

ERGODIC TRANSPORT THEORY, PERIODIC MAXIMIZING PROBABILITIES AND THE TWIST CONDITION

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ABSTRACT. The present paper is a follow up of another one by A. O. Lopes, E. Oliveira and P. Thieullen which analyze ergodic transport problems. In that paper the authors were mainly concerned with optimal pairs in the support of the optimal transport probability. Here we want to analyze optimal pairs outside the support of the optimal transport probability. Our main focus will be a more precise analysis of case where the maximizing probability is unique and is also a periodic orbit.

Consider the shift T acting on the Bernoulli space $\Sigma = \{1, 2, 3, \dots, d\}^{\mathbb{N}}$ and $A : \Sigma \rightarrow \mathbb{R}$ a Holder potential.

Denote

$$m(A) = \max_{\nu \text{ an invariant probability for } T} \int A(x) d\nu(x),$$

and, $\mu_{\infty, A}$, any probability which attains the maximum value. We assume this probability is unique (a generic property). We denote by \mathbb{T} the left-shift acting on the space of points $(w, x) \in \{1, 2, 3, \dots, d\}^{\mathbb{Z}} = \Sigma \times \Sigma = \hat{\Sigma}$. For a given potential Holder $A : \Sigma \rightarrow \mathbb{R}$, where A acts on the variable x , we say that a Holder continuous function $W : \hat{\Sigma} \rightarrow \mathbb{R}$ is a involution kernel for A (where A^* acts on the variable w), if there is a Holder function $A^* : \Sigma \rightarrow \mathbb{R}$, such that,

$$A^*(w) = A \circ \mathbb{T}^{-1}(w, x) + W \circ \mathbb{T}^{-1}(w, x) - W(w, x).$$

We say that A^* is a dual potential of A , or, that A and A^* are in involution. It is true that $m(A) = m(A^*)$. A calibrated sub-action $V : \Sigma \rightarrow \mathbb{R}$ for the potential A , is a function V such that

$$\sup_{y \text{ such that } T(y) = x} \{V(y) + A(y) - m(A)\} = V(x).$$

One can also consider V^* the calibrated subaction for A^* , and, the maximizing probability μ_{∞, A^*} for A^*

It was shown by A. O. Lopes, E. Oliveira and P. Thieullen that for any given $x \in \Sigma$, it is true the relation

$$V(x) = \sup_{w \in \Sigma} [(W(w, x) - I^*(w)) - V^*(w)],$$

where I^* is non-negative lower semicontinuous function (it can attain the value ∞ in some points) which comes from Large Deviations results of a paper by A. Baraviera, A. O. Lopes and Ph. Thieullen

This is related to the fact that the minimization transport problem for the cost $c(w, x) = -W(w, x) + I^*(w)$ has $(-V, -V^*)$, as the dual-Kantorovich pair of $(\mu_{\infty, A}, \mu_{\infty, A^*})$.

For each x we get one (or, more than one) w_x such attains the supremum above. That is, solutions of

$$V(x) = W(w_x, x) - V^*(w_x) - I^*(w_x).$$

A pair of the form (x, w_x) is called an optimal pair. If \mathbb{T} is the shift acting on $(x, w) \in \{1, 2, 3, \dots, d\}^{\mathbb{Z}}$, then, the image by \mathbb{T}^{-1} of an optimal pair is also an optimal pair.

We assume a technical condition which is crucial for results in Ergodic Transport Theory: the twist condition. When $d = 2$, a special point, called transition point, will play an important role. If this point is eventually periodic, then, the number possible optimal w_x is finite.

Our main result is:

Theorem - Generically, in the set of Holder potentials A that satisfy

- (i) the twist condition,
- (ii) uniqueness of maximizing probability which is supported in a periodic orbit,

the set of possible optimal w_x , when x covers the all range of possible elements x in Σ , is finite.

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1. INTRODUCTION

This paper is a follow up of [26]. In that paper the authors were concern, among other things, with optimal pairs of points in the support of the optimal transport probability. Here we want to analyze optimal pairs of points outside the support of the optimal transport probability. We point out that the problems in the classical transport theory do not have a dynamical meaning. It is necessary to consider a cost function with dynamical meaning in order to have the dynamics playing some role. Here, this cost function will be the involution kernel, which was previously consider in problems of Large Deviation [2]. We will assume the cost function satisfies a twist condition, sometimes called supermodular condition (see section 5.2 in[30]), which a natural hypothesis in optimization problems (see [4]).

We denote by T the action of the shift in the Bernoulli space $\{1, 2, 3, \dots, d\}^{\mathbb{N}} = \Sigma$.

We refer the reader to [24] for transport problems in continuous time with dynamical content.

In this introductory section we will describe the setting and some of the main result from [26].

We address the question of transport from a maximizing probability to another. More precisely, we would like to consider the transport problem from $\mu_{\infty, A}$ to μ_{∞, A^*} , where $A : \Sigma = \{1, 2, 3, \dots, d\}^{\mathbb{N}} \rightarrow \mathbb{R}$ is a Holder potential and A^* its dual (see [2]). We will state precise definitions soon.

Definition 1.1. *Denote*

$$m(A) = \max_{\nu \text{ an invariant probability for } T} \int A(x) d\nu(x),$$

and, $\mu_{\infty, A}$, any probability which attains the maximum value. Any one of these probabilities $\mu_{\infty, A}$ is called a maximizing probability for A .

The analysis of this kind of problem is called Ergodic Optimization Theory [22].

We denote the set of α -Holder potentials on Σ by $C^\alpha(\Sigma, \mathbb{R})$.

If A is Holder, and, the maximizing probability for A is unique, then the probability $\mu_{\infty, A}$ is the limit of the Gibbs states $\mu_{\beta A}$, for the potentials βA , when $\beta \rightarrow \infty$ [14] [15].

The metric d in Σ is defined by

$$d(\omega, \nu) := \lambda^N, \quad N := \min\{k \in \mathbb{N} \mid \omega_k \neq \nu_k\}.$$

The norm we consider in the set $C^\alpha(\Sigma, \mathbb{R})$ of α -Holder potentials A is

$$\|A\|_\alpha = \sup_{d(x, y) < \varepsilon} \frac{|A(x) - A(y)|}{d(x, y)^\alpha} + \sup_{x \in \Sigma} |A(x)|.$$

An important conjecture claims that generically, in the Holder topology, A is such that the maximizing probability for A has support in a unique periodic orbit (see [14] for partial results).

Here we make a more precise analysis of case where the maximizing probability $\mu_{\infty, A} = \mu_\infty$ for A is **unique, and, it's a periodic orbit**.

We denote $\hat{\Sigma} = \Sigma \times \Sigma = \{1, 2, 3, \dots, d\}^{\mathbb{Z}}$, and, we use the notation $\bar{x} = (\dots x_{-2}, x_{-1} \mid x_0, x_1, x_2, \dots) \in \hat{\Sigma}$, $w = (x_0, x_1, x_2, \dots) \in \Sigma$, $x = (x_{-1}, x_{-2}, \dots) \in \Sigma$. Sometimes we denote $\bar{x} = (w, x)$. We say $w = (x_0, x_1, x_2, \dots)$ are the future coordinates of \bar{x} and $x = (x_{-1}, x_{-2}, \dots) \in \Sigma$ are the past coordinates of \bar{x} . We also use the notation: σ is the shift acting in the past coordinates w and T is the shift acting in the future coordinates x . Moreover, \mathbb{T} is the (write side)-shift on $\hat{\Sigma}$, that is,

$$\mathbb{T}^{-1}(\dots x_{-2}, x_{-1} \mid x_0, x_1, x_2, \dots) = (\dots x_{-2}, x_{-1} \mid x_0, x_1, x_2, \dots).$$

As we said before we denote the action of the shift in the coordinates x by T , that is,

$$T(x_{-1}, x_{-2}, x_{-3}, \dots) = (x_{-2}, x_{-3}, x_{-4}, \dots).$$

For $w = (x_0, x_1, x_2, \dots) \in \Sigma$, $x = (x_{-1}, x_{-2}, \dots) \in \Sigma$, denote $\tau_w(x) = (x_0, x_{-1}, x_{-2}, \dots)$.

We use sometimes the simplified notation $\tau_w = \tau_{x_0}$, because the dependence on w is just on the first symbol x_0 ,

Using the simplified notation $\bar{x} = (w, x)$, we have

$$\mathbb{T}^{-1}(w, x) = (\sigma(w), \tau_w(x)).$$

Note that

$$\mathbb{T}^{-2}(w, x) = (\sigma^2(w), \tau_{\sigma(w)}(\tau_w(x))).$$

A similar expression is true for \mathbb{T}^{-n} . We use the notation

$$\mathbb{T}^{-n}(w, x) = (\sigma^n(w), \tau_{n,w}(x)).$$

Bellow we consider that A acts on the past coordinate x and A^* acts on the future coordinate w .

Definition 1.2. Consider $A : \Sigma \rightarrow \mathbb{R}$ Holder. We say that a Holder continuous function $W : \hat{\Sigma} \rightarrow \mathbb{R}$ is a involution kernel for A , if there is a Holder function $A^* : \Sigma \rightarrow \mathbb{R}$, such that,

$$A^*(w) = A \circ \mathbb{T}^{-1}(w, x) + W \circ \mathbb{T}^{-1}(w, x) - W(w, x).$$

We say that A^* is a dual potential of A , or, that A and A^* are in involution.

The above expression can be also written as

$$A^*(w) = A(\tau_w(x)) + W(\sigma(w), \tau_w(x)) - W(w, x).$$

Theorem 1.1. [2] Given A Holder there exist a Holder W which is an involution kernel for A .

We call μ_{∞, A^*} the maximizing probability for A^* (it is unique if A has a unique maximizing $\mu_{\infty, A}$, as one can see in [2])

If A depends just on two coordinates, that is $A(x) = A(x_1, x_2, x_3, \dots, x_n, \dots) = A(x_1, x_2)$, then A^* can be taken the transpose of A . In this way, we can get with the above result a generalization of this concept for potentials A which depends on all coordinates of $x = (x_1, x_2, x_3, \dots, x_n, \dots)$

We denote by $\mathbb{K} = \mathbb{K}(\mu_{\infty, A}, \mu_{\infty, A^*})$ the set of probabilities $\hat{\eta}(w, x)$ on $\hat{\Sigma}$, such that $\pi_x^*(\hat{\eta}) = \mu_{\infty, A}$, and also that $\pi_y^*(\hat{\eta}) = \mu_{\infty, A^*}$.

We are interested in the solution $\hat{\mu}$ of the Kantorovich Transport Problem for $-W$, that is, the solution of

$$\inf_{\hat{\eta} \in \mathbb{K}} \int \int -W(w, x) d\hat{\eta}.$$

Note that in the definition of W we use the dynamics of \mathbb{T} . Note also that if we consider a new cost of the form $c(x, w) = -W(x, w) + \varphi(w)$, instead of $-W$, where φ bounded and measurable, then we **do not** change the original minimization problem.

Definition 1.3. A calibrated sub-action $V : \Sigma \rightarrow \mathbb{R}$ for the potential A , is a continuous function V such that

$$\sup_{y \text{ such that } T(y) = x} \{V(y) + A(y) - m(A)\} = V(x).$$

The existence of such V could be seen as a solution of a kind of a "dynamical" additive eigenvalue problem [15] [12] [20] [3].

Given a calibrated subaction, then, adding a constant we have another calibrated subaction. If the maximizing probability is unique, then, up to an additive constant, the calibrated subaction is unique [14] [17].

We denote by V^* a calibrated subaction for A^* .

We denote by $\hat{\mu}$ the minimizing probability over $\hat{\Sigma} = \{1, 2, 3, \dots, d\}^{\mathbb{Z}}$ for the natural Kantorovich Transport Problem associated to the $-W$, where W is the involution kernel for A (see [2]).

We call $\hat{\mu}_{max}$ the natural extension of $\mu_{\infty, A}$ as described in [2] [34].

In [26] was shown (not assuming the maximizing probability is a periodic orbit) that:

Theorem 1.2. Suppose the maximizing probability for A Holder is unique (not necessarily a periodic orbit). Then, the minimizing Kantorovich probability $\hat{\mu}$ on $\hat{\Sigma}$ associated to $-W$, where W is the involution kernel for A , is $\hat{\mu}_{max}$.

Moreover, it was shown in [26]:

Theorem 1.3. *Suppose the maximizing probability is unique (not necessarily a periodic orbit). If V is the calibrated subaction for A , and V^* is the calibrated subaction for A^* , then, the pair $(-V, -V^*)$ is the dual $(-W)$ -Kantorovich pair of $(\mu_{\infty, A}, \mu_{\infty, A^*})$.*

The solution comes from the so called slackness condition [8] [36] [37] which were obtained in proposition 10 (1) [2]. We can assume that γ in [2] is equal to zero.

One can consider in the Bernoulli space $\Sigma = \{0, 1\}^{\mathbb{N}}$ the lexicographic order. In this way, $x < z$, if and only if, the first element i such that, $x_j = z_j$ for all $j < i$, and $x_i \neq z_i$, satisfies the property $x_i < z_i$. Moreover, $(0, x_1, x_2, \dots) < (1, x_1, x_2, \dots)$.

One can also consider the more general case $\Sigma = \{0, 1, \dots, d-1\}^{\mathbb{N}}$, but in order to simplify the notation and to avoid technicalities, we consider only the case $\Sigma = \{0, 1\}^{\mathbb{N}}$. We also suppose, from now on, that $\hat{\Sigma} = \Sigma \times \Sigma$.

Definition 1.4. *We say a continuous $G : \hat{\Sigma} = \Sigma \times \Sigma \rightarrow \mathbb{R}$ satisfies the twist condition on $\hat{\Sigma}$, if for any $(a, b) \in \hat{\Sigma} = \Sigma \times \Sigma$ and $(a', b') \in \Sigma \times \Sigma$, with $a' > a$, $b' > b$, we have*

$$(1) \quad G(a, b) + G(a', b') < G(a, b') + G(a', b).$$

Definition 1.5. *We say a continuous $A : \Sigma \rightarrow \mathbb{R}$ satisfies the twist condition, if its involution kernel W satisfies the twist condition.*

The involution kernel W of A is not unique (see [2]), but if the above property is true for some W -kernel of A , then it will also be true for any other one.

The twist condition in the present setting plays the role of the convexity of the Lagrangian in Aubry-Mather Theory, or, the twist condition in Frenkel-Kantorova models [1] [15]. It is natural to assume above the signal $<$, instead of $>$, in the definition of twist because we are maximizing probabilities, and, not, minimizing (see [26]).

Examples of twist potentials are presented in [26].

One of the main results in [26] is:

Theorem 1.4. *Suppose the maximizing probability is unique (not necessarily a periodic orbit) and W satisfies the twist condition on $\hat{\Sigma}$, then, the support of $\hat{\mu}_{max} = \hat{\mu}$ on $\hat{\Sigma}$ is a graph (up to one possible orbit).*

Given $A : \Sigma = \{1, 2, \dots, d\}^{\mathbb{N}} \rightarrow \mathbb{R}$ Holder, the Ruelle operator $\mathcal{L}_A : C^0(\Sigma) \rightarrow C^0(\Sigma)$ is given by $\mathcal{L}_A(\phi)(x) = \psi(x) = \sum_{T(z)=x} e^{A(z)} \phi(z)$.

We also consider \mathcal{L}_{A^*} acting on continuous functions in $\Sigma = \{1, 2, \dots, d\}^{\mathbb{N}}$.

We also denote by ϕ_A and ϕ_{A^*} the corresponding eigen-functions associated to the main common eigenvalue $\lambda(A)$ of the operators \mathcal{L}_A and \mathcal{L}_{A^*} (see [PP]).

ν_A , and, ν_{A^*} are respectively the eigen-probabilities for the dual of the Ruelle operator \mathcal{L}_A and \mathcal{L}_{A^*} .

Finally, $\mu_A = \nu_A \phi_A$ and $\mu_{A^*} = \nu_{A^*} \phi_{A^*}$ are the invariant probabilities such that they are solution of the respective pressure problems for A and A^* . The probability μ_A is called the Gibbs measure for the potential A .

If the maximizing probability is unique, then, it is easy to see that considering for any real β the potential βA and the corresponding $\mu_{\beta A}$, then, $\mu_{\beta A} \rightarrow \mu_{\infty, A}$, when $\beta \rightarrow \infty$.

In the same way, if we take any real β , the potential βA^* , and, the corresponding $\mu_{\beta A^*}$, then $\mu_{\beta A^*} \rightarrow \mu_{\infty, A^*}$, when $\beta \rightarrow \infty$.

In Statistical Mechanics $\beta = \frac{1}{T}$ where T is temperature. Then, $\mu_{\infty, A}$ is the version of the Gibbs probability at temperature zero.

One can choose c (a normalization constant) such that

$$(2) \quad \int \int e^{W(w,x)-c} d\nu_{A^*}(w) d\nu_A(x) = 1,$$

A calibrated sub-action V can be obtained as the limit [14] [15]

$$(3) \quad V(x) = \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \phi_{\beta A}(x).$$

In the same way we can get a calibrated sub-action V^* for A^* by taking

$$(4) \quad V^*(w) = \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \phi_{\beta A^*}(w).$$

Moreover, by [2]

$$(5) \quad \phi_{A^*}(w) = \int e^{W_A(w,x)-c} d\nu_A(x),$$

and,

$$(6) \quad \phi_A(x) = \int e^{W_A(w,x)-c} d\nu_{A^*}(w) = \int e^{W_A(w,x)-c} \frac{1}{\phi_{A^*}} d\mu_{A^*}(w).$$

Note that if W and A^* define an involution for A , then, given any real β , we have that βW and βA^* define an involution for the potential βA . The normalizing constant c , of course, changes with β (see [2]).

We say a family of probabilities μ_β , $\beta \rightarrow \infty$, satisfies a Large Deviation Principle, if there is function $I : \Sigma \rightarrow \mathbb{R} \cup \infty$, which is non-negative, lower semi-continuous, and such that, for any cylinder $K \subset \Sigma$, we have

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log(\mu_\beta(K)) = - \inf_{z \in K} I(z).$$

In this case we say the I is the deviation function. The function I can take the value ∞ in some points.

In [2] a Large Deviation Principle is described for the family $\mu_\beta = \mu_{\beta A}$ of equilibrium states for βA , under the assumption that the maximizing probability for A is unique (see also [29] for a different case where it is not assumed uniqueness). The function I is zero on the support of the maximizing probability for A . We point out that there are examples where I can be zero outside the support (even when the maximizing probability is unique) as we will show below in an example due to R. Leplaideur.

Applying the same to A^* we get a deviation function $I^* : \Sigma \rightarrow \mathbb{R} \cup \{\infty\}$.

The function I^* is defined by

$$I^*(w) = \sum_{n \geq 0} (V^* \circ \sigma - V^* - A^*) \circ \sigma^n(w),$$

where V^* is a fixed calibrated subaction.

We point out that in fact the claim of theorem 1.3 should say more precisely that the pair $(-V, -V^)$ is the dual $(-W + I^*)$ -Kantorovich pair of $(\mu_{\infty, A}, \mu_{\infty, A^*})$.*

Using the property (6) above, and, adapting Varadhan's Theorem [16] to the present setting, it is shown in [26], that given any $x \in \Sigma$, then, it is true the relation

$$V(x) = \sup_{w \in \Sigma} [W(w, x) - V^*(w) - I^*(w)].$$

For each x we get one (or, more than one) w_x such attains the supremum above. Therefore,

$$V(x) = W(w_x, x) - V^*(w_x) - I^*(w_x).$$

A pair of the form (x, w_x) is called an optimal pair. We can also say that w_x is optimal for x .

Given A the involution kernel W and the dual potential A^ are not unique. But, the above maximization problem is intrinsic on A . That is, if we take another A^* and the corresponding W , there is some canceling, and we get the same problem as above (given x the optimal w_x does not change).*

It is also true that (see [26]) for any w

$$V^*(w) = \sup_{x \in \Sigma} [W(w, x) - V(x) - I(x)].$$

Given w , a solution of the above maximization is denoted by x_w . **We denote the pair (x_w, w) a *-optimal pair.**

Definition 1.6. *The set of all (x, w_x) , is called the optimal set for A , and, denoted by $\mathbb{O}(A)$.*

Remark 1: Note that in [26] it was shown that, under the twist assumption, the support of the optimal transport periodic probability $\hat{\mu}_A$, for the cost $-W$, is a graph, that is, for each x , there is only one w such that (x, w) is in the support of $\hat{\mu}_A$. But, nothing is said about the graph property of the set $\mathbb{O}(A)$. Note that the support of $\hat{\mu}_A$ is contained in $\mathbb{O}(A)$.

Remark 2: Note also that the minimal transport problem for the cost $-W$, or the cost $-W + I^*$ is the same (see [26]).

Our main result is:

Theorem 1.5. *Generically, in the set of potentials Holder potentials A that satisfy*

- (i) *the twist condition,*
 - (ii) *uniqueness of maximizing probability which is supported in a periodic orbit,*
- the set of possible optimal w_x , when x covers the all range of possible elements x in Σ , is finite.*

We point out that this is a result for points outside the support of the maximizing probability.

It is known from [26] that for x in the support of the maximizing probability for A , there is a w in the support of the maximizing probability for A^* , such that, (x, w) is an optimal pair. In the same way, that for w in the support of the maximizing probability for A^* , there is a x in the support of the maximizing probability for A , such that, (x, w) is an optimal pair.

Note that W satisfies the twist condition, if, and only if, $W - V^*$ (or, $W(w, x) - V^*(w) - I^*(w)$) satisfies the twist condition.

We point out that in order to check that W is twist we just have to check properties of the potential A (see [26]). The property of been twist is stable by Holder perturbations. Examples of twist potentials A are presented in [26]. In [27] it is also considered the twist condition and results for the analytic setting are obtained.

Proposition 5 in [2] claims that if $\hat{\mu}_{max}$ is the natural extension of the maximizing probability μ_∞ , then, for all (p, p^*) in the support of $\hat{\mu}_{max}$, we have the slackness condition

$$V(p) + V^*(p^*) = W(p, p^*) - \gamma.$$

From this follows that if (p, p^*) in the support of $\hat{\mu}_{max}$ (then, $p \in \Sigma$ is in the support of μ_∞ and $p^* \in \Sigma$ is in the support of μ_∞^*), then

$$\begin{aligned} V(p) &= \sup_{w \in \Sigma} (W(w, p) - \gamma - V^*(w) - I^*(w)) = \\ &= (W(p^*, p) - \gamma - V^*(p^*) - I^*(p^*)) = (W(p^*, p) - \gamma - V^*(p^*)). \end{aligned}$$

If the potential A is twist, then for any given p in the support of μ_∞ , there is only one p^* , such that (p, p^*) is in the periodic orbit which is the support of the probability $\hat{\mu}_{max}$ (see [26]).

Remark: Given x , this maximum at w_x can not be realized where $I^*(w)$ is infinity.

Given A , we denote

$$\Delta(x, x', y) = \sum_{n \geq 1} A \circ \tau_{y, n}(x) - A \circ \tau_{y, n}(x').$$

The involution kernel W can be computed for any (ω, x) by $W(\omega, x) = \Delta_A(\omega, \bar{x}, x)$, where we choose a point \bar{x} for good.

It is known the following relation: for any $x, x', w \in \Sigma$, we have that $W(w, x) - W(w, x') = \Delta(w, x, x')$ (see [2])

2. THE TWIST PROPERTY

In order to simplify the notation we assume that $m(A) = m(A^*) = 0$.

We assume from now on that A satisfies the twist condition. It is known in this case (see [4] [28]), that $x \rightarrow w_x$ (can be multivalued) is monotonous decreasing. We recall the proof:

Proposition 2.1. *If A is twist, then $x \rightarrow w_x$ is monotonous decreasing.*

Proof: Suppose $x < x'$, and, that $(x, w_x), (x', w_{x'})$ are two optimal pairs. We will show that $w_x \geq w_{x'}$. Indeed, as

$$V(x) = \sup_{w \in \Sigma} (W(w, x) - V^*(w) - I^*(w)) = W(w_x, x) - V^*(w_x) - I^*(w_x),$$

then

$$W(w, x) - V^*(w) - I^*(w) \leq W(w_x, x) - V^*(w_x) - I^*(w_x), \quad (*)$$

for any w , and we also have that

$$V(x') = \sup_{w \in \Sigma} (W(w, x') - V^*(w) - I^*(w)) = W(w_{x'}, x') - V^*(w_{x'}) - I^*(w_{x'}).$$

Therefore,

$$W(w, x') - V^*(w) - I^*(w) \leq W(w_{x'}, x') - V^*(w_{x'}) - I^*(w_{x'}), \quad (**)$$

for any w .

Suppose, $x < x'$. Substituting $w_{x'}$ in the first expression (*), and w_x in the second one (**) we get

$$\Delta(x, x', w_{x'}) \leq \Delta(x, x', w_x),$$

where $W(x, w) - W(x', w) = \Delta(x, x', w)$. So the twist property implies that $w_{x'} \leq w_x$. ■

We define R by the expression $R(x) = V(\sigma(x)) - V(x) - A(x) \geq 0$, and, we define R^* by $R^*(w) = V^*(\sigma(w)) - V^*(w) - A^*(w) \geq 0$.

Note that given y , there is a preimage x of y , such that, $R(x) = 0$. The analogous property is true for R^* .

Given A (and a certain choice of A^* and W) the next result (which does not assume the twist condition) claims that the dual of R is R^* , and the corresponding involution kernel is $(V^* + V - W)$.

Proposition 2.2. *(Fundamental Relation)(FR)*

$$R(\tau_w x) = (V^* + V - W)(x, w) - (V^* + V - W)(\tau_w x, \sigma(w)) + R^*(w).$$

Proof:

As $R^*(w) = V^*(\sigma(w)) - V^*(w) - A^*(w)$, we get

$$V^*(w) - V^*(\sigma(w)) + R^*(w) = -A^*(w),$$

and, now using $x = T(\tau_w x)$, we get

$$V(x) - V(\tau_w x) = V(T(\tau_w x)) - V(\tau_w x) - A(\tau_w x) + A(\tau_w x) = R(\tau_w x) + A(\tau_w x).$$

Substituting the above in the previous equation we get

$$\begin{aligned} (V^* + V - W)(x, w) - (V^* + V - W)(\tau_w x, \sigma(w)) + R^*(w) &= \\ [V^*(w) - V^*(\sigma(w)) + R^*(w)] + [V(x) - V(\tau_w x)] - W(x, w) + W(\tau_w x, \sigma(w)) &= \\ -A^*(w) + R(\tau_w x) + A(\tau_w x) + W(\tau_w x, \sigma(w)) - W(x, w) &= \\ R(\tau_w x), \end{aligned}$$

because $A^*(w) = A(\tau_w x) + W(\tau_w x, \sigma(w)) - W(x, w)$. So the claim follows. ■

We know that the calibrated subaction satisfies

$$V(x) = \max_{w \in \Sigma} (-V^* - I^* + W)(x, w).$$

Then, we define

$$b(x, w) = (V^* + V + I^* - W)(x, w) \geq 0,$$

and,

$$\Gamma_V = \{(x, w) \in \Sigma \times \Sigma \mid V(x) = (-V^* - I^* + W)(x, w)\},$$

which can be written in an equivalent form

$$\Gamma_V = \{(x, w) \in \Sigma \times \Sigma \mid b(x, w) = 0\}.$$

Remark 3: Note, that $b(x, w) = 0$, if and only if, (x, w) is an optimal pair. We are not saying anything for *-optimal pairs.

If we use $R^*(w) = I^*(w) - I^*(\sigma w)$, the FR becomes

$$R(\tau_w x) = (V^* + V - W)(x, w) - (V^* + V - W)(\tau_w x, \sigma(w)) + I^*(w) - I^*(\sigma(w)),$$

or

$$R(\tau_w x) = b(x, w) - b(\tau_w x, \sigma(w)) \quad \text{FR1.}$$

From this main equation we get:

Lemma 2.1. *If $\mathbb{T}^{-1}(x, w) = (\tau_w x, \sigma(w))$, then*

- a) $b - b \circ \mathbb{T}^{-1}(x, w) = R(\tau_w x)$;
- b) *The function b it is not decreasing in the trajectories of \mathbb{T} ;*
- c) Γ_V *is backward invariant;*
- d) *when (x, w) is optimal then $R(\tau_w(x)) = 0$.*

Proof:

The first one its a trivial consequence of the definition of \mathbb{T}^{-1} . The second one it is a consequence of $R \geq 0$:

$$\begin{aligned} b - b \circ \mathbb{T}^{-1}(x, w) &= R(\tau_w x) \geq 0 \\ b(x, w) &\geq b \circ \mathbb{T}^{-1}(x, w). \end{aligned}$$

In order to see the third part we observe that

$$(x, w) \in \Gamma_V \Leftrightarrow b(x, w) = 0$$

Since

$$b(x, w) \geq b \circ \mathbb{T}^{-1}(x, w) \geq 0$$

we get $b(\tau_w x, \sigma(w)) = 0$ thus $(\tau_w x, \sigma(w)) \in \Gamma_V$. ■

From the above we get that in the case (x, w) is optimal, then, $\mathbb{T}^{-1}(x, w)$ is also optimal. Indeed, we have that

$$b(x, w) = 0 \Rightarrow b(\tau_w x, \sigma(w)) = 0.$$

This is equivalent to

$$\begin{aligned} V(x) &= -V^*(w) - I^*(w) - W(x, w) \Rightarrow \\ V(\tau_w x) &= -V^*(\sigma(w)) - I^*(\sigma(w)) - W(\tau_w x, \sigma(w)). \end{aligned}$$

In this way \mathbb{T}^{-n} spread optimal pairs.

The function b is a kind of Lyapunov function for the iteration of \mathbb{T}^{-1} on $\{1, 2, 3, \dots, d\}^{\mathbb{N}} \times \{1, 2, 3, \dots, d\}^{\mathbb{N}}$.

We denote by M the support of the maximizing probability periodic orbit.

Consider the compact set of points $P = \{w \in \Sigma, \text{ such that } \sigma(w) \in M, \text{ and } w \text{ is not on } M\}$.

Definition 2.1. *We say that A is good if for each $w \in P$, we have that $R^*(w) > 0$.*

We alternatively, say sometimes that R^* is good for A^* .

We point out that there are examples of potentials A^* (with a unique maximizing probability) where the corresponding R^* is not good (see Example 1 in the end of the present section).

Remember that ,

$$I^*(w) = \sum_{n \geq 0} (V^* \circ \sigma - V^* - A^*) \circ \sigma^n(w) = \sum_{n \geq 0} R^*(\sigma^n(w)).$$

In [28] section 5 it is shown that if $I^*(w)$ is finite, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma^j(w)} \rightarrow \mu_{\infty}^*.$$

One important assumption here is that R^* is good. We will show later that this property is true for generic potentials A .

Proposition 2.3. *Assume A is good. If $I^*(w) < \infty$, then, w is in the preimage of the maximizing probability for A^* .*

Proof:

We consider in Σ the metric d , such that $d(w_1, w_2) = \frac{1}{2^n}$, where n is the first symbol in which w_1 and w_2 disagree.

There exist a fixed $0 < \delta < 2^{-(p+1)}$, (in the case M is a periodic orbit, the p can be taken the period) for some $p > 0$, such that, if

$$\Omega_{\delta} = \{w \in \Sigma \mid d(w, P) < \delta\},$$

then

$$c_{\delta} = \min_{w \in \Omega_{\delta}} R^*(w) > 0.$$

Consider a small neighborhood A_{δ} of the set M such that $\sigma(\Omega_{\delta}) = A_{\delta}$.

We can assume the above δ is such that any point in A_{δ} has a distance smaller than 2^{-p} to a point of M , where p is the period.

Note that in order that the orbit of point w by σ enter (a new time) the set A_{δ} , it has to pass before by Ω_{δ} .

As $\mu_{\infty}^*(M) > 0$, then considering the continuous function $\chi_{A_{\delta}}$ (indicator of A_{δ}), we have that, if $I^*(w) < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{A_{\delta}}((\sigma)^j(w)) > 0.$$

Therefore, $(\sigma)^j(w)$ visits A_δ for infinitely many values of j .

Given w , suppose there exist a $N > 0$, such that for all $j > N$, we have that $(\sigma^*)^j(w) \in A_\delta$. In this case, there exist a k such that $(\sigma)^k(w) \in M$.

Now, we consider the other case.

Denote by m_1 the total amount of time the orbit $(\sigma)^k(w)$ remains in A_δ for the first time, then the trajectory goes out of A_δ , and m_2 is the total amount of time the orbit $(\sigma)^k(w)$ remains in A_δ for the second time it returns to A_δ , and so on..

We suppose from now on that the maximizing probability for A^* has support in a unique periodic orbit of period p denoted by $M = \{\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_p\} \subset \Sigma$.

We have two possibilities:

a) The times m_n , $n \in \mathbb{N}$, of visits to A_δ , satisfies $2^{-m_n} < \delta$, for infinitely many values of n . In this case, the orbit visits Ω_δ an infinite number of times, and $I^*(w) = \infty$, and we reach a contradiction.

b) The times m_n , $n \in \mathbb{N}$, are bounded by a constant N . We can consider now a new set $A_{\bar{\delta}}$, which is a smaller neighborhood of M , in such way that any point in $A_{\bar{\delta}}$ has a distance smaller than 2^{-N} , to a point of M .

As,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{A_{\bar{\delta}}}(\sigma^j(w)) > 0,$$

we reach a contradiction.

We are interested only in the case the support is periodic orbit. The shift is expanding, then by the shadowing property there is an ϵ such if the corresponding forward orbits of two points are ϵ close, for all n , then the points are the same. From this it follows that the orbit we are considering (which eventually remains indefinitely within A_δ) should be eventually periodic.

Therefore, if w is such that $I^*(w) < \infty$, then, there exists a k such that $\sigma^k(w) = \tilde{w} \in M$. ■

Proposition 2.4. *Suppose (x, w_x) is an optimal pair, and, A is good, then, there exists k , such that, $\tilde{w}_k = \sigma^k(w_x)$ is in the support of the maximizing probability for A^* . Moreover, for such k , we have that $\mathbb{T}^{-k}(x, w_x)$ is an optimal pair.*

Proof:

Suppose (x, w_x) is optimal. Therefore, $I^*(w_x) < \infty$. Then, by a previous proposition, w_x is in the pre-image of the maximizing probability for A^* , that is, there exists k such that $\tilde{w} = \sigma^k(w_x)$ is in the support of the maximizing probability for A^* .

Moreover, $b(\mathbb{T}^{-k}(x, w_x)) = (x_k, \sigma^k(w_x)) = (x_k, \tilde{w}_k)$ is also optimal. ■

Proposition 2.5. *Assume A is good, then, the set of w such that $I^*(w) < \infty$ is countable.*

Definition 2.2. *We say a continuous $A : \Sigma \rightarrow \mathbb{R}$ satisfies the countable condition, if there are a countable number of possible optimal w_x , when x ranges over the interval Σ .*

Remark: If A is good, then, it satisfies the countable condition.

We showed before that the twist property implies that for $x < x'$, if $b(x, w) = 0$ and $b(x', w') = 0$, then $w' < w$, which means that the optimal sequences are monotonous not increasing. Thus, we define the “turning point c ” as being the maximum of the point x that has his optimal sequence starting in 1:

$$c = \sup\{x \mid b(x, w) = 0 \Rightarrow w = (1 w_1 w_2 \dots)\}.$$

The main criteria is the following:

“If $x \in \sigma$ has the optimal sequence $w = (w_0 w_1 w_2 \dots)$ then

$$w_0 = \begin{cases} 1, & \text{if } x \in [0^\infty, c] \\ 0, & \text{if } x \in (c, 1^\infty] \end{cases}”$$

Starting from (x_0, w_0) we can iterate FR1 by $\mathbb{T}^{-n}(x, w) = (x_n, w_n)$ in order to obtain new points $w_1, w_2 \dots \in \Sigma$. Unless the only possible optimal point w_x , for all x , is a fixed point for σ , then, $0 < c < 1$.

Note that for c there are two optimal pairs (c, w) and (c, w') , where the first symbol of w is zero, and, the first symbol of w' is one.

We denote

$$B(w) = \{x \mid b(x, w) = 0\}.$$

Lemma 2.2. (Characterization of optimal change) *Let $c \in (0^\infty, 1^\infty)$ be the turning point then, for any $x < x'$ and $b(x, w) = 0$ and $b(x', w') = 0$, we have $w \neq w'$ if, and only if, there exists $n \geq 0$ such that $T^n(c) \in [x, x']$. Moreover, if x, x' are such that w_x and w'_x are identical until the n coordinate, then, $T^n(c) \in (x, x')$.*

Each set $B(w) = [a, b]$ is such that $a = T^n(c)$, or, a it is accumulated by a subsequence of $T^j(c)$ from the left side. Similar property is true for b (accumulated by the right side).

Proof. (Short proof - iterative)

Step 0

If $x < x' \leq c$ then $w_0 = w'_0 = 1$ else if $c < x < x'$ then $w_0 = w'_0 = 0$. Suppose $w_0 = w'_0 \in \{0, 1\}$ then applying FR1 we get $\tau_i x < \tau_i x'$ and $b(\tau_i x, (w_1 w_2 \dots)) = 0$ and $b(\tau_i x', (w'_1 w'_2 \dots)) = 0$.

Step 1

If $\tau_i x < \tau_i x' < c$ then $w_1 = w'_1 = 1$ else if $c < \tau_i x < \tau_i x'$ then $w_1 = w'_1 = 0$. Otherwise if $\tau_i x < c < \tau_i x'$ we can use the monotonicity of T in each branch in order to get $x < T(c) < x'$. Thus

$$w_1 \neq w'_1 \Leftrightarrow x < T(c) < x'.$$

The conclusion comes by iterating this algorithm.

The last claim is obvious from the above. □

A point x is called pre-periodic (or, eventually periodic) if there is $n \neq m$, such that, $T^n(x) = T^m(x)$.

We denote $[a, b]$, $a, b \in \Sigma$, $a \leq b$, the set of all points w in Σ such $a \leq w \leq b$. We call $[a, b]$ the interval determined by a and b . Each interval $[a, b]$, with $a < b$, is not countable

Lemma 2.3. *The set*

$$B(w) = \{x \mid b(x, w) = 0\}$$

is an interval (can eventually be a single point). More specifically, if $B(w) = [a, b]$, then, a and b are adherence points of the orbit of c .

In particular, if c is pre-periodic, then, for any non-empty $B(w)$, there exists n, m such that $B(w) = [T^n(c), T^m(c)]$ (unless $B(w)$ is of the form $[0, b]$, or $[a, 1]$).

Proof: Indeed, remember that τ_i , for $i = 0, 1$, are order preserving. If, $x < y$, and, $x, y \in B(w)$, then, we claim that each $z \in (x, y)$ satisfies $z \in B(w)$. Indeed, otherwise if $\tilde{w} \neq w$ is the optimal sequence for z , we know that there is $K > 0$ such that $\tilde{w}_j = w_j$ for $j = 0, \dots, k-1$ and $\tilde{w}_k \neq w_k$. On the other hand

$$\tau_{k,w}x < \tau_{k,\tilde{w}}z < \tau_{k,w}y.$$

Without lost of generality suppose $w_k = 1$, then, $\tilde{w}_k = 0$, a contradiction by the twist property, analogously, if $w_k = 0$, then, $\tilde{w}_k = 1$, and, we reach a contradiction again.

The closeness follows from the continuity on x of the function b : if, $x_n \in B(w)$, and $x_n \rightarrow \bar{x}$, we observe that

$$b(x_n, w) = 0 \Leftrightarrow V(x_n) + V^*(w) + I^*(w) - W(x_n, w) = 0,$$

and, this implies $b(\bar{x}, w) = 0$, that is, $\bar{x} \in B(w)$.

For the second part it is enough to see that, for each extreme of the interval, for example b , if the optimal w is not constant in the right side, for any $e > b$, there is an image of c , namely $T^j(c) \in [b, e]$.

■

Lemma 2.4. *Let $c \in \Sigma$ be the turning point. Let us suppose the c is isolated from his orbit, which means that, there is d, e, w^- and w^+ , such that, $b(x, w^-) = 0$, for any $x \in (d, c]$, and, $b(x, w^+) = 0$, for any $x \in [c, e)$, then, there is no accumulation points of the orbit of c . In this case c is eventually periodic.*

Proof: Take $N > 0$, such that $\frac{1}{2^{N-1}} < \delta$, and, consider the sequence $\{c, T(c), \dots, T^{N-1}(c)\}$, which gives an partition, which will be denoted by: $\{I_0, I_1, \dots, I_{N-1}\}$. Since each interval I_j does not have in its interior points of the form $T^k(c)$, $k \leq N - 1$, we get from the above that:

$$b(x, w) = 0 \rightarrow w \in \overline{i_0 i_1 \dots i_{N-1}}, \forall x \in I_j.$$

On the other hand, we claim that $I_j = [a, b]$ can have in its interior at most one point in the forward orbit of c .

Indeed, if $I_j \cap \{T^N(c), T^{N+1}(c), \dots\} = \emptyset$, then the optimal w will be constant and I_j of the form $[T^n(c), T^m(c)]$. Else, if $T^k(c) \in I_j \cap \{T^N(c), T^{N+1}(c), \dots\} \neq \emptyset$, for $k \geq N$, we denote by k the minimum one where this happens. Then, we get

$$b(x, w) = 0 \rightarrow w \in \overline{i_0 i_1 \dots i_{k-1}}, \forall x \in I_j.$$

If, we iterate the $k - 1$ times the FR1, then $c \in Z_j = \tau_{i_{k-1} \dots i_0} I_j$. By the choice of N we get $Z_j \subset (c - \delta, c + \delta)$. Dividing $I_j = [a, T^k(c)] \cup [T^k(c), b]$ we get

$$b(x, w) = 0 \rightarrow w = (i_0 i_1 \dots i_{k-1} * w^-), \forall x \in [a, T^k(c)],$$

and,

$$b(x, w) = 0 \rightarrow w = (i_0 i_1 \dots i_{k-1} * w^+), \forall x \in [T^k(c), b].$$

Therefore, there is no room for another $T^r(c)$, $r \neq k$, to belong to I_j .

■

Remark 4: The main problem we have to face is the possibility that the orbit of c is dense in $[0, 1]$.

If c is eventually periodic there exist just a finite number intervals $B(w)$ with positive length. The other $B(w)$ are reduced to points and they are also finite.

Lemma 2.5. *Suppose A satisfies the twist and the countable condition. Then there is at least one $B(w)$ with positive length of the form $(T^n(c), T^m(c))$. Moreover, for each subinterval (a, b) there exists at least one $B(w)$ with positive length of the form $(T^n(c), T^m(c))$ contained on it. Therefore, there exists an infinite number of such intervals.*

Denote the possible w , such that, $I^*(w) < \infty$, by w_j , $j \in \mathbb{N}$.

For each w^j , $j \in \mathbb{N}$, denote $I_j = B(w^j)$, the maximal interval where for all $x \in I_j$, we have that, (x, w^j) is an optimal pair. Some of these intervals could be eventually a point, but, an infinite number of them have positive length, because the set Σ is not countable. We consider from now on just the ones with positive length.

Note that by the same reason, in each subinterval (e, u) , there exists an infinite countable number of $B(w)$ with positive length.

We suppose, by contradiction, that each interval $B(w) = [a, b]$, with positive length is such that, each side is approximated by a sub-sequence of points $T^j(c)$.

Take one interval (a_1, b_1) with positive length inside Σ . There is another one (a_2, b_2) inside $(0^\infty, a_1)$, and one more (a_3, b_3) inside $(b_1, 1^\infty)$.

If we remove from the interval Σ these three intervals we get four intervals. Using our hypothesis, we can find new intervals with positive length inside each one of them. Then we do the same removal procedure as before. This procedure is similar to the construction of the Cantor set. If we proceed inductively on this way, the set of points x which remains after infinite steps is not countable. An uncountable number of such x has a different w_x . This is not possible because the optimal w_x are countable.

Then, the first claim of the lemma is true.

Given an interval $(a, b) \subset \Sigma$, we can do the same and use the fact that (a, b) is not countable.

■

Lemma 2.6. *Under the twist and the countable condition the turning point c is eventually periodic.*

Proof:

Denote by w^j , $j \in \mathbb{N}$, the countable set of pre-images of the periodic orbit maximizing A^* .

For each w^j , $j \in \mathbb{N}$, denote $I_j = B(w^j)$, the maximal interval where for all $x \in I_j$, we have that, (x, w^j) is an optimal pair. Some of these intervals could be, eventually, a point, but, an infinite number of them have positive length, because the set Σ is not countable

We will suppose c is not eventually periodic, and, we will reach a contradiction. Therefore, if $T^n(c) = T^m(c)$, then $m = n$.

From, now on we consider just j , such that, for the corresponding w_j , the interval I_j has positive length, and, it is of the form $[T^n(c), T^m(c)]$, $m, n \geq 1$. From last lemma there exist an infinite number of them.

Denote by $I_j = [a_j, b_j]$. We denote I_0 the interval of the form $[0^\infty, b_0]$, and, I_1 the interval of the form $[a_0, 1^\infty]$. From last lemma, for $j \neq 0, 1$, there is n_j and m_j , such that $a_j = T^{n_j}(c)$ and $b_j = T^{m_j}(c)$.

Consider the inverse branch $\tau_{i_1^j}$, where $i_1^j = i_1$ is such that $\tau_{i_1}((T^{n_j})(c)) = T^{n_j-1}(c)$. This i_1 **do not have to be the first symbol of the optimal w for $T^{n_j}(c)$** . Then, $\tau_{i_1}(I_j)$ is another interval, which is strictly inside a domain of injectivity of T , does not contain any forward image of c , and in its left side we have the point $T^{n_j-1}(c)$.

Then, repeating the same procedure inductively, we get i_2 , such that $\tau_{i_2}((T^{n_j-1})(c)) = T^{n_j-2}(c)$, determining another interval which does not contain any forward image of c , and in his left side we have the point $T^{n_j-2}(c)$. Repeating the reasoning over and over again, always taking the same inverse branch which contain $T^n(c)$, $0 \leq n \leq n_j$, after n_j times we arrive in an interval of the form (c, r_j) . Note that each inverse branch preserves order. It is not possible to have an iterate $T^k(c)$, $k \in \mathbb{N}$, inside this interval (c, r_j) (by the definition of I_j). Then, the optimal w for x in this interval (c, r_j) is a certain \tilde{w}_j which can be different of $\sigma^{n_j}(w_j)$.

Suppose now that $n_j > m_j$. Using the analogous procedure we get that there exists r^j , such that the optimal w_x for x in the interval (r^j, c) is $\sigma^{n_j}(w_j)$.

If both cases happen, then c is eventually periodic.

The trouble happens when just one type of inequality is true. Suppose without lost of generality that we have always $n_j < m_j$, for all possible j .

Let's fix for good a certain j .

Therefore, all we can get with the above procedure is that c is isolated by the right side

In the procedure of taking pre-image of $T^{n_j}(c)$, always following the forward orbit $T^n(x)$, $0 \leq n \leq n_j$, we will get a sequence of i_1, i_2, \dots, i_{n_j} . In the first step we have two possibilities: $\tau_{i_1}(T^{m_j-j}(c)) = T^{m_j-1}(c)$, or not.

If it happens the second case, we are done. Indeed, the interval $\tau_{i_1}[T^{n_j}(c), T^{m_j}(c)]$ does not contain forward images of c (otherwise $[T^{n_j}(c), T^{m_j}(c)]$ would also have). Now we follow the same procedure as before, but, this time following the branches which contains the orbit of $T^m(c)$, $0 \leq m \leq m_j$. In this way, we get that c is isolated by the left side.

Suppose $\tau_{i_1}(T^{m_j}(c)) = T^{m_j-1}(c)$. Consider the interval, $[T^{n_j-1}(c), T^{m_j-1}(c)]$, which do not contain forward images of c .

Now, you can ask the same question: $\tau_{i_2}(T^{m_j-1}(c)) = T^{m_j-2}(c)$? If this do not happen (called the second option), then, in the same way as before, we are done (c is also isolated by the right side). If the expression is true, then, we proceed with the same reasoning as before.

We proceed in an inductive way until time n_j . If in some time we have the second option, we are done, otherwise, we show that any $x \in (c, T^{m_j-n_j}(c))$ has a unique optimal w_x (there is no forward image of c inside it).

Denote $k = m_j - n_j$ for the j we fixed.

From the above we have that for any $B(w)$, which is an interval of the form $[T^{n_i}(c), T^{m_i}(c)]$, for any possible i , it is true that $m_i - n_i = k$. There are an infinite number of intervals of this form.

We claim that the set of points x which are extreme points of any $B(\tilde{w})$, and, such that x can be approximated by the forward orbit of c is finite. Suppose without lost of generality that x is the right point of a $B(w) = (z, x)$.

If the above happens, then, by the last lemma, we have an infinite sequence of intervals of the form $[T^{n_i}(c), T^{n_i+k}(c)]$, such that $T^{n_i}(c) \rightarrow x$, as $n_i \rightarrow \infty$. Therefore, x is a periodic point of period k . There are a finite number of points of period k . This shows our main claim. Finally, c is eventually periodic. ■

Definition 2.3. We denote by \mathbb{G} the set of Holder potentials such that

- 1) the maximizing probability is unique, and it is a periodic orbit;
- 2) The potential A satisfies the twist condition;
- 3) R^* is good for A^* .

From the above one can get:

Theorem 2.7. Suppose A is in \mathbb{G} , then, there is a finite number of possible w_x such that (x, w_x) are optimal pairs. If we denote by $I_j = (a_j, b_j)$, $j = 1, 2, \dots, n$, the maximal open intervals where for $x \in I_j$ the w_x is constant, then, just on the points $x = a_j$, or $x = b_j$, we can get two different w_x , which define points (x, w_x) in the optimal set $\mathbb{O}(A)$.

Note that if x is in the maximizing orbit for $A \in \mathbb{G}$, then, at least one of the optimal w_x is in the support of the maximizing probability for A^* . This point x can be eventually in the extreme of one of this maximal intervals I_j . This do not contradicts the graph property

Definition 2.4. We denote by \mathbb{A} the open set of Holder potentials such that

- 1) the maximizing probability is unique, and it is a periodic orbit;
- 2) The potential A satisfies the twist condition.

The next theorem shows the class we consider above is large.

Theorem 2.8. The set \mathbb{G} is generic in the open set \mathbb{A} .

The proof of this result will be done in the next two sections (see Theorem 4.1 bellow).

Corollary 2.9. For any $A \in \mathbb{G}$, the value $c = c_A$ is given by the expression

$$c = \inf\{x \mid V(x) - V(\tau_1 x) - A(\tau_1(x)) > 0\},$$

where V is any calibrated subaction for A . Moreover, c_A is locally constant as a function of A .

Proof:

The first claim follows from the fact that we have to use 0 as first symbol of the optimal w_x , when x is on the right of c .

If R^* is good the point c is in the pre-image of the support of the maximizing probability (which is locally constant by the continuous varying support property [14]). Therefore, the possible c are in a countable set.

Note that under the uniqueness hypothesis of the maximizing probability for A , the sub-action $V = V_A$ can be chosen in a continuous fashion with A . From this follows the last claim of the corollary. ■

Now we will provide a counterexample.

Example 1. The following example is due to R. Leplaideur.

We will show an example on the shift where the maximizing probability for a certain Lipchitz potential $A^* : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ is a unique periodic orbit γ of period two, denoted by $p_0 = (01010101\dots)$, $p_1 = (10101010\dots)$, but for a certain point, namely, $w_0 = (110101010\dots)$, which satisfies $\sigma(w_0) = p_1$, we have that $R^*(w_0) = 0$.

The potential A^* is given by $A^*(w) = -d(w, \gamma \cup \Gamma)$, where d is the usual distance in the Bernoulli space. The set Γ is described later.

For each integer n , we define a $2n + 3$ -periodic orbit $z_n, \sigma(z_n), \dots, \sigma^{2n+2}(z_n)$ as follows:

we first set

$$b_n = (\underbrace{01010101 \dots 01}_{2n} 101),$$

and the point z_n is the concatenation of the word b_n : $z_n = (b_n, b_n, \dots)$

The main idea here is to get a sequence of periodic points which spin around the periodic orbit $\{p_0, p_1\}$ during the time $2n$, and then pass close by w_0 (note that $d(\sigma^{2n}(z_n), w_0) = 2^{-2(n+1)}$).

Denote γ_n the periodic orbit $\gamma_n = \{z_n, \sigma(z_n), \sigma^2(z_n), \dots, \sigma^{2n+2}(z_n)\}$.

Consider the sequence of Lipchitz potentials $A_n^*(w) = -d(w, \gamma_n \cup \gamma)$. The support of the maximizing probability for A_n^* is $\gamma_n \cup \gamma$. Moreover

$$0 = m(A_n^*) = \max_{\nu \text{ an invariant probability for } \sigma} \int A_n^*(w) d\nu(w).$$

Denote by V_n^* a Lipchitz calibrated subaction for A_n^* such that $V_n^*(w_0) = 0$. In this way, for all w

$$R_n^*(w) = (V_n^* \circ \sigma - V_n^* - A_n^*)(w) \geq 0,$$

and for $w \in \gamma_n \cup \gamma$ we have that $R_n^*(w) = 0$.

We know that R_n^* is zero on the orbit γ_n , because γ_n is included in the Masur set.

Note that we not necessarily have $R_n^*(w_0) = 0$.

By construction, the Lipchitz constant for A_n^* is 1. This is also true for V_n^* . Hence the family of subactions (V_n^*) is a family of equicontinuous functions. Let us denote by V^* any accumulation point for (V_n^*) for the \mathcal{C}^0 -topology. Note that V^* is also 1-Lipschitz continuous. For simplicity we set

$$V^* = \lim_{k \rightarrow \infty} V_{n_k}^*.$$

We denote by Γ the set which is the limit of the sets γ_n (using the Hausdorff distance). $\gamma \cup \Gamma$ is a compact set. Note that Γ is not a compact set, but the set of accumulation points for Γ is the set γ . We now consider $A^*(w) = -d(w, \gamma \cup \Gamma)$.

As any accumulation point of Γ is in γ , any maximizing probability for the potential A^* has support in γ . On the contrary, the unique σ -invariant measure with support in γ is maximizing for A^* .

Remember that for any n we have $V_n^*(w_0) = 0$. We also claim that we have $A_{n_k}^*(w_0) \rightarrow 0$ and $V_{n_k}^*(\sigma(w_0)) \rightarrow 0$, as $k \rightarrow \infty$.

For each fixed w we set

$$R_{n_k}^*(w) = (V_{n_k}^* \circ \sigma - V_{n_k}^* - A_{n_k}^*)(w) \geq 0.$$

The right hand side terms converge (for the \mathcal{C}^0 -topology) as k goes to $+\infty$. Then $R_{n_k}^*$ converge, and we denote by R^* its limit. Then for every w we have:

$$R^*(w) = (V^* \circ \sigma - V^* - A^*)(w) \geq 0.$$

This shows that V^* is a subaction for A^* . Note that $R^*(w_0) = 0$. From the uniqueness of the maximizing probability for A^* we know that there exists a unique calibrated subaction for A^* (up to an additive constant).

Consider a fixed w and its two preimages w_a and w_b . For any given n , one of the two possibilities occur $R_n^*(w_a) = 0$ or $R_n^*(w_b) = 0$, because V_n^* is calibrated for A_n^* .

Therefore, for an infinite number of values k either $R_{n_k}^*(w_a) = 0$ or $R_{n_k}^*(w_b) = 0$.

In this way the limit of $V_{n_k}^*$ is unique (independent of the convergent subsequence) and equal to V^* , the calibrated subaction for A^* (such that $V^*(w_0) = 0$).

Therefore,

$$R^*(w_0) = (V^* \circ \sigma - V^* - A^*)(w_0) = 0,$$

and V^* is a calibrated subaction for $A^*(w) = d(w, \gamma \cup \Gamma)$.

3. GENERIC CONTINUITY OF THE AUBRY SET.

In this section and in the next we will present the proof of the generic properties we mention before.

We will denote the action of the shift in the points x by T and leave σ for the action of the shift in the coordinates w .

We will present our main results in great generality. First, in this section, we analyze the main properties of sub-actions and its dependence on the potential A .

First we will present the main definitions we will consider here.

$\mathcal{F} \subset C^0(\Sigma, \mathbb{R})$ denotes a complete metric space with a (topology finer than) metric larger than $d_{C^0}(f, g) = \|f - g\|_0 := \sup_{x \in \Sigma} |f(x) - g(x)|$; (for instance, Hölder functions, Lipschitz functions, etc) AND such that

$$(7) \quad \forall K \subset \Sigma \text{ compact}, \exists \psi \in \mathcal{F} \text{ s.t. } \psi \leq 0, [\psi = 0] = \{x \mid \psi(x) = 0\} = K.$$

Given $A \in \mathcal{F}$ and F a calibrated sub-action for A , remember that its *error* is denoted by $R = R_A : \Sigma \rightarrow [0, +\infty[$:

$$R(x) := F(T(x)) - F(x) - A(x) + m_A \geq 0.$$

$\mathcal{S}(A)$ denotes the set of Hölder calibrated sub-actions (it is not empty [15] [14])

Given A , the *Mañé action potential* is:

$$S_A(x, y) := \lim_{\varepsilon \rightarrow 0} \left[\sup \left\{ \sum_{i=0}^{n-1} [A(T^i(z)) - m_A] \mid n \in \mathbb{N}, T^n(z) = y, d(z, x) < \varepsilon \right\} \right].$$

Given x and y the above value describe the A -cost of going from x to y following the dynamics.

The *Aubry set* is $\mathbb{A}(A) := \{x \in \Sigma \mid S_A(x, x) = 0\}$.

The terminology is borrowed from the Aubry-Mather Theory [10].

For any $x \in \mathbb{A}(A)$, we have that $S_A(x, \cdot)$ is a sub-action (in particular, in this case, $S_A(x, y) > -\infty$, for any y), see Proposition 23 in [14].

The set of maximizing measures is

$$\mathcal{M}(A) := \left\{ \mu \in \mathcal{M}(T) \mid \int A d\mu = m_A \right\}.$$

If $F \in C^\alpha(\Sigma, \mathbb{R})$ is a Hölder function define

$$|F|_\alpha := \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x, y)^\alpha}.$$

Define the *Mañé set* as

$$\mathbb{N}(A) := \bigcup_{F \in \mathcal{S}(A)} I_F^{-1}\{0\},$$

where the union is among all the α -Hölder calibrated sub-actions F for A and

$$I_F(x) = \sum_{i=0}^{\infty} R_F(T^i(x)).$$

$I_F(x)$ is the deviation function we considered before.

For $A \in \mathcal{F}$ define the *Mather set* as

$$\mathbb{M}(A) := \bigcup_{\mu \in \mathcal{M}(A)} \text{supp}(\mu).$$

The *Peierls barrier* is

$$h_A(x, y) := \lim_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow +\infty} S_A(x, y, k, \varepsilon),$$

$$\text{where } S_A(x, y, k, \varepsilon) := \sup \left\{ \sum_{i=0}^{n-1} [A(T^i(z)) - m_A] \mid n \geq k, T^n(z) = y, d(z, x) < \varepsilon \right\}.$$

Several properties of the Mañé potential and the Peierls barrier are similar (but not all, see section 4 in [19]). We will present proofs for one of them and the other case is similar.

Lemma 3.1.

(1) If μ is a minimizing measure then

$$\text{supp}(\mu) \subset \mathbb{A}(A) = \{x \in \Sigma \mid S_A(x, x) = 0\}.$$

(2) $S_A(x, x) \leq 0$, for every $x \in \Sigma$.

(3) For any $z \in \Sigma$, the function $F(y) = h_A(z, y)$ is Hölder continuous.

(4) If, $a \in \mathbb{A}(A)$, then, $h_A(a, x) = S_A(a, x)$, for all $x \in \Sigma$.

In particular, $F(y) = S_A(a, x)$ is continuous, if $a \in \mathbb{A}(A)$.

(5) If $S_A(w, y) = h_A(w, y)$ then the function $F(y) = S_A(w, y)$ is continuous at y .

(6) If $S(x_0, T^{n_k}(x_0)) = \sum_{j=0}^{n_k-1} A(T^j(x_0))$, $\lim_k T^{n_k}(x_0) = b$ and $\lim_k n_k = +\infty$, then

$$\lim_k S(x_0, T^{n_k}(x_0)) = S(x_0, b)$$

Item (1) follows from Atkinson-Mañé's lemma which says that if μ is ergodic for μ -almost every x and every $\varepsilon > 0$, the set

$$N(x, \varepsilon) := \left\{ n \in \mathbb{N} \mid \left| \sum_{j=0}^{n-1} A(T^j(x)) - n \int A d\mu \right| \leq \varepsilon \right\}$$

is infinite (see Lemma 2.2 [31] (which consider non-invertible transformation, [10], [14] or [18] for the proof). We will show bellow just the items which are not proved in the mentioned references.

The problem with the discontinuity of $F(y) = S_A(w, y)$ is when the maximum is obtained at a *finite* orbit segment (i.e. when $S_A(w, y) \neq h_A(w, y)$), the hypothesis in item 5.

Proof.

By adding a constant we can assume that $m_A = 0$.

(2). Let F be a continuous sub-action for A . Then

$$-R_F = A + F - F \circ T \leq 0.$$

Given $x_0 \in \Sigma$, let $x_k \in \Sigma$ and $n_k \in \mathbb{N}$ be such that $T^{n_k}(x_k) = x_0$, $\lim_k x_k = x_0$ and

$$S(x_0, x_0) = \lim_k \sum_{j=0}^{n_k-1} A(T^j(x_k)).$$

We have

$$\begin{aligned} \sum_{j=0}^{n_k-1} A(T^j(x_k)) &= \left[\sum_{j=0}^{n_k-1} (A + F - F \circ T)(T^j(x_k)) \right] + F(x_0) - F(x_k) \\ &\leq F(x_0) - F(x_k). \end{aligned}$$

Then

$$S(x_0, x_0) = \lim_k \sum_{j=0}^{n_k-1} A(T^j(x_k)) \leq \lim_k [F(x_0) - F(x_k)] = 0.$$

The proofs of (3) (4) (5) can be found in [14] [18]

(6). Let τ_k be the branch of the inverse of T^{n_k} such that $\tau_k(T^{n_k}(x_0)) = x_0$. Let $b_k = \tau_k(b)$ for k sufficiently large. Then, by the expanding property of the shift, there is $\lambda < 1$, such that,

$$d(x_0, b_k) \leq \lambda^{n_k} d(T^{n_k}(x_0), b) \xrightarrow{k} 0,$$

$$\left| \sum_{i=0}^{n_k-1} A(T^i(x_0)) - \sum_{i=0}^{n_k-1} A(T^i(b_k)) \right| \leq \frac{\|A\|_\alpha}{1-\lambda^\alpha} d(T^{n_k}(x_0), b)^\alpha.$$

Write $Q := \frac{\|A\|_\alpha}{1-\lambda^\alpha}$, then

$$\begin{aligned} S(x_0, b) &\geq \limsup_k \sum_{i=0}^{n_k-1} A(T^i(b_k)) \\ &\geq \limsup_k S(x_0, T^{n_k}(x_0)) - Q d(T^{n_k}(x_0), b)^\alpha \\ &\geq \limsup_k S(x_0, T^{n_k}(x_0)). \end{aligned}$$

Now for $\ell \in \mathbb{N}$ let $b_\ell \in \Sigma$ and $m_\ell \in \mathbb{N}$ be such that $\lim b_\ell = x_0$, $T^{m_\ell}(b_\ell) = b$ and

$$\lim_\ell \sum_{j=0}^{m_\ell-1} A(T^j(b_\ell)) = S(x_0, b).$$

Let $\hat{\tau}_\ell$ be the branch of the inverse of T^{m_ℓ} such that $\hat{\tau}_\ell(b) = b_\ell$. Let $x_\ell := \hat{\tau}_\ell(T^{n_k}(x_0))$. Then

$$\begin{aligned} d(x_\ell, x_0) &\leq d(x_0, b_\ell) + d(b_\ell, x_\ell) \\ &\leq d(x_0, b_\ell) + \lambda^\ell d(T^{n_k}(x_0), b) \xrightarrow{\ell} 0. \end{aligned}$$

$$\left| \sum_{j=0}^{m_\ell-1} A(T^j(x_\ell)) - \sum_{j=0}^{m_\ell-1} A(T^j(b_\ell)) \right| \leq Q d(T^{n_k}(x_0), b)^\alpha.$$

Since $x_\ell \rightarrow x_0$ and $T^{m_\ell}(x_\ell) = T^{n_k}(x_0)$, we have that

$$\begin{aligned} S(x_0, T^{n_k}(x_0)) &\geq \limsup_\ell \sum_{j=0}^{m_\ell-1} A(T^j(x_\ell)) \\ &\geq \limsup_\ell \sum_{j=0}^{m_\ell-1} A(T^j(b_\ell)) - Q d(T^{n_k}(x_0), b)^\alpha \\ &\geq S(x_0, b) - Q d(T^{n_k}(x_0), b)^\alpha. \end{aligned}$$

And hence

$$\liminf_k S(x_0, T^{n_k}(x_0)) \geq S(x_0, b).$$

□

Proposition 3.1. *The Aubry set is*

$$\mathbb{A}(A) = \bigcap_{F \in \mathcal{S}(A)} I_F^{-1}\{0\},$$

where the intersection is among all the α -Hölder calibrated sub-actions for A .

Proof.

By adding a constant we can assume that $m_A = 0$.

We first prove that $\mathbb{A}(A) \subset \bigcap_{F \in \mathcal{S}(A)} I_F^{-1}\{0\}$.

Let $F \in \mathcal{S}(A)$ be a Hölder sub-action and $x_0 \in \mathbb{A}(A)$. Since $S_A(x_0, x_0) = 0$ then there is $x_k \rightarrow x_0$ and $n_k \uparrow \infty$ such that $\lim_k T^{n_k}(x_k) = x_0$ and $\lim_k \sum_{j=0}^{n_k-1} A(T^j(x_k)) = 0$. If $m \in \mathbb{N}$ we have that

$$\begin{aligned}
(8) \quad F(T^{m+1}(x_0)) &\geq F(T^m(x_0)) + A(T^m(x_0)) \\
&\geq F(T^{m+1}(x_k)) + \sum_{j=m+1}^{n_k+m-1} A(T^j(x_k)) + A(T^m(x_0)) \\
&\geq F(T^{m+1}(x_k)) + \sum_{j=0}^{n_k-1} A(T^j(x_k)) - \sum_{j=0}^m |A(T^j(x_k)) - A(T^j(x_0))|
\end{aligned}$$

When $k \rightarrow \infty$ the right hand side of (8) converges to $F(T^{m+1}(x_0))$, and hence all those inequalities are equalities. Therefore $R_F(T^m(x_0)) = 0$ for all m and hence $I_F(x_0) = 0$.

Now let $x_0 \in \bigcap_{F \in \mathcal{S}(A)} I_F^{-1}\{0\}$. Since Σ is compact there is $n_k \xrightarrow{k} +\infty$ such that the limits $b = \lim_k T^{n_k}(x_0) \in \Sigma$ and $\mu = \lim_k \mu_k \in \mathcal{M}(T)$, $\mu_k := \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{T^i(x_0)}$ exist and $b \in \text{supp}(\mu)$. Let G be a Hölder calibrated sub-action. For $m \geq n$ we have

$$\begin{aligned}
G(T^n(x_0)) + S_A(T^n(x_0), T^m(x_0)) &\geq G(T^n(x_0)) + \sum_{j=n}^{m-1} A(T^j(x_0)) \\
&= G(T^m(x_0)) \quad [\text{because } I_G(x_0) = 0] \\
&\geq G(T^n(x_0)) + S_A(T^n(x_0), T^m(x_0)).
\end{aligned}$$

Then they are all equalities and hence for any $m \geq n$

$$S_A(T^n(x_0), T^m(x_0)) = \sum_{j=n}^{m-1} A(T^j(x_0)).$$

Since

$$0 = \lim_k \frac{1}{n_k} S_A(T^n(x_0), T^m(x_0)) = \lim_k \frac{1}{n_k} \sum_{j=n}^{m-1} A(T^j(x_0)) = \int A d\mu,$$

μ is a minimizing measure. By lemma 3.1.(1), $b \in \mathbb{A}(A)$.

Let $F : \Sigma \rightarrow \mathbb{R}$ be $F(x) := S_A(b, x)$. Then F is a Hölder calibrated sub-action. By hypothesis $I_F(x_0) = 0$ and then

$$\begin{aligned}
F(T^n(x_0)) &= F(x_0) + S_A(x_0, T^{n_k}(x_0)). \\
S_A(b, T^{n_k}(x_0)) &= S_A(b, x_0) + S_A(x_0, T^{n_k}(x_0)).
\end{aligned}$$

By lemma 3.1.(4) and Lemma 3.1.(6), taking the limit on k we have that

$$\begin{aligned}
0 &= S_A(b, b) = S_A(b, x_0) + S_A(x_0, b) = 0. \\
0 &\geq S_A(x_0, x_0) \geq S_A(x_0, b) + S_A(b, x_0) = 0
\end{aligned}$$

Therefore $x_0 \in \mathbb{A}(A)$. □

We want to show the following result which will require several preliminary results.

Theorem 3.1. *The set*

$$(9) \quad \mathcal{R} := \{ A \in C^\alpha(\Sigma, \mathbb{R}) \mid \mathcal{M}(A) = \{\mu\}, \mathbb{A}(A) = \text{supp}(\mu) \}$$

contains a residual set in $C^\alpha(\Sigma, \mathbb{R})$.

The proof of the bellow lemma (Atkinson-Mañé) can be found in [31] and [10].

Lemma 3.2. Let (X, \mathfrak{B}, ν) be a probability space, f an ergodic measure preserving map and $F : X \rightarrow \mathbb{R}$ an integrable function. Given $A \in \mathfrak{B}$ with $\nu(A) > 0$ denote by \hat{A} the set of points $p \in A$ such that for all $\varepsilon > 0$ there exists an integer $N > 0$ such that $f^N(p) \in A$ and

$$\left| \sum_{j=0}^{N-1} F(f^j(p)) - N \int F d\nu \right| < \varepsilon.$$

Then $\nu(\hat{A}) = \nu(A)$.

Corollary 3.3. If besides the hypothesis of lemma 3.2, X is a complete separable metric space, and \mathfrak{B} is its Borel σ -algebra, then for a.e. $x \in X$ the following property holds: for all $\varepsilon > 0$ there exists $N > 0$ such that $d(f^N(x), x) < \varepsilon$ and

$$\left| \sum_{j=0}^{N-1} F(f^j(x)) - N \int F d\nu \right| < \varepsilon$$

Proof. Given $\varepsilon > 0$ let $\{V_n(\varepsilon)\}$ be a countable basis of neighbourhoods with diameter $< \varepsilon$ and let \hat{V}_n be associated to V_n as in lemma 3.2. Then the full measure subset $\bigcap_m \bigcup_n \hat{V}_n(\frac{1}{m})$ satisfies the required property. \square

Lemma 3.4. Let \mathcal{R} be as in Theorem 3.1. Then if $A \in \mathcal{R}$, $F \in \mathcal{S}(A)$ we have

- (1) If $a, b \in \mathbb{A}(A)$ then $S_A(a, b) + S_A(b, a) = 0$.
- (2) If $a \in \mathbb{A}(A) = \text{supp}(\mu)$ then $F(x) = F(a) + S_A(a, x)$ for all $x \in \Sigma$.

Proof.

(1). Let $a, b \in \mathbb{A}(A) = \text{supp}(\mu)$. Since μ is ergodic, by Corollary 3.3 there are sequences $\alpha_k \in \Sigma$, $m_k \in \mathbb{N}$ such that $\lim_k m_k = \infty$, $\lim_k \alpha_k = a$, $\lim_k d(T^{m_k}(\alpha_k), \alpha_k) = 0$,

$$\sum_{j=0}^{m_k-1} A(T^j(\alpha_k)) \geq \frac{1}{k}, \quad \text{and writing} \quad \mu_k := \frac{1}{m_k} \sum_{j=1}^{m_k-1} \delta_{T^{m_k}(\alpha_k)}, \quad \lim_k \mu_k = \mu.$$

Since $b \in \text{supp}(\mu)$ there are $n_k \leq m_k$ such that $\lim_k T^{n_k}(\alpha_k) = b$.

Let τ_k be the branch of the inverse of T^{n_k} such that $\tau_k(T^{n_k}(\alpha_k)) = \alpha_k$. Let $b_k := \tau_k(b)$. Then $T^{n_k}(b_k) = b$ and

$$\begin{aligned} d(b_k, a) &\leq d(b_k, \alpha_k) + d(\alpha_k, a) \\ &\leq \lambda^{n_k} d(b, T^{n_k}(\alpha_k)) + d(\alpha_k, a) \\ &\leq d(b, T^{n_k}(\alpha_k)) + d(\alpha_k, a) \xrightarrow{k} 0. \end{aligned}$$

We have that

$$\left| \sum_{j=0}^{n_k-1} A(T^j(b_k)) - \sum_{j=0}^{n_k-1} A(T^j(\alpha_k)) \right| \leq \frac{\|A\|_\alpha}{1 - \lambda^\alpha} d(T^{n_k}(\alpha_k), b)^\alpha.$$

$$\begin{aligned} S(a, b) &\geq \limsup_k \sum_{j=0}^{n_k-1} A(T^j(b_k)) \\ &\geq \limsup_k \sum_{j=0}^{n_k-1} A(T^j(\alpha_k)) - Q d(T^{n_k}(\alpha_k), b)^\alpha. \end{aligned}$$

Let τ_k be the branch of the inverse of $T^{m_k-n_k}$ such that $\tau_k(T^{m_k}(\alpha_k)) = T^{n_k}(\alpha_k)$. Let $a_k := \tau_k(a)$. Then $T^{m_k-n_k}(a_k) = a$ and

$$\begin{aligned} d(b, a_k) &\leq d(b, T^{n_k}(\alpha_k)) + d(T^{n_k}(\alpha_k), a_k) \\ &\leq d(b, T^{n_k}(\alpha_k)) + \lambda^{m_k-n_k} d(T^{m_k}(a_k), a) \\ &\leq d(b, T^{n_k}(\alpha_k)) + d(T^{m_k}(a_k), a) \xrightarrow{k} 0. \end{aligned}$$

Also

$$\left| \sum_{j=0}^{m_k-n_k-1} A(T^j(a_k)) - \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) \right| \leq \frac{\|A\|_\alpha}{1-\lambda^\alpha} d(a, T^{m_k}(\alpha_k)).$$

$$\begin{aligned} S(a, b) &\geq \limsup_k \sum_{j=0}^{m_k-n_k-1} A(T^j(a_k)) \\ &\geq \limsup_k \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) - Q d(a, T^{n_k}(\alpha_k)) \end{aligned}$$

Therefore

$$\begin{aligned} 0 &\geq S(a, a) \geq S(a, b) + S(b, a) \\ &\geq \limsup_k \sum_{j=0}^{n_k-1} A(T^j(\alpha_k)) + \limsup_k \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) \\ &\geq \limsup_k \left[\sum_{j=0}^{n_k-1} A(T^j(\alpha_k)) + \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) \right] \\ &\geq \limsup_k \frac{1}{k} \\ &\geq 0. \end{aligned}$$

(2). We first prove that if for some $x_0 \in \Sigma$ and $a \in \mathbb{A}(A)$ we have

$$(10) \quad F(x_0) = F(a) + S_A(a, x_0),$$

then equation (10) holds for every $a \in \mathbb{A}(A)$. If $b \in \mathbb{A}(A)$, using item 1 we have that

$$\begin{aligned} F(x_0) &\geq F(b) + S(b, x_0) \\ &\geq F(a) + S_A(a, b) + S_A(b, x_0) \\ &\geq F(a) + S_A(a, b) + S_A(b, a) + S_A(a, x_0) \\ &= F(a) + S_A(a, x_0) \\ &= F(x_0). \end{aligned}$$

Therefore $F(x_0) = F(b) + S(b, x_0)$.

It is enough to prove that given any $x_0 \in \Sigma$ there is $a \in \mathbb{A}(A)$ such that the equality (10) holds. since F is calibrated there are $x_k \in \Sigma$ and $n_k \in \mathbb{N}$ such that $T^{n_k}(x_k) = x_0$, $\exists \lim_k x_k = a$ and for every $k \in \mathbb{N}$,

$$F(x_0) = F(x_k) + \sum_{j=0}^{n_k-1} A(T^j(x_k)).$$

We have that

$$\begin{aligned}
S_A(a, x_0) &\geq \limsup_k \sum_{j=0}^{n_k-1} A(T^j(x_k)) \\
&= \limsup_k F(x_0) - F(x_k) \\
&= F(x_0) - F(a) \\
&\geq S(a, x_0).
\end{aligned}$$

Therefore equality (10) holds.

It remains to prove that $a \in \mathbb{A}(A)$, i.e. that $S_A(a, a) = 0$. We can assume that the sequence n_k is increasing. Let $m_k = n_{k+1} - n_k$. Then $T^{m_k}(x_{k+1}) = x_k$. Let τ_k be the branch of the inverse of T^{m_k} such that $\tau_k(x_k) = x_{k+1}$ and $a_{k+1} := \tau_k(a)$. We have that

$$\left| \sum_{j=0}^{m_k-1} A(T^j(a_{k+1})) - \sum_{j=0}^{m_k-1} A(T^j(x_{k+1})) \right| \leq \frac{\|A\|_\alpha}{1-\lambda^\alpha} d(a, x_k)^\alpha.$$

Since $x_k \rightarrow a$ we have that

$$\begin{aligned}
d(a_{k+1}, a) &\leq d(a_{k+1}, x_{k+1}) + d(x_{k+1}, a) \\
&\leq \lambda^{m_k} d(x_k, a) + d(x_{k+1}, a) \\
&\leq d(x_k, a) + d(x_{k+1}, a) \xrightarrow{k} 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 \geq S_A(a, a) &\geq \limsup_k \sum_{j=0}^{m_k-1} A(T^j(a_{k+1})) \\
&\geq \limsup_k \sum_{j=0}^{m_k-1} A(T^j(x_{k+1})) - Q d(a, x_k)^\alpha \\
&= \limsup_k F(x_k) - F(x_{k+1}) - Q d(a, x_k)^\alpha \\
&= 0.
\end{aligned}$$

□

The above result (2) is true for F only continuous.

Corollary 3.5. *Let \mathcal{R} be as in Theorem 3.1. Then if $A \in \mathcal{R}$, $F \in \mathcal{S}(A)$ we have*

- (1) *If $x \notin \mathbb{A}(A)$ then $I_F(x) > 0$.*
- (2) *If $x \in \mathbb{A}(A)$ and $T(x) \in \mathbb{A}(A)$ then $R_F(x) > 0$.*

Proof.

(1). By lemma 3.4.(2) modulo adding a constant there is only one Hölder calibrated sub-action F in $\mathcal{S}(A)$. Then by proposition 3.1, $\mathbb{A}(A) = [I_F = 0]$. Since $I_F \geq 0$, this proves item 1.

(2). Since $T(x) \in \mathbb{A}(A)$

$$I_F(x) = \sum_{n \geq 1} R_F(T^n(x)) = 0.$$

Since $x \notin \mathbb{A}(A)$, by item 2 and proposition 3.1, $\mathbb{A}(A) = [I_F = 0]$. Then

$$I_F(x) = \sum_{n \geq 0} R_F(T^n(x)) > 0.$$

Hence $R_F(x) > 0$.

□

Lemma 3.6.

- (1) $A \mapsto m_A$ has Lipschitz constant 1.
(2) Fix $x_0 \in \Sigma$. The set $\mathcal{S}(A)$ of α -Hölder calibrated sub-actions F for A with $F(x_0) = 0$ is an equicontinuous family. In fact

$$\sup_{F \in \mathcal{S}(A)} |F|_\alpha < \infty.$$

- (3) The set $\mathcal{S}(A)$ of α -Hölder continuous calibrated sub-actions is closed under the C^0 topology.
(4) If $\#\mathcal{M}(A) = 1$, $A_n \xrightarrow{n} A$ uniformly, $\sup_n |A_n|_\alpha < \infty$, and, $F_n \in \mathcal{S}(A_n)$, then, $\lim_n F_n = F$ uniformly.
(5) $A \leq B$ & $m_A = m_B \implies S_A \leq S_B$.
(6) $\limsup_{B \rightarrow A} \mathbb{N}(B) \subseteq \mathbb{N}(A)$, where

$$\limsup_{B \rightarrow A} \mathbb{N}(B) = \left\{ \lim_n x_n \mid x_n \in \mathbb{N}(B_n), B_n \xrightarrow{n} A, \exists \lim_n x_n \right\}$$

- (7) If $A \in \mathcal{R}$ then

$$\lim_{B \rightarrow A} d_H(\mathbb{A}(B), \mathbb{A}(A)) = 0,$$

where d_H is the Hausdorff distance.

- (8) If $A \in \mathcal{R}$ with $\mathcal{M}(A) = \{\mu\}$ and $\nu_B \in \mathcal{M}(B)$ then

$$\lim_{B \rightarrow A} d_H(\text{supp}(\nu_B), \text{supp}(\mu)) = 0.$$

- (9) If $A \in \mathcal{R}$ then

$$\lim_{B \rightarrow A} d_H(\mathbb{M}(B), \mathbb{A}(A)) = 0.$$

If X, Y are two metric spaces and $\mathbb{F} : X \rightarrow 2^Y = \mathbb{P}(Y)$ is a set valued function, define

$$\begin{aligned} \limsup_{x \rightarrow x_0} \mathbb{F}(x) &= \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{d(x, x_0) < \delta} V_\varepsilon(\mathbb{F}(x)), \\ \liminf_{x \rightarrow x_0} \mathbb{F}(x) &= \bigcap_{\varepsilon > 0} \bigcup_{\delta < 0} \bigcap_{d(x, x_0) < \delta} V_\varepsilon(\mathbb{F}(x)), \end{aligned}$$

where

$$V_\varepsilon(C) = \bigcup_{y \in C} \{z \in Y \mid d(z, y) < \varepsilon\}.$$

Proof.

- (1). We have that $A \leq B + \|A - B\|_0$, then

$$\begin{aligned} \int A \, d\mu &\leq \int B \, d\mu + \|A - B\|_0, \quad \forall \mu \in \mathcal{M}(T), \\ \int A \, d\mu &\leq \sup_{\mu \in \mathcal{M}(T)} \int B \, d\mu + \|A - B\|_0 = m_B + \|A - B\|_0, \\ m_A &\leq m_B + \|A - B\|_0. \end{aligned}$$

Similarly $m_B \leq m_A + \|A - B\|_0$ and then $|m_A - m_B| \leq \|A - B\|_0$.

See also [22] and [14] for a proof.

- (2). Let $\varepsilon > 0$ and $0 < \lambda < 1$ be such that for any $x \in \Sigma$ there is an inverse branch τ of T which is defined on the ball $B(T(x), \varepsilon) := \{z \in \Sigma \mid d(z, T(x)) < \varepsilon\}$, has Lipschitz constant λ and $\tau(T(x)) = x$.

Let $F \in \mathcal{S}(A)$. Let

$$K := |F|_\alpha := \sup_{d(x, y) < \varepsilon} \frac{|F(x) - F(y)|}{d(x, y)^\alpha}, \quad a := |A|_\alpha := \sup_{d(x, y) < \varepsilon} \frac{|A(x) - A(y)|}{d(x, y)^\alpha}$$

be Hölder constants for F and A . Given $x, y \in \Sigma$ with $d(x, y) < \varepsilon$ let $\tau_i, i = 1, \dots, m(x) \leq M$ be the inverse branches for T about x and let $x_i = \tau_i(x), y_i = \tau_i(y)$. We have that

$$\begin{aligned} |F(x_i) - F(y_i)| &\leq K; \lambda^\alpha d(x, y)^\alpha, & |A(x_i) - A(y_i)| &\leq \lambda^\alpha d(x, y)^\alpha. \\ F(x_i) + A(x_i) &\leq F(y_i) + A(y_i) + (K + a) \lambda^\alpha d(x, y)^\alpha, \\ \max_i [F(x_i) + A(x_i) - m_A] &\leq \max_i [F(y_i) + A(y_i) - m_A] + (K + a) \lambda^\alpha d(x, y)^\alpha, \\ F(x) &\leq F(y) + (K + a) \lambda^\alpha d(x, y)^\alpha, \end{aligned}$$

Then $|F|_\alpha \leq \lambda^\alpha (|F|_\alpha + |A|_\alpha)$ and hence

$$(11) \quad |F|_\alpha \leq \frac{\lambda^\alpha}{(1 - \lambda^\alpha)} |A|_\alpha.$$

This implies the equicontinuity of $\mathcal{S}(A)$.

The proof of the above result could be also get if we just assume that F is continuous.

(3). It is easy to see that uniform limit of calibrated sub-actions is a sub-action, and it is calibrated because the number of inverse branches of T is finite, i.e. $\sup_{y \in \Sigma} \#T^{-1}\{y\} < \infty$. By (2) all C^α calibrated sub-actions have a common Hölder constant, the uniform limits of them have the same Hölder constant.

(4). The family $\{F_n\}$ satisfies $F_n(x_0) = 0$ and by inequality (11)

$$|F_n|_\alpha < \frac{\lambda^\alpha}{(1 - \lambda^\alpha)} \sup_n |A_n|_\alpha < \infty.$$

Hence $\{F_n\}$ is equicontinuous. By Arzelá-Ascoli theorem it is enough to prove that there is a unique $F(x) = S_A(x_0, x)$ which is the limit of any convergent subsequence of $\{F_n\}$. Since $\sup |A_n|_\alpha < \infty$, by inequality (11), any such limit is α -Hölder. Since by lemma 3.4.(2), $\mathcal{S}(A) \cap [F(x_0) = 0] = \{F(x) = S_A(x_0, x)\}$, it is enough to prove that any limit of a subsequence of $\{F_n\}$ is a calibrated sub-action. But this follows from the continuity of $A \mapsto m_A$, the equality

$$F_n(x) = \max_{T(y)=x} F_n(y) + A_n(y) - m_{A_n}$$

and the fact $\sup_{x \in \Sigma} \#(T^{-1}\{x\}) < \infty$.

(5). The proof follows from the expression

$$S_A(x, y) := \lim_{\varepsilon \rightarrow 0} \left[\sup \left\{ \sum_{i=0}^{n-1} [A(T^i(z)) - m_A] \mid n \in \mathbb{N}, T^n(z) = y, d(z, x) < \varepsilon \right\} \right].$$

(6). Let $x_n \in B_n \rightarrow A$ be such that $x_n \rightarrow x_0$. Let $F_n \in \mathcal{S}(A)$ be such that $I_{F_n}(x_n) = 0$. Adding a constant we can assume that $F_n(x_0) = 0$ for all n . By (2), taking a subsequence we can assume that $\exists F = \lim_n F_n$ in the C^0 topology. Then F is a C^α calibrated sub-action for A . Also $R_{F_n} \rightarrow R_F$ uniformly and there is a common Hölder constant C for all the R_{F_n} . We have that

$$\begin{aligned} |R_{F_n}(T^k(x_n)) - R_F(T^k(x_0))| &\leq \\ &|R_{F_n}(T^k(x_n)) - R_{F_n}(T^k(x_0))| + |R_{F_n}(T^k(x_0)) - R_F(T^k(x_0))| \\ &\leq C d(T^k(x_n), T^k(x_0))^\alpha + \|R_{F_n} - R_F\| \xrightarrow{n} 0 \end{aligned}$$

Since for all $n, k, R_{F_n}(T^k(x_n)) = 0$, we have that $R_F(T^k(x_0)) = 0$ for any k . Hence $I_F(x_0) = 0$ and then $x_0 \in \mathbb{N}(A)$.

(7). By Lemma 3.4.(2), there is only one calibrated sub-action modulo adding a constant. Then by Proposition 3.1, $\mathbb{A}(A) = \mathbb{N}(A)$. Then by (6) $\limsup_{B \rightarrow A} \mathbb{A}(B) \subset \mathbb{A}(A)$. It is enough to prove that for any $x_0 \in \mathbb{A}(A)$ and $B_n \rightarrow A$, there is $x_n \in \mathbb{A}(B_n)$ such that $\lim_n x_n = x_0$. Let $\mu_n \in \mathcal{M}(B_n)$. Then $\lim_n \mu_n = \mu$ in the weak* topology. Given $x_0 \in \mathbb{A}(A) = \text{supp}(\mu)$ we have that

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) > 0 \quad \forall n \geq N : \quad \mu_n(B(x_0, \varepsilon)) > 0.$$

We can assume that for all $m \in \mathbb{N}$, $N(\frac{1}{m}) < N(\frac{1}{m+1})$. For $N(\frac{1}{m}) \leq n < N(\frac{1}{m+1})$ choose $x_n \in \text{supp}(\mu_n) \cap B(x_0, \frac{1}{m})$. Then $x_n \in \mathbb{A}(B_n)$ and $\lim_n x_n = x_0$.

(8). For any $B \in \mathcal{F}$ We have that

$$\text{supp}(\nu_B) \subseteq \mathbb{A}(B) \subseteq \mathbb{N}(B).$$

By item 7,

$$\limsup_{B \rightarrow A} \text{supp}(\nu_B) \subseteq \mathbb{A}(A) = \text{supp}(\mu).$$

It remains to prove that

$$\liminf_{B \rightarrow A} \text{supp}(\nu_B) \supseteq \text{supp}(\mu).$$

But this follows from the convergence $\lim_{B \rightarrow A} \nu_B = \mu$ in the weak* topology.

(9). Write $\mathcal{M}(A) = \{\mu\}$. By items (7) and (8) we have that

$$\begin{aligned} \limsup_{B \rightarrow A} \mathbb{M}(B) &\subseteq \limsup_{B \rightarrow A} \mathbb{A}(B) \subseteq \mathbb{A}(B), \\ \mathbb{A}(A) = \text{supp}(\mu) &\subseteq \liminf_{B \rightarrow A} \mathbb{M}(B). \end{aligned}$$

□

Proof of Theorem 3.1.

The set

$$\mathcal{D} := \{A \in \mathcal{F} \mid \#\mathcal{M}(A) = 1\}$$

is dense (c.f. [14]). We first prove that $\mathcal{D} \subset \overline{\mathcal{R}}$, and hence that \mathcal{R} is dense.

Given $A \in \mathcal{D}$ with $\mathcal{M}(A) = \{\mu\}$ and $\varepsilon > 0$, let $\psi \in \mathcal{F}$ be such that $\|\psi\|_0 + \|\psi\|_\alpha < \varepsilon$, $\psi \leq 0$, $[\psi = 0] = \text{supp}(\mu)$. It is easy to see that $\mathcal{M}(A + \psi) = \{\mu\} = \mathcal{M}(A)$. Let $x_0 \notin \text{supp}(\mu)$. Given $\delta > 0$, write

$$S_A(x_0, x_0; \delta) := \sup \left\{ \sum_{k=0}^{n-1} A(T^k(x_n)) \mid T^n(x_n) = x_0, d(x_n, x_0) < \delta \right\}.$$

If $T^n(x_n) = x_0$ is such that $d(x_n, x_0) < \delta$ then

$$\begin{aligned} \sum_{k=0}^{n-1} (A + \psi)(T^k(x_n)) &\leq S_A(x_0, x_0; \delta) + \sum_{k=0}^{n-1} \psi(T^k(x_n)) \\ &\leq S_A(x_0, x_0; \delta) + \psi(x_n). \end{aligned}$$

Taking $\limsup_{\delta \rightarrow 0}$,

$$S_{A+\psi}(x_0, x_0) \leq S_A(x_0, x_0) + \psi(x_0) \leq \psi(x_0) < 0.$$

Hence $x_0 \notin \mathbb{A}(A + \psi)$. Since by lemma 3.1.(1), $\text{supp}(\mu) \subset \mathbb{A}(A + \psi)$, then $\mathbb{A}(A + \psi) = \text{supp}(\mu)$ and hence $A + \psi \in \mathcal{R}$.

Let

$$\mathcal{U}(\varepsilon) := \{A \in \mathcal{F} \mid d_H(\mathbb{A}(A), \mathbb{M}(A)) < \varepsilon\}.$$

From the triangle inequality

$$d_H(\mathbb{A}(B), \mathbb{M}(B)) \leq d_H(\mathbb{A}(B), \mathbb{A}(A)) + d_H(\mathbb{A}(A), \mathbb{M}(B))$$

and items (7) and (9) of lemma 3.6, we obtain that $\mathcal{U}(\varepsilon)$ contains a neighbourhood of \mathcal{D} . Then the set

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{U}(\frac{1}{n})$$

contains a residual set.

□

4. DUALITY.

In this section we have to consider properties for A^* which depends of the initial potential A .

We will consider now the specific example described before. We point out that the results presented bellow should hold in general for natural extensions.

We will assume that T and σ are topologically mixing.

Remember that $\mathbb{T} : \hat{\Sigma} \rightarrow \hat{\Sigma}$,

$$\mathbb{T}(x, \omega) = (T(x), \tau_x(\omega)) , \quad \mathbb{T}^{-1}(x, \omega) = (\tau_\omega(x), \sigma(\omega))$$

Given $A \in \mathcal{F}$ define $\Delta_A : \Sigma \times \Sigma \times \Sigma \rightarrow \mathbb{R}$ as

$$\Delta_A(x, y, \omega) := \sum_{n \geq 0} A(\tau_{n,\omega}(x)) - A(\tau_{n,\omega}(y))$$

where

$$\tau_{n,\omega}(x) = \tau_{\sigma^n \omega} \circ \tau_{\sigma^{n-1} \omega} \circ \dots \circ \tau_\omega(x).$$

Fix $\bar{x} \in \Sigma$ and $\bar{\omega} \in \Sigma$.

The involution W -kernel can be defined as $W_A : \Sigma \times \Sigma \rightarrow \mathbb{R}$, $W_A(x, \omega) = \Delta_A(x, \bar{x}, \omega)$. Writing $\mathbb{A} := A \circ \pi_1 : \mathbb{T} \rightarrow \mathbb{R}$, we have that

$$W(x, \omega) = \sum_{n \geq 0} \mathbb{A}(\mathbb{T}^{-n}(x, \omega)) - \mathbb{A}(\mathbb{T}^{-n}(\bar{x}, \omega))$$

We can get the *dual function* $A^* : \Sigma \rightarrow \mathbb{R}$ as

$$A^*(\omega) := (W_A \circ \mathbb{T}^{-1} - W_A + A \circ \pi_1)(x, \omega).$$

Remember that we consider here the metric on Σ defined by

$$d(\omega, \nu) := \lambda^N, \quad N := \min\{k \in \mathbb{N} \mid \omega_k \neq \nu_k\}$$

Then λ is a Lipschitz constant for both τ_x , and τ_ω and also for $\mathbb{T}|_{\{x\} \times \Sigma}$ and $\mathbb{T}^{-1}|_{\Sigma \times \{\omega\}}$

Write $\mathcal{F} := C^\alpha(\Sigma, \mathbb{R})$ and $\mathcal{F}^* := C^\alpha(\Sigma, \mathbb{R})$. Let \mathcal{B} and \mathcal{B}^* be the set of coboundaries

$$\mathcal{B} := \{u \circ T - u \mid u \in C^\alpha(\Sigma, \mathbb{R})\},$$

$$\mathcal{B}^* := \{u \circ \sigma - u \mid u \in C^\alpha(\Sigma, \mathbb{R})\}.$$

Remember that

$$\|z\|_\alpha = \|z\|_0 + |z|_\alpha.$$

We also use the notation $[z]_\alpha = \|z\|_\alpha$.

Lemma 4.1.

- (1) $z \in \mathcal{B} \iff z \in C^\alpha(\Sigma, \mathbb{R}) \ \& \ \forall \mu \in \mathcal{M}(T), \int z d\mu = 0.$
- (2) *The linear subspace $\mathcal{B} \subset C^\alpha(\Sigma, \mathbb{R})$ is closed.*
- (3) *The function*

$$[z + \mathcal{B}]_\alpha = \inf_{b \in \mathcal{B}} [z + b]_\alpha$$

is a norm in \mathcal{F}/\mathcal{B} .

Proof.

(1). This follows¹ from [7], Theorem 1.28 (ii) \implies (iii).

(2). We prove that the complement \mathcal{B}^c is open. If $z \in C^\alpha(\Sigma, \mathbb{R}) \setminus \mathcal{B}$, by item (1), there is $\mu \in \mathcal{M}(T)$ such that $\int z d\mu \neq 0$. If $u \in C^\alpha(\Sigma, \mathbb{R})$ is such that

$$\|u - z\|_0 < \frac{1}{2} \left| \int z d\mu \right|$$

then $\int u d\mu \neq 0$ and hence $u \notin \mathcal{B}$.

(3). This follows from item (2). □

¹Theorem 1.28 of R. Bowen [7] asks for T to be topologically mixing.

Lemma 4.2.

- (1) If A is C^α then A^* is C^α .
- (2) The linear map $L : C^\alpha(\Sigma, \mathbb{R}) \rightarrow C^\alpha(\Sigma, \mathbb{R})$ given by $L(A) = A^*$ is continuous.
- (3) $\mathcal{B} \subset \ker L$.
- (4) The induced linear map $L : \mathcal{F}/\mathcal{B} \rightarrow \mathcal{F}^*/\mathcal{B}^*$ is continuous.
- (5) Fix one $\bar{\omega} \in \Sigma$. Similarly the corresponding linear map $L^* : \mathcal{F}^* \rightarrow \mathcal{F}$, given by

$$\begin{aligned} L^*(\psi) &= W_\psi^* \circ \mathbb{T} - W_\psi + \psi \circ \pi_2 \\ &= \sum_{n \geq 0} \Psi(\mathbb{T}^n(x, \bar{\omega})) - \Psi(\mathbb{T}^n(Tx, \bar{\omega})) \\ &= \Psi(x, \bar{\omega}) + \sum_{n \geq 0} \Psi(\mathbb{T}^n(Tx, \tau_x \bar{\omega})) - \Psi(\mathbb{T}^n(Tx, \bar{\omega})), \end{aligned}$$

with $\Psi = \psi \circ \pi_2$, is continuous and induces a continuous linear map $L^* : \mathcal{F}^*/\mathcal{B}^* \rightarrow \mathcal{F}/\mathcal{B}$, which is the inverse of $L : \mathcal{F}/\mathcal{B} \rightarrow \mathcal{F}^*/\mathcal{B}^*$.

Proof.

(1) and (2). We have that

$$\begin{aligned} A^*(\omega) &= \sum_{n \geq 0} \mathbb{A}(\mathbb{T}^{-n}(\bar{x}, \omega)) - \mathbb{A}(\mathbb{T}^{-n}(\bar{x}, \sigma \omega)) \\ &= A(\bar{x}) + \sum_{n \geq 0} \mathbb{A}(\mathbb{T}^{-n}(\tau_\omega \bar{x}, \sigma \omega)) - \mathbb{A}(\mathbb{T}^{-n}(\bar{x}, \sigma \omega)) \end{aligned}$$

Since $d(\mathbb{T}^{-n}(\tau_\omega \bar{x}, \sigma \omega), \mathbb{T}^{-n}(\bar{x}, \sigma \omega)) \leq \lambda^n d(\tau_\omega \bar{x}, \bar{x}) \leq \lambda^n$ and $\|\mathbb{A}\|_\alpha = \|A \circ \pi_1\|_\alpha = \|A\|_\alpha$, we have that

$$\|A^*\|_0 \leq \|A\|_0 + \frac{\|A\|_\alpha}{1 - \lambda^\alpha}.$$

Also if $m := \min\{k \geq 0 \mid w_k \neq \nu_k\}$

$$\begin{aligned} A^*(\omega) - A^*(\nu) &= \sum_{n \geq m-1} \mathbb{A}(\mathbb{T}^{-n}(\tau_\omega \bar{x}, \sigma \omega)) - \mathbb{A}(\mathbb{T}^{-n}(\bar{x}, \sigma \omega)) \\ &\quad - \sum_{n \geq m-1} \mathbb{A}(\mathbb{T}^{-n}(\tau_\nu \bar{x}, \sigma \nu)) - \mathbb{A}(\mathbb{T}^{-n}(\bar{x}, \sigma \nu)) \\ |A^*(\omega) - A^*(\nu)| &\leq 2 \|A\|_\alpha \frac{\lambda^{(m-1)\alpha}}{1 - \lambda^\alpha} = \frac{2 \|A\|_\alpha \lambda^{-\alpha}}{1 - \lambda^\alpha} d(\omega, \nu)^\alpha. \\ \|A^*\|_\alpha &\leq \frac{2 \|A\|_\alpha}{\lambda^\alpha(1 - \lambda^\alpha)}. \end{aligned}$$

(3). If $u \in \mathcal{F}$ and $\mathbb{U} := u \circ \pi_1$ from the formula for L (in the proof of item [1]) we have that

$$\begin{aligned} L(u \circ T - u) &= \mathbb{U}(\mathbb{T}(\bar{x}, \omega)) - \mathbb{U}(\mathbb{T}(\bar{x}, \sigma \omega)) \\ &= u(T\bar{x}) - u(T\bar{x}) = 0. \end{aligned}$$

(4). Item (4) follows from items (2) and (3).

(5). We only prove that for any $A \in \mathcal{F}$, $L^*(L(A)) \in A + \mathcal{B}$. Write

$$\begin{aligned} L^*(L(A)) &= (W_{A^*}^* \circ \mathbb{T} - W_{A^*}^* + \mathbb{A}^*)(\cdot, \bar{\omega}) \\ &= (W_{A^*}^* \circ \mathbb{T} - W_{A^*}^* + W_A \circ \mathbb{T}^{-1} - W_A) + A \end{aligned}$$

Write

$$(12) \quad B := W_{A^*}^* \circ \mathbb{T} - W_{A^*}^* + W_A \circ \mathbb{T}^{-1} - W_A.$$

Since $A, L^*(L(A)) \in \mathcal{F} = C^\alpha(\Sigma, \mathbb{R})$, then $B \in C^\alpha(\Sigma, \mathbb{R})$.

Following Bowen, given any $\mu \in \mathcal{M}(T)$ we construct an associated measure $\nu \in \mathcal{M}(\mathbb{T})$. Given $z \in C^0(\Sigma \times \Sigma, \mathbb{R})$ define $z^\sharp \in C^0(\Sigma, \mathbb{R})$ as $z^\sharp(x) := z(x, \bar{w})$. We have that

$$\|(z \circ \mathbb{T}^n)^\sharp \circ T^m - (z \circ \mathbb{T}^{n+m})^\sharp\|_0 \leq \text{var}_n z \xrightarrow{n} 0,$$

where

$$\begin{aligned} \text{var}_n z &= \sup\{|z(a) - z(b)| \mid \exists x \in \Sigma, a, b \in \mathbb{T}^n(\{x\} \times \Sigma)\} \\ &\leq \sup\{|z(a) - z(b)| \mid d_{\Sigma \times \Sigma}(a, b) \leq \lambda^n\} \xrightarrow{n} 0, \\ d_{\Sigma \times \Sigma} &= d_\Sigma \circ (\pi_1, \pi_1) + d_\Sigma \circ (\pi_2, \pi_2). \end{aligned}$$

Then

$$|\mu((z \circ \mathbb{T}^n)^\sharp) - \mu((z \circ \mathbb{T}^{n+m})^\sharp)| = |\mu((z \circ \mathbb{T}^n)^\sharp \circ T^m) - \mu((z \circ \mathbb{T}^{n+m})^\sharp)| \leq \text{var}_n z.$$

Therefore $\mu((z \circ \mathbb{T}^n)^\sharp)$ is a Cauchy sequence in \mathbb{R} and hence the limit

$$\nu(z) := \lim_n \mu((z \circ \mathbb{T}^n)^\sharp)$$

exists. By the Riesz representation theorem ν defines a Borel probability measure in $\Sigma \times \Sigma$, and it is invariant because

$$\nu(z \circ \mathbb{T}) = \lim_n \mu((z \circ \mathbb{T}^{n+1})^\sharp) = \nu(z).$$

Now let $B := L^*(L(A)) - A$ and $\mathbb{B} := B \circ \pi_1$. By formula (12) we have that \mathbb{B} is a coboundary in $\Sigma \times \Sigma$. Since $\pi_1 \circ T^n = T^n$ we have that

$$\begin{aligned} 0 = \nu(\mathbb{B}) &= \lim_n \mu((\mathbb{B} \circ \mathbb{T}^n)^\sharp) \\ &= \lim_n \mu(B \circ T^n) \\ &= \mu(B). \end{aligned}$$

Since this holds for every $\mu \in \mathcal{M}(T)$, by lemma 4.1.(1), $B \in \mathcal{B}$ and then $(L^+ \circ L)(A + \mathcal{B}) \subset A + \mathcal{B}$. □

Theorem 4.1.

There is a residual subset $\mathcal{Q} \subset C^\alpha(\Sigma, \mathbb{R})$ such that if $A \in \mathcal{Q}$ and $A^ = L(A)$ then*

$$(13) \quad \begin{aligned} \mathcal{M}(A) &= \{\mu\}, & \mathbb{A}(A) &= \text{supp}(\mu), \\ \mathcal{M}(A^*) &= \{\mu^*\}, & \mathbb{A}(A^*) &= \text{supp}(\mu^*). \end{aligned}$$

In particular

$$\begin{aligned} I_A(x) > 0 & \quad \text{if} \quad x \notin \text{supp}(\mu), \\ I_A(\omega) > 0 & \quad \text{if} \quad \omega \notin \text{supp}(\mu^*). \end{aligned}$$

and

$$\begin{aligned} R_A(x) > 0 & \quad \text{if} \quad x \notin \text{supp}(\mu) \quad \text{and} \quad T(x) \in \text{supp}(\mu), \\ R_A(\omega) > 0 & \quad \text{if} \quad \omega \notin \text{supp}(\mu^*) \quad \text{and} \quad \sigma(\omega) \in \text{supp}(\mu). \end{aligned}$$

Proof. Observe that the subset \mathcal{R} defined in (9) in theorem 3.1 is invariant under translations by coboundaries, i.e. $\mathcal{R} = \mathcal{R} + \mathcal{B}$. Indeed if $B = u \circ T - u \in \mathcal{B}$, we have that

$$\int (A + B) d\mu = \int A d\mu, \quad \forall \mu \in \mathcal{B},$$

$$S_{A+B}(x, y) = S_A(x, y) + B(x) - B(y), \quad \forall x, y \in \Sigma.$$

Then the Aubry set and the set of minimizing measures are unchanged: $\mathcal{M}(A + B) = \mathcal{M}(A)$, $\mathbb{A}(A + B) = \mathbb{A}(A)$.

For the dynamical system (Σ, σ) let

$$\mathcal{R}^* = \{ \psi \in C^\alpha(\Sigma, \mathbb{R}) \mid \mathcal{M}(\psi) = \{ \mu \}, \mathbb{A}(\psi) = \text{supp}(\nu) \}$$

By theorem 3.1, the subset \mathcal{R}^* contains a residual set in $\mathcal{F}^* = C^\alpha(\Sigma, \mathbb{R})$ and it is invariant under translations by coboundaries: $\mathcal{R}^* = \mathcal{R}^* + \mathcal{B}^*$

By lemma 4.2 the linear map $L : \mathcal{F}/\mathcal{B} \rightarrow \mathcal{F}^*/\mathcal{B}^*$ is a homeomorphism with inverse L^* . Then the set $\mathcal{Q} := \mathcal{R} \cap L^{-1}(\mathcal{R}^*) = \mathcal{R} \cap (L^*(\mathcal{R}^*) + \mathcal{B})$ contains a residual subset and satisfies (13).

By Corollary 3.5 the other properties are automatically satisfied. □

From this last theorem it follows our main result about the generic potential A to be in \mathbb{G} .

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