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Entropy and large deviation

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Abstract. We show the existence of a deviation function for the maximal measure μ of a hyperbolic rational map of degree d (see theorem 7). We relate several results of large deviation with the thermodynamic formalism of ergodic theory. The maximal measure plays a distinguished role among other invariant measures, because the stochastic process given by the rational map and the maximal measure will generate a free energy function, whose Legendre transform in the set of invariant measures will be $\log d$ minus the entropy in the sense of Shannon–Kolmogorov (see theorem 7). This result is associated with the relation between pressure and free energy given by theorem 3.

A general description of the result is as follows. Consider μ to be the maximal entropy measure and v another invariant measure. The ergodic theorem claims that the mean of the sum of Dirac measures in the orbit of a μ-almost everywhere point z, will converge to μ. Given a convex neighbourhood G of v in the set of measures, we can estimate the deviations of the mean of the sum of Dirac measures in the orbit of a μ-almost everywhere point z, with respect to this neighbourhood G. If the neighbourhood G is very small and we consider large iterates, the exponential value of decreasing of the μ-measure of points whose mean orbit is in G is approximately the entropy of v minus $\log d$. In this way, we can calculate the entropy of v as an information of large deviation related to the maximal measure μ.

We will apply this result, using a contraction principle, to measure the deviation of the Liapunov number of the maximal measure.

The same proof presented in this paper also works (with minor modifications) for shifts of finite type in the lattice $\mathbb{N}$.

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1. Introduction

The main purpose of this paper is to analyse the heuristic relationship between the concepts of pressure in the sense of ergodic theory [35] and the free energy in the sense of large deviation (as presented by Ellis [5]). By means of the Legendre transform, this relationship turns out to be a relationship between entropy (in the sense of Shannon–Kolmogorov) and the deviation function (in the sense of level-2 large deviation) (see theorem 7)).

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Free energy (respectively pressure) and deviation function (respectively entropy) are dual concepts in the formulation of the same statistical physics problem. This situation is heuristically analogous with the formulation of classical mechanics in Lagrangian and Hamiltonian mechanics.

A crucial matter in large deviation theory is the measure that one chooses to begin the consideration of the large deviation. In this sense, the maximal measure (see [6, 16, 21]) is the one that you have to consider. In this way the deviation function turns out to be, up to a constant, the entropy.

We became interested in the applications of large deviation to dynamical systems after reading the work of Collet *et al* [3] and Rand [30]. In these papers the authors explained the theory of the spectrum of dimension that was discussed initially by people working in statistical mechanics (see [7, 8] for references and applications; also see [14] for related results for the maximal measure).

All these results can be seen (following the notation of Ellis [5]) as problems in level-1. If one considers large deviations in the context of measures, then one is considering problems in level-2. The contraction principle as considered in Orey's paper [24] is a very useful instrument for transferring results from level-2 to level-1. We refer the reader to the book of Ellis [5] and the paper of Orey [24] as general references on the subject of large deviation. The reader can find general results on the maximal measure in [6, 11–13, 16, 21]. We will use in the proof several results on ergodic theory and thermodynamic formalism. Good sources of reference on the subject are [9, 17, 32, 33, 35].

It will be essential for the proof of the main result to show that any invariant measure can be approximated weakly and in entropy by other invariant measures that have a Holder-continuous Jacobian. These measures are also equilibrium states [15, 18]. These results are basically contained in [9, 10, 34]; but we will give a proof of them in the appendix.

Generally the proof proceeds by first relating the pressure and the free energy of the large deviation associated with the maximal measure (see sections 2 and 3). Then we use the result of the paragraph above to obtain a new free energy for a different large deviation system. By using the fact that equilibrium states correspond to sub-differentiability [34, 35], we will be able to conclude the proof (see section 4).

We point out that when we try to work with the mean of ∂-Diracs in the orbit of μ-almost everywhere point (also called the empirical measure [24]), we cannot avoid working with non-invariant measures. The concept of entropy is essentially designed for invariant measures, but this will not create any trouble, because, the deviation function for non-invariant measure will play no important role in the problem, as we will see in section 4.

An announcement of the results of this paper as well as new results about a kind of Donsket–Vatadhan variational formula appeared in [13].

2. Ergodic theory of rational maps

Let $M(f)$ be the set of invariant probabilities for f, i.e. the set of measures v such that $v(f^{-1}(A)) = v(A)$, for any set A in the Borel sigma-algebra of $\mathbb{R}^2$, and also $v(\mathbb{R}^2) = 1$.

The support of all these invariant measures is the Julia set of f (see [6, 19, 21] for exact definitions). We will denote such a set by J.

Definition 1. For a continuous function $g:J\to\mathbb{R}$ and u in $M(f)$ we define the pressure of u with respect to g by

$$h(u)+\int g(z)\,\mathrm{d}u(z)$$

where $h(u)$ denotes the entropy of u in the sense of Shannon–Kolmogorov (see [17, 35]). We will denote such an expression by $P(u, g)$.

Definition 2. We denote $P(g)=\sup\{P(u,g)\mid u \text{ in } M(f)\}$, the pressure (sometimes called the topological pressure) of the function g.

Let C be the set of continuous functions in J. Here we will suppose the rational map is hyperbolic (see [19, 21] for definition).

Definition 3. If there exists a unique probability, denoted by $u(g)$, such that

$$P(g)=h(u(g))+\int g(z)\,\mathrm{d}(u(g))(z)$$

then we will call $u(g)$ the maximal pressure measure for g.

In the hyperbolic case such measure always exists and is unique when g is Holder continuous [26].

Definition 4. When g is a constant equal to zero, the maximal pressure measure is called the maximal measure (sometimes also called the balanced measure) [1, 6, 11–13, 16, 21].

Let z_0 be a point in the Riemann sphere, and for each $n\in\mathbb{N}$, denote by $z(n, i, z_0)$, $i\in\{1, 2, 3, \ldots, d^n\}$ the d^n solutions (with multiplicity) of the equation

$$f^n(z)=z_0.$$

Denote the Dirac delta measure on z by $\delta(z)$.

Let $u(n, z_0)$ be the probability

$$d^{-n}\sum_{i=1}^{d^n}\delta(z(n, i, z_0)).$$

In [6, 16], it was shown that for any z_0 (but at most two exceptional points), and independent of z_0, there exists the weak limit

$$\lim_{n\to\infty} u(n, z_0)=\mu$$

and the measure μ is the maximal measure.

To obtain this result, the hyperbolicity condition is not assumed.

Given $0<\gamma<1$, denote by $C(\gamma)$ the space of Holder-continuous real-valued functions in J endowed with the metric

$$\|g\|=\|g\|_0+\sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|^{\gamma}}$$

where $\|g\|_0$ is the usual supreme norm.

Ruele [31] showed the real analyticity of the Hausdorff dimension of the Julia set as a function of the parameter for any analytic family of hyperbolic rational maps.

Theorem 1 ([26, 31, 32]). Let f be a hyperbolic rational map, then

$$C(\gamma) \ni g \to P(g)$$

is real analytic as a function of g.

A very nice characterisation of the pressure is given by the theorem stated below.

Theorem 2 ([26, 31]). Let f be a hyperbolic rational map and $g \in C(\gamma)$, then

$$P(g) = \lim_{n\to\infty} n^{-1} \log \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i))\Big),$$

where $z(n, i) = z(n, i, z_0)$ for any z_0.

Now we will give another characterisation of the pressure that is more suitable for use in large deviation theory.

Theorem 3. Let f be a hyperbolic rational map, and $g \in C(\gamma)$, then

$$P(g) = \lim_{n\to\infty} n^{-1}\Big[\log \int \exp\Big(\sum_{j=0}^{n-1} g(f^j(z))\Big)\, \mathrm{d}\mu(z)\Big] + \log d.$$

Proof. Let g be a Holder-continuous function on the compact set J.

Let us consider a fixed z_0 and call $z(n, i)$ the $z(n, i, z_0)$, $n \in N$ and $i \in \{1, 2, 3, \ldots, d^n\}$.

Now consider $n \in N$, then

$$\int \exp\Big(\sum_{j=0}^{n-1} g(f^j(z)\Big)\, \mathrm{d}\mu(z) = \lim_{m\to\infty} d^{-m} \sum_{i=1}^{d^m} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(m, i))\Big)$$
$$= \lim_{m\to\infty} d^{-m} \sum_{k=1}^{d^{m-n}} \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i, (z(m-n, k)))\Big).$$

From [6], there exist constants C_1, c_1 such that for n large enough

$$c_1 \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i, z)))\Big) \leqslant \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i)))\Big)$$
$$\leqslant C_1 \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i, z)))\Big) \tag{1}$$

for any $z \in J$.

Therefore,

$$c_1 d^{-m} d^{m-n} \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i)))\Big)$$
$$\leqslant d^{-m} \sum_{k=1}^{d^{m-n}} \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i, (z(m-n, k)))))\Big)$$
$$\leqslant C_1 d^{-m} d^{m-n} \sum_{i=1}^{d^n} \exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n, i)))\Big).$$

From this, it follows that

$$\lim_{n\to\infty} n^{-1}\log\int \exp\Big(\sum_{j=0}^{n-1} g(f^j(z))\Big)\,d\mu(z)$$
$$= \lim_{n\to\infty} n^{-1}\log\sum_{i=1}^{d^n}\exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n,i))\Big) - \log d$$

and the theorem is proved.

Remark 0. From [6] we know that such measure μ has entropy $\log d$.

Now we will consider g to be a continuous function not necessarily Holder continuous.

Theorem 4. The function $c: C\to\mathbb{R}$ is well defined for $g\in C$ as

$$c(g) = \lim_{n\to\infty}\frac{1}{n}\Big[\log\int \exp\Big(\sum_{j=0}^{n-1} g(f^j(z))\Big)\,d\mu(z)\Big]$$
$$= -\log d + \lim_{n\to\infty}\frac{1}{n}\log\sum_{i=0}^{d^n}\Big[\exp\Big(\sum_{j=0}^{n-1} g(f^j(z(n,i)))\Big)\Big]$$

and is continuous on C with the norm sup.

Proof. Consider

$$a_n = \log\sum_{i=1}^{d^n}\exp\sum_{j=0}^{n-1}(g(f^j(z(n,i)))).$$

We will show that a_n is subadditive, and therefore the function c is well defined. Suppose $m>n$, then

$$a_m - a_n = \log\sum_{i=1}^{d^m}\exp\sum_{j=0}^{m-1}(g(f^j(z(m,i)))) - \log\sum_{i=1}^{d^n}\exp\sum_{j=0}^{n-1} g(f^j(z(n,i)))$$
$$= \log\Big\{\Big[\sum_{k=1}^{d^{m-n}}\Big(\sum_{i=1}^{d^n}\exp\sum_{j=0}^{n-1} g(f^j(z(n,i,z(m-n,k))))\Big)$$
$$\times\exp\sum_{l=0}^{m-n} g(f^l(z(m-n,k)))\Big]\Big(\sum_{i=1}^{d^n}\exp\sum_{j=0}^{n-1} g(f^j(z(n,i)))\Big)^{-1}\Big\}.$$

Now from (1) for m and n large enough

$$a_m - a_n \le \log C\sum_{k=1}^{d^{m-n}}\exp\sum_{l=0}^{m-n} g(f^l(z(m-n),k)) = \log C + a_{m-n}.$$

As g is continuous in the compact set J, then g is bounded, therefore $-\infty<\inf a_n/n$. Now we will use the usual subadditive arguments. Consider $p=\log C$; then, as we had before,

$$a_m < a_n + a_{m-n} + p.$$

Consider $n_0\in\mathbb{N}$ and $n = kn_0 + i$, $0<i\le n_0$. Then

$$\frac{a_n}{n} = \frac{a_{kn_0}+i}{kn_0+1} \le \frac{a_{kn_0}}{kn_0} + \frac{a_i}{kn_0} + \frac{p}{kn_0} \le \frac{ka_{n0}}{kn_0} + \frac{a_i}{kn_0} + \frac{kp}{kn_0} = \frac{a_{n0}}{n_0} + \frac{a_i}{kn_0} + \frac{p}{n_0}.$$

As n goes to infinity then $k \to \infty$, so

$$\limsup_{n\to\infty} \frac{a_n}{n} \leqslant \frac{a_{n_0}}{n_0} + \frac{p}{n_0} \qquad n_0 \in \mathbb{N}.$$

Now

$$\limsup_{n\to\infty} \frac{a_n}{n} \leqslant \liminf_{n_0\to\infty} \left(\frac{a_{n_0}}{n_0} + \frac{p}{n_0}\right) = \liminf_{n\to\infty} \frac{a_n}{n}.$$

Therefore, there exists $\lim_{n\to\infty} a_n/n$ and $c(g)$ is well defined.

The proof that

$$\lim_{n\to\infty} \frac{1}{n}\left[\log \int \exp\left(\sum_{j=0}^{n-1} g(f^j(z))\right) \mathrm{d}\mu(z)\right] + \log d = \lim \frac{1}{n} \log \sum_{i=0}^{d^n} \left(\exp \sum_{j=0}^{n-1} g(f^j(z(n,i)))\right)$$

is the same as in theorem 3.

Now we will show that c is continuous. Suppose for $f, g \in \mathbb{C}$ that for $x \in J$, $|g(x) - h(x)| < \xi$.

Therefore, for $n \in \mathbb{N}$, $i \in \{1, \ldots, d^n\}$

$$\begin{aligned}
\exp(-\xi n) &\leqslant \exp\left(\sum_{j=0}^{n-1} g(f^j(z(n,i)))\right)\left[\exp\left(\sum_{j=0}^{n-1} h(f^j(z(n,i)))\right)\right]^{-1} \\
&= \exp\left(\sum_{j=0}^{n-1} (g-h)(f^j(z(n,i)))\right) \\
&\leqslant \exp(\xi n)
\end{aligned} \tag{2}$$

In this case

$$\begin{aligned}
&\left|\frac{1}{n}\log \sum_{i=1}^{d^n} \exp\left(\sum_{j=0}^{n-1} g(f^j(z(n,i)))\right) - \frac{1}{n}\log \sum_{i=1}^{d^n} \exp\left(\sum_{j=0}^{n-1} h(f^j(z(n,i)))\right)\right| \\
&\qquad = \frac{1}{n}\log\left\{\left[\sum_{i=1}^{d^n} \exp\left(\sum_{j=0}^{n-1} g(f^j(z(n,i)))\right)\right]\left[\sum_{i=1}^{d^n} \exp\left(\sum_{j=0}^{n-1} h(f^j(z(n,i)))\right)\right]^{-1}\right\}.
\end{aligned}$$

Now, using (2), we conclude from the above expression that

$$-\xi < \left|\frac{1}{n}\log \sum_{i=1}^{d^n} \exp\left(\sum_{j=0}^{n-1} g(f^j(z(n,i)))\right) - \frac{1}{n}\log \sum_{i=1}^{d^n} \exp\left(\sum_{j=0}^{n-1} h(f^j(z(n,i)))\right)\right| \leqslant \xi$$

and, therefore, c is continuous.

This is the end of the proof of theorem 4.

Remark 1. We point out that from theorems 2, 3 and 4, we have, for any $g \in C(\gamma)$,

$$c(g) = P(g) - \log d.$$

Now we will state some slightly different versions of some theorems presented in [35]. The proofs of these theorems are the same as presented in the theorems in sections 9.4 and 9.5 of [35], using the fact that the Holder-continuous functions are dense in the set of continuous functions, and that the entropy is lower-semicontinuous for a rational map [16, 23].

Theorem 5. Let v be in the set E of finite signed measure with support on J. Then v is an invariant measure for g if and only if

$$\int g \,\mathrm{d}v \leqslant P(g) \qquad g \in C(\gamma) \quad \left(\text{also} \int g\,\mathrm{d}v \leqslant P(g) \qquad \forall g \in C\right).$$

Theorem 6. For any $v \in M(f)$ we have

$$h(v) = -\inf\left\{P(g) - \int g \, dv \mid g \in C(\gamma)\right\}$$

(also $h(v) = -\inf\{P(g) - \int g \, dv \mid g \in C\}$) and for $v \in E - M(f)$

$$-\inf\{P(g) - \int g \, dv \mid g \in C(\gamma)\} \leqslant 0$$

(also $-\inf\{P(g) - \int g \, dv \mid g \in C\} \leqslant 0$).
Denote by M the set of probabilities on J.

3. Large deviation theory

Here we recall some basic definitions and properties stated in Ellis' book [5] and Orey's paper [24].

Let $W = \{W_n : n = 1, 2, 3, \ldots\}$ be a sequence of random vectors which are defined on probability spaces $\{(\Omega_n, F_n, P_n); n = 1, 2, 3, \ldots\}$ and which takes values in a vector space E. We will consider here $\Omega_n = J$, $n \in \mathbb{N}$.

Definition 5. If the space E considered above is $\mathbb{R}^d$, then we define for each $n \in \mathbb{N}$ and $t \in \mathbb{R}^d$ the function

$$c_n(t) = n^{-1} \log E_n \exp\langle t, W_n \rangle$$

where E_n is the expected value with respect to P_n and $\langle \, , \rangle$ is the usual inner product in $\mathbb{R}^d$.

Definition 6. In the case that E is the space of signed measures on J, we define for each $n \in \mathbb{N}$ and $t \in C$ the function

$$c_n(t) = n^{-1} \log E_n \exp\langle t, W_n \rangle$$

where, for each $t \in C$ and signed measure v, the expression $\langle t, v \rangle$ means $\int t(z) \, dv(z)$.

We will consider in this case the weak topology in the set of signed measures.

The following hypotheses are assumed to hold.

(a) Each function $c_n(t)$ is finite for all $t \in C$.

(b) $c(t) = \lim_{n \to \infty} c_n(t)$ exists for all $t \in E$ and is finite.

Definition 7. Assume that hypotheses (a) and (b) hold, and denote for each Borelean set K in E,

$$Q_n(K) = P_n\{z \in J \mid n^{-1} W_n(z) \in K\}$$

$$I(v) = \sup_{t \in C} \{\langle t, v \rangle - c(t)\} \qquad v \in E$$

then $c(t)$ is called the *free energy* of the function $t \in C$, and $I(v)$ as a function of the variable $v \in E$ is called the Legendre–Fenchel transform of the free energy $c(t)$ (i.e., a function of the variable $t \in C$). The function $I(v)$ is also called the deviation function.

Theorem 5 ([4, 5]). Assume that hypotheses (a) and (b) hold and $E = \mathbb{R}^n$.

Then the following conclusions hold.

(c) $I(z)$ is convex, closed and

$$\inf_{v \in E} \{I(v)\} = 0.$$

(d) $\lim_{n \to \infty} \sup n^{-1} \log Q_n(K) \leqslant -\inf_{v \in K}\{I(v)\}$, for any closed set K in E.

(e) $\lim_{n \leftarrow \infty} \inf n^{-1} \log Q_n(G) \geqslant -\inf_{v \in G}\{I(v)\}$, for any open set G in E, if $c(t)$ is differentiable in t.

Definition 8. If a process W satisfies conditions (d) and (e) in the above theorem, we will say it has a large deviation property (we do not assume that $c(t)$ is differentiable).

In section 4 we will consider W_n to be the sum of the Dirac measures

$$W_n(z) = \delta(z) + \delta(f(z)) + \ldots + \delta(f^{n-1}(z)) \qquad n \in \mathbb{N}$$

and we will show that this process has a large deviation property when $P_n = \mu$, the measure of maximal entropy. Conditions (a) and (b) will follow from section 3, but the problem here is that in this case E is not finite dimensional, but the set of signed measures on J. We will assume from now on that E is the space of signed measures on J.

We would like to point out that the upper bound (d) is, in general, true in infinite-dimensional cases (see [4]).

The lower bound (e) needs some extra assumptions, basically here we will need that each measure can be approximated by a measure that is the unique derivative of the free energy of a certain Holder function (in fact, a Jacobian of a measure).

The general theorem that claims that, for a generic set in C, the free energy has a unique derivative (see [9]), is not enough to prove what we want. We will also need a control of the entropy of the approximated measure. These results will be presented in section 4 and appendix 1.

If a set G is a set of I-continuity (see [5], page 37, for a definition) and

$$\lim_{n \to \infty} \frac{1}{n} \log Q_n(G) = -\inf_{v \in G} (I(v))$$

we will say that W has a large deviation property.

The assumption of I-continuity is to avoid sets with complicated boundaries and to have the above equality.

4. Large deviation properties of the maximal measure

Let $Y_1 : J \to E$ be the function that for each $z \in J$, associates a Dirac delta in the point z, denoted by $\delta(z)$. We will consider here the process given by the rational map f and μ, the maximal measure on the Borel sigma-algebra of $\mathbb{R}^2$.

In this context Y_1 is a random variable taking values in the infinite-dimensional set of signed measures on J [4, 24].

Here we will be interested in the stochastic process

$$W_n(z)=\sum_{i=0}^{n-1} Y_1(f^i(z))=\sum_{i=0}^{n-1}\delta(f^i(z))$$

$\Omega_n=J$, $F_n=$ Borel signal field of J and $P_n=\mu$, for all $n\in\mathbb{N}$, following the notation introduced in section 3.

The above system is called the empirical measure [24]. Note that $n^{-1}W_n(z)$ is a probability.

For each $n\in N$ and $t\in C$, consider

$$c_n(t)=n^{-1}\log\int\exp(\langle t, W_n(z)\rangle)\,\mathrm{d}\mu(z)$$

$$=n^{-1}\log\int\exp(t(z)+t(f(z))+\ldots+t(f^{n-1}(z)))\,\mathrm{d}\mu(z)$$

where μ is the maximal measure.

Consider now, for $t\in C$, the limit

$$c(t)=\lim_{n\to\infty}c_n(t)=\lim_{n\to\infty}n^{-1}\log\int\exp(t(z)+\ldots+t(f^{n-1}(z)))\,\mathrm{d}\mu(z).$$

As we have seen before for $t\in C(\gamma)$, this last limit is equal to $P(t)-\log d$ (see theorem 3).

Theorem 7. Let f be a hyperbolic rational map of degree d and μ the maximal measure, then under the definitions above, we have that for any open convex set G of I-continuity, such that $G\cap M(f)\neq\emptyset$, then

$$\lim_{n\to\infty}\frac{1}{n}\log\mu\{z\in J\mid n^{-1}W_n(z)\in G\}$$

exists and is equal to

$$-\inf_{v\in G}\{I(v)\}=-\inf_{v\in G\cap M(f)}\{\log d-h(v)\}.$$

In other words $I(v)=\log d-h(v)$ for $v\in M(f)$.

Proof. Note first that from theorem 4, $c(t)$ satisfies (a) and (b) in section 3, and therefore from [4, 5] we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu\{z\in J\mid n^{-1}W_n(z)\in K\}\leqslant-\inf_{z\in K}\{I(z)\}$$

for any closed set K in E.

Then we have to show the lower bound

$$\liminf_{n\to\infty}\frac{1}{n}\log\mu\{z\in J\mid n^{-1}W_n(z)\in G\}\geqslant-\inf_{z\in G}\{I(z)\}$$

for open set G.

Remark 2. Before going to the proof of this result, we point out that if v is an invariant measure, then from theorem 6 we have

$$-h(v)=\inf\left\{P(g)-\int g\,\mathrm{d}v \mid g\in C(\gamma)\right\}$$

$$=\inf\left\{c(g-\int g\,\mathrm{d}v \mid g\in C(\gamma)\right\}+\log d$$

$$=\inf\left\{c(g)-\int g\,\mathrm{d}v \mid g\in C\right\}+\log d=I(v)+\log d.$$

If v is a non-invariant signed measure, then from theorem 6

$$\inf\left\{P(g)-\int g\,\mathrm{d}v \mid g\in C(\gamma)\right\}\leqslant 0.$$

As c is continuous and $C(\gamma)$ is dense in C, it follows that

$$I(v)\geqslant \log d.$$

Therefore, if there exists a $v\in M(f)\cap G$, we do not have to worry about $x\in G-M(f)$, because $I(x)\geqslant \log d\geqslant I(v)$, and

$$\inf_{G}\{I(v)\}=\inf_{G\cap M(f)}\{I(v)\}.$$

Consider G to be open convex set of I-continuity in E such that it contains $v\in M(f)$.

We will basically follow Ellis' proof in [5]. We mention that the result (VII.4) of [5] is for the finite-dimensional case and, to obtain the above result in the infinite-dimensional case, we have to use theorem II.3.3 of [5].

Consider for $\xi>0$, $A(z,\xi)$ to be the small neighbourhood of size ξ and centre z, for $z\in M$.

Now we will state a result we will need.

Theorem 8. Suppose f is a hyperbolic rational map and v is an invariant probability for f. Then, given $\delta>0$, there exists an invariant probability p weakly close to v and a Holder continuous function ϕ such that $h(p)>h(v)-\delta$ and p is the unique solution of

$$0=P(\phi)=h(p)+\int \phi(z)\,\mathrm{d}p(z).$$

The function ϕ will be the logarithm of the Jacobian of the probability p (see [15, 18]). Jacobian here means Radon–Nikodyn derivative

$$\lim_{r\to 0}\frac{p(B(z,r))}{p(f(B(z,r)))}.$$

The proof of theorem 8 will be sketched in the appendix.

Remark 3. The above result is essentially proved in theorem 1.1 of [10] and in lemma IV, 3.2 of Israel's book [9]. Other related references can be found in [32–34]. The result in these references is shown to be true in the context of classical and quantum lattice gases [9, 10, 33]. These cases include the two-sided shift. From an easy projection argument, the analogous result for one-sided shifts follows. For a hyperbolic rational map, it is possible to transfer results from one-sided shift to results about the ergodic theory of rational maps (see [20, 27–29]). In this way, theorem 8 follows essentially from Israel and Phelps' results.

As the results stated in the context of classical and quantum gases are written in a notation that sometimes can be difficult to follow for the non-expert in the subject, we will sketch a different proof of the result in the appendix.

Now we will assume theorem 8 is already proved and we will continue the proof of theorem 7.

If the measure v is not invariant, then from theorems 5 and 6, $-I(v)$ is smaller than $-I(u)$ for any $u \in M(f)$. Therefore, if there exists a $u \in M(f) \cap G$, then we don't have to look at the case $v \in G - M(f)$ in our considerations.

Therefore, from now on we will suppose that $v \in M(f)$.

Now, from theorem 8 there exists a measure z close to v such that $h(z)$ is arbitrarily close to $h(v)$ and there exists a Holder-continuous t such that the only solution of $0 = P(t) = h(q) + \int t(y)\, dq(y)$, $q \in M(f)$, is for $z = q$.

In the context of convex functions and subdifferentials (see [34])

$$\delta P(t) = z.$$

Using the notation of Ellis [5], we will consider $Q_n(du)$ to be the distribution of the random variable $(1/n)W_n$. More precisely, we are considering the probability in the space of measures such that for open convex sets in M

$$Q_n(A) = \mu\left\{z \in J \;\middle|\; \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(z)} \in A\right\}.$$

Now consider the probabilities in E, defined by the random variable $Q_{n,t}$ where, for $u \in E$,

$$Q_{n,t}(du) = \exp n\langle t, u\rangle Q_n(du)\frac{1}{Z_n(t)}$$

where $Z_n(t) = \exp(nc_n(t))$ (see definition 6).

Now consider ξ small enough such that $A(z, \xi) \subset G$ and $-\langle t, u\rangle \geq -\langle t, z\rangle - \xi$ for $u \in A(z, \xi)$. Hence we have

$$\begin{aligned} Q_n\{G\} \geq Q_n\{A_{z,\xi}\} &= Z_n(t)\int_{A_{z,\xi}} \exp(-n\langle t, u\rangle Q_{n,t}(du) \\ &\geq \exp[n(c_n(t) - \langle t, z\rangle) - n\xi]Q_{n,t}\{A_{z,\xi}\}. \end{aligned}$$

Therefore

$$\liminf_{n\to\infty} \frac{1}{n}\log Q_n\{G\} \geq (c(t) - \langle t, z\rangle - \xi) + \liminf_{n\to\infty}\frac{1}{n}\log Q_{n,t}\{A_{z,\xi}\}.$$

Claim 1.

$$\lim_{n\to\infty} Q_{n,t}\{A_{z,\xi}\} = 1.$$

Suppose the claim is proved, then as $P(t)=0$.

$$\liminf_{n\to\infty}\frac{1}{n}\log Q_n\{G\}\geqslant c(t)-\langle t, z\rangle-\xi=h(z)-\log d-\xi.$$

Now remember that we can approximate v by z such that the entropy $h(z)$ is close to the entropy $h(v)$.

Therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\log Q_n\{G\}\geqslant h(v)-\log d=-I(v).$$

As $v\in G$ is arbitrary the result follows.

Now we will prove the claim.

Proof of claim 1. Let $W_t=\{W_{n,t}:n=1,2,\ldots\}$ be the sequence of random variables in M such that $W_{n,t}/n$ is distributed according to $Q_{n,t}$.

Let us now calculate the free energy function of W_t. For any $s\in C$

$$\begin{aligned}c_{W_t}(s)&=\lim_{n\to\infty}\frac{1}{n}\log\int\exp(n\langle s, u\rangle)Q_{n,t}(\mathrm{d}u)\\&=\lim_{n\to\infty}\frac{1}{n}\log\left(\int\exp[n\langle(s+t)u\rangle]Q_n(\mathrm{d}u)\frac{1}{\exp[nc_n(t)]}\right)\\&=c(s+t)-c(t).\end{aligned}$$

Since c is differentiable in t, $c_{W_t}(s)$ is differentiable at $s=0$, $\delta c_{W_t}(0)=\delta c(t)=z$ and z is the unique derivative of c_{W_t} in 0.

Therefore, from theorem II.3.3 of [5], it follows that

$$Q_{n,t}\{A_{z,\xi}\}\to 1\qquad\text{as } n\to\infty$$

and the claim is proved.

This is the end of the proof of theorem 7.

Note that for $Q_{n,t}\{A_{z,\xi}\}\to 1$ as $n\to\infty$, it is essential that z be the unique solution of $\delta P(t)=z$. The reason for this is that there is no other place we can miss mass of the measure $Q_{n,t}$ as $n\to\infty$ (see considerations in section II.3 of [5]). It is also essential that t be Holder continuous in order to be able to relate $c(t)$ to $P(t)$, as stated in theorem 3.

Remark 4. Therefore, if K is a convex open set that does not contain a neighbourhood of μ, we have an exponential velocity decreasing with n, of the set of points such that the mean $n^{-1}\{\delta(z)+\delta(f(z))+\ldots+\delta(f^{n-1}(z))\}$ is in K.

In the context of statistical mechanics, this is the level-2 large deviation property [5]. We will now use a result of Orey [24] to obtain a level-1 large deviation property by means of a kind of contraction principle [5, 24].

We could obtain a more general result for continuous functions t, but we will be interested here in the function

$$t(z)=-\log|f'(z)|.$$

In this case we will have information about Liapunov exponents of measures.

5. Applications to level-1

We will show the existence of a large deviation function for the random variable $W_n(z) = \sum_{i=0}^{n-1} -\log|f'(f^i(z))| = -\log|(f^{n-1})'(z)|$, for z in $\mathbb{R}^2$. In this case $E = \mathbb{R}^2$ and we use definition 5 from section 3.

We will use the notation of Orey's paper.

Let $(\Omega_1, F_1, P_1) = (J, B, \mu)$, where B is the Borel sigma field, and define

$$Y_2 : M \to R = V_2 \qquad \text{by } v \to \langle t, v\rangle$$

where $t(z) = -\log|f'(z)|$ for $z \in \mathbb{R}^2$.

Consider now $\phi = Y_1$ and $\psi = Y_2$.

The map f in $\mathbb{R}^2$ induces the function f^* on M, which we will now define. For each $z \in J$, let $f^*(\delta(z)) = \delta(f(z))$. Now extend linearly for finite sums of Dirac measures, the f^* map. As any measure in M can be obtained as a limit of finite Dirac measures [17] we can extend this map f^* in a continuous fashion to all M.

It is easy to show that for a given v the measure $q = f^*(v)$ is the only measure such that for any continuous function $g \in C$ we have

$$\langle g, q\rangle = \langle g \circ f, v\rangle.$$

Now let $T_1 = f$, $T_2 = f^*$, $\Omega_1 = J$, $\Omega_2 = M$, $V_1 = M$ and $V_2 = \mathbb{R}$, as the notation in Orey's paper.

Observe now that from the above definitions, the following diagram commutes:

$$\begin{array}{ccccc}
\Omega_2 = M & \xrightarrow{f^* = T_2} & M = \Omega_2 & \xrightarrow{Y_2} & R = V_2 \\
\delta = \phi \uparrow & & \uparrow \phi = \delta & & \uparrow \psi \\
\Omega_1 = J & \xrightarrow{f} & J = \Omega_1 & \xrightarrow{Y_1} & M = V_1.
\end{array}$$

Therefore, we can use proposition 1.2 in Orey's paper [24] and we will obtain a large deviation function for $W_n(z) = -\log|(f^{n-1})(z)|$, as

$$k(v_2) = \inf\{I(v) : \psi(v) = v_2,\ v \in V_1\}$$
$$= \inf\left\{I(v) : -\int \log|f'(z)|\,\mathrm{d}v(z) = v_2,\ v \in V_1\right\}.$$

Theorem 9. Let v_2 be a real number such that $v_2 > -\log\int|f'(z)|\,\mathrm{d}\mu(z)$.

The deviation function for $t(z) = -\log|f'(z)|$ is equal to $k(v_2) = \log d - h(\tilde{v}_2)$, where $\tilde{v}_2$ is the maximal pressure measure for $g(z) = -t\log|f'(z)|$, and t is the only real value such that

$$\left.\frac{\mathrm{d}P(-s\log|f'|)}{\mathrm{d}s}\right|_{s=t} = v_2 \in \mathbb{R}.$$

Proof. For a given v_2, the $\inf_{v\in K} I(v)$ will be obtained in $M(f)$ (see theorem 9.11 in [35]) where $K = \{v \in M \mid -\int \log|f'(z)|\,\mathrm{d}v(z) = v_2\}$.

Now we want to demonstrate the existence of a certain t in the real line with a certain property. For each real s, let $p(s)$ be equal to $P(-s\log|f'|)$, and consider $\tilde{v}_2$

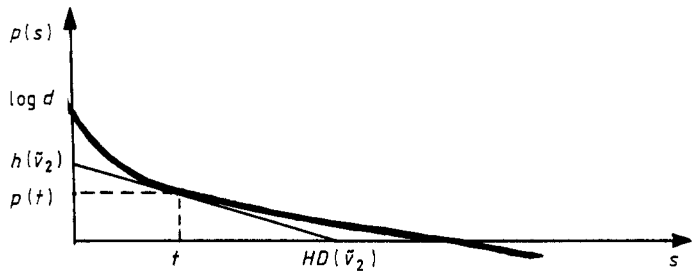

Figure 1. A plot of the function $p(s)$, showing how one can determine the values of $h(\tilde{v}_2)$ and $HD(\tilde{v}_2)$ by considering the tangent at $s = t$.

to be the only measure such that

$$p(t) = P(-t \log |f'(z)|) = h(\tilde{v}_2) - t \int \log |f'(z)| \, \mathrm{d}\tilde{v}_2(z)$$

and t is such that $p'(t) = v_2$.

In this case $\tilde{v}_2$ is the maximal pressure measure for $-t \log |f'|$.

Such t exists, because from [22, 31], $p(s)$ is a convex, differentiable function of s and (see figure 1)

$$p'(t) = -\int \log |f'(z)| \, \mathrm{d}\tilde{v}_2(z).$$

It also follows from [18, 21, 22] that

$$p'(t) = -h(\tilde{v}_2) \cdot HD(\tilde{v}_2)^{-1}$$

where $HD(\tilde{v}_2)$ is the Hausdorff dimension of the measure $\tilde{v}_2$ (see [12] for a definition).

Now that we already know that such real t exists, we will show that $k(v_2) = \log d - h(\tilde{v}_2)$.

For a given measure v in $M(f)$ such that $\int \log |f'(z)| \, \mathrm{d}v(z) = v_2$, we have that

$$\begin{aligned} -h(v) + tv_2 &= -h(v) + t \int \log |f'(z)| \, \mathrm{d}v(z) \\ &\geqslant h(\tilde{v}_2) + t \int \log |f'(z)| \, \mathrm{d}\tilde{v}_2(z) = -h(\tilde{v}_2) + tv_2. \end{aligned}$$

Therefore $-h(v) \geqslant -h(\tilde{v}_2)$. For $v = \tilde{v}_2$ we have the minimal value $I(\tilde{v}_2) = \inf\{I(v) \mid \psi(v) = v_2, \, v \in M\} = \log d - h(\tilde{v}_2)$. Remember that $I(v) = \log d - h(v)$.

Let X be the Liapunov number of μ, i.e. $X = \int \log |f'(z)| \, \mathrm{d}\mu(z)$.

Corollary 1. Define $k(A) = \inf\{k(v) \mid v \in A\}$ for a Borelean A in $\mathbb{R}^2$. Then

$$-k(A) = \lim_{n \to \infty} n^{-1} \log \mu\{z \in J \mid -n^{-1} \log |(f^n)'(z)| \in A\}.$$

Proof. The corollary is an easy consequence of the differentiability of $p(s)$ as a function of s [31, 32].

As a consequence of the above corollary, if there exists a neighbourhood of X such that it does not intercept a certain A in $\mathbb{R}$, then we have an exponential rate of decrease with n of

$$\mu\{z \in J \mid -n^{-1} \log |(f^n)'(z)| \in A\}$$

because μ is the unique measure with entropy $\log d$.

Corollary 2. For any $v_2 \in \mathbb{R}$, $v_2 > X$, we have

$$\lim_{n\to\infty} n^{-1} \log \mu\{z \in J \mid -n \log |(f^n)'(z)| > v_2\} = h(\tilde{v}_2) - \log d$$

where $\tilde{v}_2$ is the only one in $M(f)$ that is the maximal pressure measure for $-t \log |f'(z)|$ and t is such that $p'(t) = v_2$.

Proof. In order to prove corollary 2, consider $A = (v_2, \infty)$ and observe, by the convexity of the graph of $p(s)$ in figure 1, that $h(\tilde{v}_3) - \log d$ is smaller than $h(\tilde{v}_2) \log d$, where $\tilde{v}_3$ is the only one that is the maximal pressure measure for $-t_3 \log |f'|$ and $p'(t_3) = v_3$, with $v_3 > v_2$.

Therefore $-\inf\{k(v_3) \mid v_3 \in (v_2, \infty)\} = h(\tilde{v}_2) - \log d$.

Now we would like to make some final remarks.

Remark 5. We point out that we can compute the entropy of an invariant probability v in M as

$$\inf_{\substack{\text{Borel } G \\ v \in G}} \left\{ \limsup_{n\to\infty} n^{-1} \mu\{z \in J \mid n^{-1}(\delta(z) + \delta(f(z)) + \dots + \delta(f^{n-1}(z)) \in G\} \right\}.$$

Remark 6. We also mention that theorem 9 could be obtained directly, with the same procedures of theorem 7, in the more simple case of finite dimension, without using Orey's strong results in [24]. This follows from the differentiability of $p(t)$ as a function of $t \in \mathbb{R}$.

Acknowledgments

I would like to thank M Boyle and R Mañé for helping me with some specific questions which I had to deal with in the proof of theorem 8.

After this paper was written, I received preprints with related results from S Orey, S Pelikan and M Denker.

Appendix

Theorem 8. Suppose f is a hyperbolic rational map and v is an invariant probability for f. Then given $\xi > 0$, there exists an invariant probability p and a Holder-continuous function ϕ such that $h(p) > h(v) - \xi$, and p is the unique solution of

$$P(\phi) = h(p) + \int \phi(z)\, dp(z).$$

The function ϕ will be the logarithm of the Jacobian of the probability p.

Proof. The idea of the proof is to approximate v by a measure with the Jacobian constant by parts (see remark 9 in [15]), then by a new measure with Holder-continuous Jacobian. This will be done using results from one-sided shifts transferred to the dynamics of rational maps. There are two ways to translate results about invariant measures for one-sided shifts to invariant measures for rational maps. The first is to use results of Przytycki, as in lemma 2 of [27], proposition 1 in [28], and theorem 2 in [29]. As is stated in remark 3 in [28], we do not lose information about invariant measures with the coding trees for holomorphic dynamics. In [29] the general case is considered.

The other approach for translating results from one-sided shifts to rational maps is to use claim 2 below, which follows from arguments used by Mañé in [20], which we will assume proved.

We will choose the latter here, but we point out that the rest of the proof would follow basically the same lines using Przytycki coding trees.

Claim 2. There exists a set of curves $\gamma_1, \gamma_2, \ldots, \gamma_d$ in $\mathbb{C}$ containing the singularities of f such that

$$v\Big(\bigcup_{n>0} f^{-n}(\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_d)\Big) = 0.$$

The set $J - (\gamma_1 \cup \gamma_2 \ldots \cup \gamma_d)$ has d connected components $X_1, X_2, \ldots, X_d$, and if we denote by Q the partition $X_1, X_2, \ldots, X_d$, then $\bigvee_{n=0}^{\infty} f^{-n}(Q)$ coincides mod 0 with the 6-algebra of Borel in J.

Remark 7. The claim was proved in [20] for the maximal measure, but if we consider another invariant measure v, the proof works in the same way.

Now, denote by A_k^m, $k \in \mathbb{N}$, $m \in \{1, \ldots, d^k\}$, the d^k connected components of $f^{-k}(J - (\gamma_1 \cup \gamma_2, \ldots, \gamma_d))$. As $v(\bigcup_{n \in N}(f^{-n}(\gamma_1 \cup \gamma_2, \ldots, \gamma_d))) = 0$, then $\sum_{m=0}^{d^k} v(A_k^m) = 1$.

In this way the conjugacy g with the one-sided shift in d symbols σ, such that $fg = g\sigma$ is obtained by means of the position in $X_1, X_2, \ldots$ or X_d of the first n iterated values of a point in the set $J - \bigcup_{n>0} f^{-n}(\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_d)$. This set has full v-measure.

In the above notation of fixed k, each A_k^m, $m \in \{1, \ldots, d^k\}$, will denote a certain cylinder in the Julia set, encoded by the position of all the first k iterations of a point $z \in A_k^m$.

The next step, as we indicated before, is to prove the following claim.

Claim 3. For the one-sided shift, any invariant measure can be approximated by a measure that has almost the same entropy and has Jacobian constant in cylinders. In fact, we will show the result for the two-sided shift. The result for the one-sided shift follows from easy arguments using the natural projection of the two-sided shift onto the one-sided shift.

It will follow from claim 3 that the measure v can be approximated by invariant probabilities v_k, $k \in \mathbb{N}$, with Jacobian χ_k, $k \in \mathbb{N}$, constant in cylinders A_k^m, $m \in \{1, \ldots, d^k\}$. We will also obtain that the entropy of v_k converges to the entropy of v. This is obtained by transferring the results of the one-sided shift for the rational map by means of the change of coordinates g.

Now we will prove claim 3 for the two-sided shift. Let's begin with some considerations about shifts.

Suppose P is an irreducible stochastic $d \times d$ matrix. By the Perron–Frobenius theorem, P has a left eigenvector p such that $p^P = p$ and each entry of p is positive.

Define measure λ on $\tau = \{1, \ldots, d\}^Z$ by requiring for every word $i_0, i_1, \ldots, i_k$ on $\{1, 2, \ldots, d\}$, that for any $m \in Z$, $\lambda\{x \in \tau \mid x_m - i_0, x_{m+1} = i_1, \ldots, x_{m+k} = i_k\} = p(i_0)P(i_0, i_1)P(i_1, i_2) \ldots P(i_{k-1}, i_k)$. It is straightforward to check that this gives a well defined function from a finite union of sets of the given form, which extend to a unique Borel probability λ on τ, which is σ invariant. This Markov measure λ is the unique equilibrium state of the locally constant function

$$b(x) = -\log P(x_0, x_1).$$

We refer the reader to the work of Parry and Tuncel ([25], §II.4) for a short and elegant proof.

A simple calculation shows that for a point $x = (\ldots, i_{-1}, i_0, i_1, i_2, \ldots)$, the logarithm of the Jacobian of λ at x is locally constant in cylinders of a certain level and given by

$$-\log\left(\frac{p(x_0 = i_0)}{p(x_0 = i_1)} P(i_0, i_1)\right).$$

Therefore the logarithm of the Jacobian of λ is locally constant and it is homologous to $b(x)$. In particular, it has λ as a unique equilibrium state.

In general, given $k \in \mathbb{N}$ and a shift-invariant probability μ on $\{1, \ldots, d\}^Z$, let ω be the set of words $i_1, \ldots, i_k$ in $\{1, \ldots, d\}^k$ such that

$$\bar{\mu}\{x : x_1 = i_1, \ldots, x_k = i_k\} > 0.$$

Let P be the $\omega \times \omega$ stochastic matrix of transition probabilities,

$$P(i_0 i_1 \ldots i_{k-1}, j_1 j_2 \ldots j_k) = \frac{\bar{\mu}\{x : x_0 = i_0, \ldots, x_{k-1} = i_{k-1} \text{ and } x_1 = j_1, \ldots, x_k = j_k\}}{\bar{\mu}\{x : x_0 = i_0, x_1 = i_1, \ldots, x_{k-1} = i_{k-1}\}}.$$

The μ is Markov if for some k, for every word $i_0 i_1 \ldots i_n$ of length larger than k all of whose subwords lie in ω,

$$\bar{\mu}\{x : x_0 = i_0, \ldots, x_n = i_n\} = \bar{\mu}\left\{x : x_0 = i_0, \ldots, x_{k-1} = i_{k-1}\right\} \prod_{j=k}^{n} P(i_{j-k} \ldots i_{j-1}, i_{j-k+1} \ldots i_j).$$

The measure is mixing Markov if also some power of P is strictly positive. Then there is a natural shift-commuting homeomorphism from the support of $\bar{\mu}$ to ω^Z which send $\bar{\mu}$ to the measure defined on ω^Z by P (as above). Therefore, our considerations above for λ carry over to $\bar{\mu}$.

Any shift-invariant measure on $\{1, \ldots, d\}^Z$ can be approximated weakly and in entropy by a mixing Markov measure ([2], lemma 10.3). We can assume this measure to have full support because any mixing Markov measure p can be approximated weakly and in entropy by a mixing Markov q with full support.

Therefore, the considerations made earlier about the approximation of measures by equilibrium states with Jacobian constant in cylinders also apply in this situation. So we conclude the proof of claim 3 that shows that any σ-invariant measure can be approximated weakly and in entropy by another σ-invariant measure with Jacobian constant in cylinders of a certain high order.

The natural projection from the two-sided shift to the one-sided shift induces a bijection of measures that respect entropy. So this two-sided result holds also in the one-sided case.

Now, using the change of coordinates g, we transfer this result to the dynamics of the rational map. Therefore, we can assume that we have obtained a sequence of f-invariant measures v_k, $k \in \mathbb{N}$, with Jacobian $\chi_k(z)$, $z \in J$, such that for each $k \in \mathbb{N}$, $\chi_k(z)$ is constant in each cylinder A_k^m, $m \in \{1, 2, \ldots, d^k\}$. Let us denote by p_k^m, $m \in \{1, 2, \ldots, d^k\}$ such a constant value of χ_k in A_k^m. We can also assume that v_k approximates v weakly and in entropy.

From the above considerations we state

(Ii) $p_k^m > 0 \qquad \forall m \in \{1, \ldots, d^k\}$
(Iii) $\sum_{i=1}^{d^k} p_k^m = 1$
(Iiii) $h(v_k) = \sum_{k=1}^{d^k} p_k^m \log p_k^m$.

In this case we can write

$$\chi_k(z) = \sum_{m=1}^{d^k} p_k^m I_{A_k^m}(z).$$

Therefore, all we have to prove now is that each such v_k can be approximated by p such that there exists a Holder-continuous ϕ

$$P(\phi) = h(p) + \int \phi(z)\, \mathrm{d}p(z)$$

and $h(p)$ is close to the $h(v_k)$.

Denote v_k by l and χ_k by u to simplify the notation. Consider $B = f^{-k}(\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k)$.

For each $n \in \mathbb{N}$ choose Y_n Holder-continuous such that

(IIi) $Y_n(z) = u(z)$ for $z \in C_n$, where C_n is defined as

$$C_n = \left\{ z \in \bigcup_{m=1}^{d^k} A_k^m \mid d(z, B) > \frac{1}{n} \right\}$$

(IIii) $\sum_{m=1}^{d^k} Y_n(z(m, k, z_0)) = 1$ for all $z_0 \in J$
(IIiii) $Y_n(z) > 0$ for $z \in J$.

Here we are ordering the A_k^m in such a way that for

$$z_0 \in J - \{\gamma_1 \cup \ldots \cup \gamma_d\} \qquad z(m, k, z_0) \in A_k^m.$$

It follows easily from theorem 2 that $P(\log Y_n) = 0$. It also follows from [15, 18, 35] that there exists a unique ergodic invariant measure l_n such that

$$P(\log Y_n) = 0 = h(l_n) + \int \log Y_n(z)\, \mathrm{d}l_n(z).$$

The Jacobian of l_n is Y_n [15, 18].

Consider now a convergent subsequence of l_n, converging to a certain invariant measure q. For each open set V in A_k^m, with positive distance of $f^{-k}(\gamma_1 \cup \ldots \cup \gamma_d)$, consider n large enough that $V \subset C_n$ (see (IIi)). Therefore for large n

$$l_n(V) = \int_{f^k(V)} Y_n(z)\, \mathrm{d}l(z) = p_k^m l_n(f^k(V)).$$

As l_n converges to q then

$$q(V) = p_k^m q(f^k(V)).$$

From this it follows that for any Borel set K in J we have

$$q(K) = p_k^m q(f^k(K)).$$

From [1, 36], there exists a unique probability q satisfying this property and this measure is $l = v_k$.

Therefore, there exists a subsequence l_n converging weakly to l.

Now, given $\xi > 0$, as

$$l(f^{-k}(\gamma_1 \cup \ldots \cup \gamma_d)) = 0$$

there exists $s \in \mathbb{N}$ such that

$$h(l) - \xi = -\int \log u(z)\, \mathrm{d}l(z) - \xi \leqslant -\int_{C_s} \log u(z)\, \mathrm{d}l(z).$$

As l_n converges to l we can assume for n large enough that

$$h(l) - \xi \leqslant -\int_{C_s} \log u(z)\, \mathrm{d}l_n(z).$$

Now consider $n > s$, then

$$\begin{aligned} h(l) - \xi &\leqslant -\int_{C_s} \log u(z)\, \mathrm{d}l_n(z) \\ &= -\int_{C_s} \log Y_n(z)\, \mathrm{d}l_n(z) \\ &\leqslant -\int \log Y_n(z)\, \mathrm{d}l_n(z) \\ &= h(l_n). \end{aligned}$$

Therefore theorem 8 is proved.

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