conformal product measure is well defined and positive on open sets. □

Due to the constructive proof above, it is possible to obtain explicit expressions for the measure and the associated parameter λ.

Corollary 3.2. *If* $\inf_x g(x) > 0$, *then for every* $t \in \mathbb{R}$, *the function* $g(t) = g^t$ *satisfies* $\inf_x g^t(x) > 0$ *as well and the conformality parameter* λ_t *satisfies*

$$\lambda_t = \sum_{c\in\mathcal{A}} \frac{1}{\prod_{i=1}^{\infty} g(t)_i(c)}$$

for all t *where the denominator does not vanish.*

Proof. We may put $t = 1$. Inserting (3) into equation (4) yields

$$1 = \lambda \left(\sum_{b\in\mathcal{A}} g_1(b)^{-1} \right)^{-1} \prod_{i=2}^{\infty} \int g_i(u)\mu_i(du).$$

Now by equation (3)

$$\int g_n d\mu_n = \sum_{b\in\mathcal{A}} g_n(b)\mu_n(b) = \left(\sum_{b\in\mathcal{A}} \frac{\mu_{n-1}(b)}{g_n(b)} \right)^{-1}$$

and by bachward induction over m

$$\int g_n d\mu_n \dots \int g_{m-1} d\mu_{m-1} = \left(\sum_{b\in\mathcal{A}} \frac{\mu_{m-1}(b)}{g_n(b)\dots g_m(b)} \right)^{-1} \int g_{m-1} d\mu_{m-1}.$$

Using (7)

$$g_{m-1}(c)\mu_{m-1}(c) = \mu_{m-2}(c) \frac{\mu_{m-1}(b) g_{m-1}(b)}{\mu_{m-2}(b)} \quad \forall\, b \in \mathcal{A}$$

and summing over c it follows that

$$\int g_{m-1} d\mu_{m-1} = \frac{\mu_{m-1}(b) g_{m-1}(b)}{\mu_{m-2}(b)},$$

so the following identity holds

$$\begin{aligned} \int g_n d\mu_n \dots \int g_{m-1} d\mu_{m-1} &= \left(\sum_{b\in\mathcal{A}} \frac{\mu_{m-1}(b)}{g_n(b)\dots g_m(b)} \frac{\mu_{m-2}(b)}{\mu_{m-1}(b) g_{m-1}(b)} \right)^{-1} \\ &= \left(\sum_{b\in\mathcal{A}} \frac{\mu_{m-2}(b)}{g_n(b)\dots g_{m-1}(b)} \right)^{-1}. \end{aligned}$$

Taking $m = 3$ it follows that

$$\int g_n d\mu_n \dots \int g_2 dm_2 = \left(\sum_{b\in\mathcal{A}} \frac{\mu_1(b)}{g_n(b)\dots g_2(b)} \right)^{-1}.$$

Since by (6)

$$\sum_{c\in\mathcal{A}} g_1(c)^{-1} = \frac{1}{\mu_1(b) g_1(b)}$$

for every $b \in \mathcal{A}$ we obtain

$$\sum_{c\in\mathcal{A}} \frac{1}{g_1(c)} \sum_{b\in\mathcal{A}} \frac{\mu_1(b)}{g_n(b)\dots g_2(b)} = \sum_{b\in\mathcal{A}} \frac{1}{g_n(b)\dots g_1(b)},$$

and therefore the claim follows by taking $n \to \infty$. □

3.2. **Uniqueness.** Uniqueness of conformal measures requires a stronger hypothesis. We prove

Theorem 3.3. *Let the alphabet $\mathcal{A}$ be finite and suppose that $g_i : X \to \mathbb{R}_+$ ($i \geq 0$, g_0 a constant) satisfy*

$$\sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \log \max\{\|g_k\|_\infty, \|g_k^{-1}\|_\infty\} < \infty. \tag{8}$$

Then there exists exactly one conformal measure for the product type function $g(x) = g_0 \prod_{i=1}^{\infty} g_i(x_i)$. Moreover, this measure is ergodic.

Proof. Let

$$K_i = \prod_{k=i}^{\infty} \max\{\|g_k\|_\infty^2, \|g_k^{-1}\|_\infty^2\}, \quad i \geq 2.$$

Since (8) implies the existence condition for a conformal measure of product type, Theorem 3.1, guarantees a conformal measure for g which is of product type. Denote it by μ and assume there is another conformal measure ν.

We claim that both measures are equivalent, provided $\frac{\nu([a])}{\mu([a])} \in [K_1^{-1}, K_1]$. In order to show this by induction, assume that for fixed $n \in \mathbb{N}$ and all cylinder sets $[a_1, \dots, a_n]$

$$\prod_{i=1}^{n+1} K_i^{-1} \leq \frac{\mu([a_1, \dots, a_n])}{\nu([a_1, \dots, a_n])} \leq \prod_{i=1}^{n+1} K_i.$$

Then for any cylinder $[a_1, \dots, a_{n+1}]$ we have that

$$T([a_1, \dots, a_{n+1}]) = [a_2, \dots, a_{n+1}]$$

and hence

$$\begin{aligned} \nu([a_2, \dots, a_{n+1}]) &= \lambda \int_{[a_1, \dots, a_{n+1}]} \prod_{i=1}^{\infty} g_i(x_i) \nu(dx) \\ &= \lambda \prod_{i=1}^{n+1} g_i(a_i) \int_{[a_1, \dots, a_{n+1}]} \prod_{i=n+2}^{\infty} g_i(x_i) \nu(dx). \end{aligned}$$

The analogue equality holds replacing ν by μ and hence

$$\begin{aligned} \frac{\mu([a_1, \dots, a_n])}{\nu([a_1, \dots, a_n])} &= \frac{\int_{[a_1, \dots, a_{n+1}]} \prod_{i=n+2}^{\infty} g_i(x_i) \nu(dx)}{\int_{[a_1, \dots, a_{n+1}]} \prod_{i=n+2}^{\infty} g_i(x_i) \mu(dx)} \\ &\leq K_{n+2} \frac{\mu([a_1, \dots, a_{n+1}])}{\nu([a_1, \dots, a_{n+1}])}, \end{aligned}$$

and a similar lower estimate holds interchanging μ abd ν. This shows that

$$\begin{aligned} \prod_{i=1}^{n+2} K_i^{-1} &\leq K_{n+2}^{-1} \frac{\mu[a_1, \dots, a_n])}{\nu([a_1, \dots, a_n])} \leq \frac{\mu[a_1, \dots, a_{n+1}])}{\nu([a_1, \dots, a_{n+1}])} \\ &\leq K_{n+2} \frac{\mu[a_1, \dots, a_n])}{\nu([a_1, \dots, a_n])} \leq \prod_{i=1}^{n+2} K_i. \end{aligned}$$

Since $K = \prod_{i=1}^{\infty} K_i < \infty$, the claim is proved.

Next we show that a conformal measure ν satisfies $\nu([a]) > 0$ for each $a \in \mathcal{A}$. Indeed, let $b \in \mathcal{A}$ with $\nu([b]) > 0$. Then for any $a \in \mathcal{A}$

$$\nu([b]) = \nu(T[ab]) = \int_{[ab]} g(x) \mu(dx)$$

and hence $\nu([a]) \geq \nu([ab]) > 0$ since g does not vanish.

It follows that any two conformal measures are equivalent since $\mathcal{A}$ is finite.

Next we claim that every conformal measure ν is ergodic: if $A \in \mathcal{B}$ satisfies $T^{-1}(A) = A$ and $\nu(A) > 0$, then it is easy to see that $\nu(\cdot \cap A)/\mu(A)$ is a conformal measure as well. Then $T^{-1}(A^c) = A^c$ and so $\nu(\cdot \cap A^c)/\nu(A^c)$ is conformal if $\nu(A) < 1$. Both measures are singular, contradicting what has been shown so far. Hence $\nu(A) = 1$ and ν is ergodic.

Assume now there is another conformal measure ν which by the previous steps has to be absolutely continuous with respect to μ. Then there is a function $h > 0$, such that, $d\nu = h \cdot d\mu$ by the Radon-Nikodym theorem. Since

$$\begin{aligned}\nu([a_1,\dots,a_n]) &= \int_{[a_1,\dots,a_n]} h(x)\mu(dx)\\ &= \lambda\int_{[a,a_1,\dots,a_n]} h(T(x))g(x)\mu(dx)\\ &= \lambda\int_{[a,a_1,\dots,a_n]} \frac{h(T(x))}{h(x)} g(x)\nu(dx)\end{aligned}$$

and

$$\nu([a_1,\dots,a_n]) = \lambda\int_{[a,a_1,\dots,a_n]} g(x)\nu(dx)$$

we obtain, letting $n\to\infty$ that ν a.s. $h(T(x)) = h(x)$. Now, for every interval I the set $A(I) = \{x \in X : h(x) \in I\}$ is invariant. For each $\eta > 0$ there is one interval I of length η which has positive measure, hence the conditional measure of ν restricted to this set $A(I)$ is conformal, and so $\nu(A(I)) = 1$. Letting the interval shrink to a point c through a sequence of intervals $A(I)$ of measure 1, we see that $h = c$ is constant a.s., finally this implies $c = 1$ and $\nu = \mu$. ☐

Corollary 3.4. *In case the alphabet is infinite then there is only one conformal measure with*

$$0 < \inf_{a\in\mathbb{N}} \frac{\mu([a])}{\nu([a])}$$

where μ is the unique conformal measure of product type.

Proof. In this case the previous proof shows that ν is absolutely continuous with respect to μ. ☐

4. EIGENFUNCTIONS OF PRODUCT TYPE

We now analyse the (point) spectrum of the action of the Ruelle operator on functions of product type. In order to do so, we extend previous definitions to functions $g : X \to \mathbb{R}_+$ of product type. We say that a measurable function $g : X \to \mathbb{R}_+$ of product type is *ℓ_1-bounded* if

$$\sum_{k=2}^{\infty} \|\log g_k\|_\infty < \infty, \tag{9}$$

and remark that this condition implies that $\log g$ is absolutely convergent. Moreover, g is called *summable* if $\sum_{a\in\mathcal{A}} g_1(a) < \infty$. Observe that g is always summable if $\mathcal{A}$ is finite, and that ℓ_1-boundedness in combination with summability implies that $\|\mathcal{L}_{\log g}(1)\|_\infty < \infty$.

Furthermore, we use balanced forms for functions h of product type, which are defined by $h((x_i)_{i\in\mathbb{N}}) = h_0 \prod h_i(x_i)$ where $h_0 > 0$ and $\prod_{a\in\mathcal{A}} h_i(a) = 1$ for all $i \in \mathbb{N}$. Observe that, if $\mathcal{A}$ is finite and h is ℓ_1-bounded, then h always can be written in balanced form. Moreover, for a function $g = \prod_{n=1}^{\infty} g_n$ in balanced form, it follows that $\|\log g_n\|_\infty \le v_n(\log g) \le 2\|\log g_n\|_\infty$ for all $n \in \mathbb{N}$. Hence, Bowen's condition for $\log g$ with index 2 is equivalent to

$$\sum_{m=2}^{\infty}\sum_{n=m}^{\infty} \|\log g_n\|_\infty < \infty. \tag{10}$$

Recall from [16] that Bowen's condition has a variety of important consequences when X is compact, like e. g. uniqueness of the equilibrium state, the conformal measure and the eigenfunction of the Ruelle operator. Therefore, the main novelty of the following result is the fact that it is possible to explicitly determine the eigenfunction and the maximal eigenvalue. We remark that the eigenvalue coincides with the one from Corollary 3.2 for the $1/g$-conformal measure, even though the construction below relies on the hypothesis that g is in balanced form.

Theorem 4.1. *Let (X,T) be a topological Bernoulli shift over a finite or countable alphabet $\mathcal{A}$ and g a function in balanced form. Then, the Ruelle operator $\mathcal{L} = \mathcal{L}_{\log g}$ maps a balanced function $h = \prod h_k$ with $|\sum_a g_1(a)h_1(a)| < \infty$ to a balanced function.*

(1) *If $\mathcal{L}(h) = \lambda h$, for $h = \prod h_k$ in balanced form and some $\lambda > 0$, then*

$$\lambda = g_0 \sum_{a\in\mathcal{A}} \prod_{k=1}^{\infty} g_k(a), \quad h_i(a) = \prod_{k>i} g_k(a) \ \forall i \in \mathbb{N}, a \in \mathcal{A}.$$

(2) *If g satisfies Bowen's condition (10) of index 2, then the function $h(x) = \prod_{i=1}^{\infty} h_i(x_i)$, with h_i as above, is defined for all $x \in X$. Furthermore, if g is summable, then $\lambda < \infty$.*

Proof. We first show how the Ruelle operator $\mathcal{L} = \mathcal{L}_{\log g}$ acts on the set of balanced functions. In order to do so, observe that if $h = 1$ and h is in balanced form, then all the entries of h have to be equal to one. In particular, there exists at most one balanced form of a function. For h in balanced form, we have

$$\mathcal{L}(h)(x) = \sum_{a\in\mathcal{A}} g(ax)h(ax) = g_0h_0 \sum_{a\in\mathcal{A}} g_1(a)h_1(a) \prod_{i=1}^{\infty} g_{i+1}(x_i)h_{i+1}(x_i). \tag{11}$$

Hence, provided that $\sum_a g_1(a)h_1(a)$ is finite, the balanced form of $\mathcal{L}(h)$ is given by $(\mathcal{L}(h))_0 = g_0h_0 \sum_{a\in\mathcal{A}} g_1(a)h_1(a)$ and $(\mathcal{L}(h))_i = g_{i+1}h_{i+1}$ for all $i \in \mathbb{N}$.
Proof of 1. Assume that $\mathcal{L}(h) = \lambda h$, for h in balanced form with $h_0 = 1$. It follows from (11) that $\mathcal{L}(h) = \lambda h$ implies that $\lambda = g_0 \sum_{a\in\mathcal{A}} g_1(a)h_1(a)$ and $h_i = g_{i+1}h_{i+1}$ for all $i \in \mathbb{N}$. Hence, by induction,

$$h_i = \prod_{k>i} g_k \quad (\forall i \in \mathbb{N}), \quad \lambda = g_0 \sum_{a\in\mathcal{A}} \prod_{i=1}^{\infty} g_k(a).$$

Proof of 2. Bowen's condition implies that $\sum_{i\geq 2} \log g_i$ is an absolutely convergent series. Hence, $h(x)$ exists for all $x \in X$. In order to show the existence of λ, note that by summability,

$$\lambda = g_0 \sum_{a\in\mathcal{A}} \prod_{k=1}^{\infty} g_k(a) \leq g_0 \left(\textstyle\sum_{a\in\mathcal{A}} g_1(a)\right) e^{\sum_{k=2}^{\infty} \|\log g_k\|_\infty} < \infty.$$

Since $\prod_k g_k(a) > 0$, it follows from this that λ exists. □

Observe that the theorem does not state that the space of balanced functions of product type is $\mathcal{L}$-invariant due to the fact that the sum $\sum_{a\in\mathcal{A}} g_1(a)h_1(a)$ might be not well defined if $\mathcal{A}$ is infinite. In order to construct an invariant function space in this case one has to consider subclasses of potentials and functions of product type. For example, it easily follows from the argument in the first part of the above proof that, if $g = \prod_i g_i$ is summable and $\|g_i\|_\infty < \infty$ for all i, then $\mathcal{L}_{\log g}$ acts on the space

$$\{f = \textstyle\prod_i f_i : \|g_i\|_\infty < \infty \; \forall i = 1, 2, \ldots\}.$$

The main motivation of this note is to consider potentials beyond Bowen's condition. In particular, it will turn out that Bowen's condition is a sharp condition for the existence of an continuous eigenfunction h. However, the situation with respect to measures is somehow satisfactory, as it is possible to explicitly construct conformal measures and equilibrium states for ℓ_1-bounded potentials. In order to do so, we first have to introduce the action of $\mathcal{L}_{\log g}$ on measures and the notions of pressure and equilibrium states.

If g is ℓ_1-bounded and summable, then $\log g$ is locally uniformly continuous and $\|\mathcal{L}_{\log g}(1)\|_\infty < \infty$. Moreover, by a standard calculation, $\mathcal{L}_{\log g}$ acts on uniformly continuous functions. In particular, $\int h d\mathcal{L}^*_{\log g}\mu = \int \mathcal{L}_{\log g}(h) d\mu$, for continuous functions h, defines an operator $\mathcal{L}^*_{\log g}$ on the space of finite signed Borel measures on X.

We now recall the definition of the pressure for countable state Markov shifts from [13]. As it is shown in there, the pressure $P(\log g)$ defined by

$$P(\log g) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{a \in \mathcal{A}^n} \sup_{x\in[a]} \prod_{i=0}^{n-1} g \circ T^i(x)$$

exists by subadditivity, but is not necessarily finite. However, as shown below, $P(\log g) < \infty$ for ℓ_1-bounded, summable potentials g. Also recall that, if $\mathcal{A}$ is finite and $\log g$ is continuous, the variational principle ([14])

$$P(\log g) = \sup\{h_m(T) + \textstyle\int \log g \; dm : m \text{ probability with } m = m \circ T^{-1}\}$$

holds, with $h_m(T)$ denoting the Kolmogorov-Sinai entropy. Furthermore, if m is an invariant probability measure which realizes the supremum, m is referred to as an *equilibrium state*. However, note that this notion is only applicable if $\mathcal{A}$ is finite since it is unknown whether a variational principle holds for general locally bounded, summable potentials.

The construction of an equilibrium state for topological Bernoulli shifts is based on the following observation which reveals the independence from the existence of the eigenfunction h. Namely, a formal calculation gives, for $x = (x_i)_{i\in\mathbb{N}}$, that

$$\begin{aligned} \frac{g(x)\,h(x)}{\lambda \cdot h \circ T(x)} &= \frac{g(x)}{\lambda} \prod_{i=1}^{\infty} \frac{h_i(x_i)}{h_i(x_{i+1})} = \frac{g(x) h_1(x_1)}{\lambda} \prod_{i=1}^{\infty} \frac{h_{i+1}(x_{i+1})}{h_i(x_{i+1})} \\ &= \frac{g(x) h_1(x_1)}{\lambda \prod_{i=1}^{\infty} g_{i+1}(x_{i+1})} = \frac{\prod_{i=1}^{\infty} g_i(x_1)}{\sum_{a\in\mathcal{A}} \prod_{i=1}^{\infty} g_i(a)} =: \tilde{g}(x). \end{aligned} \tag{12}$$

Hence, even though the function h might not exist, the quotients $h/h \circ T$ and $\tilde{g} = gh/(\lambda h \circ T)$ are well defined for summable, locally bounded g.

The following theorem now provides explicit constructions of conformal measures and equilibrium states as well as a partial answer to the existence of the eigenfunction. If the sequence $(\log h_i)_{i\in\mathbb{N}}$, with (h_i) as above is square summable, then the eigenfunction h exists a.e. with respect to the Bernoulli measure of maximal entropy, but not necessarily with respect to the conformal measure (see the class of examples in Section 6).

The motivation for the following definition, equivalent to (2) above, is to provide a sufficient condition for this property. We say that g has *ℓ_2-bounded tails* if there exists $k \in \mathbb{N}$ such that

$$\sum_{i=k}^{\infty} \sup_{a\in\mathcal{A}} \left(\sum_{j=i}^{\infty} \log g_j(a) \right)^2 < \infty, \tag{13}$$

Theorem 4.2. *Let (X,T) be a topological Bernoulli shift over a finite or countable alphabet $\mathcal{A}$ and let g be a ℓ_1-bounded, summable potential function. Furthermore, let λ be as in Theorem 4.1 and assume that $\mu = \otimes_{n=1}^{\infty}\mu_n$ is a measure of product type and $\tilde{\mu}$ is the Bernoulli measure with weights $\{\tilde{\mu}_0(a) : a \in \mathcal{A}\}$, where*

$$\mu_n(a) := \prod_{i=1}^{n} g_i(a) \Bigg/ \sum_{b\in\mathcal{A}} \prod_{i=1}^{n} g_i(b)\ , \quad \tilde{\mu}_0(a) := \prod_{i=1}^{\infty} g_i(a) \Bigg/ \sum_{b\in\mathcal{A}} \prod_{i=1}^{\infty} g_i(b)\ .$$

1. *We have $\mathcal{L}^*_{\log g}(\mu) = \lambda\mu$, $\mathcal{L}^*_{\log \tilde{g}}(\tilde{\mu}) = \tilde{\mu}$, $\log\lambda = P(\log g)$ and*
$$P(\log g) = h_{\tilde{\mu}}(T) + \textstyle\int \log g d\tilde{\mu}.$$
If $\mathcal{A}$ is finite, then $\tilde{\mu}$ is an equilibrium state.
2. *If g is in balanced form, $\mathcal{A}$ is finite and, for some $k > 1$, (13) holds, then $h(x)$ defined as in Theorem 4.1 exists for almost every $x \in X$ with respect to the $(1/|\mathcal{A}|, \ldots, 1/|\mathcal{A}|)$-Bernoulli measure on X. Furthermore, $\mathcal{L}_{\log g}(h) = \lambda h$.*

Proof. As it is well known, $\mathcal{L}^*_{\log g}(\mu) = \lambda\mu$ if and only if μ is $1/g$-conformal. Hence, by the first part of Theorem 3.1, we have that μ is given by μ_n as in the statement of the theorem. In order to verify that λ is as in Theorem 4.1, note that by bounded convergence,

$$\begin{aligned}
\int \mathcal{L}_{\log g} 1 d\mu &= g_0 \sum_{b\in\mathcal{A}} g_1(b) \prod_{i=1}^{\infty} \int g_{i+1}(x_i) d\mu_i(x_i) \\
&= g_0 \sum_{b\in\mathcal{A}} g_1(b) \prod_{i=1}^{\infty} \frac{\sum_{a\in\mathcal{A}} g_1(a)\cdots g_{i+1}(a)}{\sum_{a\in\mathcal{A}} g_1(a)\cdots g_i(a)} \\
&= g_0 \lim_{i\to\infty} \sum_{a\in\mathcal{A}} g_1(a)\cdots g_{i+1}(a) = g_0 \sum_{a\in\mathcal{A}} \prod_{i=1}^{\infty} g_i(a).
\end{aligned}$$

Hence, $\mathcal{L}^*_{\log g}(\mu) = \lambda\mu$ with λ as in Theorem 4.1. In order to show that $\mathcal{L}^*_{\log \tilde{g}}(\tilde{\mu}) = \tilde{\mu}$, note that $\tilde{g}$ as defined in (12) only depends on the first coordinate and in particular is of product type and locally bounded. Furthermore, it follows from $\mathcal{L}_{\log \tilde{g}}(1) = 1$ that $\tilde{g}$ is summable. Hence, $\mathcal{L}^*_{\log \tilde{g}}(\tilde{\mu}) = \tilde{\mu}$ again by the first part of Theorem 3.1.

We now establish $P(\log g) = h_{\tilde{\mu}}(T) + \int \log g\ d\tilde{\mu}$ by proving that $h_{\tilde{\mu}}(T) = \log\lambda - \int \log g d\tilde{\mu}$ and $P(\log g) = \log\lambda$. As $\tilde{\mu}$ is a Bernoulli measure we obtain

$$\begin{aligned}
h_{\tilde{\mu}}(T) &= -\int \log \tilde{\mu}([x_1])\tilde{\mu}(d(x)) \\
&= -\sum_{a\in\mathcal{A}} \tilde{\mu}_0(a) \left(\log \prod_{i=1}^{\infty} g_i(a) - \log \sum_{b\in\mathcal{A}} \prod_{i=1}^{\infty} g_i(b) \right) \\
&= \log \sum_{b\in\mathcal{A}} \prod_{i=1}^{\infty} g_i(b) - \sum_{i=1}^{\infty} \sum_{a\in\mathcal{A}} \log g_i(a)\tilde{\mu}_0(a) = \log\lambda - \int \log g d\tilde{\mu}.
\end{aligned}$$

In order to show that $P(\log g) = \log \lambda$, note that ℓ_1-boundedness implies for $x, y \in [a_1, \ldots, a_n]$ that there exists $C > 0$ such that

$$\log \prod_{k=0}^{n-1} \frac{g(T^k(x))}{g(T^k(y))} \leq 2 \sum_{k=0}^{n-1} \sum_{i \geq k} \| \log g_i \|_\infty \leq Cn.$$

Hence, $\mathcal{L}^n_{\log g}(1)(x) = e^{\pm Cn} \mathcal{L}^n_{\log g}(1)(y)$ for all $x, y \in X$. Since $\log n / n \to 0$, we have

$$\begin{aligned} P(\log g) &= \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}^n_{\log g}(1)(x) = \lim_{n\to\infty} \frac{1}{n} \int \log \mathcal{L}^n_{\log g}(1) d\mu \\ &= \lim_{n\to\infty} \frac{1}{n} \log \int 1 d(\mathcal{L}^n_{\log g})^*(\mu) = \log \lambda. \end{aligned}$$

Hence, assertion 1. is proven. In order to show assertion 2., let ρ denote the $(1/|\mathcal{A}|, \ldots, 1/|\mathcal{A}|)$-Bernoulli measure on X, the measure of maximal entropy. Write $\rho = \otimes \rho_i$, the product of the equidistribution ρ_i on $\mathcal{A}$. With respect to this measure, and since g is balanced it follows that, for all $j \geq k$,

$$\int \log h_j d\rho = \int \log h_j(a) d\rho_j(a) = \sum_{i>j} \int \log g_i(a) d\rho_j(a) = 0.$$

We now consider $(h_i)_{i\in\mathbb{N}}$ as a stochastic processes on the probability space (X, ρ). In particular, the above implies that $\mathbb{E}(\log h_j) = 0$. Furthermore, for the variances of $\log h_j$, we obtain

$$\operatorname{Var}(\log h_j) = \int (\log h_j)^2 d\rho \leq \max_{a\in\mathcal{A}} (\log h_j(a))^2 = \max_{a\in\mathcal{A}} \left(\textstyle\sum_{i>j} \log g_i(a) \right)^2.$$

Hence, the summability condition implies that $\sum_{j>k} \operatorname{Var}(\log h_j) < \infty$. Hence, as a consequence of Kolmogorov's three series theorem (as in [10, Corollary 3 on p. 87]), it follows that $\log h = \sum_{j\geq 1} \log h_j$ converges ρ-a.s. The remaining assertion $\mathcal{L}_{\log g}(h) = \lambda h$ follows as in Theorem 4.1. □

The existence of h in the second part of the above theorem is based on the fact that the log of a balanced function has zero integral with respect to the measure of maximal entropy. By considering a suitable scaling of h, an analogous result holds with respect to μ. The existence of this function is equivalent to the equivalence of the measures μ and $\tilde{\mu}$.

Theorem 4.3. *Let (X, T) be a topological Bernoulli shift over a finite or countable alphabet $\mathcal{A}$, let g be a ℓ_1-bounded, summable potential function of product type and let μ and λ as in Theorem 4.2.*

(1) *There is at most one $h \in L^1(X, \mu)$ with $\mathcal{L}_{\log g}(h) = \lambda h$ and $\int h d\mu = 1$.*

(2) *If (13) holds for some $k \in \mathbb{N}$, then the function*

$$h_\mu((x_j)) = \prod_{j=1}^{\infty} \frac{\sum_{a\in\mathcal{A}} \prod_{l=1}^{j} g_l(a)}{\sum_{a\in\mathcal{A}} \prod_{l=1}^{\infty} g_l(a)} \prod_{l=1}^{\infty} g_{l+j}(x_j) \tag{14}$$

is in $L^1(X, \mu)$. Furthermore, $\int h_\mu d\mu = 1$, $\mathcal{L}_{\log g}(h_\mu) = \lambda h_\mu$ and $d\tilde{\mu} = h_\mu d\mu$.

(3) *The function h_μ exists μ-a.s.. Moreover, $\int h_\mu d\mu > 0$ if and only if μ and $\tilde{\mu}$ are equivalent. If $\int h_\mu d\mu = 0$, then $\tilde{\mu}$ and μ are singular measures with disjoint support.*

Proof. (i) In order to show uniqueness, we will identify $\lambda^{-1}\mathcal{L}_{\log g}$ with the transfer operator. As it was noted above, ℓ_1-boundedness and summability imply that $\lambda^{-1}\mathcal{L}_{\log g}$ acts on uniformly continuous functions. It now follows from the conformality of μ that $\lambda^{-1}\mathcal{L}_{\log g}$ acts as the transfer operator on $L^1(X,\mu)$, that is $\int \psi\lambda^{-1}\mathcal{L}_{\log g}(\phi)d\mu = \int \psi\circ T\cdot\phi\, d\mu$ for all $\psi \in L^\infty(X,\mu)$ and $\phi \in L^1(X,\mu)$. A further important ingredient is exactness, that is triviality of the tail σ-field $\bigcap_{n>1} T^{-n}\mathcal{B}$ modulo μ. As μ is a product measure, it follows from Kolmogorov's 0-1 law that T is exact. Hence, by Lin's criterion for exactness ([9], Th. 4.4)

$$\lim_{n\to\infty} \|\lambda^{-n}\mathcal{L}^n_{\log g}(\phi)\|_1 = 0$$

for all $\phi \in L^1(X,\mu)$ with $\int \phi\, d\mu = 0$. In particular, if $\mathcal{L}_{\log g}(h) = \lambda\phi$ and $\int h\, d\mu = 0$, then $\|h\|_1 = 0$. Hence, if h_1, h_2 satisfy $\mathcal{L}_{\log g}(h_i) = \lambda h_i$ and $\int h_i d\mu = 1$, then $\|h_1 - h_2\|_1 = 0$. This proves the uniqueness of h.
(ii) In order to show that h_μ exists, we employ Kolmogorov's three series theorem as in [10, Corollary 1 on p. 84]. Hence we have to show that $|\sum \int \log h_\mu^{(j)} d\mu_j| < \infty$ and $\sum \int (\log h_\mu^{(j)})^2 d\mu_j < \infty$, for

$$h_\mu^{(j)} := \Delta_j \prod_{l=1}^{\infty} g_{l+j}(x_j), \text{ where } \Delta_j := \frac{\sum_{a\in\mathcal{A}} \prod_{l=1}^{j} g_l(a)}{\sum_{a\in\mathcal{A}} \prod_{l=1}^{\infty} g_l(a)}.$$

By construction of μ, we have $\int h_\mu^{(j)} d\mu_j = 1$ and, by Jensen's inequality, $\int \log h_\mu^{(j)} d\mu_j \leq 0$. In order to prove summability of the first sum, it therefore suffices to obtain a lower bound which follows from

$$\begin{aligned}
\int \log h_\mu^{(j)} d\mu_j &= \int \sum_{l>j} \log g_l d\mu_j - \log \frac{1}{\Delta_j} \geq \int \sum_{l>j} \log g_l d\mu_j + 1 - \frac{1}{\Delta_j} \\
&= \int \sum_{l>j} \log g_l d\mu_j + \frac{\sum_{a\in\mathcal{A}} \prod_{l=1}^{j} g_l(a)(1 - \prod_{l>j} g_l(a))}{\sum_{a\in\mathcal{A}} \prod_{l=1}^{j} g_l(a)} \\
&= \int \sum_{l>j} \log g_l + 1 - \prod_{l>j} g_l \, d\mu_j = o\left(\sup_a (1 - \prod_{l>j} g_l(a))^2\right)
\end{aligned}$$

where we used $\log(1+x) - x = o(x^2)$ in the last identity. Hence, if (13) holds, then $\sum \int \log h_\mu^{(j)} d\mu_j$ is summable. Using a similar argument, it easily can be seen that $\log \Delta_j \sim \int \sum_{l>j} \log g_l d\mu_j$. Hence, if (13) holds, then $\sum \int (\log h_\mu^{(j)})^2 d\mu_j$ is summable. In particular, h_μ exists μ-a.s. by the three series theorem whereas it follows from $\int h_\mu^{(j)} d\mu_j = 1$ that $\int h_\mu d\mu = 1$.

In order to show that $d\tilde{\mu} = h_\mu d\mu$ it suffices to show that $\tilde{\mu}([w]) = \int_{[w]} h_\mu d\mu$, for each $n \in \mathbb{N}$ and $w = (w_1, \ldots, w_n)$ with $w_j \in \mathcal{A}$. It follows from the product structure that

$$\int_{[w]} h_\mu d\mu = \prod_{j=1}^{n} \int_{[w_j]} h_\mu^{(j)} d\mu_j = \prod_{j=1}^{n} \Delta_j \frac{\prod_{l=1}^{j} g_l(w_j)}{\sum_a \prod_{l=1}^{j} g_l(a)} \prod_{l=1}^{\infty} g_{l+j}(w_j) = \tilde{\mu}([w]).$$

Hence, $h_\mu = d\tilde{\mu}([w])/d\mu$. As $\lambda^{-1}\mathcal{L}_{\log g}$ acts as the transfer operator and $d\tilde{\mu} = h_\mu d\mu$ is invariant, it follows for each test function $\phi \in L^\infty(X,\mu)$, that

$$\int \phi\, \lambda^{-1}\mathcal{L}_f(h_\mu) d\mu = \int \phi \circ T \cdot h_m d\mu = \int \phi\, h_m d\mu.$$

Hence, $\mathcal{L}_f(h_\mu) = \lambda h_\mu$.
(iii) In order to prove the third part of the theorem, we will make use of the fact, that the shift space X is a Besicovitch space and therefore, a measure differentiation theorem holds (see [2]). That is, the function

$$D_\mu(\tilde{\mu})((x_j)) = \lim_{n\to\infty} \frac{\tilde{\mu}([x_1,\ldots,x_n])}{\mu([x_1,\ldots,x_n])}$$

exists and is finite μ-a.e.. Moreover, $D_\mu(\tilde{\mu})$ is the Radon-Nikodym derivative $d\tilde{\mu}_{\mathrm{ac}}/d\mu$, where $\tilde{\mu}_{\mathrm{ac}}$ is the absolutely continuous part of $\tilde{\mu}$ with respect to μ.

In order to apply the result, observe that $D_\mu(\tilde{\mu}) = 0$ implies that $\tilde{\mu}$ and μ are singular measures with disjoint support. However, if $\int D_\mu(\tilde{\mu})d\mu = \tilde{\mu}_{\mathrm{ac}}(X) > 0$, it follows from ergodicity of $\tilde{\mu}$ that $\tilde{\mu}_{\mathrm{ac}} = \tilde{\mu}$ and from

$$\frac{\tilde{\mu}([x_1,\ldots,x_n])}{\mu([x_1,\ldots,x_n])} = \prod_{j=1}^{n} \frac{\prod_{l=1}^{\infty} g_l(x_j)/(\sum_a \prod_{l=1}^{\infty} g_l(a))}{\prod_{l=1}^{j} g_l(x_j)/(\sum_a \prod_{l=1}^{j} g_l(a))} = \prod_{j=1}^{n} \Delta_j \prod_{l=1}^{\infty} g_{l+j}(x_j)$$

that $D_\mu(\tilde{\mu})$ and h_μ are equal μ-a.s.. It follows from ergodicity of μ that $D_\mu(\tilde{\mu}) > 0$ a.s. □

5. Eigenfunctions in L^1-spaces

The Ruelle operator $\mathcal{L}_f$ with $f \in C(X)$ acts on classes of measurable functions modulo any Bernoulli measure ρ on X of the form $\rho = \otimes_{i=1}^{\infty}\rho_0$, where ρ_0 is any probability measure on $\mathcal{A}$. Indeed, note that ρ is a shift invariant and ergodic measure on (X,T). Let ϕ, ψ be two functions which agree ρ almost surely. Let $A = \{\phi = \psi\}$. Then $\rho(A) = 1$ and because of invariance of ρ we may assume that $T^{-1}(A) \subset A$. Then by definition $\mathcal{L}_f\phi = \mathcal{L}_f\psi$ on A (if the operator is well defined for these functions), so that $\mathcal{L}_f$ maps equivalence classes of measurable functions into such classes.

In this section we always assume that the alphabet $\mathcal{A}$ is finite. Then the Ruelle operator is always well defined on measurable functions. When the Ruelle operator is well defined in case of an infinite alphabet the following results can be adapted. The first theorem is a slightly modified and extended result from Theorem 4.2, part 2.

Theorem 5.1. *Let $g = e^f = \prod_{i=0}^{\infty} g_i$ be a balanced potential function.*

1. The Ruelle operator $\mathcal{L}_f$ defines canonically a bounded linear operator on $L^p(X,\rho)$ for all $1 \leqslant p \leq \infty$, where $\rho = \otimes_{i=1}^{\infty}\rho_0$ is any stationary Bernoulli measure.

2. Assume that g has ℓ_2-summable tails, that is for some $k > 1$,

$$M := \sum_{i=k}^{\infty} \max_{a\in\mathcal{A}} \left(\sum_{j=i}^{\infty} \log g_j(a)\right)^2 < \infty,$$

and that ρ is a stationary Bernoulli measure with

$$\int \log g_k(x)\rho(dx) = 0 \qquad \forall k \geq 1.$$

Then the function $h : X \to \mathbb{R}_+ \cup \{\infty\}$ defined by

$$h((x_i)_{i\in\mathbb{N}}) = \prod_{i=1}^{\infty} h_i(x_i) \qquad h_i(a) = \prod_{k>i} g_k(a) \quad a \in \mathcal{A}$$

belongs to $L^p(X,\rho)$ *for every* $1 \leq p < \infty$ *and is an almost surely positive eigenfunction of* $\mathcal{L}_f : L^p(X,\rho) \to L^p(X,\rho)$ *with eigenvalue*

$$\lambda = g_0 \sum_{a\in\mathcal{A}} \prod_{k=1}^{\infty} g_k(a).$$

Proof. 1. We need to show that $\mathcal{L}_f$ sends $L^p(X,\rho)$ into itself. Indeed, let $1 \leqslant p < \infty$ be fixed and $\varphi \in L^p(X,\rho)$. Bounding f from above by its supremum norm and using the triangular inequality we get

$$|\mathcal{L}_f(\varphi)(x)|^p = |\sum_{a\in\mathcal{A}} \varphi(ax)g(ax)|^p \leq \|g\|_\infty^p \sum_{a\in\mathcal{A}} |\varphi(ax)|^p.$$

By the hypothesis

$$\int_X |\varphi(x)|^p d\rho(x) < +\infty$$

and since ρ is a Bernoulli measure,

$$\sum_{a\in\mathcal{A}} \int_X |\phi(ax)|^p \rho(dx) = |\mathcal{A}| \int_X |\phi(x)|^p \rho(dx) < \infty,$$

thus proving that $\mathcal{L}_f$ sends $L^p(X,\rho)$ to itself in case $1 \leq p < \infty$. The case $p = \infty$ is trivial because the Ruelle operator is just a finite sum of a product of two uniformly bounded functions.

This estimate also shows that $\mathcal{L}_f$ can be considered as a bounded operator acting on $L^p(X,\rho)$ for $1 \leq p \leq \infty$.

2. We first show that h is almost surely finite. Similarly to the proof of Theorem 4.2 the random variables $\log h_j$ satisfy $\int \log h_j d\rho = 0$ and

$$\mathrm{Var}(\log h_j) \leq \max_{a\in\mathcal{A}} \left(\sum_{i>j} \log g_i(a) \right)^2 .$$

Again by Kolmogorov's three series theorem ([10, p. 87]) $\sum_{i=1}^{\infty} \log h_i$ converges ρ a.s..

We show next that the moment generating function for $H = \sum_{n=1}^{\infty} \log h_n$ exists on $\mathbb{R}$. Since $\log h_n(x) \leq \max_{a\in\mathcal{A}} \sum_{i>n} g_i(a)$ it follows from independence of $\log h_n$ that that for $p \geq 2$

$$E|H|^p \leq (M^{2p-2})^{1/2}(EH^2)^{1/2} \leq M^{p-1} \sum_{n=1}^{\infty} E(\log h_n)^2 \leq M^p$$

whence

$$Ee^{tH} = \sum_{n=0}^{\infty} \frac{t^n}{n!} EH^n \leq \sum_{n=0}^{\infty} \frac{(tM)^n}{n!} < \infty.$$

In particular, for $p \in \mathbb{N}$

$$Eh^p = Ee^{pH} < \infty$$

and $h \in L^p(X,\rho)$.

The proof is completed similar to the one given in Theorem 4.1. □

We finally turn towards uniqueness questions of the eigenfunction h. We assume that the alphabet $\mathcal{A}$ is finite.

The uniqueness of the eigenfunction h with respect to the eigenvalue λ takes the following form. Recall that

$$\mu_0(a) = \lambda^{-1} \prod_{l=1}^{\infty} g_j(a) \qquad a \in \mathcal{A}$$

defines the equilibrium product measure. The operator

$$P_{\mu_0} \psi(x_1, x_2, ...) = \sum_{a \in \mathcal{A}} \psi(a, x_1, x_2, ...) \mu_0(a)$$

acts on measurable functions and on ρ-equivalence classes in $L^1(X, \rho)$, whence P_{μ_0} will be considered as an operator on $L^1(X, \rho)$.

Theorem 5.2. *Let $\mathcal{A}$ be a finite alphabet and ρ be a product measure as in the previous theorem and $g = e^f$ be a balanced potential with ℓ_2-summable tails. Then the Ruelle operator $\mathcal{L}_f : L^1(X, \rho) \to L^1(X, \rho)$ has (up to multiplication by constants) exactly one eigenfunction $h \in L^1(X, \rho)$ with respect to the eigenvalue*

$$\lambda = g_0 \sum_{a \in A} \prod_{k=1}^{\infty} g_k(a)$$

if and only if P_{μ_0} is ergodic (i.e. has only one eigenfunction for the eigenvalue 1 up to multiplication by constants).

Proof. We only need to show uniqueness. Let $\phi \in L^1(X, \rho)$ be an eigenfunction for the eigenvalue λ. Let $X_1, X_2, ...$ denote the i.i.d. coordinate process determining ρ. Then

$$\begin{aligned} \phi(X_1, X_2, ...) &= \lambda^{-1} \mathcal{L}_f \phi(X_1, X_2, ...) \\ &= \lambda^{-1} \sum_{a \in \mathcal{A}} \phi(a, X_1, X_2, ...) \prod_{k=1}^{\infty} g_{k+1}(X_k) g_1(a) \end{aligned}$$

and dividing by $h(X_1, X_2, ...)$ yields

$$\begin{aligned} \frac{\phi(X_1, X_2, ...)}{h(X_1, X_2, ...)} &= \lambda^{-1} \sum_{a \in \mathcal{A}} \frac{\phi(a, X_1, X_2, ...)}{h(X_1, X_2, ...)} \prod_{k=1}^{\infty} g_{k+1}(X_k) g_1(a) \\ &= \lambda^{-1} \sum_{a \in \mathcal{A}} \frac{\phi(a, X_1, X_2, ...)}{\prod_{k=1}^{\infty} \prod_{j=k+2}^{\infty} g_j(X_k)} g_1(a) \\ &= \lambda^{-1} \sum_{a \in \mathcal{A}} \frac{\phi(a, X_1, X_2, ...)}{h(a, X_1, X_2, ...)} \prod_{l=1}^{\infty} g_l(a) \\ &= P_{\mu_0} \frac{\phi}{h}(X_1, X_2, ...) \end{aligned}$$

Therefore ϕ/h is an eigenfunction for the eigenvalue 1 (note that $P_{\mu_0} 1 = 1$). Thus if P_{μ_0} is ergodic, ϕ/h is constant.

Conversely, the above equation shows that if P_{μ_0} has another eigenfunction ψ, then ψh is an eigenfunction for $\mathcal{L}_f$, proving the theorem. □

6. The leading example

We return to the class of potentials defined in Example 2. Recall that it uses the alphabet $\mathcal{A} = \{-1, 1\}$ and potentials of the form

$$f(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^\gamma}, \quad \gamma > 1. \tag{15}$$

Observe that $g(x) := e^{-f(x)}$ satisfies $\inf_x g(x) = \exp(-\sum_n n^\gamma) > 0$, whence the potential of product type g is bounded from below. We also have that g is balanced and ℓ_1-bounded. Hence, we obtain explicit expressions for the conformal measure, the equilibrium state and λ by applying Theorems 3.1, 4.1 and 4.2. In here, $\zeta(\gamma)$ refers to the Riemann ζ-function $\zeta(\gamma) := \sum_{j=1}^{\infty} j^{-\gamma}$.

(1) The conformal measure $\mu = \otimes_{i=1}^{\infty} \mu_i$ is of product type, where

$$\mu_i(\{1\}) = \frac{\exp(\sum_{j=1}^{i} j^{-\gamma})}{2\cosh(\sum_{j=1}^{i} j^{-\gamma})}, \qquad \mu_i(\{-1\}) = \frac{\exp(-\sum_{j=1}^{i} j^{-\gamma})}{2\cosh(\sum_{j=1}^{i} j^{-\gamma})}. \tag{16}$$

(2) The conformality parameter is equal to $\lambda = 2\cosh(\zeta(\gamma))$.

(3) The equilibrium state $\tilde{\mu} = \otimes_{i=1}^{\infty} \tilde{\mu}_i$ is a Bernoulli measure (that is a $\tilde{\mu}_i = \tilde{\mu}_j$ for all i, j). The measure $\tilde{\mu}_0 := \tilde{\mu}_i$ is given by

$$\tilde{\mu}_0(\{1\}) = \frac{\exp(\zeta(\gamma))}{2\cosh(\zeta(\gamma))}, \qquad \tilde{\mu}_0(\{-1\}) = \frac{\exp(-\zeta(\gamma))}{2\cosh(\zeta(\gamma))}. \tag{17}$$

6.1. Bowen's class ($\gamma > 2$).

Recall that it has been shown above that f is in Bowen's class if and only if $\gamma > 2$. In this situation, we obtain a stronger result. Namely, by Theorem 3.3, the measure μ above is the unique conformal measure. In particular, λ is also uniquely determined by $\mathcal{L}_f^*(\mu) = \lambda\mu$. Moreover, the function $h((x_i)) = \prod_{i \geq 1} h_i(x_i)$ defined by

$$h_n(x_n) := \exp(\alpha_n x_n), \qquad \alpha_n := \sum_{j=n+1}^{\infty} j^{-\gamma}, \tag{18}$$

is an eigenfunction of product type. This function is the unique function with $\mathcal{L}_f(h) = \lambda h$, and the equilibrium state is given by, as usual, $d\tilde{\mu} = h d\mu$. It is worth noting that for $\gamma > 2$, Walters showed in [16] that a Perron-Frobenius theorem holds in a more general situation. Furthermore, the main result in [4] is applicable to our example and implies polynomial decay of $\mathcal{L}_f$ for these parameters of γ.

6.2. The case $3/2 < \gamma \leq 2$.

We now consider the case of $3/2 < \gamma \leq 2$ which is related to the second case of Theorem 4.2 and Theorem 4.3. Namely, as the coefficients h_n defined in (18) satisfy $|\log h_n| \sim n^{1-\gamma}$, it follows that $\sum_{m>n} |\log h_m|^2 \sim n^{2-2\gamma}$. Hence, $\sum_n \sum_{m>n} |\log h_m|^2$ converges iff $2\gamma - 2 > 1$ which is equivalent to $\gamma > 3/2$. Therefore, if $\gamma > 3/2$, the function

$$h_\rho(x) = \exp\left(\sum_{i=1}^{\infty} \alpha_i x_i\right), \tag{19}$$

is ρ-almost surely well defined, where $\rho = \otimes_{i=1}^{\infty} \rho_0$ is the Bernoulli product measure with parameter 1/2 on $X = \{-1, 1\}^{\mathbb{N}}$. With respect to μ, it follows from Theorem

4.3 that

$$h_\mu(x) = \exp\left(\sum_{i=1}^{\infty} \alpha_i x_i + \log \frac{\cosh(\sum_{j=1}^{i} j^{-\gamma})}{\cosh(\zeta(\gamma))}\right) \tag{20}$$

is μ-almost surely well defined. Furthermore, both functions satisfy the functional equation $\mathcal{L}_f(h) = \lambda h$, for $\lambda = 2\cosh(\zeta(\gamma))$.

Theorem 6.1. *Let $1 < \gamma \leq 2$ and μ as in (16) and $\tilde{\mu}$ as in (17).*

1. *If $3/2 < \gamma \leq 2$, then $h_\rho(x) = \infty$ for μ-a.e. $x \in X$, and $h_\mu(x) = 0$ for ρ-a.e. $x \in X$.*
2. *If $\gamma > 3/2$, then μ and $\tilde{\mu}$ are absolutely continuous, and $d\tilde{\mu} = h_\mu d\mu$.*
3. *If $1 < \gamma \leq 3/2$, then μ, $\tilde{\mu}$ and ρ are pairwise singular.*
4. *If $3/2 < \gamma \leq 2$, then, for any open set A, we have*

$$\begin{aligned} \textit{ess-inf}_\rho\{h_\rho(x) : x \in A\} &= \textit{ess-inf}_\mu\{h_\mu(x) : x \in A\} = 0, \\ \textit{ess-sup}_\rho\{h_\rho(x) : x \in A\} &= \textit{ess-sup}_\mu\{h_\mu(x) : x \in A\} = \infty. \end{aligned}$$

   In particular, neither h_ρ nor h_μ can be extended to a (locally) continuous function.

Proof. The first assertion is an application of Kolmogorov's three series theorem as in [10, p. 87]. By a direct calculation,

$$\begin{aligned} E_{\mu_i}(\log h_\rho^{(i)}) &= \int \alpha_i x \, d\mu_i(x) = \alpha_i \frac{\exp(\sum_{j=1}^{i} j^{-\gamma})) - \exp(-\sum_{j=1}^{i} j^{-\gamma}))}{2\cosh(\sum_{j=1}^{i} j^{-\gamma})} \\ &= \alpha_i \tanh(\textstyle\sum_{j=1}^{i} j^{-\gamma}) \sim \frac{\tanh(\zeta(\gamma))}{(\gamma-1)} i^{1-\gamma} \\ \mathrm{Var}_{\mu_i}(\log h_\rho^{(i)}) &= \int (\alpha_i x)^2 \, d\mu_i(x) - (\alpha_i \tanh(\textstyle\sum_{j=1}^{i} j^{-\gamma}))^2 = \alpha_i^2(1 - \tanh^2(\sum_{j=1}^{i} j^{-\gamma})) \\ &= \frac{\alpha_i^2}{\cosh^2(\sum_{j=1}^{i} j^{-\gamma})} \sim \frac{i^{2-2\gamma}}{(\gamma-1)^2 \cosh^2(\zeta(\gamma))} \end{aligned}$$

For $3/2 < \gamma \leq 2$, it follows that $\sum_i E_{\mu_i}(\log h_\rho^{(i)}) = \infty$ and $\sum_i \mathrm{Var}_{\mu_i}(\log h_\rho^{(i)}) < \infty$. This then implies that $\sum_{i=1}^{\infty}(\log h_\rho^{(i)} - E_{\mu_i}(\log h_\rho^{(i)}))$ converges μ-a.s. ([10, p. 87]). Hence, $h_\rho = \infty$ μ-a.s. In order to prove the statement for h_μ with respect to ρ, we apply the same arguments. Namely, the assertion follows from

$$E_\rho\left(\log h_\rho^{(i)} + \log \frac{\cosh(\sum_{j=1}^{i} j^{-\gamma})}{\cosh(\zeta(\gamma))}\right) = \log \frac{\cosh(\sum_{j=1}^{i} j^{-\gamma})}{\cosh(\zeta(\gamma))} \sim -\frac{\tanh(\zeta(\gamma)) i^{1-\gamma}}{\gamma - 1}$$

and $\mathrm{Var}_\rho(\log h_\mu^{(i)}) = \mathrm{Var}_\rho(\log h_\rho^{(i)}) = \alpha_i^2$.

The second and the third are an application of Theorem 4.3 and the three series theorem as in [10, p. 88]. Namely, we have that

$$\mathrm{Var}_{\mu_i}(\log h_\mu^{(i)}) = \mathrm{Var}_{\mu_i}(\log h_\rho^{(i)}) \sim \frac{i^{2-2\gamma}}{(\gamma-1)^2 \cosh^2(\zeta(\gamma))}.$$

Hence, if $\gamma \leq 3/2$, then $\log h_\mu$ does not exist in $(-\infty, \infty)$. However, h_μ exists by Theorem 4.3 also in this case, but might be equal to 0. Hence, $h_\mu = 0$ and μ and $\tilde{\mu}$ are pairwise singular. Assertion (iii) then follows from the obvious fact that ρ is singular with respect to both μ and $\tilde{\mu}$. Furthermore, part (ii) is a consequence of Theorem 4.3 as $h_\mu > 0$ for $\gamma > 3/2$.

It remains to show the last part. We begin with the proof for h_ρ. As A is open, there exist $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \{-1, 1\}$ such that $[a_1, \ldots, a_m] \subset A$. In order to show that $\text{ess-sup}_\rho h_\rho(x) = \infty$, it remains to show that, for all $M > 0$,

$$\rho\left(\{x \in [a_1, \ldots, a_m] : \textstyle\sum_{i=1}^\infty \alpha_i x_i > M\}\right) > 0.$$

In order to do so, note that $\gamma \leq 2$ implies that $\sum_{i=1}^\infty \alpha_i = \infty$. Hence, for each $M > 0$, there exists $n > m$ such that $-\alpha_1 - \ldots - \alpha_m + \alpha_{m+1} + \ldots + \alpha_n > M$. For $\mathfrak{C} := \{x \in [a_1, \ldots, a_m] : x_{m+1} = \ldots = x_n = 1\}$, we have

$$\rho\left(\mathfrak{C} \cap \left\{\textstyle\sum_{i=n+1}^\infty \alpha_i x_i \geqslant 0\right\}\right) \leq \rho\left(\textstyle\sum_{i=1}^\infty \alpha_i x_i > M\right).$$

Observe that the events $\mathfrak{C}$ and $\{\sum_{i=n+1}^\infty \alpha_i x_i \geqslant 0\}$ are independent, that $\rho(\mathfrak{C}) = 2^{-n}$ and that, by symmetry, $\rho(\{\sum_{i=n+1}^\infty \alpha_i x_i \geqslant 0\}) \geq 1/2$. Hence,

$$\rho\left(\mathfrak{C} \cap \left\{\textstyle\sum_{i=n+1}^\infty \alpha_i x_i \geqslant 0\right\}\right) = \rho(\mathfrak{C}) \cdot \rho(\{\textstyle\sum_{i=n+1}^\infty \alpha_i x_i \geqslant 0\}) \geq 2^{1-n} > 0.$$

Hence, $\text{ess-sup}_\rho\{h_\rho(x) : x \in A\} \geq e^M$. The proof of $\text{ess-inf}_\rho\{h_\rho(x) : x \in A\} = 0$ follows by substituting $\mathfrak{C}$ with $\{x \in [a_1, \ldots, a_m] : x_{m+1} = \ldots = x_n = -1\}$, where n is chosen such that $\alpha_1 + \ldots + \alpha_m - (\alpha_{m+1} + \cdots + \alpha_n) < -M$.

In order to proof the local unboundedness of h_μ, we make use of (ii). Namely, in order to obtain that $\text{ess-sup}_\mu h_\mu(x) = \infty$, it suffices to show that, for $[a_1, \ldots, a_m] \subset A$ and $w_n := (a_1, \ldots, a_m, 1, 1, \ldots, 1) \in \mathcal{A}^{m+n}$, we have

$$\lim_{n\to\infty} \frac{\int_{[w_n]} h_\mu d\mu}{\mu([w_n])} = \lim_{n\to\infty} \frac{\tilde{\mu}([w_n])}{\mu([w_n])} = \lim_{n\to\infty} \frac{\tilde{\mu}([a_1 \cdots a_m])}{\mu([a_1 \cdots a_m])} \frac{\tilde{\mu}([1, \ldots, 1])}{\mu([1, \ldots, 1])} = \infty,$$

where $(1, \ldots, 1)$ stands for the word of length n with all entries equal to one. In order to verify this condition, note that $\log(\cosh(\sum_{l=1}^j l^{-\gamma})/\cosh(\zeta(\gamma))) \sim -\tanh(\zeta(\gamma)) \sum_{l=j+1}^\infty l^{-\gamma}$. Hence,

$$\begin{aligned}
\sum_{j=m+1}^{m+n} \log \frac{\tilde{\mu}_j(1)}{\mu_j(1)} &= \sum_{j=m+1}^{m+n} \log \frac{\exp \sum_{l=1}^\infty l^{-\gamma}}{2\cosh(\zeta(\gamma))} - \log \frac{\exp \sum_{l=1}^j l^{-\gamma}}{2\cosh(\sum_{l=1}^j l^{-\gamma})} \\
&= \sum_{j=m+1}^{m+n} \left(\sum_{l=j+1}^\infty l^{-\gamma} + \log \frac{\cosh(\sum_{l=1}^j l^{-\gamma})}{\cosh(\zeta(\gamma))} \right) \\
&\asymp \sum_{j=m+1}^{m+n} (1 - \tanh(\zeta(\gamma))) \sum_{l=j+1}^\infty l^{-\gamma} \xrightarrow{n\to\infty} \infty.
\end{aligned}$$

Hence, $\text{ess-sup}_\mu h_\mu(x) = \infty$. The proof of $\text{ess-inf}_\mu h_\mu(x) = 0$ is the same. □

We turn our attention to h as an element of $L^1(X, \rho)$. The following results are Theorems 5.1 and 5.2 adapted to our example.

Theorem 6.2. *For $\gamma > 3/2$ and h as in* (19)*, the following holds.*

1. *The Ruelle operator $\mathcal{L}_f$ defines canonically a bounded linear operator on $L^p(X, \rho)$ for all $1 \leqslant p \leq \infty$.*
2. *The function h defined above belongs to $L^p(X, \rho)$ for every $1 \leq p < \infty$ and is the unique eigenfunction of $\mathcal{L}_f : L^p(X, \rho) \to L^p(X, \rho)$ with eigenvalue $\lambda = 2\cosh(\zeta(\gamma))$ if and only if the operator*

$$P\phi(x_1, x_2, ...) = \frac{1}{\lambda}\left[\phi(1, x_1, x_2, ...)\exp(\zeta(\gamma)) + \phi(-1, x_1, x_2, ...)\exp(-\zeta(\gamma))\right]$$

acting on $L^1(X,\rho)$ *is ergodic.*

References

[1] A. T. Baraviera, L. Cioletti, A. O. Lopes, J. Mohr and R. R. Souza. On the general one-dimensional XY model: positive and zero temperature, selection and non-selection. *Rev. Math. Phys.*, 23(10): 1063–1113, 2011.

[2] V. Bogachev. *Measure theory. Vol. I.* Springer-Verlag, Berlin, 2007

[3] R. Bowen. Some systems with unique equilibrium states. *Math. Systems Theory*, 8: 193–202, 1974.

[4] X. Bressaud, R. Fernández and A. Galves. Decay of correlations for non-Hölderian dynamics. A coupling approach. *Electron. J. Probab.*, 4(3): 19 pp. (electronic), 1999.

[5] L. Cioletti and A. O. Lopes. Interactions, specifications, probabilities and the Ruelle operator in the one-dimensional lattice. *arXiv:1404.3232*, 2014.

[6] V. Climenhaga and D. J. Thompson. Equilibrium states beyond specification and the Bowen property. *J. Lond. Math. Soc.* (2) 87(2): 401–427, 2013.

[7] M. Denker and M. Urbański. On the existence of conformal measures. *Trans. Amer. Math. Soc.*, 328(2): 563–587, 1991.

[8] G. Iommi and M. Todd. Transience in dynamical systems. *Ergodic Theory and Dynamical Systems* 33(5): 1450–1476, 2013.

[9] M. Lin. Mixing for Markov operators. *Z. Wahrsch. u. v. Geb.* 19(3): 231–242, 1971.

[10] E. Lukacs. *Stochastic convergence.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, second edition, 1975. Probability and Mathematical Statistics, Vol. 30.

[11] N. G. Markley and M. E. Paul. Equilibrium states of grid functions. *Trans. Amer. Math. Soc.* 274: 169–191, 1982.

[12] O. Sarig. Thermodynamic formalism for null recurrent potentials. *Israel J. Math.* 12: 285–311, 2001.

[13] B. O. Stratmann and M. Urbański. Pseudo-Markov systems and infinitely generated Schottky groups. *Amer. J. Math.*, 129(4): 1019–1062, 2007.

[14] P. Walters. A variational principle for the pressure of continuous transformations. *Amer. J. Math.* 97(4): 937–971, 1975.

[15] P. Walters. Ruelle's operator theorem and g-measures. *Trans. Amer. Math. Soc.* 214: 375–387, 1975.

[16] P. Walters. Convergence of the Ruelle operator for a function satisfying Bowens condition. *Trans. Amer. Math. Soc.*, 353(1): 327–347 (electronic), 2001.

[17] P. Walters. Regularity conditions and Bernoulli properties of equilibrium states and g-measures. *J. London Math. Soc.* (2), 71(2): 379–396, 2005.

[18] P. Walters. A natural space of functions for the Ruelle operator theorem. *Ergodic Theory Dynamical Systems*, 27(4): 1323–1348, 2007.

[19] M. Yuri. Zeta functions for certain non-hyperbolic systems and topological Markov approximations. *Ergodic Theory Dynamical. Systems*, 18(6): 1589–1612, 1998.

L. Cioletti. Departamento de Matemática - UnB, Brazil.
E-mail address: `cioletti@mat.unb.br`

M. Denker. Mathematics Department, the Pennsylvania State University, U.S.A.
E-mail address: `denker@math.psu.edu`

A. O. Lopes, Dept de Matemática Pura e Aplicada- UFRGS, Brazil.
E-mail address: `arturoscar.lopes@gmail.com`

M. Stadlbauer, Departamento de Matemática - UFRJ, Brazil
E-mail address: `manuel.stadlbauer@gmail.com`