THE INFINITE DIMENSIONAL MANIFOLD OF HÖLDER EQUILIBRIUM PROBABILITIES HAS NON-NEGATIVE CURVATURE

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Abstract. Here we consider the discrete time dynamics described by a transformation $T : M \to M$, where T is either the action of shift $T = \sigma$ on the symbolic space $M = \{1, 2, ..., d\}^{\mathbb{N}}$, or, T describes the action of a d to 1 expanding transformation $T : S^1 \to S^1$ of class $C^{1+\alpha}$ (for example $x \to T(x) = d\,x$ (mod 1)), where $M = S^1$ is the unitary circle.

It is known that the infinite dimensional manifold $\mathcal{N}$ of Hölder equilibrium probabilities is an analytical manifold and carries a natural Riemannian metric. Given a certain normalized Hölder potential A denote by $\mu_A \in \mathcal{N}$ the associated equilibrium probability. The set of tangent vectors X (a function $X : M \to \mathbb{R}$) to the manifold $\mathcal{N}$ at the point μ_A coincides with the kernel of the Ruelle operator for the normalized potential A. The Riemannian norm $|X| = |X|_A$ of the vector X, which is tangent to $\mathcal{N}$ at the point μ_A, is described via the asymptotic variance, that is, satisfies

$$|X|^2 \;= < X, X > = \;\lim_{n\to\infty} \frac{1}{n} \int (\sum_{i=0}^{n-1} X \circ T^i)^2\, d\mu_A.$$

Given two unitary tangent vectors to the manifold $\mathcal{N}$ at μ_A, denoted by X and Y, we will show that the sectional curvature $K(X, Y)$ equals to $\int X^2 Y^2 d\mu_A$, so it is always non-negative. The curvature vanishes, if and only if, the supports of the functions X and Y are disjoint.

In our proof for the above expression for the curvature it is necessary in some moment to show the existence of geodesics for such Riemannian metric.

1. Introduction

We denote by $T : M \to M$ a transformation acting on the metric space M, which is either the shift σ acting on $M = \{1, 2, ..., d\}^{\mathbb{N}}$, or, T is the action of a d to 1 expanding transformation $T : S^1 \to S^1$, of class $C^{1+\alpha}$, where $M = S^1$ is the unitary circle.

For a fixed $\alpha > 0$ we denote by Hol the set of α-Hölder functions on M.

For a Hölder potential $A : M \to \mathbb{R}$ we define the Ruelle operator (sometimes called transfer operator) - which acts on Hölder functions $f : M \to \mathbb{R}$ - by

$$f \to \mathscr{L}_A f(x) = \sum_{T(y)=x} e^{A(y)} f(y)$$

It is known (see for instance [12] or [1]) that $\mathscr{L}_A$ has a positive, simple leading eigenvalue λ_A with a positive Hölder eigenfunction h_A. Moreover, the dual operator acting on measures $\mathscr{L}_A^*$ has a unique eigenprobability ν_A which is associated to the same eigenvalue λ_A.

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Given a Hölder potential A we say that the probability μ_A - acting on the Borel sigma-algebra of M - is the equilibrium probability for A, if μ_A maximizes the values

$$h(\mu) + \int A \ d\mu,$$

among Borel T-invariant probabilities μ and where $h(\mu)$ is the Shannon-Kolmogorov entropy of μ.

It is known that the probability μ_A is unique and is given by the expression $\mu_A = h_A \, \nu_A$.

In some particular cases the equilibrium probability (also called Gibbs probability) μ_A is the one observed on the thermodynamical equilibrium in the Statistical Mechanics of the one dimensional lattice $\mathbb{N}$ (under an interaction described by the potential A). As an example (where the spin in each site of the lattice $\mathbb{N}$ could be $+$ or $-$) one can take $M = \{+,-\}^{\mathbb{N}}$, $A : M \to \mathbb{R}$ and T is the shift.

We say that a Hölder potential A is normalized if $\mathscr{L}_A \, 1 \ = 1$. In this case $\lambda_A = 1$ and $\mu_A = \nu_A$.

We say that two potentials A, B in Hol are coboundary to each other, if there exists a continuous function $g : M \to \mathbb{R}$ and a constant c, such that,

$$A = B + g - g \circ T - c.$$

Note that the equilibrium probability for A, respectively B, is the same, if A and B are coboundary to each other. In each coboundary class (an equivalence relation) there exists a unique normalized potential A (see [12]). Therefore, the set of equilibrium probabilities for Hölder potentials $\mathcal{N}$ can be indexed by Hölder potentials A which are normalized. We will use this point of view here: $A \leftrightarrow \mu_A$.

The infinite dimensional manifold $\mathcal{N}$ of Hölder equilibrium probabilities μ_A is an analytical manifold (see [14], [9], [12], [5]) and it was shown in [10] that it carries a natural Riemannian structure. We will recall some definitions and properties described on [10].

The set of tangent vectors X (a function $X : M \to \mathbb{R}$) to $\mathcal{N}$ at the point μ_A coincides with the kernel of $\mathscr{L}_A$. The Riemannian norm $|X| = |X|_{\mu_A}$ of the vector X, which is tangent to $\mathcal{N}$ at the point μ_A, is described (see Theorem D in [10]) via the asymptotic variance, that is, satisfies

$$|X| \ = \ \sqrt{< X, X >} \ = \ \sqrt{\lim_{n \to \infty} \tfrac{1}{n} \int (\sum_{j=0}^{n-1} X \circ T^j)^2 \, d\mu_A}.$$

The associated bilinear form on the tangent space at the point μ_A can be described (see Theorem D in[10]) by

$$< X \, , Y \ > = \int X \, Y \, d\mu_A.$$

This bilinear form is positive semi-definite and in order to make it definite one can consider equivalence classes as described by Definition 5.4 in [10]. In this way we finally get a Riemannian structure on $\mathcal{N}$ (as anticipated some paragraphs above). Elements X on the tangent space at μ_A have the property $\int \, X \, d\mu_A = 0$.

The tangent space to $\mathcal{N}$ at μ_A is denoted by $T_A\mathcal{N}$.

Our main result is Theorem 4.10 which claims:

Theorem 1.1. *Given two unitary orthogonal vectors X, Y tangent to $\mathcal{N}$ at the point μ_A we have that the sectional curvature $K(X,Y)$ is equal to $\int X^2 Y^2 d\mu_A$.*

Therefore, the sectional curvature is non negative. Moreover, the curvature is zero, if and only if, the supports of the functions X and Y are disjoint.

We point out that section 8 in [10], which considers a simplified model for potentials that depend just on two coordinates on the symbolic space $\{1,2\}^{\mathbb{N}}$, there was an indication that the curvature should be non negative.

We will show in section 5 the existence of geodesics for such Riemannian metric.

An important tool which will be used here is item (iv) on Theorem 5.1 in [10]: for all normalized $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$ and φ a continuous function it holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, d\mu_{A+tX}\Big|_{t=0} = \int \varphi \, X \, d\,\mu_A. \tag{1}$$

We point out that in the case T is the shift, then, for each given value $n > 0$ one can get for $A = -\log d$ (in this case μ_A is the measure of maximal entropy) a pair of functions $X, Y \in T_{\mu_A}(\mathcal{N})$, such that, $K(X,Y) > n$. Therefore, the sectional curvature is not bounded above.

In [11] , [3] and [13] the authors consider a similar kind of Riemannian structure. The bilinear form considered in [11] is the one we consider here divided by the entropy of μ_A. As mentioned in section 8 in [10] in this case the curvature can be positive and also negative in some parts.

The main motivation for the results obtained on [11] (and also [3]) is related to the study of a particular norm on the Teichmüller space.

A reference for general results in infinite dimensional Riemannian manifolds is [2].

In section 6 in [10] it is explained that the Riemannian metric considered here is not compatible with the 2-Wasserstein Riemannian structure on the space of probabilities.

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2. Preliminaries of Riemannian geometry

Let us introduce some basic notions of Riemannian geometry. Given a C^∞ manifold (M,g) equipped with a smooth Riemannian metric g, let TM be the tangent bundle and T_1M be the set of unit norm tangent vectors of (M,g), the unit tangent bundle. Let $\chi(M)$ be the set of C^∞ vector fields of M. For practical purposes, we shall call *Energy* the function $E(v) = g(v,v)$, $v \in TM$, although in mechanics the energy is rather defined by $\frac{1}{2}g(v,v)$.

Given a smooth function $f : M \longrightarrow \mathbb{R}$, the derivative of f with respect to a vector field $X \in \chi(M)$ will be denoted by $X(f)$. The Lie bracket of two vector fields $X, Y \in \chi(M)$ is the vector field whose action on the set of functions $f : M \longrightarrow \mathbb{R}$ is given by $[X,Y](f) = X(Y(f)) - Y(X(f))$.

The *Levi-Civita connection* of (M,g), $\nabla : \chi(M) \times \chi(M) :\longrightarrow \chi(M)$, with notation $\nabla(X,Y) = \nabla_X Y$, is the affine operator characterized by the following properties:

(1) Compatibility with the metric g:

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for every triple of vector fields X, Y, Z.

(2) Absence of torsion:

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

(3) For every smooth scalar function f and vector fields $X, Y \in \chi(M)$ we have
- $\nabla_{fX} Y = f \nabla_X Y$,
- Leibnitz rule: $\nabla_X (fY) = X(f)Y + f \nabla_X Y$.

The expression of $\nabla_X Y$ can be obtained explicitly from the expression of the Riemannian metric, in dual form. Namely, given two vector fields $X, Y \in \chi(M)$, and $Z \in \chi(M)$ we have

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \qquad (2)\\ &- g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)), \qquad (3) \end{aligned}$$

A smooth curve $\gamma : (a,b) \longrightarrow M$ is called a *geodesic* of (M,g) if $\nabla_{\gamma'(t)}\gamma'(t) = 0$ for every $t \in (a,b)$. If M is finite dimensional, in any coordinate system the equation of geodesics gives rise to a second order, ordinary differential equation, so given any initial condition (p,v) in T_1M there exists a unique solution $\gamma_{(p,v)}(t)$ such that $\gamma_{(p,v)}(0) = p$, $\gamma'_{(p,v)}(0) = v$. If M is infinite dimensional, the existence of geodesics is a nontrivial issue usually related to the Palais-Smale condition. In our case, where M is the manifold of normalized potentials and g is the L^2 metric $g_A(X,Y) = \int XY d\mu_A$, we shall show in the last section that

Theorem 2.1. *Given $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$, there exist $\epsilon > 0$ and a unique geodesic $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{N}$ such that $\gamma(0) = A$, $\gamma'(0) = X$.*

Although we won't show the Palais-Smale condition for $\mathcal{N}$, we shall show that the manifold $(\mathcal{N}, <,>)$ has enough compactness to ensure the existence of geodesics provided that $T_A\mathcal{N}$ has a countable basis (as a Banach space). This is the case of normalized potentials of the expanding map $T(x) = 2x(mod.1)$ in S^1.

Once we have geodesics we can solve the equation of parallel transport.

Theorem 2.2. *Under the assumptions of Theorem 2.1, given a unit vector $Y \in T_A\mathcal{N}$ there exists a unique smooth vector field $Y(t) \in T_{\gamma(t)}\mathcal{N}$, $t \in (-\epsilon, \epsilon)$, such that $Y(0) = Y$ and*

$$\nabla_{\gamma'(t)} Y(t) = 0$$

for every $t \in (-\epsilon, \epsilon)$. This vector field is the parallel transport of Y along $\gamma(t)$.

The proof of this theorem is postponed to the last section, it is actually a consequence of the proof of the existence of geodesics.

2.1. **Fermi coordinates.** A parametrized local surface $S : (-\epsilon, \epsilon) \times (-\delta, \delta) \longrightarrow \mathcal{N}$, with parameters $S(t,s)$, is given in *Fermi* coordinates if

(1) $S(t, 0) = \gamma(t)$ is a geodesic,
(2) The vector field $\frac{\partial S(t,0)}{\partial s}$ is parallel along $\gamma(t)$ and is perpendicular to $\gamma'(t)$,
(3) The curves $S_t(s) = S(t,s)$, $s \in (-\delta, \delta)$ are geodesics for each given $t \in (-\epsilon, \epsilon)$.

As a consequence of Theorems 2.1 and 2.2 we have

Proposition 2.3. *Given $A \in \mathcal{N}$, $X, Y \in T_A\mathcal{N}$ with unit norms, there exists a local surface $S : (-\epsilon, \epsilon) \times (-\delta, \delta) \longrightarrow \mathcal{N}$ parametrized in Fermi coordinates such that $S(t,0) = \gamma(t)$, $\frac{\partial S(0,0)}{\partial s} = Y$, where $\gamma'(0) = X$.*

Proof. The proof goes as for Riemannian manifolds of finite dimensions. Let $X \in T_A\mathcal{N}$ with unit norm, let $\gamma(t)$ be the geodesic whose initial conditions are $\gamma(0) = A$, $\gamma'(0) = X$. Given $Y \in T_A\mathcal{N}$ with unit norm such that $< X, Y >= 0$, let $Y(t)$ be the parallel transport of Y along $\gamma(t)$. It is clear that $< \gamma'(t), Y(t) >= 0$ for every t because parallel transport is an isometry, so let us consider the local surface S defined by

$$S(t, s) = \beta_{(\gamma(t),Y(t))}(s)$$

for $s \in (-\delta, \delta)$ depending on Y, where $\beta_{(\gamma(t),Y(t))}(s)$ is the geodesic whose initial conditions are $\beta_{(\gamma(t),Y(t))}(0) = \gamma(t)$, $\beta'_{(\gamma(t),Y(t))}(0) = Y(t)$. Since $\mathcal{N}$ is analytic, the parallel transport is analytic and geodesics depend analytically on their initial conditions. So the local surface S is an analytic surface whose coordinates are Fermi coordinates according to the definition. □

Fermi coordinates will be applied to simplify the calculation of the sectional curvatures of the Riemannian manifold $(\mathcal{N}, <,>)$.

3. Curves of constant energy

Consider $\gamma(t)$ (where t ranges in a neighborhood of 0 in $\mathbb{R}$) a smooth curve in $\mathcal{N}$.

As it was mentioned in the Introduction, there is an a priori restriction on the tangent vector of a curve in the space of normalized potentials, namely, $\int X(t)d\mu_{\gamma(t)} = 0$ if $\gamma'(t) = X(t)$. The next result will be important in the sequel.

Lemma 3.1. *Let $\gamma(t)$ be a smooth curve of potentials such that $\gamma'(t) = X(t)$ has constant energy and $X(0) \in T_A\mathcal{N}$. Then $\gamma(t)$ is a curve of normalized potentials if and only if*

$$\int \frac{d}{dt}(X' + \frac{1}{2}X^2)d\mu_{\gamma(t)} = 0.$$

Proof. Let us first suppose that $\gamma(t)$ is a curve of normalized potentials. The constant energy implies that $\int X^2(t)d\mu_{\gamma(t)} = c$ for every t in the domain of $\gamma(t)$. The constraint $\int X(t)d\mu_{\gamma(t)} = 0$ for the curves in the manifold of normalized potentials gives, by taking derivatives, the equality

$$\frac{d}{dt}\int X(t)d\mu_{\gamma(t)} = 0 = \int (X' + X^2(t))d\mu_{\gamma(t)}$$

(using the definition of the inner product). So we get $\int X_t d\mu_{\gamma(t)} = 1$ and hence, taking again derivatives

$$\frac{d}{dt}\int X'(t)d\mu_{\gamma(t)} = 0 = \int (X'' + X'X)d\mu_{\gamma(t)} = \int \frac{d}{dt}(X' + \frac{1}{2}X^2)d\mu_{\gamma(t)}.$$

This proves the first statement in the Lemma.

To show the second statement, let $\gamma(t)$ be a smooth curve of potentials (not necessarily normalized) and let $\gamma'(t) = X(t)$, where $X(0) \in T_A\mathcal{N}$. Suppose that $\int X^2(t)d\mu_{\gamma(t)} = 1$ for every t in the domain of $\gamma(t)$. We know that $\int X(0)d\mu_{\gamma(t)} = 0$. If $X(t)$ satisfies the equation $\int \frac{d}{dt}(X' + \frac{1}{2}X^2)d\mu_{\gamma(t)} = 0$, for every t we have by the previous argument that

$$\frac{d}{dt}\int X'(t)d\mu_{\gamma(t)} = 0,$$

which yields that $\int X'(t) d\mu_{\gamma(t)} = c$ is constant in t. By the assumption on $\gamma(t)$, we have that $\int X(0) d\mu_{\gamma(0)} = 0$, so at $t = 0$ we have

$$\frac{d}{dt}\int X d\mu_{\gamma(0)} = 0 = \int (X'(0) + X^2(0)) d\mu_{\gamma(0)} = \int X'(0) d\mu_{\gamma(0)} + 1,$$

so $\int X'(0) d\mu_{\gamma(0)} = -1$ and therefore, $\int X'(t) d\mu_{\gamma(t)} = -1$ for every t, and the constant c equals to -1. Since the curve has energy 1 we get

$$\frac{d}{dt}\int X(t) d\mu_{\gamma(t)} = \int (X'(t) + X^2(t)) d\mu_{\gamma(t)} = -1 + 1 = 0,$$

and since by assumption $\int X(0) d\mu_A = 0$ we get that $\int X(t) d\mu_{\gamma(t)} = 0$ for every t thus characterizing a smooth curve in the manifold of normalized potentials. □

Proposition 3.2. *Given $A \in \mathcal{N}$, $X_0 \in T_A\mathcal{N}$, the differential equation*

$$X' + \frac{1}{2}X^2 = \int_0^t (\log(h_s) - \log(h_s \circ T)) ds,$$

where

(1) *$T : M \to M$ is under or main conditions,*
(2) *$\gamma(t)$ is a smooth curve of energy 1 in $\mathcal{N}$,*
(3) *$X(t) = \gamma'(t)$,*
(4) *h_t is the eigenfunction of the transfer operator $\mathcal{L}_{\gamma(t)}$,*

is such that has a unique solution in $\mathcal{N}$ with $A = \gamma(0)$, $X_0 = \gamma'(0)$.

Proof. Suppose that $\gamma(t)$ is a solution of the differential equation with $\gamma(0) = A$, $\gamma'(0) = X_0$. Integrating the differential equation in the above statement we get

$$\int (X'(t) + X^2(t)) d\mu_{\gamma(t)} = 0$$

and

$$\int \frac{d}{dt}(X'(t) + X^2(t)) d\mu_{\gamma(t)} = 0,$$

because the right hand side expression of the equation is a co-boundary as well as its derivative with respect to t. By Lemma 3.1, the vector field $X(t)$ will be tangent to $\mathcal{N}$ provided that $X(0)$ has vanishing mean with respect to $d\mu_A$.

Claim: The curve $\gamma(t)$ has constant energy.

Indeed, let us differentiate $< X, X >$ with respect to X. As noticed above,

$$\begin{aligned} X < X, X > &= \int (2XX' + X^3) d\mu_{\gamma(t)} \\ &= 2\int X(X' + \frac{1}{2}X^2) d\mu_{\gamma(t)} \\ &= 2\frac{d}{dt}\int (X' + \frac{1}{2}X^2) d\mu_{\gamma(t)} - 2\int \frac{d}{dt}(X' + \frac{1}{2}X^2) d\mu_{\gamma(t)} \\ &= 0 \end{aligned}$$

by the definition of $\gamma(t)$.

To complete the proof of the lemma it remains to show that the differential equation has a unique solution $\gamma(t)$ such that $\gamma(0) = A$ and $\gamma'(0) = X_0$. The differential equation has order 2 with respect to γ, so taking variables γ, X, it becomes equivalent to the first order system of differential equations

$$\begin{aligned} \frac{d}{dt}\gamma(t) &= X(t) \\ \frac{d}{dt}X(t) &= -\frac{1}{2}X^2 + \int_0^t (\log(h_r) - \log(h_r \circ T))dr. \end{aligned}$$

Letting $W = (\gamma, X)$ we get a differential equation of the form $W'(t) = F(W(t))$. The integral expression of this differential equation,

$$W(t) = W(0) + \int_0^t F(W(u))du$$

tells us that a fixed point of the operator $W(0) + F(W)$ would be a solution of our system.

Notice that the operator $H(\gamma(t)) = -\frac{1}{2}X^2 + \int_0^t (\log(h_r) - \log(h_r \circ T))dr$ depends analytically of $\gamma(t)$ and X since eigenfunctions of the Ruelle operator $\mathcal{L}_A$ of depend analytically on the normalized potential A (see [14], [12] or [9]), and therefore there exists $a > 0$ such that the operator $G(W) = W(0) + F(W)$ is a contraction in the set of smooth curves with $W(0) = (A, X_0)$ for every $| t |< a$. Hence, we get a fixed point by the contraction principle, and a fixed point is the unique solution of the differential equation we are looking for. It is defined in a certain interval $(-a, a)$.

Finally, combining the uniqueness of $\gamma(t)$ and Lemma 3.1 we get the Lemma. $\square$

4. On the sectional curvatures of the Riemannian metric

We assume in this section the existence of geodesics. This is a property which we will show to be true on section 5.

Let A be a normalized Hölder potential and $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{N}$ a geodesic of the Riemannian metric such that $\gamma(0) = A$, $\gamma'(t) = X(t)$, where $X(t)$ is a parallel unit vector field. By the existence of parallel transport along γ, we can define a local smooth surface $\phi(t, s)$ given in Fermi coordinates in the following way: let Y be a unit vector field in the tangent space of $\mathcal{N}$, that is perpendicular to $\gamma'(t)$ and is parallel in $\gamma(t)$, namely, $\nabla_X Y = 0$, let $\gamma_Y(t)(s)$ be the geodesic given by the initial conditions $\gamma_{Y(t)}(0) = \gamma(t)$, $\gamma'_{Y(t)}(0) = Y(t)$. Then,

$$\phi(t, s) = \gamma_{Y(t)}(s),$$

for every $| t |, | s |\leq \epsilon$. It is clear that $\phi(t, 0) = \gamma(t)$, and that the image S of $\phi : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow \text{Hol}$ is a smooth embedded surface on Hol for ϵ suitably small. Let us calculate the sectional curvature $K(X, Y)$ at the point $A = \gamma(0)$. Through the section we shall use the notation for derivatives $\frac{d}{dt}Z = Z_t$ for any vector field or function.

Let $\bar{X}$ be the vector field tangent to the t-coordinate in S, it extends the vector field X and it is not necessarily geodesic in the whole surface. Since $\bar{X}, Y$ commute, from the definition of sectional curvatures we deduce that,

$$K(X,Y) = Y(X < X,Y >) - \frac{1}{2}Y(Y(\| X \|)^2)) + X(X(\| Y \|^2)) + \| \nabla_Y X \|^2 .$$

at the points of $\gamma(t)$. Moreover, since Y is parallel along γ we have $\nabla_X Y = 0 = \nabla_Y X$ since $[\bar{X}, Y] = 0$; and since $< Y, Y >= 1$ we have $X(X(\| Y \|)^2) = 0$. Next, notice that

Lemma 4.1. *The vector fields $\bar{X}$ and Y are perpendicular in S.*

Proof. Since Y is parallel along γ and geodesic, we have

$$\begin{aligned} Y < \bar{X}, Y > = < \nabla_Y \bar{X}, Y > + < \bar{X}, \nabla_Y Y > &= < \nabla_Y \bar{X}, Y > = < \nabla_{\bar{X}} Y, Y > \\ &= \frac{1}{2}\bar{X} < Y, Y > = 0, \end{aligned}$$

where in the last equality we used the fact that $[\bar{X}, Y] = 0$. Therefore, the function $Y < \bar{X}, Y >$ vanishes in S, and hence the function $< \bar{X}, Y >$ is constant along the integral curves of Y. But at $\gamma(t)$ this function is $< X, Y >$ which vanishes by hypothesis. So $< \bar{X}, Y >$ vanishes everywhere in S thus proving our claim. □

The lemma shows that the sectional curvature is just

$$K(X,Y) = -\frac{1}{2}Y(Y(\| X \|)^2)).$$

To estimate this function we shall need some preparatory lemmas. Let X_t be the derivative of the vector field $\bar{X}$ with respect to the parameter t and $\bar{X}_s$ be the derivative of the vector field $\bar{X}$ with respect to the parameter s. The same convention applies to Y_t, Y_s.

Lemma 4.2. *We have that $\bar{X}_s = Y_t$ in the local surface S.*

Proof. This is due to the fact that the vector fields $\bar{X}, Y$ commute in S, so

$$[\bar{X}, Y] = \bar{X}(Y) - Y(\bar{X}) = Y_t - \bar{X}_s = 0.$$

□

Lemma 4.3. *The expression of $K(X,Y)$ at the point $A = \gamma(0)$ is*

$$K(X,Y) = -\int ((X_s)^2 + XX_{ss} + 2XX_sY + \frac{1}{2}X^2Y + \frac{1}{2}X^2Y^2) d\mu_A.$$

Proof. The equation is derived from the definition of the Riemannian metric. We have

$$\begin{aligned} K(\bar{X}, Y) &= -\frac{1}{2}Y(Y(\| \bar{X} \|)^2)) \\ &= -\frac{1}{2}Y(\frac{\partial}{\partial s}\int \bar{X}^2 d\mu_{A+sY}) \\ &= -\frac{1}{2}Y(\int (2\bar{X}\bar{X}_s + \bar{X}^2 Y) d\mu_A) \\ &= -\frac{1}{2}\frac{\partial}{\partial s}(\int (2\bar{X}\bar{X}_s + \bar{X}^2 Y) d\mu_{A+sY}) \end{aligned}$$

which gives exactly the expression in the statement. □

Lemma 4.4. *The equation $\bar{X}(Y<\bar{X},Y>)=0$ in S is equivalent to*

$$\int(\bar{X}_{st}Y+\bar{X}_sY_t+\bar{X}_tY_s+\bar{X}Y_{st}+\bar{X}_tY^2+2YY_tX+X^2Y^2)d\mu_A=0.$$

Proof. As in the previous lemma, we apply the definition of the Riemannian structure.

$$\begin{aligned}\bar{X}(Y<\bar{X},Y>) &= \bar{X}(\frac{\partial}{\partial s}\int \bar{X}Y d\mu_{A+sY}) \\ &= \bar{X}(\int(\bar{X}_sY+\bar{X}Y_s+\bar{X}Y^2)d\mu_A \\ &= \frac{\partial}{\partial t}\int(\bar{X}_sY+\bar{X}Y_s+\bar{X}Y^2)d\mu_{A+t\bar{X}(A)}\end{aligned}$$

that is just the expression in the statement. □

Corollary 4.5. *The equation $\bar{X}(Y<\bar{X},Y>)=0$ in S is equivalent to*

$$\int(\bar{X}_{st}Y+(\bar{X}_s)^2+\bar{X}_tY_s+\bar{X}X_{ss}+\bar{X}_tY^2+2Y\bar{X}_sX+\bar{X}^2Y^2)d\mu_A=0.$$

In particular, we get

$$\int((\bar{X}_s)^2+\bar{X}\bar{X}_{ss}+2Y\bar{X}_s\bar{X}+\frac{1}{2}\bar{X}^2Y^2)d\mu_A=-\frac{1}{2}\int\bar{X}^2Y^2d\mu_A-\int(YY_{tt}+\bar{X}_tY_s+\bar{X}_tY^2)d\mu_A.$$

Proof. The proof follows from the fact that $[\bar{X},Y]=0$ so Lemma 4.2 applies, and the fact that the derivatives with respect to t, s commute (so $Y_{st}=Y_{ts}=\bar{X}_{ss}$). □

Lemma 4.6. *The curvature at A is equal to*

$$K(X,Y)=\frac{1}{2}\int X^2(Y^2-Y_s)d\mu_A+\frac{\partial}{\partial s}\int YX_td\mu_{A+sY}.$$

Proof. Combining Lemma 4.5 and the expression of the curvature $K(X,Y)$ we get

$$K(X,Y)=\frac{1}{2}\int X^2(Y^2-Y_s)d\mu_A+\int(Y\bar{X}_{st}+\bar{X}_tY_s+\bar{X}_tY^2)d\mu_A.$$

Now, $\bar{X}=X$ along the geodesic γ, so $\bar{X}_t=X_t$. And since $\frac{\partial}{\partial s}\int Y\bar{X}_td\mu_{A+sY}=\int(Y\bar{X}_{st}+\bar{X}_tY_s+\bar{X}_tY^2)d\mu_A$ we get the formula in the statement. □

Lemma 4.7. *Suppose that the geodesic $\alpha(s)$ whose initial conditions are $\alpha(0)=0$, $\alpha'(0)=Y(0)$ satisfies the differential equation in Proposition 3.2. Then we have*

$$\int X^2(Y^2-Y_s)d\mu_A=\frac{3}{2}\int X^2Y^2d\mu_A,$$

so this term in the sectional curvature $K(X,Y)$ is always nonnegative.

Proof. The differential equation in Proposition 3.2 implies that

$$Y_s=-\frac{1}{2}Y^2+\int_0^s f(t)d\mu_{\alpha(s)}$$

for every s in the domain of α. This equation defines a differential equation at each point p in the space, namely, $Y(s)$ is actually a one parameter family of functions $Y(s,p)=Y(s)(p)$, $Y(s):\chi(\Omega)\longrightarrow\mathbb{R}$.

Replacing in the integral we get

$$\int X^2(Y^2 - Y_s)d\mu_A = \int X^2(Y^2 + \frac{1}{2}Y^2 - \int_0^s f(t)d\mu_{\alpha(s)})d\mu_A$$

and thus, evaluating at $A = \alpha(0)$ we get

$$\int X^2(Y^2 - Y_s)d\mu_A = \frac{3}{2}\int X^2Y^2 d\mu_A$$

as claimed.

□

So to show that $K(X,Y)$ is non-negative it remains to study the term

$$\frac{\partial}{\partial s}\int Y\bar{X}_t d\mu_{A+sY} = \int (Y\bar{X}_{st} + \bar{X}_t Y_s + \bar{X}_t Y^2)d\mu_A.$$

The same idea applied to calculate the first term in the curvature formula will do the job. First we shall need to prove the following technical result that follows the same line of ideas of Proposition 3.2.

Lemma 4.8. *There exists an analytic family of functions $p^t(s)$ with zero mean such that*

$$Y_t + \frac{1}{2}Y\bar{X} = \int_0^s p^t(r)dr.$$

Proof. The vector field Y is geodesic and normalized, $< Y, Y >= 1$ in the parameterized surface defined by the Fermi coordinates. Moreover, by Lemma 4.1 we have that $< \bar{X}, Y >= 0$ in the parametrized surface. So we have

$$\frac{d}{dt} < Y, Y >= 0 = 2\int Y(Y_t + \frac{1}{2}XY)d\mu_P$$

for every point P in the parametrized surface. By the definition of the Riemannian metric in $\mathcal{N}$ we get

$$0 = \int Y(Y_t + \frac{1}{2}\bar{X}Y)d\mu_P = \frac{d}{ds}\int (Y_t + \frac{1}{2}\bar{X}Y)d\mu_P - \int \frac{d}{ds}(Y_t + \frac{1}{2}\bar{X}Y)d\mu_P.$$

Claim: The vector field Y_t has vanishing mean.

Indeed, we already have that $\int Y d\mu_P = 0$ for every P in the surface, so taking derivatives with respect to t:

$$0 = \frac{d}{dt}\int Y d\mu_P = \int Y_t d\mu_P + \int \bar{X}Y d\mu_P = \int Y_t d\mu_P + < \bar{X}, Y >= \int Y_t d\mu_P,$$

thus proving the Claim.

From the previous equations, the Claim and the fact that $\int \bar{X}, Y d\mu_P =< \bar{X}, Y >_P = 0$, we deduce that

$$\int \frac{d}{ds}(Y_t + \frac{1}{2}\bar{X}Y)d\mu_P = 0$$

therefore the function $(Y_t + \frac{1}{2}\bar{X}Y)$ and its derivative with respect to s must have zero means. This proves the Lemma. □

Lemma 4.9. *We have that $\frac{\partial}{\partial s}\int Y\bar{X}_t d\mu_{A+sY} = \frac{1}{4}\int Y^2X^2 d\mu_A$ at $t = s = 0$*

Proof. Let us replace $\bar{X}_t$, Y_s by the expressions derived from Proposition 3.2 and Lemma 4.8 in the integral formula. For the second and third terms of the integral, by Proposition 3.2 we have at $t = s = 0$,

$$\begin{aligned}
\int (\bar{X}_t Y_s + \bar{X}_t Y^2) d\mu_{A(s)} \,|_{t,s=0} &= \int \bar{X}_t (Y_s + Y^2) d\mu_{A(s)} \,|_{t,s=0} \\
&= \int (-\frac{1}{2}X^2 + \int_0^t f(r)dr)(Y_s + Y^2) d\mu_{A(s)} \,|_{t,s=0} \\
&= \int (-\frac{1}{2}X^2)(-\frac{1}{2}Y^2 + \int_0^s g(r)dr + Y^2) d\mu_{A(s)} \,|_{t,s=0} \\
&= -\frac{1}{4}\int X^2 Y^2 d\mu_A.
\end{aligned}$$

The first term, by Lemma 4.8, is equal to

$$\begin{aligned}
\int Y \bar{X}_{st} d\mu_{A(s)} \,|_{t,s=0} &= \int Y Y_{tt} d\mu_{A(s)} \,|_{t,s=0} \\
&= \int Y(-\frac{1}{2}YX + \int_0^s p^t(r)dr)_t d\mu_{A(s)} \,|_{t,s=0} \\
&= \int Y(-\frac{1}{2}(Y_t X + Y X_t) + \int_0^s p^t_t(r)dr) d\mu_{A(s)} \,|_{t,s=0} \\
&= -\frac{1}{2}\int Y(Y_t X + Y X_t) d\mu_{A(s)} \,|_{t,s=0}
\end{aligned}$$

so, again by Proposition 3.2 and Lemma 4.8,

$$\begin{aligned}
-\frac{1}{2}\int Y(Y_t X + Y X_t) d\mu_{A(s)} \,|_{t,s=0} &= -\frac{1}{2}\int Y(X(-\frac{1}{2}YX + \int_0^s p^t(r)dr) \\
&+ Y(-\frac{1}{2}X^2 + \int_0^t f(r)dr)) d\mu_{A(s)} \,|_{t,s=0} \\
&= -\frac{1}{2}\int (-\frac{1}{2}Y^2X^2 - \frac{1}{2}Y^2X^2) d\mu_{A(s)} \,|_{t,s=0} \\
&= \frac{1}{2}\int Y^2 X^2 d\mu_A.
\end{aligned}$$

Adding the above two quantities we get the statement. □

Finally, combining Lemmas 4.6 and 4.9 we get the value of the sectional curvature $K(X,Y)$:

Theorem 4.10. *The sectional curvature $K(X,Y)$ equals $\int X^2Y^2 d\mu_A$, so it is always nonnegative. The curvature vanishes if and only if the supports of the functions X, Y are disjoint.*

5. The geodesics of the space of normalized potentials

The goal of the section is to show Theorems 2.1 and 2.2. Namely, given an element $A \in \mathcal{N}$, and a vector $X \in T_A\mathcal{N}$, we shall show that there exists a geodesic $\gamma(t)$ in the space such that $\gamma(0) = A$, $\gamma'(0) = X(0) = X$, and that the parallel transport of vectors along $\gamma(t)$ is well defined. Since the manifold of normalized potentials is an infinite dimensional manifold, the usual way of proving the existence of geodesics via solutions of an ordinary differential equations with coefficients in the set of Cristoffel symbols may not proceed.

One of the most common approaches to the problem of existence of geodesics in Hilbert manifolds is to show the Palais-Smale condition for the Riemannian metric. This is an issue in infinite dimensional Lagrangian calculus of variations: the Palais-Smale condition depends very much on each particular Riemannian metric and in our case it is not clear that such a condition is satisfied. However, what we shall show is in some sense a weak Palais-Smale condition for our Riemannian manifold: roughly speaking, we shall construct a sequence of approximated solutions of the Euler-Lagrange equation having as a limit a true solution of the equation.

We shall develop an strategy to prove the existence of geodesics under the following assumption: there exists a countable basis $\{v_n\}$, $n \in \mathbb{N}$, of tangent vectors in each tangent space $T_A\mathcal{N}$. We know that in every Banach space, the existence of a countable, dense subset gives a countable basis, so the above assumption holds for instance if our dynamics acts on a smooth manifold (the space of polynomial functions is dense for instance). This will do the job in the case $M = S^1$.

Remark: When $M = \{1, 2..., d\}^{\mathbb{N}}$ and μ the equilibrim probabilty for a Holder potential A it was shown in Theorem 3.5 in [8] that there exist a (countable) complete orthogonal set φ_n, $n \in \mathbb{N}$, on $\mathcal{L}^2(\mu_A)$.

5.1. **Some more estimates from Thermodynamic Formalism.** Given a potential $B \in \text{Hol}$ we consider the associated Ruelle operator $\mathcal{L}_B$ and the corresponding main eigenvalue λ_B and eigenfunction h_B.

The function

$$\Pi(B) = B + \log(h_B) - \log(h_B(T)) - \log(\lambda_B)$$

describes the projection of the space of potentials B on Hol onto the analytic manifold of normalized potentials $\mathcal{N}$.

When B is normalized the eigenvalue is 1 and the eigenfunction is equal to 1. We would like to study the geometry of the projection Π restricted to the tangent space $T_A\mathcal{N}$ into the manifold $\mathcal{N}$ (namely, to get bounds for its first and second derivatives with respect to the potential (viewed as a variable), for a given normalized potential A.

The space $T_A\mathcal{N}$ is a linear subspace of functions, so the map Π is analytic when restricted to it. The goal of the subsection is to estimate the first and second derivatives of Π restricted to $T_A\mathcal{N}$ in a small neighborhood of A in the sup norm. This is of course linked to the geometry of the transfer operator in a small neighborhood of a normalized potential A. The geometry of $\Pi\mid_{T_A\mathcal{N}}$ will be important to show the existence of geodesics as we shall see in the forthcoming subsections.

To get such estimates we recall some well known results of the analytic theory of the Ruelle operator.

Proposition 5.1. *Given a normalized potential $A \in \mathcal{N}$ and $\delta > 0$ there exists $r > 0$, such that, for every Hölder continuous function B in the ball $B_r(A)$ of radius r around A, the norms of $D_B\Pi$ and $D_B^2\Pi$ restricted to the functions in $T_A\mathcal{N}$ satisfy*

$$\| D_B\Pi\mid_{T_A\mathcal{N}} - I \| \leq \delta$$
$$\| D_B^2\Pi\mid_{T_A\mathcal{N}} \| \leq 1 + \delta,$$

Proposition 5.1 is perhaps well known, we sketch its proof for the sake of completeness. Let us recall some well known results of the theory of the transfer operator.

The following results are taken from [10], [9], [4] and [14].

Lemma 5.2. *Let $\Lambda : Hol \longrightarrow \mathbb{R}$, $H : Hol \longrightarrow Hol$ be given, respectively, by $\Lambda(B) = \lambda_B$, $H(B) = h_B$. Then we have*

1. *The maps Λ, H, and $A \longrightarrow \mu_A$ are analytic.*
2. $D_B \log(\Lambda)(\psi) = \int \psi d\mu_B$,
3. $D^2_B \log(\Lambda)(\eta, \psi) = \int \eta\psi d\mu_B$, *where ψ, η are L^2 functions.*
4. $D_A H(X) = h_A \int (I - \mathcal{L}_{T,A})^{-1}(1 - h_A) X d\mu_A$.
5. *If A is a normalized potential, then for every function $X \in T_A\mathcal{N}$ we have* $\int X d\mu_A = 0$.

Remark 1: The expression of item (4) appears in an old ArXiv version of [4] (Proposition 4.6. in the 2012 version). Note that the derivative linear operator $X \to D_A H(X)$ is zero when A is normalized.

Remark 2: Note that item (2) implies by item (5) that $D_B \log(\Lambda)(\psi) = \int \psi d\mu_B = 0$, when B is normalized and $\psi \in T_{\mu_B}(\mathcal{N})$.

Questions related to second derivatives on Thermodynamic Formalism are considered in [6] and [13].

From the above lemma we deduce the following:

Lemma 5.3. *Given a normalized potential A and $\delta \in (0, 1)$, there exists $r > 0$, such that, for every Hölder continuous B in the C^0 ball $B_r(A)$ of radius r centered at A, we have that the L^2 norms of $D_B\Lambda$, $D_B H$ and $D^2_B\Lambda$ satisfy*

1. $\| D_B\Lambda\,|_{T_A\mathcal{N}} \| \leq \delta$,
2. $\| D_B H\,|_{T_A\mathcal{N}} \| \leq \delta$,
3. $\| D_B\Pi\,|_{T_A\mathcal{N}} - I \| \leq \delta$,
4. $\| D^2_B\Lambda\,|_{T_A\mathcal{N}} \| \leq 1 + \delta$, *for every $B \in B_r(A) \cap T_A\mathcal{N}$.*

Proof. Since the map $A \longrightarrow \mu_A$ is analytic given $\epsilon > 0$ there exists $r > 0$ such that for every L^2 function $X : S^1 \longrightarrow \mathbb{R}$ with unit norm with respect to μ_A we have

$$\left| \int X d\mu_A - \int X d\mu_B \right| < \epsilon$$

for every Hölder function B in the ball $B_r(A)$ of radius r around A in the C^0 topology. Let $X \in T_A\mathcal{N}$, items (2) and (4) in the assumptions imply that $D_A \log(\Lambda) = 0$ and moreover,

$$| D_A \log(\Lambda)(X) - D_B \log(\Lambda)(X) | = \left| \int X d\mu_B \right| < \epsilon,$$

so the L^2 norm of $D_B \log(\Lambda)$ restricted to $B(1, A, L^2)$ - the L^2 ball of radius 1 with respect to the measure μ_A - is bounded above by $\epsilon \sup_{X \in B(1,A,L^2)} \int X d\mu_B$. From this assertion follows the estimate for $D_B\Lambda$.

The estimate for the second derivative of Λ follows from items (2) and (3) in Lemma 5.2, since the second derivative of $\log(\Lambda)$ at B is just the L^2 inner product with respect to the measure $d\mu_B$.

To show item (2), observe that according to item (4) in Lemma 5.2,

$$\| D_B H(X) \| \leq h_B \, \| (I - \mathcal{L}_{T,A})^{-1} \|_\infty \| (1 - h_B) \|_\infty \| X \| .$$

Since A is a normalized potential, we have $h_A = 1 = \lambda_A$ and we can suppose that in the ball $B_r(A)$ we also have $\mid 1 - h_B \mid < \epsilon$ by the analyticity of the function H. The operator $(I - \mathcal{L}_{T,A})^{-1}$ is uniformly bounded as well because of the spectral gap of the operator $\mathcal{L}_{T,A}$. This yields that the norms $\| \quad \|_\infty$, $\| \quad \|_{L^1}$, $\| \quad \|_{L^2}$ are small for $D_B H$, $B \in B_r(A)$.

The proof of item (3) is a consequence of the definition of Π and the already proved items in the lemma.

□

Notice that item (3) in the previous lemma is the first inequality of Proposition 5.1. So it remains to show the second inequality.

In a future section we will need to control the second order derivative of the function Π acting on Hölder potentials B close to a normalized potential A. On that moment we will have to use the next lemma. We point out that the continuous dependence (follows from analyticity) on all parameters which are involved on the computations.

Lemma 5.4. *Let $A \in \mathcal{N}$, $r > 0$, $B_r(A)$ be given in Lemma 5.3. Then there exists $\delta(r) > 0$ small enough, such that, the second order derivative bilinear form of the function*

$$B \to \Pi(B) \;=\; B + \log(h_B) - \log(h_B(T)) - \log \lambda(B) \tag{4}$$

restricted to $T_A\mathcal{N}$ is $\delta(r)$-close to the identity in L^2 for every $B \in B_r(A) \cap T_A\mathcal{N}$.

Proof. Remember that when A is normalized $\Pi(A) \;=\; A + \log(h_A) - \log(h_A(T)) - \log \lambda(A) = A$. Moreover, the first and second derivatives of Π on B are close to the corresponding ones of A.

It is known that for a normalized potential A we have

$$\frac{\partial}{\partial \psi} \Pi \mid_A = I,$$

where I is the identity.

Let us analyze the first derivative of Π at a point $B \in B_r(A)$ not necessarily normalized and a variable increment ψ.

By the analyticity of H, and the fact that $\log(H(A)) - \log(H(A)(T)) = 0$ if A is normalized, there exists $\delta_1 > 0$ small such that $\| \log(h_B) - \log(h_B(T)) \|_\infty < \delta_1$ for every $B \in B_r(A)$.

We get from item (3) of Lemma 5.3

$$\frac{\partial}{\partial \psi} \Pi \mid_B \sim I,$$

where I is the identity, for every $\psi \in T_A\mathcal{N}$ with unit L^2 norm, the error of this approximation is bounded above by δ in Lemma 5.3.

Moreover, for the single increment $\psi \in T_A\mathcal{A}$ we get (by the rule of the derivative of the product)

$$\frac{\partial}{\partial \psi} \Pi(B + \psi) = \int \psi d\mu_B + \int \frac{\partial}{\partial \psi} (\log h_{B+\psi} - \log h_{B+\psi} \circ T)\, d\mu_B +$$
$$\int (\log h_{B+\psi} - \log h_{B+\psi} \circ T)\; \psi\; d\mu_B\,.$$

As we mentioned before $(\log h_{B+\psi} - \log h_{B+\psi} \circ T)$ (and its first derivative) is small when ψ is small by Lemma 5.3.

Now, we analyze the second derivative. For the double increment ψ, φ, by taking derivative (and the rule of the derivative of the product) we get the bilinear form

$$(\psi, \varphi) \to \int \frac{\partial}{\partial \psi} \frac{\partial}{\partial \varphi} (\log h_B - \log h_B \circ T)\, d\mu_B +$$
$$\int \frac{\partial}{\partial \varphi} (\log h_B - \log h_B \circ T)\ \psi\ d\mu_B +$$
$$\int \frac{\partial}{\partial \psi} (\log h_B - \log h_B \circ T)\ \varphi\ d\mu_B +$$
$$\int (\log h_B - \log h_B \circ T)\ \psi\ \varphi\ d\mu_B\,.$$

The claim of the lemma follows from the following facts:

1) the first term of the sum above is zero by the coboundary property,

2) the linear derivative of $B \to (\log h_B - \log h_B \circ T)$ is δ_r small (second and third terms by Lemma 5.3),

3) $(\log h_B - \log h_B \circ T)$ is small when ψ and φ are small and close to a normalized potential (fourth term).

□

5.2. **The system of differential equations of geodesic vector fields.** Let us begin with the same ideas of the finite dimensional case. Suppose that $\gamma(t)$ exists, we are going to characterize γ in terms of a differential equation in the space $\mathcal{N}$ that has a unique solution. Let $X(t) = \gamma'(t)$, since it is geodesic, $\nabla_X X = 0$, where ∇ is the Levi-Civita connection of the Riemannian metric in $\mathcal{N}$. Actually, we have to show that this equation has a solution, we shall reduce this problem to solve another differential equation. So we have that

$$< \nabla_X X, Y >= 0$$

for every $Y \in T_{\gamma(t)}\mathcal{N}$. By the compatibility properties of the Riemannian metric and the covariant derivative

$$< \nabla_X X, Y >= X < X, Y > - \frac{1}{2} Y < X, X > - < X, [X, Y] >$$

where $X(f)$ means the derivative of a scalar function f with respect to X.

In particular, the energy of geodesics is constant,

$$< \nabla_X X, X >= 0 = \frac{1}{2} X < X, X >= \frac{1}{2} \int (2XX' + X^3) d\mu_{\gamma(t)}.$$

So let us restrict ourselves to the energy level of vector fields X with constant norm equal to 1. In this case, the equation of geodesics gives

$$0 =< \nabla_X X, Y >= X < X, Y > - < X, [X, Y] >,$$

or equivalently,

$$X < X, Y >=< X, [X, Y] >$$

for every vector field Y. In particular, if the elements of the basis v_n generate vector fields we have

$$X < X, v_n > = < X, [X, v_n] > .$$

In the case where the vector fields v_n correspond to a finite number of coordinate vector fields this set of equations might be used to show the existence of the geodesic vector field. Indeed, say that $n \leq m$, then the above system of equations is equivalent to a system of first order partial differential equations whose solution always exists by the theory of characteristics. Let us write down the system explicitly.

Let $\Phi : U_m \longrightarrow V_m$, $\Phi(t_1, t_2, .., t_m)$, be a coordinate system defined in an open neighborhood of $0 \in \mathbb{R}^m$ whose image is a smooth m-dimensional manifold in $\mathcal{N}$ containing A. Let e_n be vector fields in $\mathbb{R}^m$ tangent to the coordinates t_n, and let $v_n = D\Phi(e_n)$ define the coordinate vector fields in $\mathcal{N}$.

Let $X = \sum_{i=1}^m x_i v_i$, $\bar{x}_i = < X, v_i >$. The differential equation of the geodesic vector field X is equivalent to

$$X < X, v_n > = < X, [X, v_n] > = < X, [\sum_{i=1}^m x_i v_i, v_n] >$$

and observe that

$$[\sum_{i=1}^m x_i v_i, v_n] = \sum_{i=1}^m [x_i v_i, v_n] = \sum_{i=1}^m (x_i [v_i, v_n] - v_n(x_i) v_i)$$

and since the vector fields v_n commute we get

$$[\sum_{i=1}^m x_i v_i, v_n] = \sum_{i=1}^m -v_n(x_i) v_i = -(\sum_{i=1}^m v_n(x_i v_i)) + x_n \bar{v}_n = -v_n(X) + +x_n \bar{v}_n$$

because the derivatives $v_n(v_i) = \frac{d}{dt_n} v_i$ are equal to 0 if $n \neq i$ and $\bar{v}_n = D^2(\Phi)(v_n)$ if $n = i$. Hence we can write the differential equation for X as

$$X(\bar{x}_n) = X < X, v_n > \quad = \quad - < X, v_n(X) > + < X, x_n \bar{v}_n > .$$

In terms of $\frac{d}{dt}$, $\frac{d}{dt_n}$ we obtain a system S_m of first order partial differential equations

$$(5) \qquad S_m := \frac{d}{dt}(\bar{x}_n) \quad = \quad - < X, \frac{d}{dt_n}(X) > + < X, x_n \bar{v}_n >, \; n = 1, 2, .., m.$$

The above system of differential equations gives rise to a system of partial differential equations for the functions $\bar{x}_i$. Indeed, let $\bar{X} = (\bar{x}_1, \bar{x}_2, .., \bar{x}_m)$, and let M_m be the matrix of the first fundamental form in the basis v_i, namely,

$$(M_m)_{ij} = < v_i, v_j > .$$

We have that $\bar{X} = M_m X$, so $X = M_m^{-1} \bar{X}$. Replacing this identity in the initial system we get a system of first order, quasi-linear partial differential equations for the functions $\bar{x}_n$ whose coefficients depend on the matrices M_m^{-1} and $\frac{d}{dt_n}(M_m^{-1})$.

5.3. **Uniform bounds for the PDE geodesic systems.** A natural way to obtain geodesics from the family of systems S_m would be to solve each of the systems with a given initial condition and take the limit $m \to \infty$. A limit function would be the desired geodesic. However, the limit process might not give any limit function, this depends on uniform bounds for the coefficients of the matrices M_m. This is the subject of next lemma which consider the case $M = S^1$ where it is well know the existence of a countable basis (independent of the equilibrium probability).

For the case when M is the symbolic space we shall use the Remark just after the beginning of subsection 5.1 and the next lemma will work in a similar way.

Lemma 5.5. *Let $A : S^1 \to \mathbb{R}$ be normalized potential and $B \in B_r(A)$ be the open neighborhood of A given in Proposition 5.1. Let f_n be any countable basis of analytic functions of the circle (interval) in the L^2 norm, and let*

$$\bar{f}_n = f_n - \int f_n d\mu_A.$$

Then

1. *The set of functions $\bar{f}_n$ is a basis of $T_A\mathcal{N}$.*
2. *Let e_n be an orthonormal basis of $T_A\mathcal{N}$ obtained from $\bar{f}_n$. Then the functions*
$$v_n(B) = D_B\Pi(e_n)$$
form a basis for $T_B\mathcal{N}$ and
$$|< v_n(B), v_m(B) > -\delta_{nm} |\leq \delta$$
where δ_{nm} is the Kronecker function : $\delta_{nm} = 1$ if $n = m$, and 0 otherwise.
3. *There exists $b > 0$ such that map Π restricted to the sets*
$$U_m = \{\sum_{i=1}^{m} t_i e_i, \ \ | t_i |< b\}$$
is an embedding into a m-dimensional submanifold $V_m \subset \mathcal{N}$.

Proof. The map $f \to f - \int f d\mu_A$ is a linear map from the set of functions to $T_A\mathcal{N}$. Therefore, if f_n is a basis of the set of functions the image of the set $\{f_n\}$ by this linear map is a basis in the image of the map, that is precisely $T_A\mathcal{N}$. From the basis $\bar{f}_n$ we can of course obtain an orthonormal basis e_n by Gram-Schmidt.

From Proposition 5.1, we know that $D_A\Pi |_{T_A\mathcal{A}}= I$ and that $D_B\Pi |_{T_A\mathcal{A}}$ is close to the identity if $B \in B_r(A)$. Hence, if we chose $B = A + \sum_{i=1}^{m} t_i w_i$ in a way that $\| B - A \|< r$ then the vectors $v_n(B) = D_B\Pi(e_n)$ will be almost perpendicular at $T_B\mathcal{N}$. This yields that the vectors $v_n(B)$ are linearly independent in $T_B\mathcal{N}$ and therefore, the map Π has constant rank m in U_m. By the local form of immersions, the image $V_m = \Pi(U_m)$ is an analytic submanifold of $\mathcal{N}$ of dimension m. □

By virtue of Lemma 5.5, we shall consider the collection of m-dimensional coordinate systems given by the restrictions of Π to the sets U_m. Let us estimate the norms of the associated matrices M_m, M_m^{-1} and its derivatives.

Lemma 5.6. *There exists $C > 0$ such that the norms of the matrices M_m^{-1}, $\frac{d}{dt_n}(M^{-1})$ are uniformly bounded by C in the neighborhood $B_r(A)$.*

Proof. The coefficients of the first fundamental form M_m at a point $B \in B_r(A)$ are

$$< v_i(B), v_j(B) >= \int v_i(B) v_j(B) d\mu_B.$$

By Lemma 5.5 and Lemma 5.4, the matrix M_m is a perturbation of the identity at every point $B \in B_r(A)$. This yields that the norm of M_m^{-1} is uniformly bounded above in $B_r(A)$.

The derivative of M_m^{-1} with respect to t_n is $-M_m^{-1} \frac{d}{dt_n}(M_m) M_m^{-1}$. The derivatives at $B \in B_r(A)$ of the coefficients of M_m are determined by the derivatives of the terms $< v_i, v_j >$. We have

$$\frac{d}{dt_n} < v_i, v_j >= \int (\frac{d}{dt_n}(v_i) v_j + v_i \frac{d}{dt_n}(v_j) + v_i v_j v_n) d\mu_B$$

by the definition of the Riemannian metric. The norms of the terms in the equation are bounded by the products of the norms of $D^2(\Pi)$, v_i, v_j, v_n, which have uniform bounds according to Proposition 5.1.

□

As a consequence of Lemma 5.6 we get an existence of solutions result for the partial differential equation of geodesics.

Lemma 5.7. *Under the assumptions of Lemma 5.6, there exist $\rho > 0$, $D > 0$, such that given a unit vector $X(0) \in T_A\mathcal{N}$ and $m \in \mathbb{N}$, there exists a unique solution of the equation (1) whose initial condition is $X(0)$. The solution $X(t)$ is defined in an interval $\mid t \mid \leq \rho$, and the norms of $X(t)$, $X'(t)$ are bounded by D for every $\mid t \mid \leq \rho$.*

Proof. By the theory of first order partial differential equations, the system (1) is equivalent to a system of first order ordinary differential equations $\frac{d}{dt}Y = F_m(Y)$ where the function F depends on the first fundamental form A and its derivatives with respect to the coordinates t_n. So this function has bounded norm in the neighborhood $B(r)$ and is analytic. The theory of existence and uniqueness of solutions of ordinary differential equations implies that given $P > 0$, there exists $\rho > 0$ such that in the set of Lipschitz functions X with constant P the solution $X_m(t)$ of (1) with initial condition $X(0)$ is unique and defined in $(-\rho, \rho)$. Moreover, by shrinking ρ if necessary, we can suppose that $X(t)$ is in $B(r)$, so if $\bar{L} > 0$ is an upper bound for the norm of F in $B(r)$ we get

$$\frac{d}{dt} \parallel Y \parallel \leq \parallel F \parallel \parallel Y \parallel$$

which yields that

$$\parallel Y(t) \parallel \leq \parallel Y(0) \parallel e^{\bar{L}t} \leq \parallel Y(0) \parallel e^{\bar{L}\rho}$$

for every $\mid t \mid \leq \rho$. This clearly implies the same kind of estimate for $X_m(t)$.

Now, since the bounds for F_m are uniform in $B(r)$ we get a sequence X_m of solutions of (1) defined in $(-\rho, \rho)$ that are uniformly bounded in this interval. The integral curves $\gamma_m(t)$ of these vector fields form a family of functions which are uniformly Lipschitz and uniformly bounded in $(-\rho, \rho)$. This implies that their Hölder constants are uniformly bounded by some $L > 0$ in $(-\rho, \rho)$, and since the set of β-Hölder functions in S^1 with Hölder constant bounded above by L is a compact metric space, Arzela-Ascoli Theorem yields the existence of a convergent subsequence with limit $\gamma(t)$. The function $\gamma(t)$ is smooth and its tangent vector $X(t)$ is the limit of a convergent subsequence of the solutions $X_m(t)$ in $(-\rho, \rho)$.

Claim: The curve $\gamma(t)$ is a geodesic.

To show that, we have to prove that for every vector field Y tangent to $\mathcal{N}$ in $B(r)$ we get $< \nabla_X X, Y >= 0$. This equation is equivalent to equation (1) for the vector Y, and since v_n is a basis for the tangent space of $\mathcal{N}$ it is enough to show that $< \nabla_X X, v_m >= 0$ for every m. This is just a consequence of the fact that the solutions $X_m(t)$ and its derivatives converge uniformly to $X(t)$ and its derivatives, combined with the continuity of the differential equation (1) with respect to these quantities.

□

5.4. **Parallel transport and Fermi coordinates for local surfaces.** The proof of Theorem 2.2 is similar to the proof of Theorem 2.1, we shall sketch the proof in some steps to avoid repetition of arguments. Let $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$, $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{N}$ the geodesic such that $\gamma(0) = A$, $\gamma'(0) = X$. Let $\gamma'(t) = X(t)$, and consider a countable basis e_n of $T_A\mathcal{N}$ such that $e_1 = X$.

Let us define a family of local n-dimensional submanifolds S_n of $\mathcal{N}$ in the following way. Let $v : (-\epsilon, \epsilon) \longrightarrow T_A\mathcal{N}$ be the curve $v(t) = (\Pi_A)^{-1}(\gamma(t))$, where $\Pi_A : T_A\mathcal{N} \longrightarrow \mathcal{N}$ is the restriction of Π to $T_A\mathcal{N}$. Since Π_A is a local diffeomorphism in a small ball around $0 \in T_A\mathcal{N}$ the curve $v(t)$ is analytic and tangent to X at $t = 0$. Let us consider the subsets W_n of functions in $T_A\mathcal{N}$

$$W_n = \cup_{|t_i|<\epsilon}\{X(t_1) + \sum_{i=2}^{n} t_i e_i\}.$$

It is a n-dimensional submanifold of functions whose tangent space at A contains the vectors $X, e_2, .., e_n$. Since $D\Pi$ is close to the identity in an open neighborhood of $T_A\mathcal{N}$ we have that

$$S_n = \Pi(W_n)$$

is a family of parametrized smooth n-dimensional submanifolds in $\mathcal{N}$. Notice that this family is slightly different from the family V_n considered in the previous subsection. The point is that the geodesic $\gamma(t)$ now is a coordinate axis of S_n, $t = t_1$ is the first coordinate of the parametrization. We can suppose that the coordinate tangent vector fields $\sigma_n = D_{v(t)}\Pi(e_n)$ are perpendicular to $X(t) = \sigma_1(\gamma(t))$ for every $t \in (-\epsilon, \epsilon)$ (otherwise we just orthogonalize them along $\gamma(t)$).

To find a local surface S parametrized in Fermi coordinates we start by choosing a vector $Y \in T_A\mathcal{N}$, and we would like to solve the equation

$$\nabla_{X(t)}Y(t) = 0$$

where $Y(t)$ is a vector field defined in $\gamma(t)$ such that $Y(0) = Y$, which amounts to solve the system of equations

$$< \nabla_X Y, \sigma_n >= 0$$

in each S_n for every n. In the finite dimensional case, we can parametrize open neighborhoods of the Riemannian manifold with Fermi coordinates. We are not going to show that in our case (we do not need for the proof of Theorem 2.2). However, we shall make the following assumption on $Y(t)$ that is satisfied in the finite dimensional case: Y is a vector field defined in an open neighborhood of A

which commutes with the coordinate vector fields σ_n at $\gamma(t)$. If we show that the above system has a solution under this hypothesis we find the parallel transport of Y along $\gamma(t)$ and Theorem 2.3 proceeds.

Let us orthogonalize the vector fields σ_n to get vector fields $\bar{\sigma}_n$ that might not be coordinate vector fields, although they are along $\gamma(t)$. The vector fields $\bar{\sigma}_n$ continue to form a basis of $T_B\mathcal{N}$ for B in an open neighborhood of A. The expression of the parallel transport system in this base is, according to the equation of the Levi-Civita connection,

$$< \nabla_X Y, \bar{\sigma}_n >= 0 = \frac{1}{2}(X < Y, \bar{\sigma}_n > - \bar{\sigma}_n < X, Y >).$$

Let us consider the orthogonal projection Y_n of Y in the subspace generated by the vectors $\bar{\sigma}_i$, $i = 1, 2, .., n$. We have $Y_n = \sum_{i=1}^n y_i \bar{\sigma}_i$, for $y_i =< Y, \bar{\sigma}_i >$. Replacing in the system we get

$$X(y_n) = \sum_{i=1}^n \bar{\sigma}_n (y_i < X, \bar{\sigma}_i > .$$

The functions $< X, \bar{\sigma}_i >$ are known, and we can get from this system another system in terms of the coordinate vector fields σ_i that is close to it (let us remind that $\sigma_i = \bar{\sigma}_i$ along γ). Both systems are first order, partial differential equations systems with uniformly bounded coefficients by Proposition 5.1. As in the previous subsection, we get a family $Y_n(B)$ of solutions defined in an open neighborhood of A, and letting n tend to ∞ we get a solution $Y(t)$ for the parallel transport of $Y = Y(0)$ along $\gamma(t)$.

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