

# ERGODIC TRANSPORT THEORY AND PIECEWISE ANALYTIC SUBACTIONS FOR ANALYTIC DYNAMICS

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ABSTRACT. We consider a piecewise analytic real expanding map  $f : [0, 1] \rightarrow [0, 1]$  of degree  $d$  which preserves orientation, and a real analytic positive potential  $g : [0, 1] \rightarrow \mathbb{R}$ . We assume the map and the potential have a complex analytic extension to a neighborhood of the interval in the complex plane. We also assume  $\log g$  is well defined for this extension.

It is known in Complex Dynamics that under the above hypothesis, for the given potential  $\beta \log g$ , where  $\beta$  is a real constant, there exists a real analytic eigenfunction  $\phi_\beta$  defined on  $[0, 1]$  (with a complex analytic extension) for the Ruelle operator of  $\beta \log g$ .

Under some assumptions we show that  $\frac{1}{\beta} \log \phi_\beta$  converges and is a piecewise analytic calibrated subaction.

Our theory can be applied when  $\log g(x) = -\log f'(x)$ . In that case we relate the involution kernel to the so called scaling function.

Keywords: maximizing probability, subaction, analytic dynamics, twist condition, Ruelle operator, eigenfunction, eigenmeasure, Gibbs state, the involution kernel, ergodic transport, large deviation, turning point, scaling function.

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## 0. INTRODUCTION

We consider a piecewise real analytic expanding map  $f : [0, 1] \rightarrow [0, 1]$  of degree  $d$  which preserves orientation and a real analytic positive potential  $g : [0, 1] \rightarrow \mathbb{R}$ .

We assume the map and the potential have a complex analytic extension to a neighborhood of the interval in the complex plane. We also assume that  $\log g$  is well defined for this complex neighborhood and for the extension of  $g$ .

In our notation  $A = \log g$ , with  $A$  analytic, then we denote

$$m(A) = \max_{\nu \text{ an invariant probability for } f} \int A(x) d\nu(x),$$

and  $\mu_{\infty A}$  any probability which realizes the maximum value. Any one of these probabilities  $\mu_{\infty A}$  is called a maximizing probability for  $A$ . In general these probabilities do not necessarily give positive weight to every open set.

An important result in Complex Dynamics is the following: under the above hypothesis, for a given real analytic potential  $\beta \log g$ , where  $\beta \geq 0$  is a real constant, there exists a real analytic positive eigenfunction  $\phi_\beta$  defined on  $[0, 1]$  for the real Ruelle operator  $P_{\beta \log g}$  of the potential  $\beta \log g$  (see [55] [21] [54]). The existence of a complex analytic extension of  $g$  to the interval  $[0, 1]$  (see, for instance, section 2.5 beginning in page 96 [5], or, [54], [44]) is a key point in our proof.

We denote  $\mu_\beta$  the equilibrium state for  $\beta \log g$ . We recall that any accumulation point  $\mu_{\beta_n}$ ,  $n \rightarrow \infty$ , is a maximizing probability for the real function  $\log g$  (restricted to the interval  $[0, 1]$ , see for instance [22] [6] [17]). We will present precise definitions later.

It is known that any convergent subsequence of the equicontinuous family  $\frac{1}{\beta} \log \phi_\beta$  is a calibrated subaction (see [17]). Calibrated subactions play a very important role in the understanding of the properties of the maximizing probabilities (see [34] [17] [2]).

A pertinent question is to know if there exists a real analytic calibrated subaction? There are examples where there is no real analytic calibrated subaction (see [6]). Under what hypothesis one can find real analytic calibrated subactions? Is it possible to get piecewise real analytic calibrated subactions under some reasonable conditions? Our purpose here is to address these questions.

A natural strategy would be to consider the complex extension of  $\frac{1}{\beta} \log \phi_\beta$  to a certain complex neighborhood  $O_\beta$  of  $[0, 1]$  and then to use the criteria of normal families when  $\beta \rightarrow \infty$ . One problem we have to face in this approach is that the results in the literature concerning the existence of the eigenfunction  $\phi_\beta$  do not give a sharp information on the size of  $O_\beta$  when  $\beta$  changes. Our result shows that in general there is no uniform control of this size in the limit  $\beta \rightarrow \infty$ . This implies that the naive strategy has low chance to work.

## 1. DEFINITIONS AND STATEMENT OF THE MAIN RESULT

A calibrated subaction for  $A = \log g$  is a function  $V$  such that

$$\sup_{y \text{ such that } f(y) = x} \{ V(y) + \log g(y) - m(\log g) \} = V(x).$$

If the maximizing probability is unique the calibrated subaction is unique, up to an additive constant (see [6] (Lemme C) or [2] (Proposition 5)).

In Statistical Mechanics the parameter  $\beta \geq 0$  is associated to the inverse of the temperature. Then, one can say that the limit probability of  $\mu_\beta$ , when  $\beta \rightarrow \infty$ , corresponds to the case of the equilibrium at temperature zero (see [2], [4]). We refer the reader to [34] [22] [32] [48] [27] [36] [11] [12] and [17] for general references and definitions on Ergodic Optimization.

We recall that the Bernoulli space is the set  $\{1, 2, \dots, d\}^{\mathbb{N}} = \Sigma$ . A general element  $w$  in  $\Sigma$  is denoted by  $w = (w_0, w_1, \dots, w_n, \dots)$ .

In section 6 we will assume that  $d = 2$ .

We denote  $\hat{\Sigma}$  the set  $\Sigma \times [0, 1]$  and  $\psi_i$  indicates the  $i$ -th inverse branch of  $f$ . We also denote by  $\sigma$  the shift on  $\Sigma$ . Finally,  $\mathbb{T}^{-1}$  is the backward shift on  $\hat{\Sigma}$  given by  $\mathbb{T}^{-1}(w, x) = (\sigma(w), \psi_{w_0}(x))$ . In order to analyze the analytic properties of the dynamics of  $f$  we have to consider the underlying dynamics of the inverse branches, and, then it is natural to consider the extend system  $\mathbb{T}$  acting on  $\hat{\Sigma}$ . This kind of approach (in some sense) appears also in the study of the scaling function (see [55]).

**Definition 1.1.** Consider  $A : [0, 1] \rightarrow \mathbb{R}$  Hölder. We say that  $W : \hat{\Sigma} \rightarrow \mathbb{R}$  is a involution kernel for  $A$ , if there is a Hölder function  $A^* : \Sigma \rightarrow \mathbb{R}$  such that

$$A^*(w) = A \circ \mathbb{T}^{-1}(w, x) + W \circ \mathbb{T}^{-1}(w, x) - W(w, x).$$

We say that  $A^*$  is a dual potential of  $A$ , or, that  $A$  and  $A^*$  are in involution.

Above we denote  $A(x)$  and  $A^*(w)$  to stress the difference of the domains of each one. Note that  $A \circ \mathbb{T}^{-1}(w, x) = A(\psi_{w_0}(x))$ .

**Remark 1.1.** In order to show  $W$  is an involution kernel for  $A$  we just have to show that  $A \circ \mathbb{T}^{-1}(w, x) + W \circ \mathbb{T}^{-1}(w, x) - W(w, x)$  is continuous and just depends on  $w$  (see [2]).

Given a Hölder potential  $A = \log g$ , the existence and properties of an associated Hölder continuous involution kernel  $W$  was presented in [2], for the purpose of getting a Large Deviation Principle.

We show here the existence of  $W(w, x)$ ,  $x \in [0, 1]$ ,  $w \in \{1, \dots, d\}^{\mathbb{N}}$ , which is an analytic involution kernel for  $A(x) = \log g(x)$ , and a relation with the dual potential  $A^*(w) = (\log g)^*(w)$  defined in the Bernoulli space  $\{1, \dots, d\}^{\mathbb{N}}$ . In this case we have  $W : \{1, \dots, d\}^{\mathbb{N}} \times [0, 1] \rightarrow \mathbb{R}$ , and, by analytic we mean: for each  $w \in \{1, \dots, d\}^{\mathbb{N}}$  fixed, the function  $W(w, \cdot)$  has a complex analytic extension to a neighborhood of  $[0, 1]$ .

Here we assume that the maximizing probability for  $A$  is unique which implies the maximizing probability for  $A^*$  is also unique (see [2]). We denote by  $V^*$  the calibrated subaction for  $A^*$ .

We denote by  $I^*$  the deviation function for  $A^*$  (see [2]).

Suppose  $V$  is the limit of a subsequence  $\frac{1}{\beta_n} \log \phi_{\beta_n}$ , where  $\phi_{\beta_n}$  is an eigenfunction of the Ruelle operator for  $\beta_n A$ . Suppose  $V^*$  is obtained in an analogous way for  $A^*$ . Then, there exists  $\gamma$  such that

$$(1) \quad \gamma + V(x) = \sup_{w \in \Sigma} [W(w, x) - V^*(w) - I^*(w)].$$

This expression has interesting relations with the additive eigenvalue problem (see [5] [13])

We consider on  $\Sigma = \{0, \dots, d-1\}^{\mathbb{N}}$  the lexicographic order. We will consider, by technical reasons, the case where  $f : (0, 1) \rightarrow (0, 1)$  has positive derivative. In the

most of the cases we will consider,  $d = 2$ , in order to avoid an unnecessary heavy notation.

Following [40] we define:

**Definition 1.2.** *We say a continuous  $G : \hat{\Sigma} = \Sigma \times [0, 1] \rightarrow \mathbb{R}$  satisfies the twist condition on  $\hat{\Sigma}$ , if for any  $(a, b) \in \hat{\Sigma} = \Sigma \times [0, 1]$  and  $(a', b') \in \Sigma \times [0, 1]$ , with  $a' > a$ ,  $b' > b$ , we have*

$$(2) \quad G(a, b) + G(a', b') < G(a, b') + G(a', b).$$

**Definition 1.3.** *We say a continuous  $A : [0, 1] \rightarrow \mathbb{R}$  satisfies the twist condition, if some of its involution kernels satisfies the twist condition.*

Note that if the above is true for some involution kernel it will be also true for any involution kernel (see [40]).

We will assume the twist condition for  $W$  (sometimes called supermodular condition as in section 5.2 in [47]), which is a very natural assumption for the cost in optimization problems (see [5] and the Monge condition in [19]).

The twist condition will assure that for the lexicographic order in  $\Sigma$  (can be any lexicographic order) the multi-valuated function  $x \rightarrow w(x)$  is monotonous decreasing (to be proved later). In the case  $f$  is a two to one map (that is,  $d = 2$ ), a special point, which will be called turning point, will play an important role.

The turning point  $c$  (see fig. 1) is defined by

$$c = \sup\{x \mid w(x) = (1 w_1 w_2 \dots) \text{ for some the possible } w(x)\}.$$

All results before section 6 are for the general case of a finite  $d$ . However, our main result, which is Theorem 7.2, is for the case  $d = 2$ . It claims that:

**Theorem 1.1.** *Assume that*

- a) *the maximizing probability  $\mu_{\infty A}$  is unique,*
- b) *has support in a periodic orbit,*
- c)  *$\log g$  is twist.*

*If  $d = 2$  and the turning point  $c$  is eventually periodic for  $f$ , then the calibrated sub-action  $V : [0, 1] \rightarrow \mathbb{R}$  for the potential  $A = \log g$  is piecewise analytic, with a finite number of domains of analyticity.*

There are several examples where the hypothesis of the theorem are true (see section 7). We show that expression (1) above can be used to find explicit calibrated subactions in some cases (see Example 2 in section 7).

### Motivation and discussion on assumptions

As a motivation for the study of the above problem we mention the papers [1] [58] which consider the fat attractor. For a fixed potential  $A = \log g$  (called  $\tau$  in the notation of [1]) there exist an extra-parameter  $\lambda$ . In [1] it is shown that the boundary of this attractor is related the graph of a certain function  $u_\lambda$ . When  $\lambda \rightarrow 1$ , we have that this  $u_\lambda$  (normalized) converges to a calibrated subaction for  $A$  (see [7] [3]). One of the conjectures presented in [1], when translated to our language, claims that, if  $A$  is  $C^2$ -generic, then the  $u_\lambda$  is piecewise differentiable. The function denoted by  $S$  in [58] corresponds to the involution kernel here. The techniques we consider here, namely, duality and the involution kernel, will be used on that context in a forthcoming paper in order to understand the unstable manifold of some special points in the boundary of the attractor.

In the setting of the fat attractor [58] [1] the turning point corresponds to the projection on  $S^1$  of the intersection of certain unstable manifolds in the boundary of the attractor [41].

The theory described here can be applied when  $\log g(x) = -\log f'(x)$ . In that case we relate the involution kernel  $W$  to the scaling function (see [55] [31] [44]). The dual potential  $A^*$  of  $A = -\log f'(x)$  will be the scaling function. The dual relation, via the involution kernel, we consider here is a generalization of the relation of  $-\log f'(x)$  and the scaling function. More precisely, in this case,  $e^{W(w,x)}$ ,  $(w,x) \in \Sigma \times [0, 1]$ , coincides with the function  $|D\psi_w(x)|$  on the variables  $(w,x)$  of [44].

The twist condition (see [26]) on the involution kernel (it is a condition that depends just on  $A$ ) plays the same role in Ergodic Transport Theory than the convexity hypothesis in Aubry-Mather Theory (see [45] [16] [25] [43]). Here we will assume this hypothesis which was first considered in [37] and [40]. Examples of potentials  $A$  such that the corresponding involution kernel satisfies the twist condition appear there. The twist condition is an open property in the variation of the analytic potential  $A = \log g$  defined in a fixed open complex neighborhood of the interval  $[0, 1]$ .

It will be clear from our proof that in the case the support of the maximizing probability is not a periodic orbit (a Cantor set for instance), then, one gets an infinite number of distinct domains of analyticity. In this case the turning point will not be eventually periodic.

We point out that a main conjecture in Ergodic Optimization claims that generically (in the Hölder topology) on the potential  $A$  the maximizing probability has support in a periodic orbit (see [17] for related results). Therefore, the assumption that the maximizing probability is a periodic orbit makes sense.

We point out that in the case  $f$  reverses orientation (like  $-, -2x \pmod{1}$ ), then there is no potential  $A = \log g$  which is twist for the dynamics on  $\Sigma \times [0, 1]$ . A careful analysis (for different types of Baker maps) of when it is possible for  $A$  to be twist for a given dynamics  $f$  is presented in [40]. We will not consider this case here.

### Strategy of the proof

By compactness, for each  $x$  there exists **at least one**  $w(x)$  such that

$$\begin{aligned} \gamma + V(x) &= \sup_{w \in \Sigma} [W(w, x) - V^*(w) - I^*(w)] = \\ & [W(w(x), x) - V^*(w(x)) - I^*(w(x))]. \end{aligned}$$

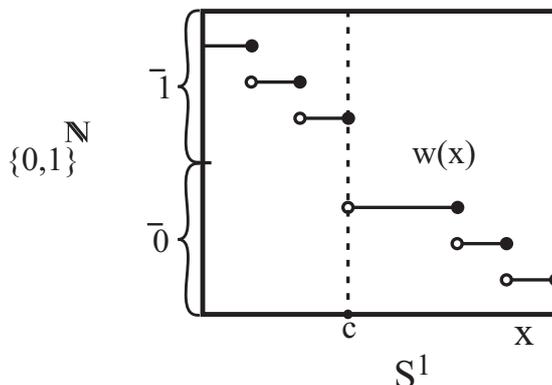
For each fixed  $w$  we will prove that  $W(w, x)$  is analytic in  $x$  (in a complex neighborhood of  $[0, 1]$ ).

As for a fixed  $w$ ,  $W(w, x)$  is analytic on  $x$  (see corollary 5.3), a result on piecewise analyticity of  $V$  is obtained if we are able to assume conditions to assure that  $w(x) \in \Sigma$  is locally constant as a function of  $x \in [0, 1]$  (up to a finite set of points  $x$ ). In some case there exist just a finite number of possible points  $w(x)$  (see fig 2).

### Section by section description of the proof

In Section 2 we present some more basic definitions and in Section 3 we show the existence of a certain function  $h_w(x) = h(w, x)$  which defines by means of

$\log(h(w, x))$  an involution kernel for  $\log g$ . In Section 4 we present some basic results in Ergodic Optimization, and, we describe the main strategy for getting the piecewise analytic sub-action  $V$ . Section 4 shows the relation of the scaling function (see [56] [31]) with the involution kernel, and, the potential  $\log g = -\log f'$ . In fact, we consider in this section a more general setting considering any given potential  $\log g$ . A main point we will need later is the proof of the analyticity on the variable  $x$  for  $w$  fixed. This is the purpose of Section 5. In Section 6 (and also 4) we consider Gibbs states for the potential  $\beta \log g$ , where  $\beta$  is a real parameter. In Section 7 and 8 we show the existence of the piecewise complex analytic calibrated sub-action. The main idea is to get the piecewise analyticity for the subaction from the analyticity of the involution kernel. We need in this moment a finiteness condition for the set optimal points. The turning point will play an essential rule in this analysis. In the end of this section an example shows that using our technique it is possible to get explicit computations and to be able to exhibit a calibrated subaction in some complicated examples.



2. ONTO ANALYTIC EXPANDING MAPS

We will consider a complex analytic extension of the real Ruelle operator  $P_{\log g}$  and general references for this topic are [50] [54] [53] page 14. We describe briefly below the extension of the Ruelle operator to an action in complex functions defined in a small neighborhood of  $I$ .

The results we state below can be found basically in [52] section 2 pages 165-167 adapted to the present situation.

Denote  $I = [0, 1]$ . We say that  $f: I \rightarrow I$  is an onto map if there exists a finite partition of  $I$  by closed intervals

$$(3) \quad \{I_i\}_{i \in \{1, 2, \dots, d\}},$$

with pairwise disjoint interiors, such that

- For each  $i$  we have that  $f(I_i) = I$ ,
- $f_i$  is monotone on each  $I_i$ .

**Definition 2.1.** *We say that  $f$  is expanding if  $f$  is  $C^1$  on each  $I_i$  and there exists  $\tilde{\lambda} > 1$  such that*

$$\inf_i \inf_{x \in I_i} |Df(x)| \geq \tilde{\lambda}.$$

Denote by

$$\psi_i: I \rightarrow I_i$$

the inverse branch of  $f$  satisfying

$$\psi_i \circ f(x) = x$$

for each  $x \in I_i$ .

We will say that an expanding onto map is analytic if there exists an simply connected, precompact open set  $O \subset \mathbb{C}$ , with  $I \subset O$ , such that, each  $\psi_i$  has a univalent extension

$$\psi_i: O \rightarrow \psi_i(O).$$

**We assume we can choose  $O$  such that**

- $\psi_i$  has a continuous extension

$$\psi_i: \bar{O} \rightarrow \mathbb{C}.$$

- We have

$$\psi_i(\bar{O}) \subset O.$$

- Moreover

$$\sup_i \sup_{x \in O} |D\psi_i(x)| \leq \lambda = \frac{\tilde{\lambda}^{-1} + 1}{2} < 1.$$

Consider a finite word

$$\gamma = (i_1, i_2, \dots, i_k),$$

where  $i_j \in \{1, 2, \dots, d\}$ . Denote  $|\gamma| = k$ . Define the univalent maps

$$\psi_\gamma: O \rightarrow \mathbb{C}$$

as

$$\psi_\gamma = \psi_{i_k} \circ \psi_{i_{k-1}} \circ \dots \circ \psi_{i_1},$$

We will denote

$$I_\gamma := \psi_\gamma(I).$$

Given either an infinite word

$$\omega = (i_1, i_2, \dots, i_k, \dots) \in \Sigma := \{1, 2, \dots, d\}^{\mathbb{N}},$$

or a finite word with  $|\omega| \geq k$ , define its  $k$ -truncation as

$$\omega_k = (i_1, i_2, \dots, i_k).$$

Note that for  $k \geq 1$

$$\psi_{\omega_k} = \psi_{i_k} \circ \psi_{\omega_{k-1}}.$$

For every finite word  $\gamma$  we can define the cylinder

$$(4) \quad C_\gamma = \{\omega \in \{1, 2, \dots, d\}^{\mathbb{N}} : \omega_{|\gamma|} = \gamma\}.$$

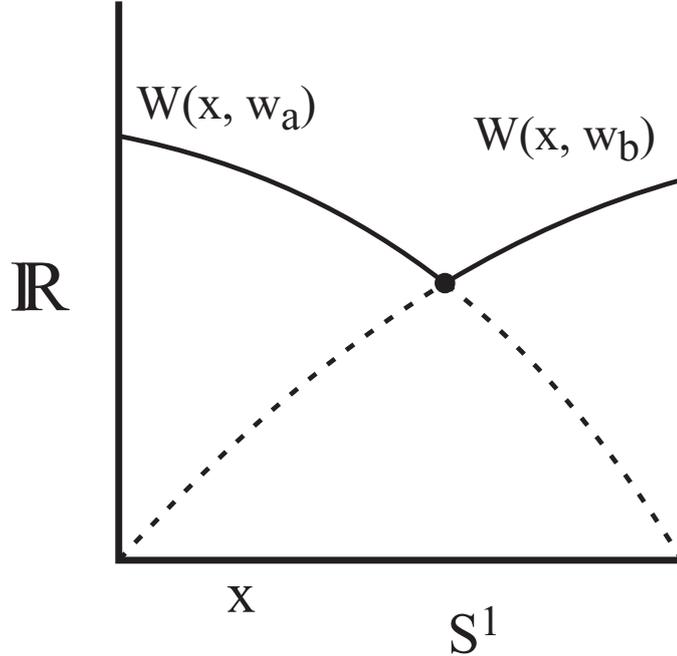


Fig 2) The graph of an specific example of a piecewise analytic subaction associated to a maximizing probability which is an orbit of period 2. It is the maximum of  $W(\cdot, w_a)$  and  $W(\cdot, w_b)$ , where  $\{w_a, w_b\} \subset \{1, 2\}^{\mathbb{N}}$  is an orbit of period 2 for the shift.

### 3. ANALYTIC POTENTIALS, SPECTRAL PROJECTIONS AND INVARIANT DENSITIES

Some of the results presented in this section extend some of the ones in [44]. We say that a function

$$g: \cup_i \text{int } I_i \rightarrow \mathbb{R}$$

is a complex analytic potential if there are complex analytic functions  $g_i: \psi_i(O) \rightarrow \mathbb{C}$  such that

- The functions  $g_i$  and  $g$  coincides in the interior of  $I_i$ .
- The functions  $g_i$  have a continuous extension to  $\overline{\psi_i(O)}$ .

- There exists  $\theta < 1$  such that

$$0 < \inf_{x \in \psi_i(O)} |g'_i(x)| \leq \sup_{x \in \psi_i(O)} |g'_i(x)| \leq \theta.$$

- We have

$$g_i(\mathbb{R} \cap \psi_i(O)) \subset \mathbb{R}^+.$$

Denote

$$\tilde{h}_i(x) = g_i(\psi_i(x)).$$

For every finite word  $\gamma$  we will define by induction on the lengths of the words the function

$$\tilde{h}_\gamma: O \rightarrow \mathbb{C}$$

in the following way: Let  $\gamma = (i_1, i_2, \dots, i_{k+1})$ . If  $|\gamma| = k + 1 = 1$  define  $\tilde{h}_\gamma(x) = g_{i_1}(\psi_{i_1}(x))$ , otherwise

$$\tilde{h}_\gamma(x) = \tilde{h}_{\gamma_k}(x) \cdot g_{i_{k+1}} \circ \psi_{\gamma_{k+1}}(x) = \tilde{h}_{\gamma_k}(x) \cdot \tilde{h}_{i_{k+1}} \circ \psi_{\gamma_k}(x).$$

As the functions we consider have complex analytic extensions, then,  $\tilde{h}_\gamma$  is complex analytic, but it is real when restricted to the interval  $I$ .

**Definition 3.1.** Define the Perron-Frobenius operator

$$P_{\log g}: C(I) \rightarrow C(I).$$

as

$$(P_{\log g} q)(x) = \sum_i \tilde{h}_i(x) q(\psi_i(x)).$$

Note that

$$(P_{\log g}^n q)(x) = \sum_{|\gamma|=n} \tilde{h}_\gamma(x) q(\psi_\gamma(x)).$$

From [50] there exists a probability  $\tilde{\mu}$ , with no atoms and whose support is  $I$ , a Hölder-continuous and positive function  $v$  and  $\alpha > 0$  such that

$$(5) \quad P_{\log g}^n v = \alpha^n v, \quad \tilde{\mu}(v) = 1,$$

and

$$\tilde{\mu}(P_{\log g}^n q) = \alpha^n \tilde{\mu}(q)$$

for every  $q \in C(I)$ . Let  $v\tilde{\mu}$  be the measure absolutely continuous with respect to  $\tilde{\mu}$  and whose Radon-Nikodym derivative with respect to  $\tilde{\mu}$  is  $v$ , that is, for every Borel set  $A$  we have

$$v\tilde{\mu}(A) = \int_A v(x) d\tilde{\mu}(x).$$

Then the probability  $v\tilde{\mu}$  is  $f$ -invariant. Let  $\omega$  be either an infinite word  $\omega = (i_1, i_2, \dots, i_k, \dots)$  or a finite word with  $|\omega| \geq k + n$ . Then

$$(6) \quad \tilde{\mu}(I_{\omega_{k+n}}) = \frac{1}{\alpha^n} \int_{I_{\omega_k}} \tilde{h}_{\omega_{n+k}-\omega_k}(x) d\tilde{\mu}(x),$$

where  $\omega_{n+k} - \omega_k$  is the word

$$(i_{k+1}, i_{k+2}, \dots, i_{k+n}).$$

The above expression is sometimes called the conformality of the probability  $\tilde{\mu}$ .

For every finite word  $\gamma$ , define

$$h_\gamma = \frac{\tilde{h}_\gamma}{\alpha^{|\gamma|} \tilde{\mu}(I_\gamma)}.$$

$h_\gamma$  is complex analytic but it is real when restricted to  $I$ .

Note that for  $|\omega| \geq k+1$

$$(7) \quad h_{\omega_{k+1}}(x) = h_{\omega_k}(x) \cdot g_{i_{k+1}} \circ \psi_{\omega_{k+1}}(x) \frac{\tilde{\mu}(I_{\omega_k})}{\alpha \tilde{\mu}(I_{\omega_{k+1}})} = h_{\omega_k}(x) \cdot \tilde{h}_{i_{k+1}} \circ \psi_{\omega_k}(x) \frac{\tilde{\mu}(I_{\omega_k})}{\alpha \tilde{\mu}(I_{\omega_{k+1}})}.$$

Let  $U \subset \mathbb{C}$  be a pre-compact open set. Consider the Banach space  $\mathcal{B}(U)$  of all complex analytic functions

$$h: U \rightarrow \mathbb{C}$$

that have a continuous extension on  $\overline{U}$ , endowed with the sup norm.

The following lemma (see theorem 2.3.2 in page 15 in [?]) is a well-known result on holomorphic functions which is very much used in complex dynamics [55] [46].

**Lemma 3.1.** *If  $U, U_1 \subset \mathbb{C}$  are relatively compact open sets such that  $\overline{U_1} \subset U$  then the inclusion  $\iota: \mathcal{B}(U) \rightarrow \mathcal{B}(U_1)$  is a compact linear operator. So every bounded sequence  $f_n \in \mathcal{B}(U)$  has a subsequence  $f_{n_i}$  such that  $f_{n_i}$  converges uniformly on  $\overline{U_1}$  to a continuous function that is complex analytic in  $U_1$ . Moreover if  $U_n$  is a sequence of open sets such that  $\overline{U_n} \subset U$  and*

$$\cup_n U_n = U,$$

*we can use a diagonal argument to show that we can find a subsequence  $f_{n_i}$  and a bounded complex analytic function  $f$  on  $U$  such that  $f_{n_i}$  converges uniformly to  $f$  on each compact subset of  $U$ .*

**Theorem 3.1.** *There exists  $K > 0$  with the following property: For every infinite word  $\omega$  the sequence  $h_{\omega_k}$  is a Cauchy sequence in  $\mathcal{B}(O)$ . Let  $h_\omega$  be its limit. For every  $\omega$  and  $x \in O$  we have*

$$\frac{1}{K} \leq |h_\omega(x)| \leq K.$$

*Proof.* Indeed since

$$\psi_{i_{k+1}}(I_{\omega_k}) = I_{\omega_{k+1}},$$

we have from the conformality property (6)

$$(8) \quad \alpha \tilde{\mu}(I_{\omega_{k+1}}) = \int_{I_{\omega_k}} g_{i_{k+1}} \circ \psi_{i_{k+1}}(y) d\tilde{\mu}(y).$$

Since  $g_i$  is analytic and

$$\text{diam } \psi_{\omega_{k+1}}(O) \leq C\lambda^{k+1},$$

by Eq. (7) we have that if  $\delta_{k,x,y}$  is defined by

$$\frac{g_{i_{k+1}} \circ \psi_{i_{k+1}}(y)}{g_{i_{k+1}} \circ \psi_{i_{k+1}}(x)} = 1 + \delta_{k,x,y},$$

then,

$$|\delta_{k,x,y}| \leq C\lambda^{k+1}.$$

for every  $x, y \in \psi_{\omega_k}(O)$ . Here  $C$  does not depend on either  $x, y \in O$ ,  $k \geq 1$ , or  $\omega$ . In particular, if  $\tilde{\delta}_{k,x}$  is defined by

$$g_{i_{k+1}} \circ \psi_{\omega_{k+1}}(x) \frac{\tilde{\mu}(I_{\omega_k})}{\alpha \tilde{\mu}(I_{\omega_{k+1}})} = 1 + \tilde{\delta}_{k,x},$$

then, by conformality of  $\tilde{\mu}$  and the usual bounded distortion argument (for instance [42] page 169)

$$|\tilde{\delta}_{k,x}| \leq C\lambda^k.$$

for  $x \in O$ . This implies that for  $m > n$ , if  $\epsilon_{n,m}$  is defined by

$$\frac{h_{\omega_m}(x)}{h_{\omega_n}(x)} = 1 + \epsilon_{n,m},$$

then,

$$(9) \quad |\epsilon_{n,m}| \leq C_1\lambda^n$$

for some  $C_1$ . Here  $C_1$  does not depend on  $x, y \in O$ ,  $k \geq 1$ , or  $\omega$ .

Let  $m_0$  large enough such that  $C_1\lambda^{m_0} < 1$ . Then

$$\inf_{y \in O, |\gamma| < m_0} |h_\gamma(y)| \prod_{k=m_0}^{\infty} (1 - C_1\lambda^k) \leq |h_{\omega_k}(x)| \leq \sup_{y \in O, |\gamma| < m_0} |h_\gamma(y)| \prod_{k=m_0}^{\infty} (1 + C_1\lambda^k)$$

for every  $x \in O$ , infinite word  $\omega$  and  $k \geq 1$ . In particular there exists  $K > 0$  such that

$$(10) \quad \frac{1}{K} \leq |h_{\omega_k}(x)| \leq K$$

for every  $k \geq 1$ ,  $x \in O$  and infinite word  $\omega$ . The family  $h_{\omega_k}$  is equicontinuous. Indeed, by estimate (9) we have that

$$\frac{h_{\omega_m}(x) - h_{\omega_n}(x)}{h_{\omega_n}(x)} = \epsilon_{n,m},$$

and by (10) we have that  $h_{\omega_n}$  is bounded above and below. Then, we conclude that  $h_{\omega_k}$  converges.

Denote

$$h_\omega = \lim_k h_{\omega_k}.$$

It follows from Eq. (10) that

$$(11) \quad \frac{1}{K} \leq |h_\omega(x)| \leq K$$

for every  $x \in O$  and infinite word  $\omega$ .

For each  $\omega$  the function  $h_\omega$  is complex analytic. It is the extension of a strictly positive real function defined on  $I$ . □

**Corollary 3.1.** *For each  $\omega \in \Sigma$  the function  $\log h_\omega(\cdot): I \rightarrow \mathbb{R}$  has a complex analytic extension to  $O$ .*

*Proof.* Since  $O$  is a simply connected open set, the functions  $h_\omega$  are complex analytic, and  $h_\omega(x) \neq 0$  for every  $x \in O$ , the result follows from the property of the normal families in Complex Analysis (see [14] Cor. 6.17). □

We use the notation  $h_\omega(x) = h(\omega, x)$ ,  $h_{\omega_k}(x) = h(\omega_k, x)$ , for  $x \in [0, 1]$  and  $\omega \in \{1, 2, \dots, d\}^{\mathbb{N}}$ , according to convenience.

For every  $\tilde{\mu}$ -integrable function  $z: I \rightarrow \mathbb{R}$  we can define the signed measure  $z\tilde{\mu}$  as

$$(z\tilde{\mu})(A) = \int_A z(x)\tilde{\mu}(x)$$

for every Borel set  $A \subset I$ .

**Theorem 3.2.** *Let*

$$z: I \rightarrow \mathbb{R}$$

*be a positive Hölder-continuous function. Then, the sequence*

$$\rho_z(x) := \lim_k \sum_{|\gamma|=k} h_\gamma(x) [(z\tilde{\mu})(I_\gamma)] = \lim_k \sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu},$$

*converges for each  $x \in O$ . This convergence is uniform on compact subsets of  $O$ . Indeed*

$$\rho_z(x) = v(x) \int z d\tilde{\mu},$$

*where  $v$  is the complex analytic extension of the function  $v$  defined in (5). Furthermore, there exists a probability  $\mu$  over the Borel sigma algebra in the space of infinite words such that*

$$(12) \quad v(x) = \rho_v(x) = \int h_\omega(x) d\mu(\omega).$$

*Proof.* Define  $\rho(k): O \rightarrow \mathbb{C}$  as

$$\rho(k)(x) := \sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu}.$$

Firstly we will prove that

$$(13) \quad \rho(k)(x) \rightarrow_k v(x) \int z d\tilde{\mu},$$

for each  $x \in I$ . Indeed for  $x \in I$

$$\begin{aligned} \sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu} &= \sum_{|\gamma|=k} h_\gamma(x) z(\psi_\gamma(x))(1 + \epsilon_{x,\gamma})\tilde{\mu}(I_\gamma) \\ &= \sum_{|\gamma|=k} h_\gamma(x) z(\psi_\gamma(x))\tilde{\mu}(I_\gamma) + \tilde{\epsilon}_{x,k} \\ &= \alpha^{-k} \sum_{|\gamma|=k} \tilde{h}_\gamma(x) z(\psi_\gamma(x)) + \tilde{\epsilon}_{x,k} \\ &= \alpha^{-k} (P_{\log g}^k z)(x) + \tilde{\epsilon}_{x,k}. \end{aligned}$$

Here,

$$|\epsilon_{x,\gamma}|, |\tilde{\epsilon}_{x,k}| \leq C\eta^k,$$

for some  $\eta < 1$ .

It is a well know fact that

$$\lim_k \alpha^{-k} (P_{\log g}^k z)(x) = v(x) \int z d\tilde{\mu}.$$

So

$$\lim_k \rho(k)(x) = v(x) \int z d\tilde{\mu}.$$

for  $x \in I$ .

Next we claim that  $\rho(k)$  converges uniformly on compact subsets of  $O$  to a complex analytic function  $\rho_z$ . Note that by Eq. (10) we have

$$\left| \sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu} \right| \leq K \sup_{x \in I} |z(x)| \sum_{|\gamma|=k} \tilde{\mu}(I_\gamma) \leq K \sup_{x \in I} |z(x)|,$$

for every  $x \in O$ , so in particular the complex analytic functions  $\rho(k)$  are uniformly bounded in  $O$ . By Lemma 3.1 every subsequence of  $\rho(k)$  has a subsequence that converges uniformly on compact subsets of  $O$  to a complex analytic function defined in  $O$ , so to prove the claim it is enough to show that every subsequence of  $\rho(k)$  that converges uniformly on compact subsets of  $O$  converges to the very same complex analytic function. Indeed we already proved that such limit functions must coincide with

$$v(x) \int z d\tilde{\mu}$$

on  $I$ . Since the limit functions are complex analytic, if they coincide on  $I$  they must coincide everywhere in  $O$ . This finishes the proof of the claim. In particular taking  $z(x) = 1$  everywhere, this proves that  $v: I \rightarrow \mathbb{R}$  has a complex analytic extension  $v: O \rightarrow \mathbb{C}$ . Consequently for every function  $z$

$$\rho_z(x) = v(x) \int z d\tilde{\mu},$$

once we already know that these functions coincide on  $I$ . For any given  $z$  we have that  $\rho_z(x) = v(x) \int z d\tilde{\mu}$  is an eigenfunction of the Ruelle operator. So we got a spectral projection in the space of eigenfunctions.

Now we will prove the second statement. Consider the unique probability  $\mu$  defined on the space of infinite words such that on the cylinders  $C_\gamma$ ,  $|\gamma| < \infty$ , it satisfies

$$\mu(C_\gamma) = (v\tilde{\mu})(I_\gamma) = \int_{I_\gamma} v d\tilde{\mu}.$$

Note that  $\mu$  extends to a measure on the space of infinite words because  $v\tilde{\mu}$  is  $f$ -invariant and it has no atoms. For each fixed  $x \in O$ , the functions  $\omega \rightarrow h_{\omega_k}(x)$  are constant on each cylinder  $C_\gamma$ ,  $|\gamma| = k$ . So

$$\int h_{\omega_k}(x) d\mu(\omega) = \sum_{|\gamma|=k} h_\gamma(x) \mu(C_\gamma).$$

By the Dominated Convergence Theorem

$$\begin{aligned} \int h_\omega(x) d\mu(\omega) &= \lim_k \int h_{\omega_k}(x) d\mu(\omega) = \lim_k \sum_{|\gamma|=k} h_\gamma(x) \mu(C_\gamma) = \\ &= \lim_k \sum_{|\gamma|=k} h_\gamma(x) (v\tilde{\mu})(I_\gamma). \end{aligned}$$

□

**Corollary 3.2.** *The function  $\rho_z = v(x) \int z d\tilde{\mu}$  is a  $\alpha$ -eigenfunction of  $P_{\log g}$*

$$P_{\log g}(\rho_z) = \alpha \cdot \rho_z.$$

Therefore, any  $\rho_z$  is an eigenfunction for the Ruelle operator for  $A = \log g$ . Later we will consider a real parameter  $\beta$  and we will denote by  $\phi_\beta(x)$  a specific normalized eigenfunction of the Ruelle operator for  $\beta \log g$ .

The two results described above are in some sense similar to the ones in [55] section 9, [44], [2]. We explain this claim in a more precise way in the next section.

The results described in this section correspond in [44] to the potential  $\log g = A = -\log f'$ .

#### 4. MAXIMIZING PROBABILITIES, THE DUAL POTENTIAL AND SCALING FUNCTIONS

From Corollary 3.2, given  $A = \beta \log g$ , there exists  $\alpha_\beta$  and  $\rho_\beta$ , such that,  $P_{\beta \log g}(\rho_\beta) = \alpha_\beta \rho_\beta$ , where  $\rho_\beta$  has a complex analytic extension to a neighborhood  $O_\beta$ . The  $\phi_{\beta A}$  is colinear with  $\rho_\beta$  and satisfies the normalization described above. Therefore, we get from Corollary 3.2 the expression

$$\rho_\beta(x) = \int h_\omega(x) d\mu(\omega).$$

Our main purpose in this section is to get the following:

**Proposition 4.1.** *For any  $\beta$  we have that  $\log h_\omega(x) = \log h_\beta(\omega, x)$  is well defined and is an involution kernel for  $\beta \log g$ . For  $\omega$  and  $\beta$  fixed, the function  $\log h_\beta(\omega, \cdot)$  has a complex analytic extension to a complex neighborhood  $O$  of  $[0, 1]$ .*

Given a finite word  $\gamma = (i_1, i_2, \dots, i_k)$ ,  $k > 1$ , define  $\sigma^*(\gamma) = (i_2, \dots, i_k)$ . For infinite words we define  $\sigma^*$  as the usual shift function. The *scaling function*  $s: \Sigma \rightarrow \mathbb{R}$  of the potential  $g$  is defined as

$$s(\omega) = \lim_{k \rightarrow \infty} \frac{\tilde{\mu}(I_{\omega_k})}{\tilde{\mu}(I_{\sigma^*(\omega_k)})}.$$

This definition is the natural generalization of the scaling function in [56] and [31]. If we take  $\log g = -\log f'$  then we get their result. It will follow from our results the existence of an involution kernel which provides a co-homology between the scaling function  $[\log(\alpha s)](\omega)$  and  $\log g(x) = -\log f'(x)$ . The constant  $\alpha$  is the eigenvalue defined before in section 1.

To verify that the above limit indeed exists, note that by Eq. (8) and since  $g$  is a Hölder-continuous function we have that

$$\frac{\tilde{\mu}(I_{\omega_{k+1}})}{\tilde{\mu}(I_{\sigma^*(\omega_{k+1})})} = \frac{\int_{I_{\omega_k}} g \circ \psi_{i_{k+1}}(y) d\tilde{\mu}(y)}{\int_{I_{\sigma^*(\omega_k)}} g \circ \psi_{i_{k+1}}(y) d\tilde{\mu}(y)} = (1 + \epsilon_k) \frac{\tilde{\mu}(I_{\omega_k})}{\tilde{\mu}(I_{\sigma^*(\omega_k)})},$$

where  $|\epsilon_k| \leq C\lambda^k$ . So  $s(\omega)$  is well defined.

Note that, since  $v > 0$  is a Hölder function and  $I_{\omega_k} \subset I_{\sigma(\omega_k)}$ ,

$$s(\omega) = \lim_{k \rightarrow \infty} \frac{(v\tilde{\mu})(I_{\omega_k})}{(v\tilde{\mu})(I_{\sigma^*(\omega_k)})} = \lim_{k \rightarrow \infty} \frac{\mu(C_{\omega_k})}{\mu(C_{\sigma^*(\omega_k)})},$$

so the the scaling function  $s$  is the Jacobian of the measure  $\mu$ .

The *dual potential*  $g^*$  is defined as

$$g^*(\omega) := \alpha s(\omega).$$

**Lemma 4.1.** *We have that*

$$\frac{g^*(\omega)}{g(\psi_{i_0}(x))} = \frac{h(\sigma(\omega), \psi_{i_0}(x))}{h(\omega, x)}.$$

*Proof.* Indeed

$$\begin{aligned} & \frac{h(\sigma(\omega), \psi_{i_0}(x))}{h(\omega, x)} \\ &= \lim_k \frac{h(\sigma(\omega_k), \psi_{i_0}(x))}{h(\omega_k, x)}. \\ & \lim_k \frac{\tilde{h}(\sigma(\omega_k), \psi_{i_0}(x))}{\tilde{h}(\omega_k, x)} \frac{\alpha^k \tilde{\mu}(I_{\omega_k})}{\alpha^{k-1} \tilde{\mu}(I_{\sigma^*(\omega_k)})} \\ &= \lim_k \frac{\alpha}{g(\psi_{i_0}(x))} \frac{\tilde{\mu}(I_{\omega_k})}{\tilde{\mu}(I_{\sigma(\omega_k)})} \\ &= \frac{\alpha}{g(\psi_{i_0}(x))} s(\omega) \end{aligned}$$

□

From the above we finally get Proposition 4.1.

## 5. ANALYTICITY OF THE INVOLUTION KERNEL

From last section we get that for each value  $\beta \geq 0$

$$\rho_{\beta A}(x) = \int e^{W_\beta(w, x)} d\nu_{\beta A^*}(w) = \int h_\beta(w, x) d\nu_{\beta A^*}(w),$$

is an eigenfunton for the Ruelle operator of the potential  $\beta \log g$ . The involution kernel  $W_\beta$  depends of the variable  $\beta$ .

**Remark:** There is a main difference from the reasoning of this section to the procedures in [2]. We will explain this. Suppose  $W_1$  is an involution kernel for  $\log g$  (that is,  $\beta = 1$ ). Therefore, given a real value  $\beta$  we have

$$\beta (\log g)^*(w) = \beta \log g \circ \mathbb{T}^{-1}(w, x) + \beta W_1 \circ \mathbb{T}^{-1}(w, x) - \beta W_1(w, x).$$

The involution kernel is not unique (see [2]). We point out that  $\log(h_\beta(w, x))$  is not necessarily equal to  $\beta W_1$ . This will require an extra work. We will need to show the existence of a  $H_\infty(w, x)$  (complex analytic on  $x$ ), such that,  $h_\beta(w, x) \sim e^{\beta H_\infty(w, x)}$  (in the sense that  $\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log h_\beta(w, x) = H_\infty(w, x)$ ). In other words, we want to replace  $W_\beta$  by a  $\beta H_\infty$  (in the notation that will be followed later).

We will show in Corollary 5.3 that for each fixed  $w$  the family  $\frac{1}{\beta} \log h_\beta(w, x)$ ,  $\beta > 0$ , is normal.

Remember that for a given  $w \in \Sigma$ , we have  $h_w = \lim_k h_{\omega_k}$ .

**Proposition 5.1.** *Let  $K \subset O$  be a compact. There exists  $C$  such that the following holds:*

- A. *For every  $\beta \geq 1$  and  $x \in K$ ,  $\omega \in \Sigma$ , we have*

$$(14) \quad e^{-\beta C} \leq |h_\beta(\omega_1, x)| \leq e^{\beta C}$$

B. For every  $\beta \geq 1$ ,  $x \in K$ ,  $\omega \in \Sigma$  and  $k \geq 1$  we have

$$(15) \quad e^{-C\beta\lambda^k} \leq \left| \frac{h_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} \right| \leq e^{C\beta\lambda^k}.$$

C. For every finite word  $\gamma$  there is a function

$$q_\gamma: \mathbb{R} \times O \rightarrow \mathbb{C},$$

that is holomorphic on  $x$ , real valued for  $x \in \mathbb{R}$  and which does not depend on  $K$ , such that for every  $x \in O$ ,  $\beta \geq 1$ ,  $\omega \in \Sigma$  we have

$$(16) \quad h_\beta(\omega_{k+1}, x) = e^{q_{\omega_{k+1}}(\beta, x)}.$$

Furthermore

$$(17) \quad |q_{\omega_1}(\beta, x)| \leq C\beta$$

and

$$(18) \quad |q_{\omega_{k+1}}(\beta, x) - q_{\omega_k}(\beta, x)| \leq C\beta\lambda^k$$

for every  $\beta \geq 1$ ,  $x \in K$ ,  $\omega \in \Sigma$  and  $k \geq 1$ .

*Proof of Claim A.* Recall that for  $i \in \{1, \dots, d\}$

$$(19) \quad h_\beta(i, x) = \frac{g_i^\beta(\psi_i(x))}{\alpha \tilde{\mu}_\beta(I_i)} = \frac{g_i^\beta(\psi_i(x))}{\int_I g_i^\beta(\psi_i(y)) \tilde{\mu}_\beta(y)},$$

so

$$|h_\beta(i, x)| = \frac{1}{\int_I \frac{g_i^\beta(\psi_i(y))}{|g_i^\beta(\psi_i(x))|} \tilde{\mu}_\beta(y)}.$$

Since  $g_i$  are holomorphic on  $\psi_i(O)$ ,  $g_i \neq 0$  in  $\psi_i(O)$ , for every compact  $K \subset O$  there exists  $C_1$  such that

$$(20) \quad e^{-C_1} \leq \frac{|g_i(\psi_i(x))|}{|g_i(\psi_i(y))|} \leq e^{C_1}$$

for every  $x, y \in K$  and  $i$ . Since  $\tilde{\mu}_\beta(I) = 1$ , it is now easy to obtain Eq. (14).  $\square$

*Proof of Claim B.* Since  $g_i$  are holomorphic on  $\psi_i(O)$ ,  $g_i \neq 0$  in  $\psi_i(O)$ , for every compact  $K \subset O$  there exists  $C_2$  such that

$$(21) \quad e^{-C_2|x-y|} \leq \left| \frac{g_i(\psi_i(x))}{g_i(\psi_i(y))} \right| \leq e^{C_2|x-y|}$$

for every  $x, y \in K$  and  $i$ . Note that every such compact is contained in a larger compact set  $\tilde{K} \subset O$  such that  $\psi_i(\tilde{K}) \subset \tilde{K}$  for every  $i$ , so we can assume that  $K$

has this property. Let  $x \in K$ . By Eq. (8)

$$\begin{aligned}
 \frac{h_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} &= \frac{\tilde{h}^\beta(\omega_{k+1}, x)}{\tilde{h}^\beta(\omega_k, x)} \frac{\alpha_\beta^k \tilde{\mu}_\beta(I_{\omega_k})}{\alpha_\beta^{k+1} \tilde{\mu}_\beta(I_{\omega_{k+1}})} \\
 &= \frac{g_{i_{k+1}}^\beta(\psi_{\omega_{k+1}}(x))}{\alpha_\beta} \frac{\tilde{\mu}_\beta(I_{\omega_k})}{\tilde{\mu}_\beta(I_{\omega_{k+1}})} \\
 &= \frac{g_{i_{k+1}}^\beta(\psi_{\omega_{k+1}}(x))}{\alpha_\beta} \frac{\alpha_\beta \tilde{\mu}_\beta(I_{\omega_k})}{\int_{I_{\omega_k}} g_{i_{k+1}}^\beta \circ \psi_{i_{k+1}}(y) d\tilde{\mu}_\beta(y)} \\
 (22) \quad &= \frac{g_{i_{k+1}}^\beta(\psi_{\omega_{k+1}}(x)) \tilde{\mu}_\beta(I_{\omega_k})}{\int_{I_{\omega_k}} g_{i_{k+1}}^\beta \circ \psi_{i_{k+1}}(y) d\tilde{\mu}_\beta(y)}
 \end{aligned}$$

$$(23) \quad = \frac{\tilde{\mu}_\beta(I_{\omega_k})}{\int_{I_{\omega_k}} \frac{g_{i_{k+1}}^\beta \circ \psi_{i_{k+1}}(y)}{g_{i_{k+1}}^\beta(\psi_{\omega_{k+1}}(x))} d\tilde{\mu}_\beta(y)}.$$

In particular

$$\left| \frac{h_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} \right| = \frac{\tilde{\mu}_\beta(I_{\omega_k})}{\int_{I_{\omega_k}} \frac{g_{i_{k+1}}^\beta \circ \psi_{i_{k+1}}(y)}{|g_{i_{k+1}}^\beta(\psi_{\omega_{k+1}}(x))|} d\tilde{\mu}_\beta(y)}.$$

For every  $y \in I_{\omega_k}$  we have

$$\psi_{i_{k+1}}(y), \psi_{\omega_{k+1}}(x) \in \psi_{\omega_{k+1}}(O)$$

From Eq. (34) we obtain

$$e^{-C\beta\lambda^k} \leq e^{-C\beta \text{diam } \psi_{\omega_{k+1}}(O)} \leq \frac{g_{i_{k+1}}^\beta \circ \psi_{i_{k+1}}(y)}{|g_{i_{k+1}}^\beta(\psi_{\omega_{k+1}}(x))|} \leq e^{C\beta \text{diam } \psi_{\omega_{k+1}}(O)} \leq e^{C\beta\lambda^k}$$

So

$$e^{-C\beta\lambda^k} \leq \frac{|h_\beta(\omega_{k+1}, x)|}{|h_\beta(\omega_k, x)|} \leq e^{C\beta\lambda^k}.$$

□

*Proof of Claim C.* Since  $g_i \circ \psi_i : O \rightarrow \mathbb{C}$  does not vanish and  $O$  is a simply connected domain, there exists a (unique) function  $r_i : O \rightarrow \mathbb{C}$  such that  $g_i \circ \psi_i = e^{r_i}$  on  $O$  and  $\text{Im } r_i(x) = 0$  for  $x \in \mathbb{R}$ . Since  $\psi_\gamma(O) \cap I \neq \emptyset$  and  $\text{diam } \psi_\gamma(O) \leq \lambda^{|\gamma|}$  we have that

$$(24) \quad |\text{Im } r_i(\psi_\gamma(x))| \leq C_3 \lambda^{|\gamma|}$$

for every  $x \in O$  and every finite word  $\gamma$ .

Define

$$q_i(\beta, x) = \beta r_i(x) + \log \frac{1}{\int_{I_i} g_i^\beta \circ \psi_i(y) d\tilde{\mu}_\beta(y)}.$$

and  $q_\gamma$ , with  $\gamma = (i_1, \dots, i_{k+1})$ , by induction on  $k$ , as

$$q_\gamma(\beta, x) = q_{\gamma_k}(\beta, x) + \beta r_{i_{k+1}}(\psi_{\gamma_k}(x)) + \log \frac{\tilde{\mu}_\beta(I_{\gamma_k})}{\int_{I_{\gamma_k}} g_{i_{k+1}}^\beta \circ \psi_{i_{k+1}}(y) d\tilde{\mu}_\beta(y)}.$$

It follows from Eq. (22) that  $q_\gamma$  satisfies Eq. (16), so

$$\operatorname{Re} q_\gamma(\beta, x) = \log |h_\beta(\gamma, x)|,$$

in particular by Eq. (14) e (15) we have

$$(25) \quad |\operatorname{Re} q_{\omega_1}(\beta, x)| \leq C_4 \beta$$

and

$$(26) \quad |\operatorname{Re} q_{\omega_{k+1}}(\beta, x) - \operatorname{Re} q_{\omega_k}(\beta, x)| \leq C_5 \beta \lambda^k$$

for  $\beta \geq 1$ . Furthermore for every  $\beta \in \mathbb{R}$ ,  $\omega \in \Sigma$  and  $k \geq 1$

$$|\operatorname{Im} q_{\omega_{k+1}}(\beta, x) - \operatorname{Im} q_{\omega_k}(\beta, x)| = |\beta \operatorname{Im} r_{i_{k+1}}(\psi_{\omega_k}(x))| \leq C_6 |\beta| \lambda^k.$$

Moreover for  $\beta > 0$  we have

$$|\operatorname{Im} q_i(\beta, x)| = |\beta| |\operatorname{Im} r_i(\psi_{\omega_k}(x))| \leq C_7 |\beta|.$$

□

For every  $x \in O$  define

$$H_{\beta,k}(\omega, x) := \frac{1}{\beta} q_{\omega_k}(\beta, x).$$

In particular, if  $x \in I$  we have that  $h_\beta(\omega_k, x)$  is a nonnegative real number by our choice of the branches  $r_i$ , so

$$H_{\beta,k}(\omega, x) = \frac{1}{\beta} \log h_\beta(\omega_k, x)$$

for  $x \in I$ . It follows from Proposition 5.1 that for every compact  $K \subset O$  there exists  $D$  such that

$$(27) \quad |H_{\beta,1}(\omega, x)| \leq D,$$

$$(28) \quad |H_{\beta,k+1}(\omega, x) - H_{\beta,k}(\omega, x)| \leq D \lambda^k$$

for  $x \in K$ , and every  $k$  and  $\omega$ . So there exists some constant  $C_8$  such that

$$|H_{\beta,k}(\omega, x)| \leq C_8$$

for every  $k, \omega, x \in K$ . This implies that the family of functions

$$\mathcal{F}_1 = \{H_{\beta,k}(\omega, \cdot)\}_{k,\omega,\beta \geq 1}$$

is a normal family on  $O$ , that is, every sequence of functions in this family admits a subsequence that converges uniformly on every compact subset of  $O$ . In Theorem 3.1 we showed that for every  $x \in I$  we have

$$\lim_k h_\beta(\omega_k, x) = h_\beta(\omega, x) > 0,$$

so

$$\lim_k H_{\beta,k}(\omega, x) = \frac{1}{\beta} \log h_\beta(\omega, x),$$

for  $x \in I$ . It follows from the normality of the family  $\mathcal{F}$  that the limit

$$H_\beta(\omega, x) := \lim_k H_{\beta,k}(\omega, x)$$

exists for every  $x \in O$  and that this limit is uniform on every compact subset of  $O$ . Moreover

$$\mathcal{F}_2 = \{H_\beta(\omega, \cdot)\}_{\omega,\beta \geq 1}$$

is also a normal family on  $O$ .

We consider in  $\Sigma$  the metric  $d$ , such that  $d(\omega, \gamma) = 2^{-n}$ , where  $n$  is the position of the first symbol in which  $\omega$  and  $\gamma$  disagree.

**Corollary 5.1.** *For every compact  $K \subset O$  there exists  $C_9$  such that*

$$(29) \quad |H_\beta(\omega, x) - H_\beta(\gamma, y)| \leq C_9|x - y| + C_9d(\omega, \gamma)$$

for every  $x, y \in K$ .

*Proof.* Since the family  $\mathcal{F}_2$  is uniformly bounded on each compact set  $K \subset O$ , we have that the family of functions

$$\mathcal{F}_3 := \{H'_\beta(\omega, \cdot)\}_{\omega, \beta \geq 1}$$

has the same property, so it is easy to see that for every compact  $K \subset O$  there exists  $C$  such that

$$|H_\beta(\omega, x) - H_\beta(\omega, y)| \leq C_{10}|x - y|.$$

Note also that Eq. (28) implies

$$|H_\beta(\omega, x) - H_\beta(\omega_k, x)| \leq C_{11}\lambda^k,$$

Let  $k + 1 = \log(d(\gamma, \omega)) / \log \lambda$ . Then  $\gamma_k = \omega_k$  and we have

$$|H_\beta(\omega, y) - H_\beta(\gamma, y)| \leq |H_\beta(\omega, y) - H_\beta(\omega_k, y)| + |H_\beta(\omega_k, y) - H_\beta(\gamma, y)| \leq C_{12}d(\omega, \gamma).$$

□

**Corollary 5.2.** *There exists a sequence  $\beta_n > 0$  satisfying  $\beta_n \rightarrow \infty$  when  $n \rightarrow \infty$  such that the limit*

$$(30) \quad H_\infty(\omega, x) = \lim_{n \rightarrow \infty} H_{\beta_n}(\omega, x),$$

exists for every  $(\omega, x)$  in

$$\{1, \dots, d\}^{\mathbb{N}} \times O.$$

Moreover for every compact  $K \subset O$  there exist  $C_{13}$  such that

$$(31) \quad |H_\infty(\omega, x) - H_\infty(\gamma, y)| \leq C_{13}|x - y| + Cd(\omega, \gamma)$$

and the limit in Eq. (30) is uniform with respect to  $(\omega, x)$  on

$$(32) \quad \{1, \dots, d\}^{\mathbb{N}} \times K$$

In particular for each  $\omega$  we have that  $x \rightarrow H_\infty(\omega, x)$  is holomorphic on  $O$ .

*Proof.* By Corollary 5.1, the family of functions  $H_\beta$  is equicontinuous on each set of the form (32), where  $K$  is a compact subset of  $O$ . So given a compact  $K \subset O$  and any sequence  $\beta_j \rightarrow +\infty$ , as  $j \rightarrow \infty$ , there is a subsequence  $\beta_{j_i}$  such that the limit

$$\lim_{i \rightarrow \infty} H_{\beta_{j_i}}(\omega, x)$$

exists and it is uniform on the set of the form (32). Then, choosing an exhaustion by compact sets of  $O$  and using Cantor's diagonal argument we can find a sequence  $\beta_n \rightarrow +\infty$  such that the limit

$$H_\infty(\omega, x) = \lim_{n \rightarrow +\infty} H_{\beta_n}(\omega, x)$$

exists and it is uniform on every set of the form (32), with compact  $K \subset O$ . Eq. (31) follows directly from Eq. (29). □

This shows the main result in this section:

**Corollary 5.3.** *For any  $w$  fixed,  $H_\infty(\omega, x)$  is analytic on  $x$ .*

From Corollary 5.2 (the convergence is uniform) and from (12)

$$\rho_v(x) = \int h_\omega(x) d\mu(\omega),$$

we get that for any  $x \in [0, 1]$

$$V(x) = \lim_{\beta \rightarrow +\infty} \frac{1}{\beta} \log \phi_{\beta_n}(x) = \sup_{w \in \Sigma} (H_\infty(w, x) - I^*(w)).$$

**Proposition 5.2.** *The function  $H_\infty(w, x)$  is an involution kernel for  $g$ .*

*Proof.* Consider  $g$  fixed. Let  $\beta_n$  be a sequence as in Corollary 5.2. For any  $\beta_n$  we have

$$\frac{(g^{\beta_n})^*(\omega)}{g^{\beta_n}(\psi_{i_0}(x))} = \frac{h_{\beta_n}(\sigma(\omega), \psi_{i_0}(x))}{h_{\beta_n}(\omega, x)}.$$

Taking  $\frac{1}{\beta_n}$  log in both sides and taking the limit  $n \rightarrow +\infty$  we get that

$$g(\mathbb{T}^{-1}(\omega, x)) + H_\infty(\mathbb{T}^{-1}(\omega, x)) - H_\infty(\omega, x)$$

depends only in the variable  $w$ .

Therefore,  $H_\infty(w, x)$  is an involution kernel (see Remark 1.1).  $\square$

## 6. A PIECEWISE ANALYTIC SUBACTION

We suppose in this section that the maximizing probability for  $A = \log g$  is unique (then the same happen for  $A^*$ , see [17]) in order we can define the deviation function  $I^*$ .

Given the analytic involution kernel  $H_\infty(w, x)$  and a fixed calibrated  $V^*$  (unique up to additive constant) define  $W(w, x) = H_\infty(w, x) + V^*(w)$ . We point out that  $W$  is also analytic on the variable  $x \in (0, 1)$  for each  $w$  fixed).

The reason for the introduction of such  $W$  (and not  $H_\infty$ ) is that, in this section, instead of

$$\gamma + V(x) = \sup_{w \in \Sigma} [H_\infty(w, x) - I^*(w)],$$

it will be more convenient the expression

$$\gamma + V(x) = \sup_{w \in \Sigma} [(W(w, x) - I^*(w)) - V^*(w)].$$

We assume without lost of generality that the above  $\gamma$  (see [2] [40]) is zero.

For each  $x$  we get one (or, more)  $w(x)$  such attains the supremum above by compactness. Therefore,

$$V(x) = W(w(x), x) - V^*(w(x)) - I^*(w(x)).$$

If there exists  $\tilde{w}$  such that for all  $x \in (a, b)$

$$V(x) = \sup_{w \in \Sigma} (H_\infty(w, x) - I^*(w)) = H_\infty(\tilde{w}, x) - I^*(\tilde{w}) = W(\tilde{w}, x) - V^*(\tilde{w}) - I^*(\tilde{w}),$$

then  $V$  is analytic on  $(a, b)$ .

Let us consider for a moment the general case ( $A$  not necessarily twist) .

We denote by  $M$  the support of  $\mu_{\infty A}^*$ .

As  $I^*$  is lower semicontinuous and  $W - V^*$  is continuous, then for each fixed  $x$ , the supremum of  $H_{\infty}(w, x) - I^*(w)$  in the variable  $w$  is achieved, and we denote (one of such  $w$ ) it by  $w(x)$ . In this case we say  $w(x)$  is optimal for  $x$ . We also say that  $(w(x), x)$  is an optimal pair of points  $x \in [0, 1], w(x) \in \{0, 1\}^{\mathbb{N}}$ . One can ask if this  $w(x)$  is independent of  $x$ , and equal to a fixed  $\tilde{w}$ . This would imply that  $V$  is analytic. If for all  $x$  in a certain open interval  $(a, b)$ , the  $w(x)$  is the same, then  $V$  is analytic in this interval. We will show under some restrictions that given any  $x$  we can find a neighborhood  $(a, b)$  of  $x$  where this is the case. The number of possible intervals can be infinite. We will give later a characterization when it is finite or infinite.

Note that given  $x$ , any optimal  $w(x)$  satisfies  $I^*(w(x))$  is finite (otherwise a  $w$  with finite  $I^*(w)$  will be better). This is a strong restriction in the set of possible  $w(x)$ , because if  $I^*(w)$  is finite, then the  $\omega$ -limit of  $w$  have to be in the support of  $\mu_{\infty A^*}$  (see section 5 [37]).

**Example 1.** *We present examples of optimal pairs.*

If  $\hat{\mu}_{max}$  is the natural extension of the maximizing probability  $\mu_{\infty A}$ , then for all  $(p^*, p)$  in the support of  $\hat{\mu}_{max}$  we have the following expression taken from Proposition 5 in [2]

$$V(p) + V^*(p^*) = W(p^*, p).$$

If  $(p^*, p)$  in the support of  $\hat{\mu}_{max}$  (then,  $p \in [0, 1]$  is in the support of  $\mu_{\infty A}$  and  $p^* \in \Sigma$  is in the support of  $\mu_{\infty A^*}$ ), then

$$V(p) = \sup_{w \in \Sigma} W(w, p) - V^*(w) - I^*(w) =$$

$$W(p^*, p) - V^*(p^*) - I^*(p^*) = W(p^*, p) - V^*(p^*).$$

Therefore,  $(p^*, p)$  is an optimal pair if  $(p^*, p)$  is in the support of  $\hat{\mu}_{max}$ . That is,  $w(p) = p^*$ .

If the potential  $\log g$  is twist, then for any given  $p$  in the support of  $\mu_{\infty A}$ , there is only one  $p^*$ , such that  $(p, p^*)$  is in the support of  $\hat{\mu}_{max}$  (see [40]) up to one orbit. If the maximizing probability for  $A$  is a periodic orbit, then the  $p^*$  associated to a  $p$  is unique.

In order to simplify the notation we assume that  $m(A^*) = 0$ .

If we denote

$$(33) \quad R^*(w) = V^* \circ \sigma(w) - V^*(w) - A^*(w),$$

then we know that  $R^* \geq 0$  because  $V^*$  is calibrated.

Note that the main result in [2] claims that the explicit expression of the deviation function is

$$(34) \quad I^*(w) = \sum_{n \geq 0} R^*(\sigma^n(w)).$$

Given  $A$ , we denote for  $x, x' \in [0, 1]$  and  $w \in \Sigma$

$$\Delta(x, x', w) = \sum_{n \geq 1} A \circ \psi_{w,n}(x) - A \circ \psi_{w,n}(x').$$

The involution kernel  $W$  can be computed for any  $(w, x)$  by  $W(w, x) = \Delta_A(x, x', w)$ , where we choose a point  $x'$  for good [CLT].

Note that for any  $x, x', w$ , we have that  $W(w, x) - W(w, x'') = \Delta(w, x, x'')$ .

Given  $A$ , suppose  $R$  satisfies

$$R(x) = V \circ f(x) - V(x) - A(x),$$

where  $V$  is a calibrated subaction. Consider a fixed involution kernel  $W$ . The next result (which does not assume the twist condition) claims that the dual of  $R$  is  $R^*$ , and the corresponding involution kernel is  $(V^* + V - W)$ .

**Proposition 6.1.** (*Fundamental Relation*)(FR)

$$R(\psi_w(x)) = (V^* + V - W)(w, x) - (V^* + V - W)(\sigma(w), \psi_w(x)) + R^*(w).$$

*Proof.* As  $R^*(w) = V^*(\sigma(w)) - V^*(w) - A^*(w)$ , we get

$$V^*(w) - V^*(\sigma(w)) + R^*(w) = -A^*(w),$$

and, now using  $x = f(\psi_w x)$ , we get

$$V(x) - V(\psi_w x) = V(f(\psi_w x)) - V(\psi_w x) - A(\psi_w x) + A(\psi_w x) = R(\psi_w x) + A(\psi_w x).$$

Substituting the above in the previous equation we get

$$\begin{aligned} (V^* + V - W)(x, w) - (V^* + V - W)(\psi_w x, \sigma(w)) + R^*(w) &= \\ [V^*(w) - V^*(\sigma(w)) + R^*(w)] + [V(x) - V(\psi_w x)] - W(x, w) + W(\psi_w x, \sigma(w)) &= \\ -A^*(w) + R(\psi_w x) + A(\psi_w x) + W(\psi_w x, \sigma(w)) - W(x, w) &= \\ R(\psi_w x), \end{aligned}$$

because  $A^*(w) = A(\psi_w x) + W(\psi_w x, \sigma(w)) - W(x, w)$ . So the claim follows.  $\square$

Note that  $R \geq 0$ , because  $V$  is a calibrated subaction.

Note also that given  $w = (w_0, w_1, \dots)$ , then,  $\psi_w(x)$  depends only of  $w_0$ . We can use either notation  $\psi_w(x)$ , or  $\psi_{w_0}(x)$ .

We know that the calibrated subaction satisfies

$$V(x) = \max_{w \in \Sigma} (-V^* - I^* + W)(w, x).$$

Then, we define

$$b(w, x) = (V^* + V + I^* - W)(w, x) \geq 0,$$

and,

$$\Gamma_V = \{(w, x) \in \Sigma \times [0, 1] \mid V(x) = (-V^* - I^* + W)(w, x)\},$$

which can be written in an equivalent form

$$\Gamma_V = \{(w, x) \in \Sigma \times [0, 1] \mid b(w, x) = 0\}.$$

**Remark 6.1.** *Note, that  $b(w, x) = 0$ , if and only if,  $(w, x)$  is an optimal pair.*

Using  $R^*(w) = I^*(w) - I^*(\sigma(w))$  (it follows from (34)), the FR becomes  
 $R(\psi_w x) = (V^* + V - W)(w, x) - (V^* + V - W)(\sigma(w), \psi_w(x)) + I^*(w) - I^*(\sigma(w))$ ,  
 or

$$R(\psi_w x) = b(w, x) - b(\sigma(w), \psi_w(x)) \quad (\text{FR1}).$$

From this main equation we get:

**Lemma 6.1.** *If  $\mathbb{T}^{-1}(w, x) = (\sigma(w), \psi_w(x))$ , then*

- a)  $b - b \circ \mathbb{T}^{-1}(w, x) = R(\psi_w x)$ ;
- b) *The function  $b$  it is non-decreasing in the trajectories of  $\mathbb{T}$ ;*
- c)  $\Gamma_V$  *is backward invariant;*
- d) *when  $(w, x)$  is optimal then  $R(\psi_w(x)) = 0$ .*

**Proof:** see [18].

In this way  $\mathbb{T}^{-n}$  spread optimal pairs.

As  $R \geq 0$ , then the function  $b$  is a kind of Lyapunov function for the iteration of  $\mathbb{T}^{-1}$ .

From now on we assume  $d = 2$ .

It is known that if  $A$  is twist, then  $x \rightarrow w_x$  (can be multi-valuated) is monotonous non-increasing (see [5] [37] [18]). We recall the proof:

**Proposition 6.2.** *If  $A$  is twist, then  $x \rightarrow w_x$  is monotonous non-increasing.*

*Proof.* Suppose  $x < x'$ , and, that  $(w_x, x), (w_{x'}, x')$  are two optimal pairs. We will show that  $w_x \geq w_{x'}$ .

Indeed, as

$$V(x) = \sup_{w \in \Sigma} (W(w, x) - V^*(w) - I^*(w)) = W(w_x, x) - V^*(w_x) - I^*(w_x),$$

then

$$W(w, x) - V^*(w) - I^*(w) \leq W(w_x, x) - V^*(w_x) - I^*(w_x), \quad (*)$$

for any  $w$ , and we also have that

$$V(x') = \sup_{w \in \Sigma} (W(w, x') - V^*(w) - I^*(w)) = W(w_{x'}, x') - V^*(w_{x'}) - I^*(w_{x'}).$$

Therefore,

$$W(w, x') - V^*(w) - I^*(w) \leq W(w_{x'}, x') - V^*(w_{x'}) - I^*(w_{x'}), \quad (**)$$

for any  $w$ .

Suppose,  $x < x'$ . Substituting  $w_{x'}$  in the first expression (\*), and  $w_x$  in the second one (\*\*) we get

$$\Delta(x, x', w_{x'}) \leq \Delta(x, x', w_x),$$

where  $W(x, w) - W(x', w) = \Delta(x, x', w)$ . So the twist property implies that  $w_{x'} \leq w_x$ . □

We showed before that the twist property implies that for  $x < x'$ , if  $b(w, x) = 0$  and  $b(w', x') = 0$ , then  $w' < w$ , which means that the optimal sequences are monotonous non-increasing. Remember, that we define the “turning point  $c$ ” as being the maximum of the point  $x$  that has his optimal sequence starting in 1:

$$c = \sup\{x \mid b(w, x) = 0 \Rightarrow w = (1 w_1 w_2 \dots)\}.$$

The main criteria is the following:

“If  $x \in [0, 1]$  has the optimal sequence  $w = (w_0 w_1 w_2 \dots)$  then

$$w_0 = \begin{cases} 1, & \text{if } x \in [0, c] \\ 0, & \text{if } x \in (c, 1] \end{cases}$$

Starting from  $(x^0, w^0)$  we can iterate FR1 by  $T^{-n}(w, x) = (w^n, x^n)$  in order to obtain new points  $w^1, w^2 \dots \in \Sigma$ . Unless the only possible optimal point  $w(x)$ , for all  $x$ , is a fixed point for  $\sigma$ , then,  $0 < c < 1$ .

Note that for  $c$  there are two optimal pairs  $(w, c)$  and  $(w', c)$ , where the first symbol of  $w$  is zero, and, the first symbol of  $w'$  is one.

The next lemma shows an interesting property of optimal pairs. If the maximizing measure for  $A$  is supported in a periodic orbit, then the optimal pair  $(w_p, p)$ , for such points  $p$  in the periodic orbit, could not be unique (that is, there exists more the one  $w_p$  for a fixed  $p$ ). This can happen (and there examples) in the case the turning point  $c$  belongs to the pre-image of the maximizing periodic orbit.

**Lemma 6.2.** *If  $A$  satisfies the twist property, then  $c$  is solution of*

$$V(\psi_1 x) + A(\psi_1 x) = V(\psi_0 x) + A(\psi_0 x).$$

*Proof.* As for  $y < c$ , we have  $b(w = (1 \dots), y) = 0$ , taking limit of  $y$  on the left side of  $c$ , then, we have from FR1, that  $R(\psi_1 y) = 0$ . From this follows  $R(\psi_1 c) = 0$ , which means  $V(c) = V(\psi_1 c) + A(\psi_1 c)$ . Analogously, taking limit of  $y$  on the right of  $c$ , we get  $V(c) = V(\psi_0 c) + A(\psi_0 c)$ . Thus,  $V(\psi_1 c) + A(\psi_1 c) = V(\psi_1 c) + A(\psi_1 c)$ .  $\square$

A point  $x$  is called eventually periodic (or, pre-periodic), if there is  $n \neq m$ , such that,  $f^n(x) = f^m(x)$ .

**Lemma 6.3.** *(Characterization of optimal change) Let  $c \in (0, 1)$  be the turning point then, for any  $x < x'$ , such that,  $b(w, x) = 0$  and  $b(w', x') = 0$ , we have  $w \neq w'$ , if, and only if, there exists  $n \geq 0$  such that  $f^n(c) \in [x, x']$ . Moreover, if  $x, x'$  are such that  $w(x)$  and  $w(x')$  are identical until the  $n$  coordinate, then,  $f^n(c) \in (x, x')$ .*

*Proof.*

Step 0

If  $x < x' \leq c$ , then,  $w_0 = w'_0 = 1$ , else if  $c < x < x'$ , then  $w_0 = w'_0 = 0$ . Suppose  $w_0 = w'_0 = i \in \{0, 1\}$  then applying FR1 we get  $\psi_i x < \psi_i x'$  and  $b((w_1 w_2 \dots), \psi_i x) = 0$  and  $b((w'_1 w'_2 \dots), \psi_i x') = 0$ .

Step 1

If  $\psi_i x < \psi_i x' \leq c$ , then,  $w_1 = w'_1 = 1$ , else, if  $c < \psi_i x < \psi_i x'$ , then  $w_1 = w'_1 = 0$ . Otherwise, if  $\psi_1 x < c < \psi_1 x'$  we can use the monotonicity of  $f$  in each branch in order to get  $x < f(c) < x'$ . Thus

$$w_1 \neq w'_1 \Leftrightarrow x < f(c) < x'.$$

The conclusion comes by iterating this algorithm. □

**Lemma 6.4.** *The set*

$$B(w) = \{x \mid b(w, x) = 0\}$$

*is closed and connected, that is, an interval (could be a single point). More specifically, if  $B(w) = [a, b]$ , then,  $a$  and  $b$  are adherence points of the orbit of  $c$ .*

*In particular, if  $c$  is pre-periodic, then, for any non-empty  $B(w)$ , there exists  $n, m$  such that  $B(w) = [f^n(c), f^m(c)]$  (unless  $B(w)$  is of the form  $[0, b]$ , or  $[a, 1]$ ).*

*Proof.* Indeed, remember that  $\psi_i$ , for  $i = 0, 1$ , are order preserving. If,  $x < y$ , and,  $x, y \in B(w)$ , then, we claim that each  $z \in (x, y)$  satisfies  $z \in B(w)$ . Indeed, otherwise if  $\tilde{w} \neq w$  is the optimal sequence for  $z$ , we know that there is  $K > 0$  such that  $\tilde{w}_j = w_j$  for  $j = 0, \dots, k - 1$  and  $\tilde{w}_k \neq w_k$ . On the other hand

$$\psi_{k,w}x < \psi_{k,\tilde{w}}z < \psi_{k,w}y.$$

Without lost of generality suppose  $w_k = 1$  then  $\tilde{w}_k = 0$  a contradiction by twist property, analogously if  $w_k = 0$  then  $\tilde{w}_k = 1$  a contradiction again.

The closeness follows from the continuity on  $x$  of the function  $b$ : if,  $x_n \subset B(w)$ , and  $x_n \rightarrow \bar{x}$ , we observe that

$$b(w, x_n) = 0 \Leftrightarrow V(x_n) + V^*(w) + I^*(w) - W(w, x_n) = 0,$$

and this implies  $b(w, \bar{x}) = 0$ , that is,  $\bar{x} \in B(w)$ .

For the second part it is enough to see that, for each extreme of the interval, for example  $b$ , if the optimal  $w$  is not constant in the right side, for any  $\delta > 0$  there is an image of  $c$ , namely  $f^j(c) \in [b, b + \delta)$  taking  $\delta \rightarrow 0$  we get  $f^j(c) \rightarrow b^+$ . □

**Remark 6.2.** *Each set  $B(w) = [a, b]$  is such that  $a = f^n(c)$ , or,  $a$  it is accumulated by a subsequence of  $f^j(c)$  from the left side. Similar property is true for  $b$  (accumulated by the right side).*

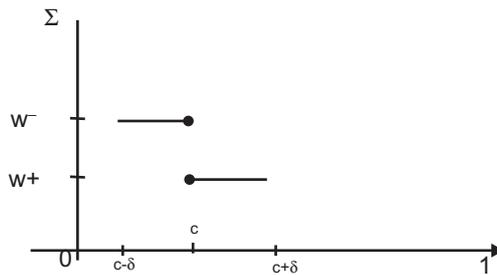


Fig. 3

**Lemma 6.5.** *Let  $c \in (0, 1)$  be the turning point. Let us suppose the  $c$  is isolated from his orbit, which means that, there is  $\delta > 0$ ,  $w^-, w^+$ , such that,  $b(w^-, x) = 0$ , for any  $x \in (c - \delta, c]$ , and,  $b(w^+, x) = 0$ , for any  $x \in [c, c + \delta)$ , then, there is no accumulation points of the orbit of  $c$ . In this case  $c$  is pre-periodic.*

*Proof.* Take  $N > 0$ , such that  $\frac{1}{2^{N-1}} < \delta$ , and, consider the sequence

$$\{c, f(c), \dots, f^{N-1}(c)\},$$

which gives an partition, which will be denoted by:  $\{I_0, I_1, \dots, I_{N-1}\}$ . Note that the points  $f^j(c)$ ,  $j = 0, \dots, N-1$ , are not order by  $j$ . A typical interval would be of the form  $I_k = (f^{j_k}, f^{j_{k+1}})$ . One of the  $I_j$  contains the point 0 in the boundary, and one contains the point 1 in the boundary. It may happen that a certain  $f^r(c) \in I_j$ , but, then  $r > N-1$ .

Since each interval  $I_j$  does not have in its interior points of the form  $f^k(c)$ ,  $k \leq N-1$ , we get from Lemma 6.3 above that:

$$b(w, x) = 0 \rightarrow w \in \overline{i_0 i_1 \dots i_{N-1}}, \forall x \in I_j,$$

where  $\overline{i_0 i_1 \dots i_{N-1}} \subset \Sigma$  denotes the cylinder with the corresponding symbols. That is, the discrepancy of the corresponding  $w$  have to be at order bigger than  $N-1$ .

If  $c$  is eventually periodic there exist just a finite number intervals  $B(w)$  with positive length. The other  $B(w)$  are reduced to points and they are also finite.

On the other hand, we claim that  $I_j = [a, b]$  can have in its interior at most one in the forward orbit of  $c$ .

Indeed, if  $I_j \cap \{f^N(c), f^{N+1}(c), \dots\} = \emptyset$ , then the optimal  $w$  will be constant and  $I_j$  of the form  $[f^n(c), f^m(c)]$ . Else, if  $f^k(c) \in I_j \cap \{f^N(c), f^{N+1}(c), \dots\} \neq \emptyset$ , for  $k \geq N$ , we denote by  $k$  the minimum one where this happens. Then, we get

$$b(w, x) = 0 \rightarrow w \in \overline{i_0 i_1 \dots i_{k-1}}, \forall x \in I_j.$$

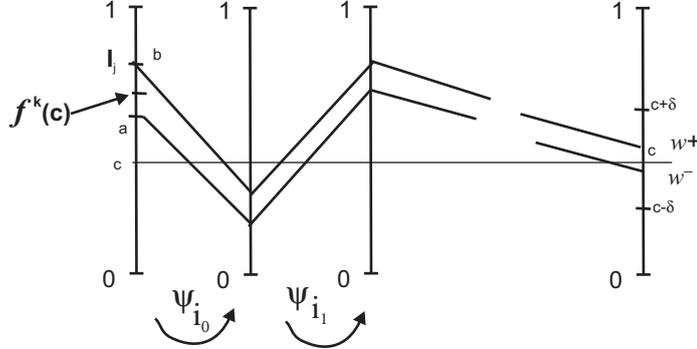


Fig. 4

If, we iterate the  $k-1$  times the FR1, then  $c \in Z_j = \psi_{i_{k-1}} \dots \psi_{i_0} I_j$ . By the choice of  $N$  we get  $Z_j \subset (c - \delta, c + \delta)$  (see Fig. 4). Dividing  $I_j = [a, f^k(c)] \cup [f^k(c), b]$  we get

$$b(w, x) = 0 \rightarrow w = (i_0 i_1 \dots i_{k-1} * w^-), \forall x \in [a, f^k(c)],$$

and,

$$b(w, x) = 0 \rightarrow w = (i_0 i_1 \dots i_{k-1} * w^+), \forall x \in [f^k(c), b].$$

Therefore, there is no room for another  $f^r(c)$ ,  $r \neq k$ , to belong to  $I_j$ . □

**Remark 6.3.** *The main problem we have to face is the possibility that the orbit of  $c$  is dense in  $[0, 1]$ .*

In the case  $f$  is  $d$  to one, we have to consider a finite number of turning points, and, similar results can also be obtained.

## 7. THE COUNTABLE AND THE GOOD CONDITIONS

We can see from last section that the subaction  $V$  will be analytic, up to a finite number of points, if and only if, the point  $c$  is eventually periodic. We would like to have sufficient conditions for this happen.

We point out that if the maximizing probability for  $A$  is a periodic orbit, then, the same happen for  $A^*$  (see [40] [2]).

Remember that a necessary condition for  $w$  to be optimal for a some  $x$  is that  $I^*(w) < \infty$ .

In [37] proposition 19 page 40 it is shown that, if  $I^*(w)$  is finite, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma^j(w)} = \mu_{\infty}^* A.$$

In principle, it can exist an uncountable number of points  $w$  such that the above limit can occur.

**Definition 7.1.** *We say a continuous  $A : [0, 1] \rightarrow \mathbb{R}$  satisfies the countable condition, if there are a countable number of possible optimal  $w(x)$ , when  $x$  ranges over the interval  $[0, 1]$ .*

We denote by  $M$  the support of the maximizing probability periodic orbit for  $A^*$ .

Consider the compact set of points  $P = \{w \in \Sigma, \text{ such that } \sigma(w) \in M, \text{ and } w \text{ is not on } M\}$ .

**Definition 7.2.** *We say that  $A$  is good, if for each  $w \in P$ , we have that  $R^*(w) > 0$ , where  $A^*$  is a dual of  $A$ .*

If  $A$  is good, according to [18], a point  $w$  satisfies  $I^*(w) < \infty$ , if, and only if,  $w$  is in the pre-image of the maximizing periodic orbit. Such set of  $w$  is countable, therefore, if  $A$  is good, then  $A$  satisfies the countable condition. The good condition, in principle, is more easy to be checked.

**Lemma 7.1.** *Suppose  $A$  satisfies the twist and the countable condition. Then there is at least one  $B(w)$  with positive length of the form  $(f^n(c), f^m(c))$ . Moreover, for any subinterval  $(a, b)$  there exists at least one  $B(w)$  with positive length of the form  $(f^n(c), f^m(c))$  inside  $(a, b)$ .*

*Proof.* Denote the possible  $w$ , such that,  $I^*(w) < \infty$ , by  $w_j$ ,  $j \in \mathbb{N}$ .

For each  $w^j$ ,  $j \in \mathbb{N}$ , denote  $I_j = B(w^j)$ , the maximal interval where for all  $x \in I_j$ , we have that,  $(x, w^j)$  is an optimal pair. Some of these intervals could be eventually a point, but, an infinite number of them have positive length, because the set  $[0, 1]$  is not countable. We consider from now on just the ones with positive length.

Note that by the same reason, in each subinterval  $(e, u)$ , there exists an infinite countable number of  $B(w)$  with positive length.

We suppose, by contradiction, that each interval  $B(w) = [a, b]$ , with positive length is such that, each side is approximated by a sub-sequence of points  $f^j(c)$ .

Take one interval  $(a_1, b_1)$  with positive length inside  $(0, 1)$ . There is another one  $(a_2, b_2)$  inside  $(0, a_1)$ , and one more  $(a_3, b_3)$  inside  $(b_1, 1)$ .

If we remove from the interval  $[0, 1]$  these three intervals we get four intervals. Using our hypothesis, we can find new intervals with positive length inside each one of them. Then we do the same removal procedure as before. This procedure is similar to the construction of the Cantor set. If we proceed inductively on this way, the set of points  $x$  which remains after infinite steps is not countable. An uncountable number of such  $x$  has a different  $w(x)$ . This is not possible because the optimal  $w(x)$  are countable.

Then, the first claim of the lemma is true.

Given an interval  $(a, b) \subset (0, 1)$ , we can do the same and use the fact that  $(a, b)$  is not countable. □

**Lemma 7.2.** *Suppose  $A$  satisfies the twist and the countable condition. If  $c$  is the turning point, then, there is  $\delta > 0$ ,  $w^-, w^+$ , such that,  $b(w^-, x) = 0$ , for any  $x \in (c - \delta, c]$ , **and**,  $b(w^+, x) = 0$ , for any  $x \in [c, c + \delta)$ .*

*That is,  $c$  is isolated of its forward orbit by both sides.*

*Proof.* If there exist just a finite number of intervals, then  $c$  is eventually periodic. We will suppose  $c$  is not eventually periodic, and, we will reach a contradiction. Therefore, if  $f^n(c) = f^m(c)$ , then  $m = n$ .

Denote by  $I_j = [a_j, b_j]$ . We denote  $I_0$  the interval of the form  $[0, b_0]$ , and,  $I_1$  the interval of the form  $[a_0, 1]$ . From last lemma, there is  $j \neq 0, 1$ , and  $n_j$  and  $m_j$ , such that  $a_j = f^{n_j}(c)$  and  $b_j = f^{m_j}(c)$ .

Suppose first that  $n_j < m_j$ .

Consider the inverse branch  $\psi_{i_1^j}$ , where  $i_1^j = i_1$  is such that  $\psi_{i_1}((f^{n_j})(c)) = f^{n_j-1}(c)$ . This  $i_1$  do not have to be the first symbol of the optimal  $w$  for  $f^{n_j}(c)$ . Then,  $\psi_{i_1}(I_j)$  is another interval, which is strictly inside a domain of injectivity of  $f$ , does not contain any forward image of  $c$ , and in its left side we have the point  $f^{n_j-1}(c)$ .

Then, repeating the same procedure inductively, we get  $i_2$ , such that

$$\psi_{i_2}((f^{n_j-1})(c)) = f^{n_j-2}(c),$$

determining another interval which does not contain any forward image of  $c$ , and in his left side we have the point  $f^{n_j-2}(c)$ . Repeating the reasoning over and over again, always taking the same inverse branch which contain  $f^n(c)$ ,  $0 \leq n \leq n_j$ , after  $n_j$  times we arrive in an interval of the form  $(c, r_j)$ . Note that each inverse branch preserves order. It is not possible to have an iterate  $f^k(c)$ ,  $k \in \mathbb{N}$ , inside this interval  $(c, r_j)$  (by the definition of  $I_j$ ). Then, the optimal  $w$  for  $x$  in this interval  $(c, r_j)$  is a certain  $\tilde{w}_j$  which can be different of  $\sigma^{n_j}(w_j)$ .

Suppose now that  $n_j > m_j$ .

Using the analogous procedure we get that there exists  $r^j$ , such that the optimal  $w(x)$  for  $x$  in the interval  $(r^j, c)$  is a certain  $\hat{w}_j$ .

If both cases happen, then  $c$  is eventually periodic.

The trouble happens when just one type of inequality is true. Suppose without loss of generality that we have always  $n_j < m_j$ .

Let's fix for good a certain  $j$ .

Therefore, all we can get with the above procedure is that  $c$  is isolated by the right side

In the procedure of taking pre-image of  $f^{n_j}(c)$ , always following the forward orbit  $f^n(x)$ ,  $0 \leq n \leq n_j$ , we will get a sequence of  $i_1, i_2, \dots, i_{n_j}$ . In the first step we have two possibilities:  $\psi_{i_1}(f^{m-j}(c)) = f^{m_j-1}(c)$ , or not.

If it happens the second case, we are done. Indeed, the interval  $\psi_{i_1}[f^{n_j}(c), f^{m_j}(c)]$  does not contain forward images of  $c$  (otherwise  $[f^{n_j}(c), f^{m_j}(c)]$  would also have). Now we follow the same procedure as before, but, this time following the branches which contains the orbit of  $f^m(c)$ ,  $0 \leq m \leq m_j$ . In this way, we get that  $c$  is isolated by the left side.

Suppose  $\psi_{i_1}(f^{m_j}(c)) = f^{m_j-1}(c)$ . Consider the interval,  $[f^{n_j-1}(c), f^{m_j-1}(c)]$ , which do not contain forward images of  $c$ .

Now, you can ask the same question:  $\psi_{i_2}(f^{m_j-1}(c)) = f^{m_j-2}(c)$ ? If this do not happen (called the second option), then, in the same way as before, we are done ( $c$  is also isolated by the right side). If the expression is true, then, we proceed with the same reasoning as before.

We proceed in an inductive way until time  $n_j$ . If in some time we have the second option, we are done, otherwise, we show that any  $x \in (c, f^{m_j-n_j}(c))$  has a unique optimal  $w(x)$  (there is no forward image of  $c$  inside it).

Denote  $k = m_j - n_j$  for the  $j$  we fixed.

From the above we have that for any  $B(w)$ , which is an interval of the form  $[f^{n_i}(c), f^{m_i}(c)]$ , for any possible  $i$ , it is true that  $m_i - n_i = k$ .

We claim that the set of points  $x$  which are extreme points of any  $B(\tilde{w})$ , and, such that  $x$  can be approximated by the forward orbit of  $c$  is finite. Suppose without loss of generality that  $x$  is the right point of a  $B(w) = (z, x)$ .

If the above happens, then, by the last lemma, applied to  $(a, b) = (x, \epsilon)$ ,  $\epsilon$  small, we have an infinite sequence of intervals of the form  $[f^{n_i}(c), f^{n_i+k}(c)]$ , such that  $f^{n_i}(c) \rightarrow x$ , as  $n_i \rightarrow \infty$ . Therefore,  $x$  is a periodic point of period  $k$ . There are a finite number of points of period  $k$ . This shows our main claim. Finally,  $c$  is eventually periodic. □

**Theorem 7.1.** *Suppose  $A$  satisfies the twist and the countable condition, that the maximizing probability is unique, and, also that it is a periodic orbit, then  $V$  is analytic, up to a finite number of points.*

*Proof.* It follows from Lemma 7.2 and Lemma 6.5. □

Therefore, we get:

**Theorem 7.2.** *Suppose  $A$  satisfies the twist condition and that the turning point is eventually periodic, that the maximizing probability is unique, and, also that it is a periodic orbit, then  $V$  is analytic, up to a finite number of points.*

The next example shows that the theory we just presented above allows one to compute, via an algorithm, the calibrated sub-action  $V$ . By this, we mean that, if

we know  $W$ , and, we have some information about the combinatorics of the position of the maximizing orbit, then, we can get the subaction  $V$ .

**Example 2.** We assume that  $m(A) = 0$  and  $T(x) = 2x \pmod{1}$ .

In this example we consider an measure supported in a periodic orbit of period 4,  $x_0 = \frac{1}{15}$  it is easy to see that in this case the optimal measure on  $\hat{\Sigma}$  is supported in:

$$\left(\frac{1}{15}, \overline{1000}\dots\right), \left(\frac{2}{15}, \overline{0100}\dots\right), \left(\frac{4}{15}, \overline{0010}\dots\right), \text{ and } \left(\frac{8}{15}, \overline{0001}\dots\right).$$

By definition the turning point  $c$  should be between  $\frac{1}{15}$  and  $\frac{2}{15}$ . Let us consider two cases:

**Case 1:** Suppose that  $c = \frac{2}{16}$ , that is a eventually periodic point, but more than that,  $c$  is a pre-image of order 4 of the fix point 1.

The orbit of  $c$ , which is given by  $c = \frac{2}{16}$ ,  $f(c) = \frac{4}{16}$ ,  $f^2(c) = \frac{8}{16}$ ,  $f^3(c) = 1$ . In this way the  $w(x)$  of the optimal pairs should be constant in the intervals:

$$I_1 = [0, c], I_2 = [c, f(c)], I_3 = [f(c), f^2(c)], I_4 = [f^2(c), f^3(c) = 1].$$

In this case the  $c$  is pre-periodic.

Note that the  $f^k(c)$  are monotonous in  $k$ . This is not always the case for other examples.

Since there is one only periodic point in each interval we get:

$$b(x, \overline{1000}\dots) = 0, \forall x \in I_1$$

$$b(x, \overline{0100}\dots) = 0, \forall x \in I_2$$

$$b(x, \overline{0010}\dots) = 0, \forall x \in I_3$$

$$b(x, \overline{0001}\dots) = 0, \forall x \in I_4.$$

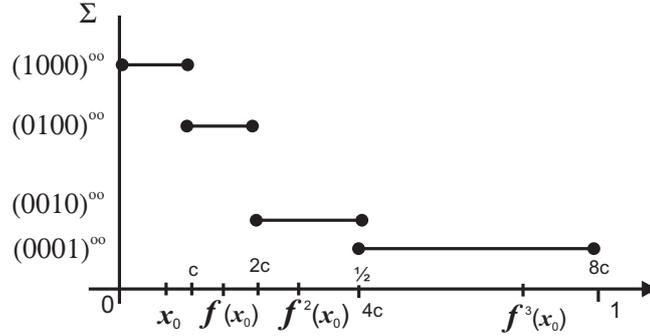


Fig. 5

Using the definition of  $b$  we get

$$V(x) = \begin{cases} -V^*(\overline{1000}\dots) - 0 + W(x, \overline{1000}\dots), & \forall x \in I_1 \\ -V^*(\overline{0100}\dots) - 0 + W(x, \overline{0100}\dots), & \forall x \in I_2 \\ -V^*(\overline{0010}\dots) - 0 + W(x, \overline{0010}\dots), & \forall x \in I_3 \\ -V^*(\overline{0001}\dots) - 0 + W(x, \overline{0001}\dots), & \forall x \in I_4 \end{cases}$$

The fundamental relation allow us to write:

$$V(x) = \begin{cases} V(\psi_{\overline{1000}\dots}x) + A(\psi_{\overline{1000}\dots}x), & \forall x \in I_1 \\ V(\psi_{\overline{0100}\dots}x) + A(\psi_{\overline{0100}\dots}x), & \forall x \in I_2 \\ V(\psi_{\overline{0010}\dots}x) + A(\psi_{\overline{0010}\dots}x), & \forall x \in I_3 \\ V(\psi_{\overline{0001}\dots}x) + A(\psi_{\overline{0001}\dots}x), & \forall x \in I_4 \end{cases}$$

In particular, applying in  $x_0$  we have:

$$V(x_0) = V(\psi_{\overline{1000}\dots}x_0) + A(\psi_{\overline{1000}\dots}x_0) = V(f^3x_0) + A(f^3x_0)$$

$$V(f(x_0)) = V(\psi_{\overline{0100}\dots}f(x_0)) + A(\psi_{\overline{0100}\dots}f(x_0)) = V(x_0) + A(x_0)$$

$$V(f^2(x_0)) = V(\psi_{\overline{0010}\dots}f^2(x_0)) + A(\psi_{\overline{0010}\dots}f^2(x_0)) = V(f(x_0)) + A(f(x_0))$$

$$V(f^3(x_0)) = V(\psi_{\overline{0001}\dots}f^3(x_0)) + A(\psi_{\overline{0001}\dots}f^3(x_0)) = V(f^2(x_0)) + A(f^2(x_0))$$

Since  $V$  is unique up to constants we can choose  $V(x_0) = 0$  for instance an solve the system finding:

$$V(x_0) = 0, \quad V(f(x_0)) = A(x_0),$$

$$V(f^2x_0) = A(x_0) - A(f^2(x_0)) \text{ and } V(f^3x_0) = -A(f^3x_0)$$

Using this results in the previous formula for  $V$  we get the values of  $V^*$ :

$$1- V(x_0) = 0 = -V^*(\overline{1000}\dots) - 0 + W(x, \overline{1000}\dots), \text{ thus}$$

$$V^*(\overline{1000}\dots) = W(x_0, \overline{1000}\dots).$$

$$2- V(f(x_0)) = A(x_0) = -V^*(\overline{0100}\dots) - 0 + W(f(x_0), \overline{0100}\dots), \text{ thus}$$

$$V^*(\overline{0100}\dots) = -A(x_0) + W(f(x_0), \overline{0100}\dots).$$

$$3- V(f^2(x_0)) = A(x_0) + A(f(x_0)) = -V^*(\overline{0010}\dots) - 0 + W(f^2(x_0), \overline{0010}\dots), \text{ thus}$$

$$V^*(\overline{0010}\dots) = -A(x_0) - A(f(x_0)) + W(f^2(x_0), \overline{0010}\dots)$$

$$4- V(f^3(x_0)) = -A(f^3x_0) = -V^*(\overline{0001}\dots) - 0 + W(f^3(x_0), \overline{0001}\dots), \text{ thus}$$

$$V^*(\overline{0001}\dots) = A(f^3x_0) + W(f^3(x_0), \overline{0001}\dots)$$

So the explicit formula for  $V$  depends on  $A$  and is given by

$$V(x) = \begin{cases} W(x, \overline{1000}\dots) - W(x_0, \overline{1000}\dots), & \forall x \in I_1 \\ A(x_0) + W(x, \overline{0100}\dots) - W(f(x_0), \overline{0100}\dots), & \forall x \in I_2 \\ A(x_0) + A(f(x_0)) + W(x, \overline{0010}\dots) - W(f^2(x_0), \overline{0010}\dots), & \forall x \in I_3 \\ -A(f^3x_0) + W(x, \overline{0001}\dots) - W(f^3(x_0), \overline{0001}\dots), & \forall x \in I_4 \end{cases}$$

Or

$$V(x) = \begin{cases} \Delta(x, x_0, \overline{1000}\dots), & \forall x \in I_1 \\ A(x_0) + \Delta(x, f(x_0), \overline{0100}\dots), & \forall x \in I_2 \\ A(x_0) + A(f(x_0)) + \Delta(x, f^2(x_0), \overline{0010}\dots), & \forall x \in I_3 \\ -A(f^3x_0) + \Delta(x, f^3(x_0), \overline{0001}\dots), & \forall x \in I_4 \end{cases}$$

8. THE OPTIMAL SOLUTION WHEN THE MAXIMIZING PROBABILITY IS NOT A PERIODIC ORBIT

We are going to analyze now the variation of the optimal point when the support of the maximizing probability is not necessarily a periodic orbit. What can be said in the general case?

Consider the subaction defined by,

$$V(x) = \sup_{w \in \Sigma} (H_\infty(w, x) - I^*(w))$$

Remember that as  $I^*$  is lower semicontinuous and  $H_\infty = W - V^*$  is continuous, then for each fixed  $x$ , the supremum of  $H_\infty(w, x) - I^*(w)$  in the variable  $w$  is achieved, and we denote (one of such  $w$ ) it by  $w(x)$ . In this case we say  $w(x)$  is an optimal point for  $x$ .

We want to show that  $w(x)$  is unique for the generic  $x$

Define the multi-valuated function  $U : [0, 1] \rightarrow \Sigma$  given by:

$$U(x) = \{w(x) | x \in [0, 1]\}$$

As graph  $(U)$  is closed in each fiber, and  $\Sigma$  is compact we can define:

$$u^+(x) = \max U(x), \text{ and } u^-(x) = \min U(x).$$

Since the potential  $A$  is twist we know that  $U$  is a monotone not-increasing multi-valuated function, that is,

$$u^-(x) \geq u^+(x + \delta),$$

when  $x < x + \delta$ . In particular are monotone not-increasing single-valuated functions.

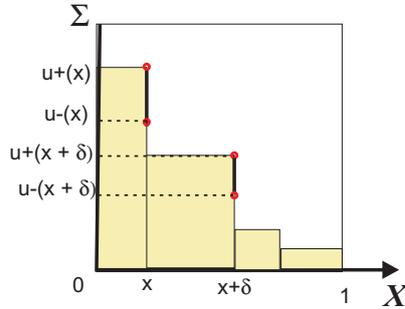


fig. 6 The graph of  $U$

We claim that  $u^+$  is left continuous. In order to conclude that, take a sequence  $x_n \rightarrow x$  on the left side. Consider, the sequence  $u^+(x_n) \in \Sigma$ , so its set of accumulation points is contained in  $U(x)$ . Indeed, suppose  $\liminf u^+(x_n) \rightarrow \tilde{w} \in \Sigma$ . In one hand, we have,  $V(x_n) = H_\infty(u^+(x_n), x_n) - I^*(u^+(x_n))$ . Taking limits on this equation and using the continuity of  $V$  and  $H_\infty$  and the lower semicontinuity of  $I^*$  we get,

$$V(x) \leq H_\infty(\tilde{w}, x) - I^*(\tilde{w}).$$

Because  $\liminf I^*(u^+(x_n)) \geq I^*(\tilde{w})$ . So  $\tilde{w} \in U(x)$ . On the other hand,  $u^+$  is monotone not-increasing, so  $u^+(x_n) \geq u^+(x)$ . From the previous we get

$$\limsup u^+(x_n) \geq u^+(x) \geq \tilde{w} = \liminf u^+(x_n),$$

that is,

$$\lim_{x_n \rightarrow x^-} u^+(x_n) = u^+(x).$$

Now consider a sequence  $x_n \rightarrow x$  on the right side. Take, the sequence  $u^+(x_n) \in \Sigma$ , so its set of accumulation points is not necessarily contained in  $U(x)$ . However it is the case. Let  $x_{n_k}$  be a subsequence such that,  $u^+(x_{n_k}) \rightarrow \tilde{w}$ .

We know that  $V(x_{n_k}) = H_\infty(u^+(x_{n_k}), x_{n_k}) - I^*(u^+(x_{n_k}))$ . Taking limits on this equation and using the uniform continuity of  $V$  and  $H_\infty$  we get

$$\begin{aligned} I^*(\tilde{w}) &\leq \liminf_{k \rightarrow \infty} I^*(u^+(x_{n_k})) = \\ &= \liminf_{k \rightarrow \infty} H_\infty(u^+(x_{n_k}), x_{n_k}) - V(x_{n_k}) = H_\infty(\tilde{w}, x) - V(x). \end{aligned}$$

In other words,  $V(x) \leq H_\infty(\tilde{w}, x) - I^*(\tilde{w})$ , that is,  $\tilde{w} \in U(x)$ . So

$$cl(u^+(x_n)) \subseteq U(x).$$

Since  $u^+$  is monotone not-increasing,  $u^+(x_n) \leq u^+(x)$ , thus

$$\limsup u^+(x_n) \leq u^+(x),$$

that is,  $u^+$  is right upper-semicontinuous.

It is known that for any USC function defined in a complete metric space the set of points of continuity is generic.

Therefore, we get that:

**Theorem 8.1.** *For a generic  $x$  we have that  $U(x) = \{u^+(x) = u^-(x)\}$  and  $w(x)$  is unique.*

*Proof.* Indeed, suppose that there is a point in the set of continuity of  $u^+(x)$  such that,  $u^+(x) > u^-(x)$  so the monotonicity of  $U$  implies that

$$u^+(x) > u^-(x) \geq u^+(x + \delta),$$

for all  $\delta > 0$ . Contradicting the continuity. □

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