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THE DIMENSION SPECTRUM OF THE MAXIMAL MEASURE*

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Abstract. A variety of complicated fractal objects and strange sets appears in nonlinear physics. In diffusion-limited aggregation, the probability of a random walker landing next to a given site of the aggregate is of interest. In percolation, the distribution of voltages across different elements in a random-resistor network (see [T. Halsey et al., *Phys. Rev. A* (3), 33 (1986), pp. 1141-1151]) may be of interest. These examples can be better analyzed by dividing certain objects in pieces labeled by indexes, but that leads to working with fractal sets and the notion of dimension [Halsey et al. (1986)].

The dimension spectrum of a system has been introduced and measured experimentally, and a substantial literature in physics addresses this topic. In several important cases, rigorous proofs of the results presented in [Halsey et al. (1986)] have been established.

Here, rigorous mathematical proofs of some results in this theory are given, specifically for the maximal entropy measure of a hyperbolic rational map in the complex plane. In this case the fractal object is the Julia set (see [H. Brolin, *Ark. Mat.*, 6 (1966), pp. 103-114], [A. Freire, A. Lopes, and R. Mañé, *Bol. Soc. Brasil Mat.*, 14 (1983), pp. 45-62]), which has been extensively studied in the physics literature.

Key words. Hausdorff dimension, entropy, maximal measure, rational maps, pressure, spectrum of dimension, large deviation

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0. Introduction. In recent years the role of the concept of dimension has been investigated by several authors in trying to understand nonconservative dynamical systems.

The possibility of an infinite number of generalized dimensions of fractals appears in a natural way in the context of relevent physical problems of critical phenomena. This topic is particularly active in the physics literature. Such problems appear in the configuration of Ising models, percolation clusters, and fully developed turbulence. In general, we can describe such models by dividing the object into pieces and rescaling. In this situation we very often obtain several different values of dimension.

We are interested in developing the thermodynamic formalism for chaotic repellers obtained from hyperbolic rational maps in the complex plane and its relation to the spectrum of dimensions.

The same problem for attractors has been investigated in [9]. In general, an attractor can have an arbitrarily fine-scaled interwoven structure of hot and cold spots (high and low probability densities). By hot and cold spots we mean points on the attractor for which the frequency of visitation to the region for typical orbits is either much greater than average (a hot spot) or much less than average (a cold spot). In these several different points we can have different local values of dimension, and the aim of this theory is to understand the situation globally.

Now we will explain more carefully the situation we are going to consider. We will analyze the dimension spectrum of the maximal measure (sometimes called the balanced measure) [1], [8], [17] of a hyperbolic rational map f on the complex plane

$$f(z)=\frac{P(z)}{Q(z)}$$

where P and Q are complex polynomials.

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The dimension spectrum of a system was introduced and measured experimentally by Halsey et al. [10], Hentschel and Procaccia [11], and Jensen et al. [12]. See also [2], [5], [9], [26], and [27] for analyses of important cases.

In [2] and [27] the theory is applied to several different systems, among them cookie-cutter maps, and it is related to the measure of maximal entropy. In [5] critical mappings of the circle with golden rotation number are considered.

We will use thermodynamic formalism as in [27] and also classical large deviation theory as in [5] to obtain our result for the maximal measure of a hyperbolic rational map.

For each complex number z and positive real number ξ, denote by $B(z, \xi)$ the ball of center z and radius ξ in the usual norm of $\mathbf{R}^2$.

We will say that a certain measure v has exponent α on z if

$$v(B(z, \xi)) \approx \xi^{\alpha}$$

for ξ small enough. (Here $\approx$ means $\lim_{\xi \to 0} (\log v(B(z, \xi))/\log \xi) = \alpha$.)

We will also say that z scales with exponent α.

Given a measure v, one of the main goals of the Dimension Spectrum Theory is to understand the set of points that scales with exponent α.

For α fixed, the structure of such a set of points can be very complicated, and this set can also have v-measure zero and two-dimensional Lebesgue measure zero. The Hausdorff dimension gives more detailed information on how small the sets of the plane with two-dimensional Lebesgue measure zero are. When the Hausdorff dimension of a set is a noninteger number, we say that this set is a fractal. It is natural to ask, in terms of Hausdorff dimension, how small these sets are with respect to the variable α.

Experimental results in [10] and [12] have suggested that the Hausdorff dimension of such sets is a differentiable function of α, in the case of a certain measure of critical mappings of the circle with rotation number equal to the golden-mean.

We point out, as has been done in [5], that without some restrictions on the measure v, nothing interesting can be said about the problem.

A given probability v is called invariant for a map f if

$$v(f^{-1}(E)) = v(E)$$

for any set E, where the probability is defined.

If we are working in the context of statistical physics with problems in the one-dimensional $\mathbf{Z}$ lattice, and in each position we have two possibilities of spin, let us say $+$ and $-$, then the natural space to consider is the Bernoulli model $\{+, -\}^{\mathbf{Z}}$. As we do not have any reason to consider a distinguished position for the value zero in our lattice, then in our problem we will consider only probabilities that are invariant by the shift map (see [3] and [29] for more references). This is a simple motivation for considering invariant probabilities in general problems.

In cases where f is a rational map, the support of any invariant probability is the Julia set (see [4], [6], [8], [21] for definitions). In almost all the cases this set is of fractal dimension [6], [14]. There are no smooth invariant measures to consider in this situation.

Consider, for example, the map $f_\xi = z^2 + \xi$ when ξ is small. In this case the Julia set is a nowhere-differentiable Jordan curve for $\xi \neq 0$. In fact, the Julia set is a fractal Jordan curve for ξ inside the main cardioid of the Mandelbrot set (and $\xi \neq 0$) [6].

The Julia set can also be a Cantor set or even a combination of parts that are locally disconnected and locally connected. The Julia set can even be the all complex plane for some nonhyperbolic rational maps. In the case of hyperbolic rational maps anyway, the Julia set always has two-dimensional Lebesgue measure zero.

There is an important conjecture that claims that the hyperbolic rational maps are dense in the set of rational maps (see [21]).

In [8] and [17] it has been shown that among all invariant probabilities, there exists a special one that obtains the maximal value of the entropy (see [19], [33] for exact definitions). We will call this probability the maximal measure.

The entropy of an invariant probability is a measure of the degree of randomness of the system given by the action of the map f and the invariant probability we are considering. In this case the maximal measure is the more chaotic one.

Following the principle that in the absence of external thermal sources nature tends to maximize entropy, we can see the maximal measure as some kind of Gibbs state. If we must take into account external sources, we are then led to consider maximal pressure probabilities (see [3], [29] for interesting considerations about this). In § 1, for some other reasons, we will have to consider maximal pressure probabilities.

We will denote by u the maximal measure for a hyperbolic rational map. We point out that, in [8] and [17], the results are for general rational maps, and hyperbolicity is not assumed.

Here we will develop all the theory to show the following theorem.

THEOREM. *Consider u the maximal measure of a hyperbolic rational map; then the Hausdorff dimension of the set of points that scale with exponent α is a real analytic function of the variable α.*

We will relate these concepts of scaling exponents with the pressure, the Legendre transform of the pressure, entropy, and large deviation. In fact, one of the main ingredients of the proof is the close relation of pressure and free-energy (see § 1 for definitions). This relationship is explored in a more general context in [16].

The analogous claim for nonhyperbolic rational maps is not always true. In [26] an example of a quadratic polynomial is shown such that there exists a point α where there is no differentiability. In this situation we can say, using an analogy with statistical physics, that phase transition exists.

The theorem stated here can also be seen as a statement concerning the nonexistence of phase transitions for the maximal entropy measure of a hyperbolic rational map.

A natural question to ask is, why do large deviation techniques appear in the understanding of the problem? The reason is that there exists a certain δ such that for a u-almost-everywhere point z in the Julia set, the point z scales with exponent δ (see [22], [20]). This follows basically from properties related to the Birkhoff Ergodic Theorem [19]. If we want to consider a certain fixed α different from the above-mentioned δ and look for the set of points z that scales with exponent α, then we are in part not covered by the Birkhoff Ergodic Theorem. The above-mentioned theorem is a result on mean values and, therefore, in considering deviations of the mean, we must use large deviation techniques. We refer the reader to Ellis [7] for references concerning large deviation. We can find the general theory of ergodic theory and thermodynamic formalism in Walters [33], Mañé [19], and Ruelle [29], [30].

Several results are known for the maximal measure [1], [8], [13]-[15], [17], [18], [20], [22]. In particular, the moments of this measure can be obtained by a three-term relation from the coefficients of the rational map (see [1], [13]). The three-term relation is a consequence of the functional equation that the complex potential generated by the maximal measure satisfies around infinity [13]. This functional equation is known as the Bochner equation in the case of polynomial maps [1]. In the case where the rational map is not a polynomial, the functional equation has another form (see [13], [14]). There are several connections of such results with Classical Potential Theory [31], [4], [13]. In particular, [4] and [13] show that this maximal measure is the charge distribution in the Julia set if and only if f is a polynomial.

The degree of the rational map f will be denoted by d. We will also denote by J the Julia set. The entropy of the maximal measure is $\log d$ [8], [17].

We will show here that, in fact, the set of points that scale with exponent α can be considered as the support of another measure (different from u, and we will not lose dimensionality with this procedure (see the proof of Theorem 4)). We refer the reader to [9]-[12] and [26] for some applications of the spectrum of dimension theory to statistical mechanics.

It is worthwhile mentioning the following heuristic analogy. In problems of physics, when we can apply renormalization techniques, in general, it is because we have some good self-similar properties. We can take a partition of the object we want to consider, and from this partition, by some well-defined procedure, we can obtain another with some additional coarse information. Now, the procedure is repeated with the new partition. If we have some good self-similar properties, we can expect to have with this procedure microscopic information from the macroscopic information. In this case, scaling properties appear in a natural way. The spectrum of dimension techniques are suitable for application in this situation. Perhaps one reason this theory works well for a rational map f is because we can think of the inverse branches of f as a natural way to obtain new partitions. Because these inverse branches are holomorphic, we have good self-similarity properties that come from the conformality and from the Koebe Distortion Theorem (see [8]).

Here is the structure of this paper. In § 1 we will introduce the main properties of ergodic theory and large deviations that we will use. In § 2 we will present the main theorem and give an outline of the proof. In § 3 we will give the formal proof of the main theorem.

1. Ergodic theory and large deviation. Let $M(f)$ be the set of invariant probabilities for f, that is, the set of measures v such that $v(f^{-1}(A)) = v(A)$ for any set A in the Borel sigma-field of $\mathbf{R}^2$. The support of all these measures is J.

DEFINITION 1. For a Hölder continuous $g: J \to \mathbf{R}$ and $v \in M(f)$, we will define the pressure of v with respect to g by

$$h(v) + \int g(z)\, dv(z)$$

where $h(v)$ denotes the entropy of v. We will denote such an expression by $P(v, g)$.

DEFINITION 2. We will call $P(g) = \sup\{P(v, g) | v \in M(f)\}$ the topological pressure of the function g.

In the case where f is hyperbolic, there exists a unique measure that attains such supremum. This measure is ergodic. These measures are sometimes called Gibbs measures [3], [29], [30]. There exist examples of C^k maps such that this supremum is not attained (see references in [19], [33]).

DEFINITION 3. In the case where there exists a unique probability in $M(f)$, denoted by $v(g)$, such that $P(g) = h(v(g)) + \int g(z) d(v(g))(z)$ we will call this measure the maximal pressure measure for $g: J \to \mathbf{R}$. When f is hyperbolic, this is always the case [3], [28].

DEFINITION 4. For g constant and equal to zero, the maximal pressure measure is called the maximal measure.

Let z_0 be a point in the Riemann sphere, and for each $n \in \mathbf{N}$, let us denote by $z(n, i, z_0)$, $i \in \{1, 2, \cdots, d^n\}$ the d^n-solutions (with multiplicity) of the equation

$$f^n(z) = z_0.$$

We denote the delta Dirac measure on z by $\delta(z)$.

Let $u(n, z_0)$ be the probability

$$d^{-n} \sum_{i=1}^{d^n} \delta(z(n, i, z_0)).$$

In [8] and [17], it has been shown that for any z_0 (but at most two exceptional points), and independent of z_0, there exists the weak limit

$$\lim_{n \to \infty} u(n, z_0) = u,$$

and the measure u is the maximal measure of the rational map f. Hyperbolicity is not assumed to obtain this result. Also, u is ergodic and has entropy $\log d$. We will denote z_i^n the $z(n, i, z_0)$ for a certain fixed z_0.

DEFINITION 5. For any real $t \in \mathbf{R}$ we will denote $P(t) = P(g)$, when $g(z) = -t \log |f'(z)|$.

From [23] and [24] it is known that $P(t)$ is convex and real analytic in the variable t when f is hyperbolic.

DEFINITION 6. For a given probability v we will call the Hausdorff dimension of the measure v, denoted by $\mathrm{HD}(v)$, the value $\inf\{\mathrm{HD}(A) | v(A) = 1,\ A \text{ a Borel set in } J\}$. Here $\mathrm{HD}(A)$ is the Hausdorff dimension of the set A.

DEFINITION 7. For any real $t \in \mathbf{R}$ we will denote $u(t)$ as the maximal pressure measure for $g(z) = -t \log |f'(z)|$.

It also follows from [20], [22], and [24] that if f is hyperbolic, then

$$P'(t) = -\int \log |f'(z)| d(u(t))(z) = -h(u(t)) \cdot (\mathrm{HD}(u(t)))^{-1}.$$

THEOREM 1 [20]. *Let f be a rational map and let $v \in M(f)$ be an ergodic probability; then there exists a Borel set A such that $v(A) = 1$, and for all $z \in A$,*

$$\lim_{r \to 0} \frac{\log v(B(z, r))}{\log r} = h(v) \left(\int \log |f'(z)|\, dv(z) \right)^{-1} = \mathrm{HD}(v)$$

where $B(z, r)$ denotes the disk of radius r and center at z.

THEOREM 2 [28]. *If f is a hyperbolic rational map, then*

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^{d^n} |(f^n)'(z_i^n)|^{-t}.$$

In § 2 we will explain why we need the pressure in this formulation.

Let $W = \{W_n : n = 1, 2, 3, \cdots\}$ be a sequence of random variables that are defined on probability spaces $\{(\tau_n, \mathcal{F}_n, P_n), n = 1, 2, \cdots\}$ and that take values in $\mathbf{R}$, where τ_n is a set, $\mathcal{F}_n$ a σ-field, and P_n a probability.

Here we will consider $\tau_n = J$, and $\mathcal{F}_n$ the Borel σ-field on J, $n \in \mathbf{N}$.

DEFINITION 8. For each $n \in \mathbf{N}$ define

$$c_n(t) = n^{-1} \log E_n\{\exp tW_n\}$$

where E_n is the expected value with respect to P_n.

We will consider in this case the weak topology in the space of signed-measures in J.

The following hypotheses are assumed to hold:

(a) Each function $c_n(t)$ is finite for all $t \in \mathbf{R}$.

(b) $c(t) = \lim_{n\to\infty} c_n(t)$ exists and is finite for all $t \in \mathbf{R}$.

(c) $c(t)$ is differentiable as a function of $t \in \mathbf{R}$.

THEOREM 3 [7]. *Assume hypotheses* (a), (b), *and* (c) *hold, and denote for each compact Borel set K in E*

$$Q_n(K) = P_n\{z \in J \mid n^{-1} W_n \in K\} \quad \text{and} \quad I(z) = \sup_{t\in\mathbf{R}} \{zt - c(t)\}, \quad z \in \mathbf{R}.$$

Then the following conclusion holds:

$$\lim_{n\to\infty} n^{-1} \log Q_n(K) = -\inf_{z\in K} \{I(z)\}.$$

DEFINITION 9. The function $c(t)$ is called the free energy of W_n.

DEFINITION 10. The function $I(z)$ is called the deviation function of the process [7]. In fact it is the Legendre transform of $c(t)$. The function $I(z)$ contains information about the deviations of the mean of the process.

2. An outline of the proof of the main theorem. Some ideas presented here were adapted from ideas in [5] and [27].

It follows from [8] that in the hyperbolic case, for any $z \in J$ (this is not a u-almost-everywhere statement), there exists

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} n^{-1} \log u(B(z, n, \varepsilon)) = -\log d$$

where $B(z, n, \varepsilon) = \{y \in J \mid j \in \{0, 1, \cdots, n-1\}, |f^j(y) - f^j(z)| < \varepsilon\}$.

It is also true in the hyperbolic case that the diameter $d(z, \nu, \varepsilon)$ of $B(z, n, \varepsilon)$ is of the order $|(f^n)'(z)|^{-1}$, for any $z \in J$, for n large and ε small [8], [22].

Therefore, if we ask whether z is such that $u(B(z, \xi)) \approx \xi^\alpha$, it is natural to consider the above definition.

DEFINITION 11. Let $J(\alpha)$ be the set of points $z \in J(f)$ such that there exists the limit

$$\lim_{n\to\infty} n^{-1} \log |(f^n)'(z)|^{-\alpha} = -\log d.$$

In this case, $u(B(z, n, \varepsilon))$ is of order $d(z, n, \varepsilon)^\alpha$. When we use arguments of [20] it follows that this is equivalent to requiring that z satisfies

$$\lim_{\xi\to 0} \frac{\log u(B(z, \xi))}{\log \xi} = \alpha.$$

DEFINITION 12. Let $\mathfrak{f}(\alpha)$ be the Hausdorff dimension of the set $J(\alpha)$.

THEOREM 4. *Suppose f is a hyperbolic rational map and u the maximal measure. Then for a given α, there exists a unique* $t \in \mathbf{R}$ *such that* $P'(t) = -\log d/\alpha$, *and* $\mathfrak{f}(\alpha) =$ HD$(u(t))$, *where* $u(t)$ *is the maximal pressure measure for* $-t \log |f'(z)|$. *The function* $\mathfrak{f}$ *is real analytic on* α.

We will now give an outline of the proof of Theorem 4 as we mentioned in the Introduction.

The proof is divided in two parts; this is very characteristic of large deviation results [7]. We must deal with the lower bound and the upper bound in separate cases.

In the first part, we show $f(\alpha) \geqq$ HD$(u(t))$, where t satisfies a Legendre condition of the form $P'(t) = -\log d/\alpha$. This part can be seen as an application of the formula

$\mathrm{HD}(v) = h(v)/\int \log|f'(z)|dv(z)$ (that is true for any invariant measure v [20], [22]) and the Manning-McCluskey picture, which means in our case that $\mathfrak{f}(\alpha)$ is the Legendre transform of the pressure [24].

The pressure contains information about $u(t)$ in the form

$$P'(t) = -\int \log|f'(z)|d(u(t))(z).$$

This information is about the Lyapunov number of $u(t)$. Using this information we obtain a set with dimension $\mathrm{HD}(u(t))$ such that for any point z on it, the measure u scales with exponent α in z. This set is the support of the measure $u(t)$. In this way we show $\mathfrak{f}(\alpha) \geqq \mathrm{HD}(u(t))$.

Now, in the second part, it is more difficult to show that $\mathfrak{f}(\alpha) \leqq \mathrm{HD}(u(t))$.

We will try to give a heuristic idea of the proof, even under the risk of oversimplifying some more difficult and subtle parts of the demonstration. First, to have a geometrical picture of the problem, consider for simplification $f(z) = z^2 + \xi z$, when ξ is small. In this case $d = 2$. Note that zero is a fixed point of f. The main ideas of the proof are presented in this simplified case. The Julia set in this situation is a nowhere-differentiable Jordan curve. This curve is very close to the unitary circle and the dynamics of f is very similar to that of z^2 on the unitary circle (they are in fact topologically conjugated). Now consider a nonself-intersecting curve γ_1^0, from zero to ∞, cutting the Julia set in the unique fixed point in this set. Taking pre-images of this curve, we obtain the new curves γ_1^1 and γ_2^1. The Julia set without these two curves has two connected components denoted by A_1^1 and A_2^1, each one with u-measure $d^{-1} = \frac{1}{2}$.

Now consider γ_1^2, γ_2^2, γ_3^2, and γ_4^2, the pre-images of the curves γ_1^1 and γ_2^1. Now the Julia set without these four curves has four connected components denoted by A_1^2, A_2^2, A_3^2, and A_4^2. Each of these components has measure $d^{-2} = 2^{-2}$. Repeating the procedure inductively, we obtain at level n, a total of $d^n = 2^n$ curves $\gamma_1^n, \gamma_2^n, \cdots, \gamma_{2^n}^n$. The Julia set without these 2^n curves has 2^n connected components denoted by $A_1^n, A_2^n, \cdots, A_{2^n}^n$, each one with u-measure $2^{-n} = d^{-n}$. If we select an initial point z_0 not in γ_1^0, then we can suppose that in each A_i^n, $i \in \{1, 2, \cdots, 2^n\}$ there exists one and only one $z(n, i, z_0)$ (see the notation in § 1).

Now we look at level n, which has the elements of the partition $A_1^n, A_2^n, \cdots, A_{2^n}^n$ that contains elements z such that $|f^{n'}(z)|^{-\alpha}$ is of order d^{-n}. By the Koebe Distortion Theorem (see [8]) we conclude (in fact, we have to consider subsets of the A_i^n, $i \in \{1, \cdots, 2^n\}$, but we do not want to be too technical here in § 2) that if A_i^n contains a z such as the one above, then the $z(n, i, z_0)$ contained in A_i^n also has this property.

Note that from the Birkhoff Ergodic Theorem (concerning mean values) and the Shannon-McMillan-Breiman Theorem (about entropy of partitions), almost all the $z(n, i, z_0)$ should satisfy

$$|f^{n'}(z(n, i, z_0))|^{-\mathrm{HD}(u)}$$

and be of order d^{-n}. This is a simplified way to look at the formula

$$\mathrm{HD}(u) = \frac{h(u)}{\int \log|f'(z)|\, du(z)}.$$

Therefore, the large deviation here appears to give information on how many elements $z(n, i, z_0)$, $i \in \{1, 2, \cdots, 2^n\}$ deviate from the mean and satisfy that $|(f^n)'(z(n, i, z_0))|^{-\alpha}$ is of order d^{-n}.

Here it becomes clear why we must consider the pressure $P(t)$ in the formulation given by Theorem 2. We must consider the random variable given by $-\log|f^{n'}(z)|$ in the pre-orbits of z_0 at level n. At this moment the close relation of $c(t)$ and $P(t)$, which we will explain in § 3, is essential.

The diameter of each element A_i^n of the partition is of order $|f^{n'}(z(n,i,z_0))|^{-1}$, where $z(n,i,z_0)$ is the only pre-image of z_0 at level n in A_i^n.

From the considerations above, we can cover the set of points that scale with exponent α with a controlled number of elements of the partition, and we also have control of the diameter of the elements of the partition that we are using to cover the set $J(\alpha)$. This partition can be obtained with a diameter as small as we want. The value $\mathrm{HD}(u(t))$ (it appears here as information that comes from a Legendre transform) is exactly the value that we must consider for the Hausdorff measure to be finite. In this way we prove finally that $\mathfrak{f}(\alpha) \leqq \mathrm{HD}(u(t))$.

The above explanation is not exactly as the proof will be done, but it gives a good idea of the main ingredients of the demonstration.

3. Proof of the main theorem. Here we will show the proof of the following theorem.

THEOREM. *Suppose f is a hyperbolic rational map and u is the measure of maximal entropy. Then for a given α, there exists a unique $t \in \mathbf{R}$ such that $P'(t) = -\log d/\alpha$ and $\mathfrak{f}(\alpha) = \mathrm{HD}(u(t))$, where $u(t)$ is the maximal pressure measure for $-t\log|f'(z)|$. The function $\mathfrak{f}$ is real analytic in the variable α.*

Proof. (a) $\mathfrak{f}(\alpha) \geqq HD(u(t))$.

For a given α, from the convexity and analyticity of $P(t)$ (see [28]), we have that there exists a unique t such that $P'(t) = -\log d/\alpha$. For this value of t, consider $u(t)$ the maximal pressure measure for the function $g = -t\log|f'|$. From the ergodic theorem we have that for a set A such that $u(t)(A) = 1$, for all $z \in A$,

$$\lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|^{-1} = -\int \log|f'(z)|\,d(u(t))(z) = P'(t) = -\frac{\log d}{\alpha}.$$

Therefore

$$\lim_{n\to\infty} n^{-1} \log|f^{n'}(z)|^{-\alpha} = -\log d$$

and $A \subset J(\alpha)$.

As the Hausdorff dimension of $u(t)$ is infimum of the Hausdorff dimension of all sets of measure zero, we have $\mathfrak{f}(\alpha) \geqq \mathrm{HD}(u(t))$.

(b) $\mathfrak{f}(\alpha) \leqq \mathrm{HD}(u(t))$.

Now we will use a large deviation property as introduced in § 1.

Consider for each $n \in \mathbf{N}$ the measure $u(n, z_0)$ as defined in § 1.

We will denote z_i^n the $z(n,i,z_0)$ to make the notation simpler. To apply Theorem 3, consider $\tau_n = J$, $\mathscr{F}_n =$ Borel σ-field on J, and $P_n = u(n, z_0)$. Consider also the random variable $W_n = -\log|f^{n'}(z)|$.

From Theorem 3 we have that

$$\lim_{n\to\infty} n^{-1} \log \sum_{i=1}^{d^n} |f^{n'}(z_i^n)|^{-t} = P(t)$$

$$= \sup_{v\in M(f)} \left\{ h(v) - t\int \log|f'(z)|\,dv(z) \right\}.$$

Therefore

$$c(t)=\lim_{n\to\infty} n^{-1}\log E_n\{\exp t W_n\}$$

$$=\lim_{n\to\infty} n^{-1}\log d^{-n}\left(\sum_{i=1}^{d^n}|f^{n'}(z_i^n)|^{-t}\right)=P(t)-\log d.$$

This relation of pressure and free energy is essential for the rest of the proof.

From the differentiability with respect to t of $P(t)$ [16], [19], we have that for any $\beta\in\mathbf{R}$ and $\xi>0$

$$\lim_{n\to\infty} n^{-1}\log P_n\{n^{-1}W_n\in(\beta-\xi,\beta+\xi)\}$$

is almost equal to $-I(\beta)$, where $I(\beta)=\sup_{s\in R}\{s\beta-c(s)\}$.

Therefore for $\beta=-\log d/\alpha$, we have $I(\beta)=t\beta-c(t)$, where $c'(t)=-\log d/\alpha$.

As $P'(s)=c'(s)$ for any $s\in\mathbf{R}$, we remark that t is the same one obtained in part (a) of this proof.

Therefore, $I(\beta)=-t\int\log|f'(z)|d(u(t))(z)-h(u(t))+t\int\log|f'(z)|d(u(t))(z)+\log d=\log d-h(u(t))$.

Therefore

$$\lim_{n\to\infty}\log P_n\{n^{-1}W_n\in(-\log d\alpha^{-1}-\xi,\,-\log d\alpha^{-1}+\xi)\}$$

is approximately equal to $h(u(t))-\log d$. In this case

$$d^{-n}\#\{z_i^n|i\in\{1,\cdots,d^n\},\,n^{-1}\log|f^{n'}(z_i^n)|^{-1}\in(-\log d\alpha^{-1}-\xi,\,-\log d\alpha^{-1}+\xi)\}$$

is of order $\exp((h(u(t))-\log d)n)$, and finally

$$\#\left\{z_i^n|i\in(1,\cdots,d^n\}\text{ and }|f_i^{n'}(z_i^n)|^{-1}\in\left(\exp\left(\left(-\frac{\log d}{\alpha}-\xi\right)n\right),\,\exp\left(\left(-\frac{\log d}{\alpha}+\xi\right)n\right)\right)\right\}$$

is of order

$$(*)\qquad \exp(h(u(t))-(\log d)n)\exp(\log d^n)=\exp(h(u(t))n).$$

As mentioned in § 2, this information allows us to control the number of points with a certain deviation of the mean.

Now we will state some properties of hyperbolic rational maps that are proved in [8] and [18].

Considering perhaps a finite iterate of f, we know from [18] that there exists a curve δ containing all the critical values of f such that:

(a) $u(\delta)=0$.

(b) $\hat{X}=\mathbf{C}-\delta$ is a topological disk.

(c) There exist branches $\phi_i:\hat{X}\to\bar{\mathbf{C}}$, $i=1,\cdots,d$, of $f^{-1}|\hat{X}$ that are injective and $f(\hat{X}_i)=\hat{X}$ where $\hat{X}_i=\phi_i(\hat{X})$.

(d) The set $X=(\bigcap_{n\geqq 0}f^{-n}(\bigcap_{m\geqq 0}f^m(\hat{X}^c)))^c$ has u-measure zero, and satisfies

$$f^{-1}(X)=X.$$

(e) Set $X_i=\hat{X}_i\cap X$; then the disjoint union $X=\bigcup_{i=1}^d X_i$ is such that if $n\leqq 1$, $1\leqq i_j\leqq d$, $j=1,\cdots,n$, we have d^n sets of the form $(\bigcap_{j=1}^n f^{-j}(X_{i_j}))$. Let us denote each such set by A_i^n, $i\in\{1,\cdots,d^n\}$.

From [18] we have $u(A_i^n)=d^n$.

(f) We can suppose there exists just one z_i^n in each A_i^n because we can obtain u as $\lim_{n\to\infty} u(n, z_0)$ and this limit does not depend on z_0.

Now let us return to the proof of the theorem. First we will show that $J(\alpha)\cap X$ has dimension smaller than $\mathrm{HD}(u(t))$, where X depends on the curve δ. Then we move the curve δ a little and we obtain the same result. By the injectivity (c) we have that these $J(\alpha)\cap X$ cover $J(\alpha)$ when we consider several different disjoint curves δ, and from this it follows that $f(\alpha)\leqq \mathrm{HD}(u(t))$.

Now we will show that $J(\alpha)\cap X$ has dimension smaller than $\mathrm{HD}(u(t))$ for any curve δ. This will be obtained in the following way. Consider a conformal representation $\phi: X\to D_1$ and $X(r)=\phi^{-1}(D_r)$ (where $D_r=\{z\in \mathbf{C} \mid |z|<r\}$, $0\leqq r\leqq 1$). Consider also for each A_i^n, $i\in\{1, 2, \cdots, d^n\}$, the corresponding $A_i^n(r)$ such that $A_i^n(r)=f^{-n}(X_r)$ for some branch f^{-n} and $A_i^n(r)\subset A_i^n$, and assume $\phi(z_i^n)=0$.

We will first show $J(\alpha)\cap\{\bigcap_{n\geqq 0} f^{-n}(\hat{X}(r))\}$ has dimension smaller than $HD(u(t))$. Note that $J(\alpha)$ is invariant by f. In this case, using the same proof presented in [32] for Theorem 4 and in [25] for Theorem 1.1, we conclude that $\lim_{r\to 1}\mathrm{HD}(J(\alpha)\cap(\bigcap_{n\geqq 0} f^{-n}(\hat{X}(r))))=\mathrm{HD}(J(\alpha)\cap X)\leqq \mathrm{HD}(u(t))$. Therefore, it is enough to show that $\mathrm{HD}(J(\alpha)\cap(\bigcap_{n\geqq 0} f^{-n}(\hat{X}(r)))\leqq \mathrm{HD}(u(t))$, and we will show this now.

From the distortion theorem for univalent functions [8], there exist c_r, $C_r>0$ such that for n large enough

$$(**) \qquad c_r<|(f^n)'(t)|\,|(f^{n'})(z)|^{-1}<C_r$$

for any t, z in $A_i^n(r)$. It also follows from [8] that for any $\xi>0$, there exists $K>0$ such that for n large enough, if $D(n, i, \xi)$ is the ball of center z_i^n and radius $K|(f^n)'(z_i^n)|^{-(1-\xi)}$, then

$$(***) \qquad D(n, i, \xi)\supset A_i^n.$$

Consider $Y=J(\alpha)\cap X$. Then for each $z\in Y\cap f^{-n}(\hat{x}(r))$ such that $|(f^n)'(z)|^{-\alpha}$ is of order d^{-n} we have that z is in a certain $A_{i_j}^n(r)$ and therefore from $(**)$

$$|(f^n)'(z_i^n)|^{-\alpha} \text{ and } |(f^n)'(z)|^{-\alpha} \text{ are of order } d^{-n}.$$

The cardinal of such possible z_i^n is of the order $\exp(h(u(t))n)$ from $(*)$. It is also true that such z is in $D(n, i, \xi)$ from $(***)$.

Now let us remember some properties of $\mathrm{HD}(Y)$. From each $\mathbf{T}>0$ consider

$$\mathrm{HD}_{\mathbf{T}}(Y)=\lim_{\delta\to 0}\mathrm{HD}_{\mathbf{T},\delta}(Y)=\inf_{\substack{Y\subset UB_i\\ \operatorname{diam} B_i\leqq\delta}}\sum(\operatorname{diam} B_i)^T$$

where B_i are balls in $\mathbf{C}$.

We also know that if for all $\mathbf{T}>\mathrm{HD}(u(t))$ we have $H_{\mathbf{T}}(\mathrm{Y})$ finite, then $\mathrm{HD}(Y)\leqq \mathrm{HD}(u(t))$.

Now observe that for each $n\in\mathbf{N}$, Y is contained in $\exp(h(u(t)n))$ balls of radius $K|(f^n)'(z_i^n)|^{-(1-\xi)}$, and $|(f^n)'(z_i^n)|^{-1}$ is of order $\exp(-\log d\cdot n\cdot\alpha^{-1})$. For each n the sum

$$\begin{aligned}\sum(\operatorname{diam} D(n, i, \varepsilon))^T &\leqq K\exp(h(u(t))n)\exp(-T\log d\cdot n\cdot\alpha^{-1}(1-\varepsilon))\\ &=K\exp(h(u(t))-\mathbf{T}\log d\alpha^{-1}(1-\varepsilon)n).\end{aligned}$$

For

$$\mathbf{T} > \mathrm{HD}(u(t)) = h(u(t)) \left(\int \log |f'(t)| du(t) \right)^{-1} = -h(u(t)) P'(t)^{-1}$$
$$= h(u(t)) \cdot \alpha (\log d)^{-1}$$

we have that the above sum is uniformly bounded. As the diameter $D(n, i, \varepsilon)$ goes to zero [8] because f is hyperbolic, we have

$$\mathrm{HD}\left(J(\alpha) \cap \left(\bigcap_{n \geqq 0} f^{-n}(\hat{X}(r)) \right) \right) \leqq \mathrm{HD}(u(t))$$

and finally the theorem is proved.

The analyticity of $\mathfrak{f}(\alpha)$ follows from the analyticity of $P(t)$ [23], [28].

As we have mentioned in the Introduction, no dimensionality is lost by considering the support of $u(t)$ instead of $J(\alpha)$ because, as we have just shown, the two sets have the same Hausdorff dimension.

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