A Ruelle Operator for continuous time Markov Chains

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Abstract

We consider a finite state set S and a continuous time Markov Chain X_t, $t \geq 0$, taking values on S. We denote by Ω the set of paths w taking values on S (the elements w are locally constant with left and right limits and are also right continuous on t). P will denote the associated probability on $(\Omega, \mathcal{B})$ which we assume that is stationary. All functions f we consider bellow are in the set $\mathcal{L}^\infty(P)$.

From P we are able to define a Ruelle operator $\mathcal{L}^t, t \geq 0$, acting on functions $f : \Omega \to \mathbb{R}$ of $\mathcal{L}^\infty(P)$. Given $V : \Omega \to \mathbb{R}$, such that is constant in sets of the form $\{X_0 = c\}$, we define a modified Ruelle operator $\mathcal{L}^t_V, t \geq 0$, and we are able to show the existence of an eigenfunction and an eigen-probability ρ_V on Ω associated to $\mathcal{L}^t_V, t \geq 0$.

We also show the follow property for the probability ρ_V: for any integrable $f \in \mathcal{L}^\infty(P)$ and any real and positive t

$$\int e^{-\int_0^t (V \circ \Theta_s)(.) ds} \, [\, (\mathcal{L}^t \, (e^{\int_0^t (V \circ \Theta_s)(.) ds} f)\,) \circ \theta_t] d\rho_V = \int f d\rho_V$$

This equation generalize for continuous time a similar one for discrete time systems (and which is quite important for understanding the KMS states of certain C^*-algebras).

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1 Introduction

We would like to consider a continuous time stochastic process that maps the positive real line $\mathbb{R}_+ = \{t \in \mathbb{R} \colon t \geq 0\}$ on a finite set S with n elements, that we can simply write as $S = \{1, 2, \ldots, n\}$. Now take a n by n real matrix L such that:

1) $0 < -L_{ii}$, for all $i \in S$,

2) $L_{ij} \geq 0$, for all $i \neq j$, $i \in S$,

3) $\sum_{i=1}^{n} L_{ij} = 0$ for all fixed $j \in S$.

We point out that, by convention, we are considering column stochastic matrices and not line stochastic matrices (see [N] section 2 and 3 for general references).

We denote by $P^t = e^{tL}$ the semigroup generated by L. The left action of the semigroup can be identified with an action over functions from S to $\mathbb{R}$ (vectors in $\mathbb{R}^n$) and the right action can be identified with action on measures on S (also vectors in $\mathbb{R}^n$).

The matrix e^{tL} is column stochastic, since from the assumptions on L follows that

$$(1, \ldots, 1)e^{tL} = (1, \ldots, 1)(I + tL + \frac{1}{2}t^2L^2 + \cdots) = (1, \ldots, 1)$$

It is well known that there exist a vector of probability $p_0 = (p_0^1, p_0^2, ..., p_0^n) \in \mathbb{R}^n$ such that $e^{tL}(p_0) = P^t p_0 = p^0$ for all $t > 0$. The vector p_0 is a right eigenvector of e^{tL}. All entries p_0^i are *strictly positive*, as a consequence of hypothesis 1.

Now let us consider the space $\tilde{\Omega} = \{1, 2, ..., n\}^{\mathbb{R}_+}$ of all functions from $\mathbb{R}_+$ to S. In principle it could be enough for our purposes, but technical details in the construction of probability measures on such a space force us to use a restriction of it: We consider the space $\Omega \subset \tilde{\Omega}$ as the set of right-continuous functions from $\mathbb{R}_+$ to S. In this set we take the sigma algebra $\mathcal{B}$ generated by the cylinders of the form

$$\{w_0 = a_0, w_{t_1} = a_1, w_{t_2} = a_2, ..., w_{t_r} = a_r\},$$

where $t_i \in \mathbb{R}_+$, $r \in \mathbb{Z}^+, a_i \in S$ and $0 < t_1 < t_2 < \ldots < t_r$. It is possible to endow Ω with a metric, the Skorohod-Stone metric d, which makes Ω complete and separable ([EK] section 3.5) but the space is not compact.

Now we can introduce a continuous time version of the shift map as follows: we define for each fixed $s \in \mathbb{R}_+$ the $\mathcal{B}$-measurable transformation $\Theta_s : \Omega \to \Omega$ given by $\Theta_s(w_t) = w_{t+s}$ (we remark that Θ_s is also a continuous transformation with respect to the Skorohod-Stone metric d).

For L and p_0 fixed as above we denote by P the probability on the sigma-algebra $\mathcal{B}$ defined for cylinders by

$$P(\{w_0 = a_0, w_{t_1} = a_1, ..., w_{t_r} = a_r\}) = P^{t_r - t_{r-1}}_{a_r a_{r-1}} ... P^{t_2 - t_1}_{a_2 a_1} P^{t_1}_{a_1 a_0} p_0^{a_0}.$$

For details of the construction of this measure the reader is refered to [B].

The probability P on $(\Omega, \mathcal{B})$ is stationary in the sense that for any integrable function f and any $s \geq 0$

$$\int f(w)dP(w) = \int (f \circ \Theta_s)dP(w).$$

From now on the Stationary Process defined by P is denoted by X_t and all functions f we consider are in the set $\mathcal{L}^\infty(P)$.

There exist a version of P such that for a set of full measure all elements w are locally constant on t on the right side with left and right limits and w is right continuous on t. We consider from now on such P.

From P we are able to define a continuous time Ruelle operator $\mathcal{L}^t$, $t > 0$, acting on functions $f : \Omega \to \mathbb{R}$ of $\mathcal{L}^\infty(P)$. It is also possible to introduce the endomorphism $\alpha_t \colon \mathcal{L}^\infty(P) \to \mathcal{L}^\infty(P)$ defined as

$$\alpha_t(\varphi) = \varphi \circ \Theta_t \ , \qquad \forall \varphi \in \mathcal{L}^\infty(P)$$

Given $V : \Omega \to \mathbb{R}$, such that it is constant in sets of the form $\{X_0 = c\}$ (i.e., V depends only on the value of $x(0)$), we are able to show the existence of a probability ρ_V on Ω which is absolutely continuous with respect to P and satisfies:

Theorem A. *For any integrable $f \in \mathcal{L}^\infty(P)$ and any positive t*

$$\int e^{-\int_0^t (V \circ \Theta_s)(.)ds} \, [\, (\mathcal{L}^t \, (e^{\int_0^t (V \circ \Theta_s)(.)ds} f) \,) \circ \theta_t \,] d\rho_V = \int f d\rho_V$$

The above functional equation is a natural generalization (for continuous time) of the similar one presented in [EL1] and [EL2]. We believe it will be important in the analysis of certain C^* algebras, generated by the operators α and $\mathcal{L}$, specially concerning the characterization of KMS states. We point out however that we are able to show this property of ρ_V just for a quite simple function V as above.

With the operators α and $\mathcal{L}$ we can rewrite the theorem above as

$$\rho_V(G_T^{-1} E_T(G_T \varphi)) = \rho_V(\varphi)$$

for all $\varphi \in \mathcal{L}^\infty$ and all $T > 0$, where, as usual, $\rho_V(\varphi) = \int \varphi d\rho_V$, $E_T = \alpha_T \mathcal{L}^T$ is in fact a projection on a subalgebra of $\mathcal{B}$ and $G_T \colon \Omega \to \mathbb{R}$ is given by

$$G_T(x) = \exp\,(\int_0^T V(x(s))ds)$$

For the map $V : \Omega \to \mathbb{R}$, which is constant in cylinders of the form $\{w_0 = i\}$, $i \in \{1, 2, ..., n\}$, we denote by V_i the corresponding value. We denote also by V the diagonal matrix with the i-diagonal element equal to V_i.

We denote by $P_V^t = e^{t(L+V)}$. The Perron-Frobenius Theorem for such semigroup will be one of the main ingredients of the proof.

A related and more general result will appear in [LNT].

As usual we denote by $\mathcal{F}_s$ the sigma-algebra generated by X_s. We also denote by $\mathcal{F}_s^+$ the sigma-algebra generated $\sigma(\{X_u, s \leq u\})$. Note that a $\mathcal{F}_s^+$-measurable function $f(w)$ on Ω does depend of the value w_s.

We also denote by I_A the indicator function of a measurable set A in Ω.

2 A continuous time Ruelle Operator

The infinitesimal generator L define a stochastic process taking values in $S = \{1, 2, .., n\}$. Taking the stationary vector of probability we obtain a probability on the Skorohod space Ω which is denoted by P.

Definition 2.1. *For t fixed we define the operator $\mathcal{L}^t : \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P)$ as follows:*

$$\mathcal{L}^t(\varphi)(x) = \int_{\bar{y}\in\Theta_t^{-1}(x)} \varphi(\bar{y}) d\mu_t^x(\bar{y})$$

Remark 2.2. *The definition above can be rewritten as*

$$\mathcal{L}^t(\varphi)(x) = \int_{y\in D[0,t)} \varphi(yx) d\mu_t^x(yx)$$

where the symbol yx means the concatenation of the path y with the translation of x:

$$xy(s) = \begin{cases} y(s) & \text{if } s \in [0, t) \\ x(s-t) & \text{if } s \geq t \end{cases}$$

and $D[0, t)$ is the set of right-continuous functions from $[0, t)$ to S. This follows simply from the fact that, in this notation, $\Theta_t^{-1}(x) = \{yx \colon y \in D[0, t)\}$.

It is possible to shed some light on the meaning of this operator applying it to some simple functions. For example, we can see the effect of $\mathcal{L}^t$ on some indicator of a given cylinder: Consider the sequence $0 = t_0 < t_1 < .. < t_{j-1} < t \leq t_j < ... < t_r$ and then take $f = I_{\{X_0=a_0, X_{t_1}=a_1,...,X_{t_r}=a_r\}}$. Then, for a path $z \in \Omega$ such that $z_{t_j-t} = a_j, ..., z_{t_r-t} = a_r$ (the future condition) we have

$$\mathcal{L}^t(f)(z) = \frac{1}{p_0^{z_0}} P_{z_0 a_{j-1}}^{t-t_{j-1}} ... P_{a_2 a_1}^{t_2-t_1} P_{a_1 a_0}^{t_1} p_0^{a_0},$$

otherwise (i.e., if the path z does not satisfy the condition above) we get $\mathcal{L}^t(f)(z) = 0$.

Note that if $t_r < t$, then $\mathcal{L}^t(f)(z)$ depends only on z_0. For example, if $f = I_{\{X_0=i_0\}}$ then

$$\mathcal{L}^t(f)(z) = \int_{y\in D[0,t)} I_{\{X_0=i_0\}}(yx) d\mu_t^z(yx) = \mu_t^z([X_0 = i_0]) = \frac{1}{p_0^{z_0}} P_{z_0,i_0}^t p_0^{i_0}$$

In the case $f = I_{\{X_0=i_0, X_t=j_0\}}$, then $\mathcal{L}^t(f)(z) = P_{z_0,i_0}^t \frac{p_0^{i_0}}{p_0^{z_0}}$, if $z_0 = j_0$, and $\mathcal{L}^t(f)(z) = 0$ otherwise.

Now we can show some properties of $\mathcal{L}^t$.

Proposition 2.3. $\mathcal{L}^t(1) = 1$, *where* 1 *is the function that maps every point in* Ω *to* 1.

Proof: Indeed

$$\mathcal{L}^t(1)(x) = \int_{y\in D[0,t)} 1(yx)d\mu_t^x(yx) = \int d\mu_t^x(yx) = \mu_t^x([X_t = x(0)]) =$$

$$\sum_{a=1}^{n} \mu_t^x([X_0 = a, X_t = x(0)]) = \frac{1}{p_0^{x(0)}} \sum_{a=1}^{n} P_{x(0)a}^t p_0^a = 1$$

□

We can also define the dual of $\mathcal{L}^t$, denoted by $(\mathcal{L}^t)^*$, acting on the measures. Then we get:

Proposition 2.4. *For any positive* t *we have that* $(\mathcal{L}^t)^*(P) = (P)$

Proof: For a fixed t we have that $(\mathcal{L}^t)^*(P) = (P)$ because for any f of the form $f = I_{\{X_0=a_0, X_{t_1}=a_1,\ldots,X_{t_r}=a_r\}}$, $0 = t_0 < t_1 < .. < t_{j-1} < t \leq t_j < ... < t_r$. we have

$$\int \mathcal{L}^t(f)(z)dP(z) = \sum_{b=1}^{n} \int_{\{X_0=b\}} \mathcal{L}^t(f)(z)dP(z) =$$

$$\sum_{b=1}^{n} \int I_{\{X_0=b, X_{t_j-t}=a_j,\ldots,X_{t_r-t}=a_r\}}(z)dP(z)\frac{1}{p_0^b} P_{ba_{j-1}}^{t-t_{j-1}} \ldots P_{a_2a_1}^{t_2-t_1} P_{a_1a_0}^{t_1} p_0^{a_0} =$$

$$\sum_{b=1}^{n} P(\{X_0 = b, X_{t_j-t} = a_j, \ldots, X_{t_r-t} = a_r\})\frac{1}{p_0^b} P_{ba_{j-1}}^{t-t_{j-1}} \ldots P_{a_2a_1}^{t_2-t_1} P_{a_1a_0}^{t_1} p_0^{a_0} =$$

$$\int f(w)dP(w).$$

□

Proposition 2.5. *Given* $t \in \mathbb{R}_+$ *and the functions* $\varphi, \psi \in \mathcal{L}^\infty(P)$ *then we have*

$$\mathcal{L}^t(\varphi \times (\psi \circ \Theta_t))(z) = \psi(z) \times \mathcal{L}^t(\varphi)(z).$$

Proof:

$$\mathcal{L}^t(\varphi(\psi \circ \Theta_t))(x) = \int_{i\in D[0,t)} \varphi(ix)(\psi \circ \Theta_t)(ix)d\mu_t^x(i) =$$

$$\psi(x) \int \varphi(ix)d\mu_t^x(i) = (\psi\mathcal{L}^t(\varphi))(x) = \psi(x)\mathcal{L}^t(\varphi)(x)$$

since $\psi \circ \Theta_t(ix) = \psi(x)$, independently of i. □

We just recall that the last proposition can be restated as

$$\mathcal{L}^t(\varphi\alpha_t(\psi)) = \psi\mathcal{L}^t(\varphi)$$

Then we get:

Proposition 2.6. *α_t is the dual of $\mathcal{L}^t$ on $\mathcal{L}^2(P)$.*

Proof: From last two propositions

$$\int \mathcal{L}^t(f)g\,dP = \int \mathcal{L}^t(f \times (g \circ \Theta_t))\,dP = \int f \times (g \circ \Theta_t)\,dP = \int f\alpha_t(g)dP$$

as claimed. □

Now we would like to obtain conditional expectations. For a given f recall that the function $Z(w) = E(f|\mathcal{F}_t^+)$ is the Z (almost everywhere defined) $\mathcal{F}_t^+$-measurable function such that for any $\mathcal{F}_t^+$-measurable set B we have $\int_B Z(w)dP(w) = \int_B f(w)dP(w)$.

Proposition 2.7. *The conditional expectation is given by*

$$E(f|\mathcal{F}_t^+)(x) = \int f d\mu_t^x$$

Proof: For t fixed, consider a $\mathcal{F}_t^+$-measurable set B. Then we have

$$\int_B E(f|\mathcal{F}_t^+)dP = \int_B \int f d\mu_t^w dP(w) = \int (I_B(w) \int f d\mu_t^w)dP(w) =$$

$$\int \int (fI_B)d\mu_t^w dP(w) = \int f(w)I_B(w)dP(w) = \int f dP,$$

and the proposition is concluded. □

Now we can relate the conditional expectation with respect to the σ-algebras $\mathcal{F}_t^+$ with the operators cL^t and α_t as follows:

Proposition 2.8. $[\mathcal{L}^t(f)](\Theta_t) = E(f|\mathcal{F}_t^+)$ *(i.e. $E = \alpha_t\mathcal{L}^t$).*

Proof: This follows from the fact that for any $B = \{X_{s_1} = b_1, X_{s_2} = b_2, ..., X_{s_u} = b_u\}$, with $t < s_1 < ... < s_u$, we have $I_B = I_A \circ \Theta_t$ for some measurable A and

$$\int_B \mathcal{L}^t(f)(\Theta_t(w))dP(w) = \int I_B(w)\mathcal{L}^t(f)(\Theta_t(w))dP(w) =$$

$$\int (I_A \circ \Theta_t)(w)\mathcal{L}^t(f)(\Theta_t(w))dP(w) = \int I_A(w)\mathcal{L}^t(f)(w)dP(w)$$

$$\int \mathcal{L}^t(f(I_A \circ \Theta_t))(w)dP(w) = \int f(w)I_A(\Theta_t(w))dP(w) = \int_B f(w)dP(w)$$

□

3 The modified operator

We are interested in the perturbation by V (defined above) of the $\mathcal{L}^t$ operator.

Definition 3.1. *We define* $G_t\colon \Omega \to \mathbb{R}$ *as*

$$G_t(x) = \exp\big(\int_0^t V(x(s))ds\big)$$

Definition 3.2. *We define the G-weigthed transfer operator* $\mathcal{L}_V^t : \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P)$ *acting on measurable functions* f *(of the above form) by*

$$\mathcal{L}_V^t(f)(w) := \mathcal{L}^t(G_t f) =$$

$$= \mathcal{L}^t(e^{\int_0^t (V\circ\Theta_s)(.)ds}\, f\,) = \sum_{b=1}^n \mathcal{L}^t(e^{\int_0^t (V\circ\Theta_s)(.)ds}\, I_{\{X_t=b\}}\, f\,)(w)$$

Note that $e^{\int_0^t (V\circ\Theta_s)(.)ds}\, I_{\{X_t=b\}}$ does not depend on information larger then t. In the case f is such that $t_r \leq t$ (in the above notation), then $\mathcal{L}_V^t(f)(w)$ depends only on $w(0)$.

The integration on s above is over the open interval $(0,t)$.

We will consider soon an eigenfunction and an eigen-measure for such operator $\mathcal{L}_V^t$. But, first we need the following:

Theorem 1. *([S] page 111) We assume S is finite. One can prove that for L, p_0 and V fixed as above there exists*

a) a unique positive function $u_V : \Omega \to \mathbb{R}$, constant equal to the value u_V^i in each cylinder $X_0 = i$, $i \in \{1, 2, .., n\}$, (we can see u_V as $u_V : S \to \mathbb{R}$, or, as a vector in $\mathbb{R}^n$),

b) a unique probability vector μ_V in $\mathbb{R}^n$ (a probability over over the set $\{1, 2, .., n\}$ such that $\mu_V(\{i\}) > 0$, $\forall i$), such that

$$\sum_{i=1}^n u_V^i (\mu_V)_i = 1,$$

c) a real positive value $\lambda(V)$,

such that

d) for any positive s

$$e^{-s\lambda(V)} u_V\, e^{s(L+V)} = u_V.$$

Moreover, for any $v = (v_1, ., v_n) \in \mathbb{R}^n$

$$\lim_{t\to\infty} e^{-t\lambda(V)}\, v\, e^{t(L+V)} = (\sum_{i=1}^n v_i (\mu_V)_i)\, u_V,$$

e) for any positive t

$$(P_V^t)^*\,\mu_V = e^{\lambda(V)\,t}\mu_V.$$

From property e) it follows that

$$(L+V)^*\,\mu_V = \lambda(V)\,\mu_V.$$

From d) it follows that

$$u_V\,(L+V) = \lambda(V)\,u_V.$$

Note that when $V=0$, then $\lambda(V)=0$, $\mu_V = p^0$ and u_V is constant equal to 1.

In order to show the existence of u_V, such that, $u_V(L+V) = \lambda(V)\,u_V$ one add a constant to V in such way that all the entries of $(L+V)$ are positive. This will imply d). For the case the space S is not finite see [LNT].

Now we return to our setting: for each i_0 and t fixed one can consider the probability $\mu_{i_0}^t$ defined over the sigma-algebra $\mathcal{F}_t^- = \sigma(\{X_s | s\leq t\})$ with support on $\{X_0 = i_0\}$ such that for cylinder sets with $0 < t_1 < ... < t_r \leq t$

$$\mu_{i_0}^t(\{X_0 = i_0, X_{t_1} = a_1, ..., X_{t_{r-1}} = a_{r-1}, X_t = j_0\}) = P_{j_0 a_r}^{t-t_r} ... P_{a_2 a_1}^{t_2 - t_1} P_{a_1 i_0}^{t_1}.$$

The probability $\mu_{i_0}^t$ is not stationary.
We denote by $Q(j,i)_t$ the i,j entry of the matrix $e^{t(L+V)}$, that is $(e^{t(L+V)})_{i,j}$.
It is known ([K] page 52 or [S] Lemma 5.15) that

$$Q(j_0,i_0)_t = E_{\{X_0=i_0\}}\{e^{\int_0^t (V\circ\Theta_s)(w)ds}\,;\, X(t) = j_0\} =$$

$$\int I_{\{X_t = j_0\}}\; e^{\int_0^t (V\circ\Theta_s)(w)ds} d\mu_{i_0}^t(w).$$

For example,

$$\int I_{\{X_t=j_0\}} e^{\int_0^t (V\circ\Theta_s)(w)ds} dP = \sum_{i=1,2,..,n} Q(j_0,i)_t\, p_i^0$$

In the particular case where V is constant equal 0, then $p^0 = \mu_V$ and $\lambda(V) = 0$.

Proposition 3.3. *$f(w) = \frac{\mu_V(w)}{p^0(w)} = \frac{(\mu_V)_{w(0)}}{(p^0)_{w(0)}}$ is an eigenfunction for $\mathcal{L}_V^t$ with eigenvalue $e^{t\lambda(V)}$.*

Proof: Note that $\frac{\mu_V}{p^0} = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)}\, I_{\{X_0=c\}}$.
For a given w, denote $w(0)$ by j_0, then conditioning

$$\mathcal{L}_V^t(\frac{\mu_V}{p^0})(w) = \sum_{c=1}^n \sum_{b=1}^n \mathcal{L}_V^t\,(\,\frac{\mu_V(c)}{p^0(c)}\, I_{\{X_0=c\}}\, I_{\{X_t=b\}}\,)\,(w).$$

Consider c fixed, then for $b = j_0$ we have

$$\mathcal{L}_V^t\,(\,I_{\{X_0=c\}}\,I_{\{X_t=b\}}\,)\,(w) = \frac{Q(j_0,c)_t\;p_c^0}{p_{j_0}^0},$$

and for $b \neq j_0$, we have $\mathcal{L}_V^t\,(\,I_{\{X_0=c\}}\,I_{\{X_t=b\}}\,)\,(w) = 0$.

Finally,

$$\mathcal{L}_V^t(\frac{\mu_V}{p^0})(w) = \sum_{c=1}^{n}\frac{\mu_V(c)}{p^0(c)}\,Q(j_0,c)_t\;\frac{p_c^0}{p_{j_0}^0} = e^{t\lambda(V)}\frac{(\mu_V)_{j_0}}{p_{j_0}^0} = e^{t\lambda(V)}(\frac{\mu_V}{p^0})(w),$$

because $e^{t(L+V)}\,(\mu_V) = e^{t\lambda(V)}(\mu_V)$.

Therefore for any $t > 0$ the function $\frac{\mu_V}{p^0}$ (that depends only on $w(0)$) is an eigenfunction for the operator $\mathcal{L}_V^t$ associated to the eigenvector $e^{t\lambda(V)}$. □

Definition 3.4. *Consider now for each t the operator acting on g by*

$$\hat{\mathcal{L}}_V^t(g)(w) = [\,\frac{p^0}{\mu_V}\mathcal{L}^t(e^{\int_0^t (V-\lambda(V))\circ\Theta_s)(.)ds}\;g\;\frac{\mu_V}{p^0})\,]\,(w)$$

From the above $\hat{\mathcal{L}}_V^t(1) = 1$ for all positive t.

Note that by conditioning, if $g = I_{\{X_0=a_0,X_{t_1}=a_1,X_{t_2}=a_2,X_t=a_3\}}$, with $0 < t_1 < t_2 < t$, then

$$\hat{\mathcal{L}}_V^t(g)(w) = \frac{\mu_V(a_0)}{\mu_V(a_3)}e_{a_3a_2}^{(t-t_2)\,(L+V-\lambda\,I)}\;e_{a_2a_1}^{(t_2-t_1)\,(L+V-\lambda\,I)}\;e_{a_1a_0}^{t_1\,(L+V-\lambda\,I)},$$

for w such that $w_0 = a_3$, and $\hat{\mathcal{L}}_V^t(g)(w) = 0$ otherwise.

Moreover, for $g = I_{\{X_0=a_0,X_{t_1}=a_1,X_t=a_2,X_{t_3}=a_3\}}$, with $0 < t_1 < t < t_3$, then

$$\hat{\mathcal{L}}_V^t(g)(w) = \frac{\mu_V(a_0)}{\mu_V(a_2)}e_{a_2a_1}^{(t-t_1)\,(L+V-\lambda\,I)}\;e_{a_1a_0}^{t_1\,(L+V-\lambda\,I)},$$

for w such that $w_0 = a_2, w_{t_3-t} = a_3$, and $\hat{\mathcal{L}}_V^t(g)(w) = 0$ otherwise.

Consider now the dual operator $(\hat{\mathcal{L}}_V^t)^*$.

For t fixed consider the transformation in the set of probabilities μ on Ω given by $(\hat{\mathcal{L}}_V^t)^*(\mu) = \nu$.

Theorem 2. *There exists a fixed probability measure ν_V on $(\Omega,\mathcal{B})$ for such transformation $(\hat{\mathcal{L}}_V^t)^*$. The stationary probability ν_V does not depend on t.*

Proof: Denote by $\nu = \nu_V$ the probability obtained in the following way, for

$$g = I_{\{X_0=a_0,X_{t_1}=a_1,X_{t_2}=a_2,\ldots,X_{t_{r-1}}=a_{r-1},X_r=a_r\}},$$

with $0 < t_1 < t_2 < ... < t_{s-1} < t \leq t_s < .. < t_r$, we define

$$\int g(w)\, d\nu(w) = e^{(t_r - t_{r-1})\,(L+V-\lambda\, I)}_{a_r a_{r-1}} \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0}\, \mu_V(a_0).$$

This probability satisfies the Kolmogorov compatibility conditions because is defined via a semigroup (see chapter IV. 2 [BW])

In order to show that ν is a probability we have to use the fact that $\sum_{c\in \tilde{S}} \mu_V(c) = 1$

On the other hand,

$$z(w) = \hat{\mathcal{L}}^t_V(g)(w) = \frac{\mu_V(a_0)}{\mu_V(w_0)} e^{(t-t_{s-1})\,(L+V-\lambda\, I)}_{w_0 a_{s-1}} \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0},$$

for w such that $w_{t_s - t} = a_s, w_{t_{s+1}-t} = a_{s+1}, ..., w_{t_r - t} = a_r$, and $\hat{\mathcal{L}}^t_V(g)(w) = 0$ otherwise. Note that $z(w) = \hat{\mathcal{L}}^t_V(g)(w)$ depends only on $w_0, w_{t_{s+1}-t}, ..., w_{t_r-t}$.

We have to show that for any g we have $\int g\, d\nu = \int \hat{\mathcal{L}}^t_V(g) d\nu$.

Now,

$$\int z(w)\, d\nu(w) = \int \sum_{c\in S} I_{\{X_0 = c, X_{t_s - t} = a_s, X_{t_{s+1}-t} = a_{s+1}, ..., X_{t_r - t} = a_r\}}\, z(w)\ d\nu(w) =$$

$$\sum_{c\in S} \nu(\{X_0 = c, X_{t_s-t} = a_s, X_{t_{s+1}-t} = a_{s+1}, ..., X_{t_r-t} = a_r\})$$

$$\frac{\mu_V(a_0)}{\mu_V(c)} e^{(t-t_{s-1})\,(L+V-\lambda\, I)}_{c a_{s-1}} \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0} =$$

$$\sum_{c\in S} \mu_V(c)\, e^{(t_r-t_{r-1})\,(L+V-\lambda\, I)}_{a_r a_{r-1}} \ldots e^{(t_{s+1}-t_s)\,(L+V-\lambda\, I)}_{a_{s+1} a_s}\, e^{t_s - t\,(L+V-\lambda\, I)}_{a_s c}$$

$$\frac{\mu_V(a_0)}{\mu_V(c)}\, e^{(t-t_{s-1})\,(L+V-\lambda\, I)}_{c a_{s-1}} \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0} =$$

$$\sum_{c\in S} e^{(t_r-t_{r-1})\,(L+V-\lambda\, I)}_{a_r a_{r-1}} \ldots e^{(t_{s+1}-t_s)\,(L+V-\lambda\, I)}_{a_{s+1} a_s}\, e^{t_s - t\,(L+V-\lambda\, I)}_{a_s c}$$

$$\mu_V(a_0)\, e^{(t-t_{s-1})\,(L+V-\lambda\, I)}_{c a_{s-1}} \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0} =$$

$$\mu_V(a_0)\, e^{(t_r-t_{r-1})\,(L+V-\lambda\, I)}_{a_r a_{r-1}} \ldots e^{(t_{s+1}-t_s)\,(L+V-\lambda\, I)}_{a_{s+1} a_s}$$

$$(\sum_{c\in S} e^{(t_s - t)\,(L+V-\lambda\, I)}_{a_s c} e^{(t-t_{s-1})\,(L+V-\lambda\, I)}_{c a_{s-1}}) \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0} =$$

$$\mu_V(a_0)\, e^{(t_r-t_{r-1})\,(L+V-\lambda\, I)}_{a_r a_{r-1}} \ldots e^{(t_{s+1}-t_s)\,(L+V-\lambda\, I)}_{a_{s+1} a_s}$$

$$e^{(t_s-t_{s-1})\,(L+V-\lambda\, I)}_{a_s a_{s-1}} \ldots e^{(t_2-t_1)\,(L+V-\lambda\, I)}_{a_2 a_1}\, e^{t_1\,(L+V-\lambda\, I)}_{a_1 a_0} =$$

$$\int g\, d\nu.$$

The claim for the general g follows from the above result. □

Definition 3.5. *Consider the probability $\rho_V = (g_V)^{-1} \nu_V$, where g_V is chosen colinear to $\frac{\mu_V}{p^0}$, in such way ρ_V is a probability (not necessarily invariant) on Ω.*

It easily follows that $(\mathcal{L}_V^t)^*(\rho_V) = e^{t\lambda_V} \rho_V$. The probability ν_V is invariant for θ_s with $s \geq 0$.

From last theorem follows easily:

Proposition 3.6. *For any integrable $f, g \in \mathcal{L}^\infty(P)$ and any positive t*

$$\int g \mathcal{L}_V^t(f) d\rho_V = \int \mathcal{L}_V^t(f(g \circ \theta_t)) d\rho_V = e^{t\lambda_V} \int f(g \circ \theta_t) d\rho_V.$$

Now we are in position to prove our main result. From $(\mathcal{L}_V^t)^*(\rho_V) = e^{t\lambda_V} \rho_V$ it follows that the measure ρ_V satisfies the important equation:

Theorem A. *For any integrable $f \in \mathcal{L}^\infty(P)$ and any positive t*

$$\int e^{-\int_0^t (V \circ \Theta_s)(.) ds} \left[\left(\mathcal{L}^t \left(e^{\int_0^t (V \circ \Theta_s)(.) ds} f \right) \right) \circ \theta_t \right] d\rho_V = \int f d\rho_V$$

Proof:

$$\int e^{-\int_0^t (V \circ \Theta_s)(.) ds} \left[\left(\mathcal{L}^t \left(e^{\int_0^t (V \circ \Theta_s)(.) ds} f \right) \right) \circ \theta_t \right] d\rho_V =$$

$$e^{-t\lambda_V} \int [\mathcal{L}^t (e^{-\int_0^t (V \circ \Theta_s)(.) ds} e^{\int_0^t (V \circ \Theta_s)(.) ds})] \left[\mathcal{L}^t \left(e^{\int_0^t (V \circ \Theta_s)(.) ds} f \right) \right] d\rho_V =$$

$$e^{-t\lambda_V} \int \mathcal{L}^t \left(e^{\int_0^t (V \circ \Theta_s)(.) ds} f \right) d\rho_V =$$

$$\int f d\rho_V$$

□

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