The Dirac operator for the pair of Ruelle and Koopman operators, and a generalized Boson formalism

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Abstract

Denote by μ the maximal entropy measure for the shift acting on $\Omega = \{0,1\}^{\mathbb{N}}$, by $\mathcal{L}$ the associated Ruelle operator and by $\mathcal{K} = \mathcal{L}^*$ the Koopman operator, both acting on $L^2(\mu)$. It is natural to call $\mathfrak{a} = \mathcal{L}$ the annihilation operator, and $\mathfrak{a}^\dagger = \mathfrak{c} = \mathcal{K}$ the creation operator (see (*)). We call $\mathcal{K}\mathcal{L} = \mathfrak{c}\mathfrak{a}$ the generalized (dynamical) boson Number operator.

In our mathematical framework we consider the following complete orthogonal family of functions in $L^2(\mu)$: given a finite word $w = w_1 w_2 \cdots w_l$, $w_j \in \{0,1\}$, denote $e_w \frac{1}{\sqrt{\mu([w])}} \left(\mathbb{1}_{[w1]} - \mathbb{1}_{[w0]}\right)$, and add $e^1_\emptyset = \sqrt{2}\,\mathbb{1}_{[1]}, e^0_\emptyset = -\sqrt{2}\,\mathbb{1}_{[0]}$ to the family.

Claim: $\mathfrak{c}(e_w) = \frac{1}{\sqrt{2}}(e_{0w} + e_{1w})$ and $\mathfrak{a}(e_{w_1 w_2 \cdots w_n}) = \frac{1}{\sqrt{2}} e_{w_2 w_3 \cdots w_n}$ (*), justifying the above terminology. Given m, n, $w_{n+1} \cdots w_k$, denoting by $l(w)$ the size of the word w:

$$\mathcal{K}^m \mathcal{L}^n (\textstyle\sum_{l(v)=n} e_{v w_{n+1} \cdots w_k}) = 2^{m/2 - n/2} \sum_{l(v)=n} e_{v w_{n+1} \cdots w_k}.$$

Therefore, $2^{m/2-n/2}$ are eigenvalues of $\mathfrak{c}^m \mathfrak{a}^n = \mathcal{K}^m \mathcal{L}^n$. It our setting $[\mathcal{K}, \mathcal{L}] = I$ (the CCR) is not true; however, we show a dynamical CCR version for generalized bosons systems. Here the Dirac operator is: $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{pmatrix}$; introducing a natural diagonal representation π on the set of bounded operators L acting on $L^2(\mu)$, we set a certain natural spectral triple, and we estimate $\| \, [\mathcal{D}, \pi(L)] \, \|$ in several cases; $\| \, [\mathcal{D}, \pi(L)] \, \| \leq 1$ corresponds to saying that the Lipchitz constant of L is ≤ 1. We show that $\| \, [\mathcal{D}, \pi(\mathcal{K}\mathcal{L})] \, \| = 1$.

In another direction, considering the case where $L = \hat{\psi}$ is the projection in $\psi \in L^2(\mu)$, and $|\psi| = 1$, we get that $\frac{3}{2\sqrt{2}} \geq \| \, [\mathcal{D}, \pi(\hat{\psi})] \, \| \geq 1$. When ψ is in the Kernel of the Ruelle operator we get $\| \, [\mathcal{D}, \pi(\hat{\psi})] \, \| = 1$. From the value $\langle \mathcal{K}(\psi), \psi \rangle$ we get explicitly $\| \, [\mathcal{D}, \pi(\hat{\psi})] \, \|$.

We show that $\| \, [\mathcal{D}, \pi(\mathcal{K}^n \mathcal{L}^n)] \, \| = 1, \forall n \in \mathbb{N}$. We also estimate $\| \, \left[\mathcal{D}, \pi(\mathcal{M}_f)\right] \, \|$, when $\mathcal{M}_f$ is the multiplication operator $g \mapsto \mathcal{M}_f(g) = gf$, for a continuous function, $f : \Omega \to \mathbb{R}$. We explore an interpretation of $\mathcal{D}$ (a form of derivative for operators) via $f \to \left[\mathcal{D}, \pi(\mathcal{M}_f)\right]$, either as related to the associated forward discrete time dynamical derivative $f - f \circ \sigma$:

$$|f - f \circ \sigma|_\infty = |\mathcal{K}f - f|_\infty \geq \| \, \left[\mathcal{D}, \pi(\mathcal{M}_f)\right] \, \| \geq |f - \mathcal{L}f|_\infty$$

(if f is coboundary to zero then $\left[\mathcal{D}, \pi(\mathcal{M}_f)\right] = 0$); or the backward discrete time derivative $\sqrt{\frac{|f(x) - f(0x)|^2}{2} + \frac{|f(x) - f(1x)|^2}{2}}$; in this direction $\| \, \left[\mathcal{D}, \pi(\mathcal{M}_f)\right] \, \| = \left| \sqrt{\mathcal{L}\, |\mathcal{K}f - f|^2} \right|_\infty$.

From all above we get lower bounds for Connes distance between C^*-states.

1 Introduction

In the physics of elementary particles, the bosons (the same is true for fermions) are particles that are indistinguishable from each other. For example, every electron (which is a fermion particle) is exactly the same as every other electron. In this way, given a state with two electrons, you can exchange the two electrons and this will not change anything physically observable from that state. For bosons, the number of indistinguishable particles can range in the set of natural numbers $0, 1, 2, \cdots, n, \cdots$; several identical bosons can simultaneously occupy the same quantum state. One of the main issues on the topic is the understanding of the statistics of a collection of non-interacting identical particles that may occupy a set of available discrete energy states at thermodynamic equilibrium. Half of the particles of the universe are bosons. Our mathematical formalism corresponds to a single bosonic mode.

For fermions, the Pauli exclusion principle claims: only one fermion can occupy a particular quantum state at a given time. A very detailed study of fermions appears in [3]. Our focus here will be on the boson formalism.

In [19] the authors analyze the so-called generalized boson systems where the classical canonical commutation relation (CCR) is not true; they extend boson sampling protocols to a larger class of quantum systems, that include, among others, interacting bosons. The generalized point of view of [19] will be in significant consonance with our dynamical setting (see (15), (16), and more details in Remark 36). In an abstract point of view denote b and $b^\dagger$, respectively, the annihilation and creation operator acting in an infinite dimensional Hilbert space. The classical CCR corresponds to

$$I = \left[b, b^\dagger\right] = b\, b^\dagger - b^\dagger\, b, \tag{1}$$

which in the general setting of [19] is not always true. We will call the case where (1) applies classic, to distinguish it from the generalized case that we will deal with here; in the classical case (1) the eigenvalues of $b^\dagger b$ are in $\mathbb{N}$.

General references for classical bosons and fermions are [1], [30], [27] and [14]. A function $n \to f(n)$ plays an important role in [19], and the main relationship would be

$$b^\dagger |n\rangle = \frac{f(n+1)}{f(n)} |n+1\rangle \quad \text{and} \quad b|n\rangle = \frac{f(n)}{f(n-1)} |n-1\rangle, \tag{2}$$

where $|n\rangle$ describe the state with n-particles, $n = 0, 1, 2, \cdots$ (see page 043096-5 in [19]). $b^\dagger$ and b can take different expressions depending of the case.

The state $|0\rangle$, which satisfies

$$b(|0\rangle) = 0, \tag{3}$$

is called the vacuum.

For the classical case $f(n) = \sqrt{n!}$, for the boson pair $f(n) = \sqrt{(2\,n)!}$ (see page 043096-2 in [19] and also [5]), and here will consider the case where $f(n) = 2^{-n/2}$ (see (16) and (123)).

In the classical case, that is $f(k) = \sqrt{k!}$, the operators $\mathcal{C} = b^\dagger$ and $\mathcal{A} = b$ act on an infinite-dimensional Hilbert space and the elements $|k\rangle$, $k = 0, 1, 2, \cdots$, are eigenfunctions for $\mathcal{C}\,\mathcal{A}$, associated to eigenvalues which range in the set of natural numbers. More precisely,

$$\mathcal{C}\,\mathcal{A}(|k\rangle) = k\,|k\rangle = (\frac{f(k)}{f(k-1)})^2\,|k\rangle. \tag{4}$$

In this case, the self-adjoint operator

$$\mathcal{C}\,\mathcal{A} \tag{5}$$

is called the (classical) *number operator*. The terminology *number operator* is due to such a property: the count of particles at a given state (see Section 5 in [1]). The eigenvalues of $\mathcal{C}\,\mathcal{A}$ are in $\mathbb{N}$.

In our setting, expression (124) corresponds to (4).

The quantization of the harmonic oscillator can be put in a form that satisfies (1) and also the above classical expression (4) (see Section 11 in [14]). In this case, $|n\rangle$ is the nth eigenfunction of the corresponding quantized Hamiltonian operator $\mathbf{H}$ acting on functions on $L^2(dx)$ defined on the real line (see details in Example 38 in section 3). In Quantum Mechanics the self-adjoint operators are the observables and their eigenvalues corresponds to the real values that can be observed when measuring (see [21]). The eigenvalues of the number operator $b^\dagger\, b$ of a fermion can be just 0 or 1; $I = \left[b, b^\dagger\right]$ is not true. The value 1 means occupied and 0 not occupied. The eigenvalues of the number operator of a boson can be any natural number $n \in \mathbb{N}$.

We refer the reader to the beginning of Section II (or (A3)) in [19] (which deals with the generalized case), where denoting by $|0\rangle$ the vacuum, one gets

$$(b^\dagger)^k|0\rangle = f(k)\,|k\rangle. \tag{6}$$

In the case of the harmonic oscillator

$$\mathcal{C}^k|0\rangle = \sqrt{k!}\,|k\rangle. \tag{7}$$

In this case, $|0\rangle$ corresponds to the eigenfunction associated with the smallest eigenvalue of the Hamiltonian operator $\mathbf{H}$, and

$$\mathcal{A}(|0\rangle) = 0. \tag{8}$$

For this reason $|0\rangle$ is called the vacuum (eigenfunction) for the harmonic oscillator (see (3)).

Moreover, when $k > m > n$, in the classical case

$$\mathcal{C}^m\,\mathcal{A}^n(|k\rangle) = (k-n)\cdots(k-1)\,\sqrt{k}\cdots\sqrt{k+m-n-1})\,|(k+m-n)\rangle; \tag{9}$$

in particular

$$\mathcal{C}^m\,\mathcal{A}^m(|k\rangle) = (k-m)\cdots(k-1)\,|k\rangle. \tag{10}$$

This should be compared with our future expression (123).

The classical CCR would be given by $[\mathcal{A},\mathcal{C}] = I$, but the corresponding relation is not true in our setting (see (88)); however, in the generalized case, there exists a CCR version that can be obtained by introducing a new form of commutator $\lceil\cdot,\cdot\rceil$, in the sense of (2) and (A5) in [19], which here will correspond to (15) (see (127) for details).

More precisely, (2) in [19] requires the existence of a naturally defined function $F:\mathbb{N}\to\mathbb{R}$ (to be defined according to the specific quantum problem), such that by definition

$$\lceil b, b^\dagger\rceil := \sum_n F(n)|n\rangle\,\langle n| = I. \tag{11}$$

The classical CCR corresponds to take $F = 1$.

The general equation (11) corresponds to CCR for the generalized case. Taking into account (A4) and the left inequality on (A5) in [19], the authors set a natural choice of F and get

$$\lceil b, b^\dagger\rceil = \sum_n F(n)|n\rangle\langle n| \sum_n \langle n|\,\lceil b, b^\dagger\rceil\,|n\rangle\;\; |n\rangle\langle n|. \tag{12}$$

In our case we will choose a special form of the above expression for defining $\lceil\cdot,\cdot\rceil$ (when taking an special F, as we will see later on (15) and Proposition 37).

We will consider here the Hilbert space $L^2(\mu)$, when μ is the measure of maximal entropy for the action of the shift $\sigma:\Omega\to\Omega$, where $\Omega = \{0,1\}^{\mathbb{N}}$.

In our mathematical framework we will introduce an orthogonal family of Hölder functions $e_w : \Omega \to \mathbb{R}$, indexed by finite words $w = w_1 w_2 \cdots w_l$, $w_j \in \{0,1\}$, which will play an essential role in part of our reasoning ($l = l(w)$ is the size of w). See (92) for definition of e_w.

Adding two more functions $e^1_\emptyset, e^0_\emptyset$ to the family, we get an orthonormal family (a Hilbert basis) on $L^2(\mu)$ (see (93) for definition).

The Ruelle operator $\mathcal{L} : L^2(\mu) \to L^2(\mu)$ is defined for a continuous function f_1, by $\mathcal{L}(f_1) = f_2$, if for all $x \in \{0,1\}^{\mathbb{N}}$, $f_2(x) = \frac{1}{2}(f_1(0,x) + f_1(1,x))$.

We will show that is natural to call $\mathfrak{a} = \mathcal{L}$ the annihilation operator and $\mathfrak{a}^\dagger = \mathfrak{c} = \mathcal{K}$, where $\mathcal{L} : L^2(\mu) \to L^2(\mu)$ is the Ruelle operator and $\mathcal{K} = \mathcal{L}^*$ the Koopman operator (see (95) and (102)). Indeed, we will prove in Section 3 that

$$\mathfrak{c}(e_w) = \frac{1}{\sqrt{2}}(e_{0w} + e_{1w}) \text{ and } \mathfrak{a}(e_{w_1 w_2 \cdots w_n}) = \frac{1}{\sqrt{2}} e_{w_2 w_3 \cdots w_n}. \tag{13}$$

Note that $\frac{f(k)}{f(k-1)} = \frac{1}{\sqrt{2}}$ (in consonance with (2)).

We can express the commutator $[\mathfrak{a}, \mathfrak{c}]$ in terms of the basis (see (126)). Given $e_{w_1, w_2 \cdots w_l}$ denote $\hat{e}_{w_1, w_2 \cdots w_l}$ the element $e_{v_1, w_2 \cdots w_l}$, where $v_1 = 1$ if $w_1 = 0$, and $v_1 = 0$ if $w_1 = 1$.

Then, given w

$$[\mathfrak{a}, \mathfrak{c}]\,(e_w) = \frac{1}{2} e_w - \frac{1}{2} \hat{e}_w \neq e_w. \tag{14}$$

In our generalized dynamical boson setting we are interested, among other things, in the Number operator $\mathcal{K}\mathcal{L}$ which is our version of (5); and also in $\mathcal{K}^m \mathcal{L}^m$, which corresponds to the action of (10). In (19) we introduce a Dirac operator $\mathcal{D}$, a representation π and we introduce a special spectral triple (see (20) and Definition 1). Our main motivation here was to show that $\| \, [\mathcal{D}, \pi(\mathcal{K}\mathcal{L})] \, \| = 1$ (see Section 2.2). In the *sense of Connes* (see [7], [8]), this would *mean* here that the self-adjoint operator Number operator $\mathcal{K}\mathcal{L}$ has Lipschitz constant equal to 1 (see Remark 5). We leave for future investigation the question of whether our mathematical formalism eventually corresponds to any application in physics.

We point out that here we will introduce a diagonal representation, which is a natural generalization of the setting in [20] (a finite dimensional non-dynamical case), where the authors were able to compute explicitly the Connes spectral distance between one-qubit states.

The function $\frac{1}{\sqrt{2}}(e^1_\emptyset + e^0_\emptyset)$ represents the vacuum (see Section 3). This is so because $\mathfrak{a}(e^1_\emptyset + e^0_\emptyset) = \mathcal{L}(e^1_\emptyset + e^0_\emptyset) = 0$.

Denote by $\mathcal{F}$ the set of functions $f : \{0,1\}^{\mathbb{N}}$ not depending on the first coordinate. For the dynamical generalized fermion setting the Number operator $\mathcal{KL}$ acts on the subspace $\mathcal{V} \subset L^2(\mu)$ generated by $\{\mathcal{F}, e_\emptyset^1 + e_\emptyset^0\}$ (see Section 3). The only possible eigenvalues of $\mathcal{KL} : \mathcal{V} \to \mathcal{V}$ are 0 and 1 (see (84)).

For the dynamical generalized boson setting the Number operator $\mathcal{KL}$ acts on the $L^2(\mu)$ space (see Section 3). In this case $\mathcal{K}^m\mathcal{L}^n$ may have an infinite number of eigenvalues, when $m \neq n$ (see (123)). We point out that the eigenvalues of $\mathcal{KL} : L^2(\mu) \to L^2(\mu)$ can be just 0 or 1 (not an infinite number of them). In our framework the boson formalism manifests itself through the expressions in (13).

We will present later (see Section 3) a generalized CCR relation (see (127) and Proposition 37) of the form:

$$\lceil \mathfrak{a}, \mathfrak{c} \rceil = \sum_w 2\langle e_w, [\mathfrak{a}, \mathfrak{c}]\,(e_w)\rangle |e_w\rangle\langle e_w| = I, \tag{15}$$

which means in some sense to take $F(w) = 2\langle e_w, [\mathfrak{a}, \mathfrak{c}]\,(e_w)\rangle$.

$\frac{1}{\sqrt{2}}(e_\emptyset^1 + e_\emptyset^0)$ corresponds to the vacuum, and in this case (see (110))

$$\mathfrak{c}^n\Big(\frac{1}{\sqrt{2}}(e_\emptyset^1 + e_\emptyset^0)\Big) = f(n) \sum_{a_1,a_2,\cdots,a_n=0}^{1} e_{a_1a_2\cdots a_n}, \tag{16}$$

corresponds to (6).

For the case of classical fermions, the Canonical Anticommutative Relation (CAR) should be valid, and this would correspond to

$$I = \{b, b^\dagger\} = b\, b^\dagger + b^\dagger\, b, \tag{17}$$

which in the general case is not always true. In our setting, we briefly mention the CAR for fermions in (86). Denote $\hat{f} = \frac{1}{\sqrt{2}}\mathcal{L}$ and $\hat{f}^\dagger = \frac{1}{\sqrt{2}}\mathcal{K}$, and by $\mathcal{F}$ the space of functions ϕ in $L^2(\mu)$ that do not depend on the first coordinate.

Is it natural to consider bounded operators acting on the $\mathcal{F}$. Then, under such restriction we will show (see Section 3) in such C^*-algebra the CAR

$$\{\hat{f}, \hat{f}^\dagger\} = I. \tag{18}$$

Concerning all the above claims, in Section 3 we will present explicit computations in order to get the main results we just described; summarized in Propositions 36 and 37.

In the second part of the paper (see Section 2) we follow a related but different path. Motivated by [20], which deals with a finite-dimensional case, we will introduce a Dirac operator $\mathcal{D}$ and a special diagonal representation π (see (5) in [20] for the case of qubits): here we will set

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{pmatrix}. \tag{19}$$

Different kinds of (dynamical) diagonal representations was considered in [28], [29] and in Section 6 in [6].

Definition 1. *A spectral triple is an ordered triple (A, H, D), where*

1. *H is a Hilbert space;*
2. *A is a C^*-algebra, π is a representation, where for each $a \in A$, we can associate a bounded linear operator $\pi(a) : H \to H$;*
3. *D is an essentially self-adjoint unbounded linear operator on H, such that $\{a \in A : \| [D, \pi(a)] \| < +\infty\}$ is dense in A, where $[D, \pi(a)]$ is the commutator operator. The operator D is called momentum (or, Dirac) operator.*

Remark 2. *Note that we do not require that D has compact resolvent.*

Here, for the spectral triple, we will take the Hilbert space $H = L^2(\mu) \times L^2(\mu)$. We denoted $\mathfrak{c} : L^2(\mu) \to L^2(\mu)$ the Koopman operator $\mathcal{K}$, which is the (Hilbert) dual of the Ruelle operator $\mathcal{L}$, which was denoted by $\mathfrak{a}$.

We denote $\mathcal{B} = \{L : L^2(\mu) \to L^2(\mu) \mid L \text{ Linear bounded operator}\}$.

We consider the diagonal representation

$$\pi : \mathcal{B} \to \{G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : L^2(\mu) \times L^2(\mu) \to L^2(\mu) \times L^2(\mu)\}, \tag{20}$$

in such way that for $L \in \mathcal{B}$ and $(\psi_1, \psi_2) \in L^2(\mu) \times L^2(\mu)$ we get

$$\pi(L)(\psi_1, \psi_2)) = (L(\psi_1), L(\psi_2)) = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

For bounded operators, we consider the norm given by the spectral radius $\rho(G)$.

A is the C^*-algebra of bounded operators in $L^2(\mu)$, π is the diagonal representation given by (20).

$D : H \to H$ is given by (19) and we will show that the commutator $[D, \pi(L)]$ can be expressed by (32).

Remark 3. *A hermitian operator L in A is called an observable. In Quantum Mechanics the values obtained by measuring the observable L is the set of (real) eigenvalues of L.*

We denote by $\mathfrak{H}$ the set of Hermitian operators on $L^2(\mu)$.

$\| \, [\mathcal{D}, \pi(L)] \, \| \leq 1$ corresponds to saying that the Lipchitz constant of the self-adjoint operator $L \in \mathfrak{H}$ is smaller than 1.

Consider the C^*-algebra $\mathcal{U}$ of bounded operators acting on $L^2(\mu)$, and denote by ρ a general C^*-state acting on $\mathcal{U}$.

Definition 4. *The Connes distance between the C^*-dynamical states ρ_1 and ρ_2 is*

$$d_C(\rho_1, \rho_2) = \sup_{L \in \mathfrak{H}, \, \| \, [\mathcal{D}, \pi(L)] \, \| \leq 1} \{ w_1(L) - w_2(L) \} \tag{21}$$

Remark 5. *The Connes distance corresponds to the* 1*-Wasserstein distance among probabilities (see [7], [8], [16], [17] and [6]). The operator norm (spectral radius) $\| \, [\mathcal{D}, \pi(L)] \, \|$ being less than or equal to one, should be in some sense, analogous to saying that L has Lipschitz constant smaller than or equal to one. We will show that the value $\| \, [\mathcal{D}, \pi(L)] \, \|$ will be given by expression* (33)*.*

For a given L in expression (21) it is quite important to be able to determine if either $\| \, [\mathcal{D}, \pi(L)] \, \| \leq 1$ or $\| \, [\mathcal{D}, \pi(L)] \, \| > 1$. One of the main issues here is to exhibit explicit examples where one can determine if either of the two options is true. This will help to provide lower bounds for $d_C(\rho_1, \rho_2)$.

Remark 6. *On the issue of the estimation of the Connes distance (via the sup in* (21)*), note that $\| \, [\mathcal{D}, \pi(L)] \, \|$ is linear on L, and for practical purposes, a form of normalization on L is natural to be considered. It follows from* (32)*,* (33)*, and triangle inequality that in the case $||L|| \leq 1/2$, we get that $\| \, [\mathcal{D}, \pi(L)] \, \| \leq 1$ (see Proposition 39).*

Below in Proposition 20, Theorems 21 and 22, and Example 23, we study the properties of $[D, \pi(L)]$ when L is a projection operator. As we are able to estimate for several operators L (for instance the projection operators) when $\| [D, \pi(L)] \| \leq 1$, we get indeed lower bounds for $d(w_1, w_2)$.

For the general case $L \in \mathfrak{H}$ we need a more detailed analysis.

Given a continuous function f, we investigate a possible interpretation of the concept $\| [\mathcal{D}, \pi(\mathcal{M}_f)] \|$, regarding either the associated forward discrete time dynamical derivative or the associated backward discrete time dynamical derivative in Subsection 2.2.

We will highlight now some of the main results obtained in our text.

One of our main purposes is to estimate the expression

$$\| [\mathcal{D}, \pi(\mathcal{K}\mathcal{L})] \|, \text{ and more generally } \| [\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)] \|,$$

$n \in \mathbb{N}$. In Theorem 34 we show that $\| [\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)] \| = 1$.

Given a continuous function $f : \Omega \to \mathbb{R}$, the multiplication operator $\mathcal{M}_f$ is the one satisfying $g \to \mathcal{M}_f(g) = gf$. The operators $\mathcal{K}^n\,\mathcal{L}^n$ and $\mathcal{M}_f$ are generators of the Exel-Lopes C^*-algebra introduced in [11] (see also [10]). In Subsection 2.2 we also estimate $\| [\mathcal{D}, \pi(\mathcal{M}_f)] \|$.

In other direction, in Subsection 2.1, we get some explicit results concerning $\| [\mathcal{D}, \pi(L)] \|$, for operators L of different kind, like projection operators. For instance, taking into account the orthogonal family of Hlder functions $e_w : \Omega \to \mathbb{R}$, indexed by finite words $w = w_1 w_2 \cdots w_l$, $w_j \in \{0, 1\}$, in Theorem 18 we get

Theorem 7. *Given a fixed word $w = w_1 w_2 w_3 \cdots w_n$, $l(w) \geq 2$, denote by $\hat{e}_w$ the projection operator on e_w. Then, we get for the operator norm:*

$$\| [\mathcal{D}, \pi(\hat{e}_w)] \| = 1. \tag{22}$$

More generally, we get in Theorems 21 and 22 in Subsection 2.1:

Theorem 8. *Given a non-constant $L^2(\mu)$ function $\psi : \Omega \to \mathbb{R}$ such that $|\psi| = 1$, denote by $\hat{\psi}$ the associated projection operator. Then, we get for the operator norm:*

$$\begin{aligned} \| \left[\mathcal{D}, \pi(\hat{\psi})\right] \| &= \| (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \| \\ &= \sup_{|\phi|=1} \sqrt{\langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{K}\phi, \psi \rangle \langle \mathcal{K}\psi, \psi \rangle + \langle \mathcal{K}\phi, \psi \rangle^2}. \end{aligned} \tag{23}$$

Expanding $\psi = \sum_w b_w e_w + \beta_0 e^0_\emptyset + \beta_1 e^1_\emptyset$ via the orthonormal family e_w, where w ranges in the set of finite words, one can get an explicit expression for $\| \left[\mathcal{D}, \pi(\hat{\psi})\right] \|$ in terms of coefficients b_w, which will be described by (62).

In another line of reasoning we will get Theorem 20 in Subsection 2.1:

Theorem 9. *Given a non-constant $L^2(\mu)$ function $\psi : \Omega \to \mathbb{R}$ such that $|\psi| = 1$, denote by $\hat{\psi}$ the associated projection operator. Then, from the value $\langle \mathcal{K}(\psi), \psi \rangle$ we will be able to get the explicit value $\| \left[\mathcal{D}, \pi(\hat{\psi})\right] \|$ which satisfies*

$$\frac{3}{2\sqrt{2}} \geq \| \left[\mathcal{D}, \pi(\hat{\psi})\right] \| \geq 1. \tag{24}$$

Note that for a Hlder function ψ we get $\langle \mathcal{K}(\psi), \psi \rangle = 1$ only when ψ is constant.

Following a different rationale in Proposition 22 we get:

Proposition 10. *Suppose ψ with norm 1 is a Hlder function in the kernel of the Ruelle operator $\mathcal{L}$, then*

$$\| \left[\mathcal{D}, \pi(\hat{\psi})\right] \| = 1. \tag{25}$$

If ψ is of the form $\psi = \mathcal{K}^k(f)$, $k \geq 1$, where f is in the kernel of the Ruelle operator $\mathcal{L}$, we get the same equality.

Moreover, from Proposition 23

Proposition 11. *For a generic Hlder function ψ with norm 1*

$$\| \left[\mathcal{D}, \pi(\hat{\psi})\right] \| > 1. \tag{26}$$

We get (25) just when $\langle \mathcal{K}(\psi), \psi \rangle = 0$.

We address the issue of possible interpretation of $\| [\mathcal{D}, \pi(\mathcal{M}_f)] \|$ regarding forms of discrete time dynamical derivatives for f. In Subsection 2.2 we show for the multiplication operator $\mathcal{M}_f$ the following result:

Theorem 12. *$\forall f \in C(\Omega)$:*

$$|f - f \circ \sigma|_\infty = |\mathcal{K}f - f|_\infty \geq \| [\mathcal{D}, \pi(\mathcal{M}_f)] \| \geq |f - \mathcal{L}f|_\infty . \tag{27}$$

The left-hand side of (27) concerns a form of the supremum of dynamical mean forward derivative for f. We will get equality on both sides when f does not depend on the first coordinate.

Moreover,

Proposition 13. *$\forall f \in C(\Omega)$:*

$$\| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| = \left| \sqrt{\mathcal{L} \left| \mathcal{K} f - f \right|^2} \right|_\infty$$

$$= \sup_{x \in \Omega} \sqrt{\frac{\left| f(x) - f(0x) \right|^2}{2} + \frac{\left| f(x) - f(1x) \right|^2}{2}}. \qquad (28)$$

The right hand side of (28) *is a special form of supremum of mean backward derivative for f.*

In Proposition 43 in Section 4 we estimate expression $\| \ [\mathcal{D}, \pi(H)] \ \|$ for a compact self-adjoint operator H.

Proposition 14. *Consider the compact hermitian operator $H : L^2(\mu) \longrightarrow L^2(\mu)$ in the form*

$$H = \sum_i \lambda_i \hat{\psi}_i \ = \sum_i \lambda_i \left| \psi_i \right\rangle \left\langle \psi_i \right| , \qquad (29)$$

where the vectors $\{\psi_i\}$ form an orthonormal basis for $L^2(\mu)$, and λ_i are the real eigenvalues.

Then,

$$\| \ [\mathcal{D}, \pi(H)] \ \|^2 \leq 2 \left(\| \, H \, \|^2 + \sum_{i,j} |\lambda_i \lambda_j| \right). \qquad (30)$$

In [20] the authors compute explicitly the Connes distance in a finite-dimensional case for the 2D fermionic space (a non-dynamical setting).

Results for spectral triples in a dynamical context can be found in [6], [15], [16], [17], [28] and [29].

2 The Dirac operator

For normal (and in particular, both self-adjoint and anti-self-adjoint) elements, the norm considered here becomes

$$\| \, G \, \| = \sup_{\substack{(\phi,\psi) \in L^2(d\mu) \times L^2(d\mu) \\ |\phi|^2 + |\psi|^2 = 1}} |G(\phi, \psi)| \, .$$

Further, notice that concerning operators of the form:

$$T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \qquad \text{or of the form:} \qquad T^{'} = \begin{pmatrix} A^{'} & 0 \\ 0 & B^{'} \end{pmatrix},$$

the operator norm (not necessarily equal to the spectral radius norm) becomes simply:

$$\begin{aligned}
\| T \| &= \sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} |T(\phi,\psi)| \\
&= \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} |T(\phi,\psi)|^2} \\
&= \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} |A\psi|^2 + |B\phi|^2} \\
&\leq \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} \| A \|^2 \, |\psi|^2 + \| B \|^2 \, |\phi|^2} \\
&\leq \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} \max\{\| A \|^2, \| B \|^2\} \, |\psi|^2 + |\phi|^2} \\
&= \max\{\| A \|, \| B \|\}
\end{aligned}$$

or

$$\begin{aligned}
\| T^{'} \| &= \sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} \left|T^{'}(\phi,\psi)\right| \\
&= \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} |T^{'}(\phi,\psi)|^2} \\
&= \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} |A^{'}\phi|^2 + |B^{'}\psi|^2} \\
&\leq \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} \| A^{'} \|^2 \, |\phi|^2 + \| B^{'} \|^2 \, |\psi|^2}
\end{aligned}$$

$$\leq \sqrt{\sup_{\substack{(\phi,\psi)\in L^2(d\mu)\times L^2(d\mu)\\ |\phi|^2+|\psi|^2=1}} \max\{\| A' \|^2, \| B' \|^2\} |\phi|^2 + |\psi|^2}$$
$$= \max\left\{\| A^{'} \|, \| B^{'} \|\right\}.$$

Finally, one may use a simple approximation argument to show the opposite inequality. We will do it here for an operator of the first form, assuming, without loss of generality, that $\| A \| \geq \| B \|$.

Let $(\psi_n)_n$ be a sequence of functions such that $|\psi_n| = 1$, and $\lim_n |A\psi_n| = \| A \|$. Then $((0,\psi_n))_n$ is a sequence of pairs of functions such that $|(0,\psi_n)| = 1$ and

$$\begin{aligned}\lim_n |T(0,\psi_n)| &= \lim_n |(A\psi_n, 0)| \\ &= \lim_n |A\psi_n| \\ &= \| A \|.\end{aligned}$$

This implies $\| T \| \geq \| A \|$ and consequently $\| T \| = \| A \|$.

Thus:

$$\| T \| \ = \max\{\| A \|, \| B \|\}, \quad \text{and: } \| T^{'} \| \ = \max\left\{\| A^{'} \|, \| B^{'} \|\right\}. \tag{31}$$

Notice our momentum operator, which is self-adjoint, is then bounded, with a norm equal to the maximum of the norms of $\mathcal{K}$ and $\mathcal{L}$. These two operators being adjoint to one another have the same norm, and thus, $\| \mathcal{D} \| = \| \mathcal{K} \| = \| \mathcal{L} \| = 1$. But if D has a spectral radius equal to 1, its spectrum is bounded, and thus it cannot have compact resolvent.

We get that for each L

$$\begin{aligned}[\mathcal{D}, \pi(L)] &= \sqrt{2}\begin{pmatrix} 0 & \hat{f}^\dagger L - L\hat{f}^\dagger \\ \hat{f}L - L\hat{f} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathcal{K}L - L\mathcal{K} \\ \mathcal{L}L - L\mathcal{L} & 0 \end{pmatrix}.\end{aligned} \tag{32}$$

If L is self-adjoint, it will follow from (31), (32) and Lemma 15 that

$$\| \ [\mathcal{D}, \pi(L)] \ \| = \| (\mathcal{L}L - L\mathcal{L}) \| = \| (\mathcal{K}L - L\mathcal{K}) \|. \tag{33}$$

Proposition 18 will exhibit non-trivial operators L such that $\| \, [\mathcal{D}, \pi(L)] \, \| = 1$.

We will consider later for each word w the projection $\hat{e}_w$ on e_w:

$$\psi \to \hat{e}_w(\psi) = \langle \psi, e_w \rangle \, e_w.$$

In this way,

$$[\mathcal{D}, \pi(\hat{e}_w)] = \begin{pmatrix} 0 & \mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K} \\ \mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L} & 0 \end{pmatrix}. \tag{34}$$

Our purpose from now on is to estimate the value of $\| \, [\mathcal{D}, \pi(\hat{e}_w)] \, \|$.

Lemma 15. *If L is self-adjoint, then*

$$\| \, (\mathcal{L}L - L\mathcal{L}) \, \| = \| \, (\mathcal{K}L - L\mathcal{K}) \, \|.$$

In particular, if $\hat{\psi}$ is the projection on the unitary vector $\psi \in L^2(\mu)$, then

$$\| \, (\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L}) \, \| = \| \, (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \, \|.$$

The main conclusion is

$$\begin{aligned} \| \, \left[D, \pi(\hat{\psi}) \right] \, \| &= \max \left\{ \| \, (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \, \|, \| \, (\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L}) \, \| \right\} \\ &= \| \, (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \, \|. \end{aligned} \tag{35}$$

Proof. Notice that:

$$\begin{aligned} \| \, A \, \|^2 &= \sup_{|\phi|=1} |A\phi|^2 \\ &= \sup_{|\phi|=1} \langle A\phi, A\phi \rangle \\ &= \sup_{|\phi|=1} \langle A^*A\phi, \phi \rangle \\ &\leq \sup_{|\phi|=1} |A^*A\phi| \, |\phi| \\ &\leq \sup_{|\phi|=1} \| \, A^*A \, \| \, |\phi|^2 \\ &\leq \| \, A^* \, \| \| \, A \, \|, \end{aligned}$$

Which implies:

$$\| A \| \leq \| A^* \|.$$

If we take $A = A^*$, this means:

$$\| A^* \| \leq \| A^{**} \| = \| A \|,$$

And thus:

$$\| A \| = \| A^* \|.$$

Then, if T is a self-adjoint operator, it follows that:

$$\begin{aligned}
\| \mathcal{L}T - T\mathcal{L} \| &= \| \ (\mathcal{L}T - T\mathcal{L})^* \ \| \\
&= \| T\mathcal{L}^* - \mathcal{L}^*T \| \\
&= \| T\mathcal{K} - \mathcal{K}T \| \\
&= \| \ -(T\mathcal{K} - \mathcal{K}T) \ \| \\
&= \| \mathcal{K}T - T\mathcal{K} \|.
\end{aligned}$$

By taking $T = \hat{\psi}$, a projection on the element ψ of norm 1 we get:

$$\| \mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L} \| = \| \mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K} \|.$$

□

2.1 Estimates in the case of projection operators

In this subsection, we consider operators L such that $L = \hat{\psi}$ is a projection on a Hlder element ψ with L^2 norm equal to 1. We want to estimate

$$\| \ \left[D, \pi(\hat{\psi})\right] \ \|. \tag{36}$$

Proposition 16. *Suppose* $e_w = e_{w_1 w_2 \cdots w_n}$, $l(w) > 1$, *then*

$$\begin{aligned}
(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(\phi) &= \sqrt{\frac{1}{2}}[\,(\int e_w\,\phi d\mu)(e_{0w} + e_{1w}) - \int e_{w_2 w_3 \cdots w_n}\,\phi\, d\mu \ \ e_w \ ] \\
&= \frac{1}{\sqrt{2}}\left[\langle e_w, \phi\rangle\,(e_{0w} + e_{1w}) - \langle e_{\sigma(w)}, \phi\rangle e_w\right],
\end{aligned}$$

and

$$(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(\phi) = \sqrt{\frac{1}{2}}\,[\int e_w\,\phi\,d\mu\; e_{w_2\cdots w_n} - \int (e_{0w} + e_{1w})\,\phi\;d\mu\;\; e_w\,]$$
$$= \frac{1}{\sqrt{2}}\left[\langle e_w, \phi\rangle e_{\sigma(w)} - \langle e_{0w} + e_{1w}, \phi\rangle e_w\right].$$

Moreover,

$$(\mathcal{K}\hat{e}_{e^0_\emptyset} - \hat{e}_{e^0_\emptyset}\mathcal{K})(\phi) = -\sqrt{2}\,\langle e^0_\emptyset, \phi\rangle(\mathbf{1}_{00} + \mathbf{1}_{10}) + \langle e^0_\emptyset, (\phi\circ\sigma)\rangle e^0_\emptyset,$$
$$(\mathcal{K}\hat{e}_{e^1_\emptyset} - \hat{e}_{e^1_\emptyset}\mathcal{K})(\phi) = \sqrt{2}\,\langle e^1_\emptyset, \phi\rangle(\mathbf{1}_{01} + \mathbf{1}_{11}) - \langle e^1_\emptyset, (\phi\circ\sigma)\rangle e^1_\emptyset,$$
$$(\mathcal{L}\hat{e}_{e^0_\emptyset} - \hat{e}_{e^0_\emptyset}\mathcal{L})(\phi) = -\sqrt{\frac{1}{2}}\,\langle e^0_\emptyset, \phi\rangle + \sqrt{2}\langle(\mathbf{1}_{00} + \mathbf{1}_{10}), \phi\rangle\, e^0_\emptyset,$$

and

$$(\mathcal{L}\hat{e}_{e^1_\emptyset} - \hat{e}_{e^1_\emptyset}\mathcal{L})(\phi) = \sqrt{\frac{1}{2}}\,\langle e^1_\emptyset, \phi\rangle - \sqrt{2}\langle(\mathbf{1}_{01} + \mathbf{1}_{11}), \phi\rangle\, e^1_\emptyset.$$

For the proof see Proposition 44.

Proposition 17. *For a word w satisfying $l(w) > 1$, given a generic word $\tilde{w}$, and the corresponding element $e_{\tilde{w}}$, we get that*

$$|(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}})|^2 = \int [\,(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}})\,]^2\,d\mu \tag{37}$$

is equal to 0, if $w \neq \tilde{w} \neq w_2w_3\cdots w_n$, is equal to $1/2$ if $w \neq \tilde{w} = w_2w_3\cdots w_n$, and is equal to 1 if $\tilde{w} = w$. Moreover,

$$|(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(e_{\tilde{w}})|^2 = \int [\,(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(e_{\tilde{w}})\,]^2\,d\mu \tag{38}$$

is equal to 0, if $\tilde{w} \neq w \neq \sigma\tilde{w}$, and is equal to $1/2$ if either $\tilde{w} \neq w = \sigma\tilde{w}$ or $\tilde{w} = w$.

For the proof see Proposition 45 in the Appendix.

Using the above results we get:

Theorem 18. *Given a fixed word $w = w_1 w_2 w_3 \cdots w_n$, $l(w) \geq 2$, we get for the operator norm*

$$\| (\mathcal{K}\hat{e}_w - \hat{e}_w \mathcal{K}) \| = 1, \tag{39}$$

and also:

$$\| (\mathcal{L}\hat{e}_w - \hat{e}_w \mathcal{L}) \| = 1; \tag{40}$$

Therefore:

$$\| \; [\mathcal{D}, \pi(\hat{e}_w)] \; \| = 1. \tag{41}$$

For proof see Proposition 46 in the Appendix.

We denote by $\mathcal{B}$ the set

$$\mathcal{B} = \{ L \in \mathcal{A} \; | \; \| \; [\mathcal{D}, \pi(L)] \; \| \; \leq 1 \}. \tag{42}$$

Given ψ such that $|\psi| = 1$, we consider the projection $\hat{\psi}$:

$$\phi \to \hat{\psi}(\phi) = \langle \phi, \psi \rangle \, \psi.$$

Assume that

$$\psi = \sum_u b_u e_u + \beta_0 e^0_\emptyset + \beta_1 e^1_\emptyset, \tag{43}$$

in such way that $\sum_u b_u^2 + \beta_0^2 + \beta_1^2 = 1$, and

$$\phi = \sum_u a_u e_u + \alpha_0 e^0_\emptyset + \alpha_1 e^1_\emptyset, \tag{44}$$

in such way that $\sum_u a_u^2 + \alpha_0^2 + \alpha_1^2 = 1$.

One can show that

$$\mathcal{K}(\psi) = \sum_u b_u \frac{1}{\sqrt{2}} (e_{0u} + e_{1u}) - \sqrt{2}\beta_0 (\mathbf{1}_{00} + \mathbf{1}_{10}) + \sqrt{2}\beta_1 (\mathbf{1}_{01} + \mathbf{1}_{11}). \tag{45}$$

Note also that:

$$\mathcal{L}(\phi) = \sum_{l(v) \geq 2} a_v \frac{1}{\sqrt{2}} e_{\sigma(v)} + \mathcal{L} \left(a_0 e_0 + a_1 e_1 + \alpha_0 e^0_\emptyset + \alpha_1 e^1_\emptyset \right)$$

$$= \sum_{l(v)\geq 2} a_v \frac{1}{\sqrt{2}} e_{\sigma(v)} + \frac{1}{2}(a_0 + a_1)\left(e^0_\emptyset + e^1_\emptyset\right) + \frac{1}{\sqrt{2}}(\alpha_1 - \alpha_0)$$

$$= \sum_{l(v)\geq 2} a_v \frac{1}{\sqrt{2}} e_{\sigma(v)} + \frac{1}{2}(a_0 + a_1)\left(e^0_\emptyset + e^1_\emptyset\right) + \frac{1}{\sqrt{2}}(\alpha_1 - \alpha_0)\left(e^1_\emptyset - e^0_\emptyset\right)$$

$$= \sum_{l(v)\geq 2} a_v \frac{1}{\sqrt{2}} e_{\sigma(v)} + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(a_0 + a_1) - (\alpha_1 - \alpha_0)\right) e^0_\emptyset +$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(a_0 + a_1) + (\alpha_1 - \alpha_0)\right) e^1_\emptyset.$$

Denote

$$B_\psi \langle \mathcal{K}(\psi), \psi \rangle = \sum_u \frac{b_u}{\sqrt{2}}(b_{0u} + b_{1u}) + \frac{1}{2}(b_0 + b_1)(\beta_0 + \beta_1). \tag{46}$$

Note that $|\mathcal{K}(\psi)| = 1$ and $|B_\psi| \leq 1$.

Given ϕ, ψ, an estimation of the value

$$\langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{K}\phi, \psi \rangle \langle \mathcal{K}\psi, \psi \rangle + \langle \mathcal{K}\phi, \psi \rangle^2 \tag{47}$$

will be important regarding the future Theorem 21 (see expression (57)). We will address this issue soon and this requires to estimate B_ψ.

Example 19. *As an example, take a fixed finite word w on the symbols $\{0,1\}$, with $l(w) > 1$, and ψ of the form*

$$\psi = b_w e_w + b_{0w} e_{0w} + b_{1w} e_{1w}, \tag{48}$$

where $b_w^2 + b_{0w}^2 + b_{1w}^2 = 1$. Then,

$$\mathcal{K}(\psi) = \frac{1}{\sqrt{2}}[(e_{0w} + e_{1w})b_w + (e_{00w} + e_{10w})b_{0w} + (e_{01w} + e_{11w})b_{1w}], \tag{49}$$

and

$$B_\psi = \langle \mathcal{K}(\psi), \psi \rangle = \frac{b_w}{\sqrt{2}}(b_{0w} + b_{1w}). \tag{50}$$

Considering this kind of ψ, for the future expression (57) *we get*

$$\sqrt{b_w^2 + \frac{1}{2}(b_{0w} + b_{1w})^2 - 2b_w \frac{1}{\sqrt{2}}(b_{0w} + b_{1w}) B_\psi} =$$

$$\sqrt{b_w^2 + \frac{1}{2}\left(b_{0w} + b_{1w}\right)^2 - b_w^2\left(b_{0w} + b_{1w}\right)^2},$$

which can be shown to be smaller than or equal to 1*, and to be equal to* 1 *only when* $b_w = 1, b_{0w} = 0 = b_{1w}$*. In this case it follows that* ψ *is of the form* $\psi = e_w$*.*

We thank M. Denker for a suggestion we used on the proof of the next result (it applies when $\mathcal{W} = \mathcal{K}$).

Theorem 20. *Consider a real Hilbert space* $\mathcal{H}$ *and the action of a linear operator* $\mathcal{W} : \mathcal{H} \to \mathcal{H}$ *that preserves the inner product. Then for a fixed* $\psi \in \mathcal{H}$ *with norm* 1*, and any variable* ϕ *with norm* 1*, we get that*

$$\frac{9}{8} \geq \langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{W}\phi, \psi \rangle \langle \mathcal{W}\psi, \psi \rangle + \langle \mathcal{W}\phi, \psi \rangle^2 \geq 1. \tag{51}$$

There exists an element ψ*, and a particular* ϕ *in the two dimensional linear space* $\{\psi, \mathcal{W}^*(\psi)\}$*, where the maximal value* $9/8$ *is attained.*

The upper bound in claim (51) *is true for either the Koopman operator* $\mathcal{K}$*, or a unitary operator.*

In order to get the value 1 *in* (51) *we have to take* $\langle \mathcal{W}(\psi), \psi \rangle = 1$ *or* $\langle \mathcal{W}(\psi), \psi \rangle = 0$*. For the Koopman operator* $\langle \mathcal{K}(\psi), \psi \rangle = 1$*, only when* $\psi = 1$*.*

Proof. Given ψ of norm 1, consider the linear subspace space Y generated by the basis $\{\psi, \mathcal{W}^*(\psi)\}$.

Denote $c = \cos(\theta) = \langle \psi, \mathcal{W}^*(\psi) \rangle = \langle \mathcal{W}(\psi), \psi \rangle \in [-1, 1]$. Note that $\mathcal{W}(\psi)$ does not necessarily belong to Y.

For fixed ψ with norm 1, given a vector $\phi \in \mathcal{H}$ of norm 1, denote $\phi = \alpha\phi_1 + \beta\phi_2$, where $|\psi_1| = 1 = |\psi_2|$, $\phi_1 \in Y$ and ϕ_2 is orthogonal to the linear subspace Y. In this case $\alpha^2 + \beta^2 = 1$, $\langle \phi_2, \psi \rangle = 0$ and $\langle \phi_2, \mathcal{W}^*(\psi) \rangle = 0$.

Denote $\phi_1 = a\,\psi + b\,\hat{\psi}$, where $|\psi| = 1 = |\hat{\psi}|$, $\langle \psi, \hat{\psi} \rangle = 0$, $\hat{\psi} \in Y$, $a, b \neq 0$ $a^2 + b^2 = 1$. Also denote $d = \langle \hat{\psi}, \mathcal{W}^*(\psi) \rangle = \langle \mathcal{W}(\hat{\psi}), \psi \rangle = \cos(\theta \pm \frac{\pi}{2}) = \pm \sin(\theta) \in [-1, 1]$. Note that $\mathcal{W}(\hat{\psi})$ does not necessarily belongs to Y.

Then,

$$\langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{W}\phi, \psi \rangle \langle \mathcal{W}\psi, \psi \rangle + \langle \mathcal{W}\phi, \psi \rangle^2 =$$

$$\langle \alpha\phi_1 + \beta\phi_2, \psi \rangle^2 - \langle \alpha\phi_1 + \beta\phi_2, \psi \rangle \langle \alpha\mathcal{W}(\phi_1) + \beta\mathcal{W}(\phi_2), \psi \rangle\, c +$$
$$\langle \alpha\mathcal{W}(\phi_1) + \beta\mathcal{W}(\phi_2), \psi \rangle^2 =$$

$$\langle \alpha\phi_1, \psi\rangle^2 - \langle \alpha\phi_1, \psi\rangle \langle \alpha\mathcal{W}(\phi_1), \psi\rangle \, c + \langle \alpha\mathcal{W}(\phi_1), \psi\rangle^2 =$$
$$\langle \alpha(a\,\psi + b\,\hat{\psi}), \psi\rangle^2 - \langle \alpha(a\,\psi + b\,\hat{\psi}), \psi\rangle \langle \alpha(a\mathcal{W}(\psi) + b\mathcal{W}(\hat{\psi})), \psi\rangle \, c +$$
$$\langle \alpha(a\mathcal{W}(\psi) + b\mathcal{W}(\hat{\psi})), \psi)\rangle^2 =$$
$$\alpha^2 a^2 - \alpha a\, \alpha(ac + b\langle \mathcal{W}(\hat{\psi})), \psi\rangle) \, c + \alpha^2 (ac + b\langle \mathcal{W}(\hat{\psi})), \psi\rangle)^2 =$$
$$\alpha^2 \, [a^2 - a\,(ac+bd)\,c + (ac+bd)^2], \tag{52}$$

where

$$|\alpha| < 1, a^2 + b^2 = 1 = c^2 + d^2. \tag{53}$$

Given a fixed α, the maximal value of $a^2 - a\,(ac+bd)\,c + (ac+bd)^2$, under the constraints (53), is the value $9/8$ attained when

$$a = -\frac{\sqrt{3}}{2}, c = -\frac{1}{2}, \text{ or } c = \frac{1}{2}.$$

When $\phi = \phi_1 \in Y$, we get get $\alpha = 1$, which maximizes (52). Taking $\alpha = 1$ means $a = \langle \phi, \psi\rangle$.

For the case of the Koopman operator $\mathcal{K}$ (associated with the maximal entropy measure on $\{0,1\}^{\mathbb{N}}$) the maximal value can be realized. Indeed, given any finite word w, consider in Example 19 the values $b_w = \frac{1}{\sqrt{2}}, b_{0w} = -1/2 = b_{1w}$. Then, we get $\langle \mathcal{K}\psi, \psi\rangle = -\frac{1}{2} = c$, when ψ is of the form $\psi = b_w e_w + b_{0w} e_{0w} + b_{1w} e_{1w}$. For such ψ, the function $\hat{\psi} = \frac{1}{\sqrt{2}} e_w + \frac{1}{2} e_{0w} + \frac{1}{2} e_{1w}$ is orthogonal to ψ and has norm 1. Take $\phi = \phi_1 \in Y$, and moreover set

$$\phi_1 = -\frac{\sqrt{3}}{2}\psi + \frac{1}{2}\hat{\psi}.$$

In this case, for such ψ, taking such ψ_1 we get the maximal value

$$\langle \phi, \psi\rangle^2 - 2\langle \phi, \psi\rangle \langle \mathcal{W}\phi, \psi\rangle \langle \mathcal{W}\psi, \psi\rangle + \langle \mathcal{W}\phi, \psi\rangle^2 \; = 9/8.$$

Consider the function $G(a,c) = a^2 - a\,(ac+bd)\,c + (ac+bd)^2$.

For a fixed value c the partial derivative

$$\frac{G(a,c)}{\partial a} = \frac{\partial\,[\,a^2 - a\,(ac+bd)\,c + (ac+bd)^2\,]}{\partial a}$$
$$= \frac{c\;\;(2\,a\,c\sqrt{1-a^2}\;+\sqrt{1-c^2} - 2\,a^2\sqrt{1-c^2})}{\sqrt{1-a^2}}. \tag{54}$$

Given c the value $a = a_c$ such that $2\,a\,c\,\sqrt{1-a^2} + \sqrt{1-c^2} - 2\,a^2\,\sqrt{1-c^2} = 0$, producing the largest value $G(a_c, c)$ is

$$a_c = \frac{\sqrt{1+c}}{\sqrt{2}},$$

and the largest value is

$$G(a_c, c) = \frac{1}{2}(2 + c - c^2) \geq 1.$$

Therefore, given ψ we get c, and then we choose $\phi = a\,\psi + b\,\hat{\psi}$ such that $\alpha = 1$ an $a = a_c$. □

Theorem 21. *Using the notation of* (43) *and* (44)*, take a non-constant* $\psi = \sum_u b_u e_u + \beta_0 e^0_\emptyset + \beta_1 e^1_\emptyset \in L^2(\mu)$*, such that* $|\psi| = 1$*, and denote by* B_ψ *the value given by* (46)*.*

Then, if $\hat{\psi}$ *denotes the projection operator, we get the operator norm:*

$$1 \leq \|\,\left[D, \pi(\hat{\psi})\right]\,\| = \max\left\{\|\,(\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K})\,\|, \|\,(\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L})\,\|\right\} \leq \frac{3}{2\sqrt{2}} \tag{55}$$

This is so because

$$\frac{3}{2\sqrt{2}} \geq \|\,(\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K})\,\| = \tag{56}$$

$$= \sup_{|\phi|=1} \sqrt{\langle\phi,\psi\rangle^2 - 2\langle\phi,\psi\rangle\langle\mathcal{K}\phi,\psi\rangle\langle\mathcal{K}\psi,\psi\rangle + \langle\mathcal{K}\phi,\psi\rangle^2} \tag{57}$$

$$\geq \sup_w \sqrt{b_w^2 + \frac{1}{2}\left(b_{0w} + b_{1w}\right)^2 - 2b_w \frac{1}{\sqrt{2}}\left(b_{0w} + b_{1w}\right) B_\psi}, \tag{58}$$

and

$$\frac{3}{2\sqrt{2}} \geq \|\,(\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L})\,\| = \tag{59}$$

$$= \sup_{|\phi|=1} \sqrt{\langle\phi,\psi\rangle^2 - 2\langle\phi,\psi\rangle\langle\mathcal{L}\phi,\psi\rangle\langle\mathcal{L}\psi,\psi\rangle + \langle\mathcal{L}\phi,\psi\rangle^2} \tag{60}$$

$$\geq \sup_w \sqrt{b_w^2 + \frac{1}{2} b_{\sigma(w)}^2 - 2b_w \frac{1}{\sqrt{2}} b_{\sigma(w)} B_\psi}. \tag{61}$$

The inequality comes from Theorem 20.

In terms of the elements of the basis, we get for ϕ of the form (44)

$$|\mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi)|^2 = \langle \mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi), \mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi)\rangle =$$

$$\langle \phi, \psi\rangle^2 + (\frac{1}{\sqrt{2}}\sum_v a_v\,(b_{0v} + b_{1v}) + \frac{1}{2}(\alpha_1 + \alpha_0)(b_1 + b_0) + \frac{1}{2}(\alpha_1 - \alpha_0)(\beta_1 - \beta_0))^2$$

$$-2\,\langle \phi, \psi\rangle\,\{[\,\frac{1}{\sqrt{2}}\sum_v a_v\,(b_{0v} + b_{1v}) + \frac{1}{2}(\alpha_1 + \alpha_0)(b_1 + b_0) + \frac{1}{2}(\alpha_1 - \alpha_0)(\beta_1 - \beta_0)\,]$$

$$[\sum_{l(u)>1} \frac{b_u}{\sqrt{2}}(b_{0u} + b_{1u}) + \frac{1}{2}(b_0 + b_1)(\beta_0 + \beta_1)]\}. \tag{62}$$

For the proof see Proposition 47 in the Appendix.

Theorem 22. *Suppose ψ with norm 1 is a Hlder function in the kernel of the Ruelle operator $\mathcal{L}$, then*

$$\|\,\left[D, \pi(\hat{\psi})\right]\,\| = \|\,(\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K})\,\| = \|\,(\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L})\,\| = 1. \tag{63}$$

Moreover, take ψ a Hlder function of the form $\psi = \mathcal{K}^k(f)$, $k \geq 1$, where f is in the kernel of the Ruelle operator $\mathcal{L}$, and has norm equal to 1. Then $\|\,\left[D, \pi(\hat{\psi})\right]\,\| = 1$.

Proof. Given a function ϕ with norm 1

$$\langle \phi, \psi\rangle^2 - 2\langle \phi, \psi\rangle\langle \mathcal{K}\phi, \psi\rangle\langle \mathcal{K}\psi, \psi\rangle + \langle \mathcal{K}\phi, \psi\rangle^2 =$$

$$\langle \phi, \psi\rangle^2 - 2\langle \phi, \psi\rangle\langle \mathcal{K}\phi, \psi\rangle\langle \psi, \mathcal{L}\psi\rangle + \langle \phi, \mathcal{L}\psi\rangle^2 = \langle \phi, \psi\rangle^2 \leq 1.$$

Taking $\phi = \psi$ above we get

$$\langle \psi, \psi\rangle^2 - 2\langle \psi, \psi\rangle\langle \mathcal{K}\psi, \psi\rangle\langle \mathcal{K}\psi, \psi\rangle + \langle \mathcal{K}\psi, \psi\rangle^2 = 1.$$

Therefore,

$$\|\,(\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K})\,\| =$$

$$\sup_{|\phi|=1} \sqrt{\langle \phi, \psi\rangle^2 - 2\langle \phi, \psi\rangle\langle \mathcal{K}\phi, \psi\rangle\langle \mathcal{K}\psi, \psi\rangle + \langle \mathcal{K}\phi, \psi\rangle^2} = 1. \tag{64}$$

For the second claim, assume that ψ is a Hlder function of the form $\psi = \mathcal{K}^k(f)$, $k \geq 1$, where f in the kernel of the Ruelle operator $\mathcal{L}$ has norm equal to 1. Then, $\psi = \mathcal{K}^k(f)$ also has norm equal to 1.

Then, taking $\phi = \mathcal{K}^k(f) = \psi$, we get

$$\langle \mathcal{K}\phi, \psi\rangle = \langle \mathcal{K}\mathcal{K}^k(f), \mathcal{K}^k(f)\rangle = \langle \mathcal{K}(f), f\rangle = \langle f, \mathcal{L}(f)\rangle = 0. \tag{65}$$

Therefore,

$$\begin{aligned}\langle \phi, \psi\rangle^2 - 2\langle \phi, \psi\rangle\langle \mathcal{K}\phi, \psi\rangle\langle \mathcal{K}\psi, \psi\rangle + \langle \mathcal{K}\phi, \psi\rangle^2 = \\ \langle \phi, \psi\rangle^2 + \langle \mathcal{K}\phi, \psi\rangle^2 = \langle \phi, \psi\rangle^2 = 1.\end{aligned} \tag{66}$$

Note that in the notation of Theorem 20 we get that

$$c = \cos(\theta) = \langle \mathcal{K}(\psi), \psi\rangle = 0,$$

and

$$\begin{aligned}\langle \phi, \psi\rangle^2 - 2\langle \phi, \psi\rangle\langle \mathcal{K}\phi, \psi\rangle\langle \mathcal{K}\psi, \psi\rangle + \langle \mathcal{K}\phi, \psi\rangle^2 = \\ \alpha^2\,[a^2 - a\,(ac+bd)\,c + (ac+bd)^2] = \alpha^2\,[a^2 + (bd)^2],\end{aligned} \tag{67}$$

where

$$|\alpha| < 1, a^2 + b^2 = 1 = c^2 + d^2. \tag{68}$$

As $\alpha^2\,[a^2 + b^2 d^2] \leq 1$, because $|d| \leq 1$, we get (63). □

We denote $\mathfrak{N}$ the kernel of the Ruelle operator $\mathcal{L}$. From Wold's Theorem (see Section 7 in [2]), it is known that the following orthogonal expression is true:

$$L^2(\mu) = \mathcal{H}_\infty \oplus \mathfrak{N} \oplus \mathcal{K}(\mathfrak{N}) \oplus \cdots \oplus \mathcal{K}^k(\mathfrak{N}) \oplus \cdots \tag{69}$$

where $\mathcal{H}_\infty$ is the set of almost everywhere constant functions.

As we will see one can get explicit values for the action of the Dirac operator on general projections.

Proposition 23. *For Hlder functions ψ with norm 1 such that*

$$\psi \in \mathfrak{N} \oplus \mathcal{K}(\mathfrak{N}) \oplus \cdots \oplus \mathcal{K}^k(\mathfrak{N}) \oplus \cdots,$$

take $c = \langle \mathcal{K}(\psi), \psi\rangle$. Then

$$\|\ \left[D, \pi(\hat{\psi})\right]\ \| = \frac{1}{2}(2 + c - c^2). \tag{70}$$

The value c can be obtained explicitly and in most of the cases $\frac{1}{2}(2 + c - c^2) > 1$. This will happen for a generic ψ.

Proof. Assume ψ with norm 1 satisfies

$$\psi = \alpha_0 f_0 + \alpha_1 \mathcal{K}(f_1) + \alpha_1 \mathcal{K}^2(f_2) + \cdots + \alpha_k \mathcal{K}^k(f_k) + \cdots, \tag{71}$$

where $\sum_{k=0}^{\infty} \alpha_k^2 = 1$, and $|f_k| = 1, \forall k$. By exchanging the signal of f_k we can assume that all $\alpha_k \geq 0$.

From Proposition 20, if $c = \langle \mathcal{K}(\psi), \psi \rangle > 0$, then $\| \left[D, \pi(\hat{\psi}) \right] \| > 1$.

One can show that

$$c = \langle \mathcal{K}\psi, \psi \rangle = \alpha_0 \alpha_1 \langle f_0, f_1 \rangle + \alpha_1 \alpha_2 \langle f_1, f_2 \rangle + \alpha_2 \alpha_3 \langle f_2, f_3 \rangle + \cdots,$$

and then we get one gets the value $\| \left[D, \pi(\hat{\psi}) \right] \|$.

Take for example $\psi = \alpha_0 f_0 + \alpha_1 \mathcal{K}(f_1)$, $\alpha_0 \neq 0, 1$, where $\langle f_0, f_1 \rangle > 0$. Then,

$$c = \langle \mathcal{K}\psi, \psi \rangle = \alpha_0 \alpha_1 \langle f_0, f_1 \rangle > 0.$$

If in (71) we take $\langle f_{j+1}, f_j \rangle = 0$, $j \geq 0$, we get $c = 0$, and then $\| \left[D, \pi(\hat{\psi}) \right] \| = 1$. □

2.2 Estimates in the case of elements in the Exel-Lopes C^*-algebra

In this section we are interested in estimates of $\| [\mathcal{D}, \pi(L)] \|$ for operators L of the form $L = \mathcal{M}_f$ or $L = \mathcal{K}^n \mathcal{L}^n$, considered in [11] (see also [10]). We are also interested in an interpretation of $\| [\mathcal{D}, \pi(\mathcal{M}_f)] \|$ as a form of the supremum of a discrete time dynamical derivative of f. Note that $|f - f \circ \sigma|_\infty \leq 1$ could be seen as a dynamical form of saying that f has Lipschitz constant smaller than or equal to 1. We will explore here such a point of view.

We denote by $\mathcal{B}$ the Borel sigma-algebra on $\{0,1\}^{\mathbb{N}}$ and μ the measure of maximal entropy.

First, we consider multiplication operators of the form:

$$\begin{aligned} \mathcal{M}_f : L^2(\mu) &\longrightarrow L^2(\mu) \\ g &\longmapsto fg, \end{aligned}$$

for a given continuous function $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ and measurable functions $g \in L^2(\mu)$. Here, we will always use $f \in C(\Omega)$ to denote a continuous function and $g \in L^2(\mu)$ to denote a square-integrable function. We will use the expression $|\cdot|$ for the $L^2(\mu)$-norm and $|\cdot|_\infty$ for the supremum norm.

Among other things, we will be interested in a dynamical interpretation for the action of the Dirac operator $\mathcal{D}$ which was introduced before; more precisely, for the action of the operator

$$[\mathcal{D}, \pi(\mathcal{M}_f)]\,. \tag{72}$$

As mentioned before, one important issue is to know when the norm of this operator (acting on $L^2(\mu) \times L^2(\mu)$) is smaller than or equal to 1. Concerning the Connes distance, a kind of normalization condition on f would be then natural, for instance, $|f|_\infty = 1$, or perhaps, $f(0^\infty) = 0$. But, we will not address this issue here.

One of our main results in this section is Theorem 29 which claims

Theorem 24. *$\forall f \in C(\Omega)$:*

$$|f - f \circ \sigma|_\infty = |\mathcal{K}f - f|_\infty \geq \|\; [\mathcal{D}, \pi(\mathcal{M}_f)] \;\| \geq |f - \mathcal{L}f|_\infty\,. \tag{73}$$

We will get equality on both sides when f does not depend on the first coordinate.

Another important result is the estimate of Proposition 25:

Proposition 25. *$\forall f \in C(\Omega)$:*

$$\|\; [\mathcal{D}, \pi(\mathcal{M}_f)] \;\| = \left| \sqrt{\mathcal{L}\,|\mathcal{K}f - f|^2} \right|_\infty. \tag{74}$$

That is, instead of projection operators (the case analyzed in a previous section) we will be now interested in results for the multiplication operator $\mathcal{M}_f$. The operator $\mathcal{D}$ is dynamically defined, and then, any form of interpretation should also carry this structure. It seems natural to us to presume that $f - f \circ \sigma$ could mean a derivative in a dynamical sense. The next Lemma is in consonance with this point of view.

The expression

$$f - f \circ \sigma \tag{75}$$

is a dynamical form of *discrete time forward derivative.*

Note that from (73), if $|f - f \circ \sigma|_\infty \leq 1$, then $\|\; [\mathcal{D}, \pi(\mathcal{M}_f)] \;\| \leq 1$.

Later in (82) we will present a dynamical form of *discrete time mean backward derivative*:

$$\sqrt{\frac{|f(x)-f(0x)|^2}{2}+\frac{|f(x)-f(1x)|^2}{2}}. \tag{76}$$

In this direction in Proposition 25 we get the expression

$$\| \, [\mathcal{D},\pi(\mathcal{M}_f)] \, \| = \sup_{x\in\Omega}\sqrt{\frac{|f(x)-f(0x)|^2}{2}+\frac{|f(x)-f(1x)|^2}{2}}. \tag{77}$$

Lemma 26. *Given any $f \in C(\Omega)$, we get that $[\mathcal{D},\pi(\mathcal{M}_f)] = 0$ is equivalent to $f - f\circ\sigma = 0$. The latter implies that f is constant.*

Proof. To prove this, first remember that since $\mathcal{M}_f$ is self-adjoint:

$$\| \, \mathcal{K}\mathcal{M}_f - \mathcal{M}_f\mathcal{K} \, \| = \| \, \mathcal{L}\mathcal{M}_f - \mathcal{M}_f\mathcal{L} \, \|.$$

Notice if $f = f\circ\sigma$, then:

$$\begin{aligned}\mathcal{K}\mathcal{M}_f(g) - \mathcal{M}_f\mathcal{K}(g) &= \mathcal{K}(fg) - f\mathcal{K}g\\ &= (\mathcal{K}f)\,(\mathcal{K}g) - f\mathcal{K}g\\ &= (\mathcal{K}f - f)\,\mathcal{K}g\\ &= 0.\end{aligned}$$

Thus, $\mathcal{L}\mathcal{M}_f - \mathcal{M}_f\mathcal{L} = 0$, and consequently $[\mathcal{D},\pi(\mathcal{M}_f)] = 0$. On the other hand, if $[\mathcal{D},\pi(\mathcal{M}_f)] = 0$, then:

$$\| \, \mathcal{K}\mathcal{M}_f - \mathcal{M}_f\mathcal{K} \, \| = \| \, \mathcal{L}\mathcal{M}_f - \mathcal{M}_f\mathcal{L} \, \| = 0,$$

and in particular:

$$\begin{aligned}|\mathcal{K}\mathcal{M}_f(1) - \mathcal{M}_f\mathcal{K}(1)| &= |\mathcal{K}f - f|\\ &= 0,\end{aligned}$$

which means $f - f\circ\sigma = 0$. □

The above motivates us to say that being constant with respect to (72) is the same as f being constant. But, of course, not all continuous functions are invariant for the shift map, and now we will analyze this more general family of functions.

Remark 27. *As μ is σ-invariant:*

$$|\mathcal{K}g| = \left(\int |g \circ \sigma|^2 \, d\mu\right)^{\frac{1}{2}} = \left(\int |g|^2 \, d\mu\right)^{\frac{1}{2}} = |g| \, .$$

This means that there exists a correspondence of the values on the left side (functions that are $\sigma^{-1}(\mathcal{B})$ measurable), and on the right side (functions that are $\mathcal{B}$-measurable).

It also follows that

$$\sup_{|g|=1} |f\mathcal{K}g| = \sup_{|\mathcal{K}g|=1} |f\mathcal{K}g| \, . \tag{78}$$

With the above Remark in mind, we look at the identity:

$$\begin{aligned} \| \, [\mathcal{D}, \pi(\mathcal{M}_f)] \, \| &= \| \, \mathcal{K}\mathcal{M}_f - \mathcal{M}_f\mathcal{K} \, \| \\ &= \sup_{|g|=1} |(\mathcal{K}f - f) \, \mathcal{K}(g)| \, . \end{aligned} \tag{79}$$

It is known that if f is continuous

$$\sup_{|g|=1} |fg| = |f|_\infty \, .$$

It is also known that $\mathcal{K} \, \mathcal{L}(f)$ is the conditional expectation of f given the sigma-algebra $\sigma^{-1}(\mathcal{B})$.

Lemma 28. $\forall f \in C(\Omega)$*:*

$$|f|_\infty \geq \sup_{|g|=1} |f\mathcal{K}g| \geq |\mathcal{L}f|_\infty \, . \tag{80}$$

Furthermore, if $f \in C(\Omega)$ does not depend on the first coordinate (that is, if f is $\sigma^{-1}(\Sigma)$-measurable), then all above inequalities are equalities.

Proof. First, we will show (80). Note that from Jensen's inequality for conditional expectation

$$\begin{aligned} |f|_\infty &= \sup_{|g|=1} |fg| \\ &\geq \sup_{|\mathcal{K}g|=1} |f\mathcal{K}g| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{|g|=1} |f\mathcal{K}g| \\
&= \sup_{|g|=1} \left(\int |f\mathcal{K}g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int \mathcal{K}\mathcal{L} \, |f\mathcal{K}g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int \mathcal{K}\mathcal{L} \left(|f|^2 \, |\mathcal{K}g|^2 \right) d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int \mathcal{K}\mathcal{L} \left(|f|^2 \right) |\mathcal{K}g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&\geq \sup_{|g|=1} \left(\int |\mathcal{K}\mathcal{L}\,(f)|^2 \, |\mathcal{K}g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int \mathcal{K} \left(|\mathcal{L}\,(f)|^2 \, |g|^2 \right) d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int |\mathcal{L}\,(f)|^2 \, |g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int |\mathcal{L}\,(f)\, g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} |(\mathcal{L}f)\, g| \\
&= |\mathcal{L}f|_\infty .
\end{aligned}
$$

Now, we will show the next claim. If f does not depend of the first coordinate we get that $|f|_\infty = |\mathcal{L}f|_\infty$.

Indeed, note that given any $x = (x_1, x_2, \cdots)$, the two preimages $y_0 = (0, x_1, x_2, \cdots), y_1 = (1, x_1, x_2, \cdots)$ are such that $f(y_0) = f(y_1)$. Denote by $z = (a_1, a_2, \cdots, a_n, \cdots)$ the point where the continuous function f assumes the maximal value $|f|_\infty$. Then,

$$
\mathcal{L}(f)(\sigma(z)) = \frac{1}{2}(f(1, a_2, a_3, \cdots, a_n, \cdots) + f(0, a_2, a_3, \cdots, a_n, \cdots)) = |f|_\infty ,
$$

which is clearly the maximal value $|\mathcal{L}f|_\infty$. $\square$

Theorem 29. *Replacing f by $\mathcal{K}f - f$ in* (80)*, in view of* (79)*, we get $\forall f \in C(\Omega)$:*

$$|\mathcal{K}f - f|_\infty \geq \| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| \geq |f - \mathcal{L}f|_\infty .$$

Moreover, if f does not depend on the first coordinate, then the same is true for $\mathcal{K}f - f$, and we get the equalities:

$$|\mathcal{K}f - f|_\infty = \| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| = |f - \mathcal{L}f|_\infty .$$

Proposition 30. *$\forall f \in C(\Omega)$:*

$$\| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| = \left| \sqrt{\mathcal{L} \left| \mathcal{K}f - f \right|^2} \right|_\infty \tag{81}$$

Expression (74) *can be written as*

$$\left| \sqrt{\mathcal{L} \left| \mathcal{K}f - f \right|^2} \right|_\infty = \sup_{x \in \Omega} \sqrt{\frac{\left| f(x) - f(0x) \right|^2}{2} + \frac{\left| f(x) - f(1x) \right|^2}{2}}. \tag{82}$$

The right-hand side of (82) *is a form of the supremum of mean backward derivative.*

Proof. We have:

$$\begin{aligned}
\sup_{|g|=1} |f\mathcal{K}g| &= \sup_{|g|=1} \left(\int |f\mathcal{K}g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int |f|^2 \, |\mathcal{K}g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int |f|^2 \left(\mathcal{K} \, |g|^2 \right) d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left(\int \left(\mathcal{L} \, |f|^2 \right) |g|^2 \, d\mu \right)^{\frac{1}{2}} \\
&= \sup_{|g|=1} \left| \left(\sqrt{\mathcal{L} \, |f|^2} \right) g \right| \\
&= \left| \sqrt{\mathcal{L} \, |f|^2} \right|_\infty ,
\end{aligned}$$

then we substitute f for $\mathcal{K}f - f$. □

Corollary 31. *$\forall f \in C(\Omega)$:*

$$|\mathcal{K}f - f|_\infty \leq 1 \implies \| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| \leq 1.$$

Remark 32. *The converse is not true. Take for example the function $f = \sqrt{2}\mathbf{1}_{[0]} \in C(\Omega)$. Then*

$$\begin{aligned} |\mathcal{K}f - f|_\infty &= \left|\sqrt{2}(\mathbf{1}_{[00]} + \mathbf{1}_{[10]} - \mathbf{1}_{[0]})\right|_\infty \\ &= \left|\sqrt{2}(\mathbf{1}_{[10]} - \mathbf{1}_{[01]})\right|_\infty \\ &= \sqrt{2} > 1. \end{aligned}$$

On the other hand, (74) *allows us to show that $\| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| = 1$. That is because:*

$$\begin{aligned} \| \ [\mathcal{D}, \pi(\mathcal{M}_f)] \ \| &= \sup_{x\in\Omega} \sqrt{\frac{|f(x) - f(0x)|^2}{2} + \frac{|f(x) - f(1x)|^2}{2}} \\ &= \sup_{x\in\Omega} \sqrt{\frac{\sqrt{2}^2}{2}} \\ &= 1. \end{aligned}$$

Notice that expression (82) is a form RMS (root mean square), also called quadratic mean, and as such, a generalized mean in the sense of Kolomogorov (see [4]), concerning the differences $|f(x) - f(0x)|$ and $|f(x) - f(1x)|$. Thus, we may say the (operator) norm of the momentum of a given continuous function f, as we have previously defined, measures the supremum of a Kolmogorov mean. In particular, it satisfies the following inequalities, which follow from the inequalities for generalized means of different *orders* as in [24, Subsection 2.14.2; Theorem 1]. Here, in the notation of [24] we are considering *orders* $-\infty$, -1, 0, 1, 2, and $+\infty$ respectively:

$$\begin{aligned} \sup_{x\in\Omega} \min\left\{|f(x) - f(0x)|, |f(x) - f(1x)|\right\} &\leq \sup_{x\in\Omega} \frac{2}{\frac{1}{|f(x)-f(0x)|} + \frac{1}{|f(x)-f(1x)|}} \\ &\leq \sup_{x\in\Omega} \sqrt{|f(x) - f(0x)|\,|f(x) - f(1x)|} \\ &\leq \sup_{x\in\Omega} \frac{|f(x) - f(0x)|}{2} + \frac{|f(x) - f(1x)|}{2} \end{aligned}$$

$$
\begin{aligned}
&\leq \| \, [\mathcal{D}, \pi(\mathcal{M}_f)] \, \| \\
&\leq \sup_{x\in\Omega} \max \{ |f(x) - f(0x)| \, , |f(x) - f(1x)| \} \\
&= |\mathcal{K}f - f|_\infty \, ,
\end{aligned}
$$

plus in general for all $-\infty < p \leq 2 \leq q < +\infty$:

$$
\begin{aligned}
\sup_{x\in\Omega} \left(\frac{|f(x) - f(0x)|^p}{2} + \frac{|f(x) - f(1x)|^p}{2} \right)^{\frac{1}{p}} &\leq \| \, [\mathcal{D}, \pi(\mathcal{M}_f)] \, \| \\
&\leq \sup_{x\in\Omega} \left(\frac{|f(x) - f(0x)|^q}{2} + \frac{|f(x) - f(1x)|^q}{2} \right)^{\frac{1}{q}} .
\end{aligned}
$$

The next result provides estimates for the the $L^2(\mu)$-norm (not for the uniform norm).

Lemma 33. *$\forall f \in C(\Omega)$:*

$$
\| \, [\mathcal{D}, \pi(\mathcal{M}_f)] \, \| \leq 1 \implies \begin{array}{c} |\mathcal{K}f - f| \leq 1 \\ \& \\ |\mathcal{L}f - f| \leq 1. \end{array}
$$

Proof. Suppose $\| \, [\mathcal{D}, \pi(\mathcal{M}_f)] \, \| \leq 1$. This implies that:

$$
\| \, \mathcal{K}\mathcal{M}_f - \mathcal{M}_f\mathcal{K} \, \| = \| \, \mathcal{L}\mathcal{M}_f - \mathcal{M}_f\mathcal{L} \, \| \leq 1.
$$

The constant function $1 \in L^2(\mu)$ has norm equal to 1. It follows that

$$
|\mathcal{K}\mathcal{M}_f(1) - \mathcal{M}_f\mathcal{K}(1)| = |\mathcal{K}f - f| \leq 1,
$$

and

$$
|\mathcal{L}\mathcal{M}_f(1) - \mathcal{M}_f\mathcal{L}(1)| = |\mathcal{L}f - f| \leq 1.
$$

□

Now we will consider estimates of the value $\| \, [\mathcal{D}, \pi(L)] \, \|$ for the class of operators $L = \mathcal{K}^n\mathcal{L}^n$, $n \geq 1$.

It is known that $L = \mathcal{K}^n\mathcal{L}^n$ is the conditional expectation operator on the sigma-algebra $\sigma^{-n}(\mathcal{B})$ (functions which do not depend on the n first coordinates). First notice:

$$
\begin{aligned}
[\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)] &= \begin{pmatrix} 0 & \mathcal{K}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{K} \\ \mathcal{L}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{L} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{K}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^{n-1} \\ \mathcal{K}^{n-1}\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{L} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{K}\left(\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^{n-1}\mathcal{L}^{n-1}\right) \\ \left(\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n\right)\mathcal{L} & 0 \end{pmatrix} .
\end{aligned}
$$

The operator $\mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^n\mathcal{L}^n$ is the difference between two projections, where the range of one is contained in the range of the other, and as such, is also a projection.

Note that

$$\begin{aligned}\left(\mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^n\mathcal{L}^n\right)^2 &= \begin{array}{c}\mathcal{K}^{n-1}\mathcal{L}^{n-1}\mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^{n-1}\mathcal{L}^{n-1}\mathcal{K}^n\mathcal{L}^n\\ -\mathcal{K}^n\mathcal{L}^n\mathcal{K}^{n-1}\mathcal{L}^{n-1}+\mathcal{K}^n\mathcal{L}^n\mathcal{K}^n\mathcal{L}^n\end{array}\\ &= \begin{array}{c}\mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^n\mathcal{L}^n\\ -\mathcal{K}^n\mathcal{L}^n+\mathcal{K}^n\mathcal{L}^n\end{array}\\ &= \mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^n\mathcal{L}^n.\end{aligned}$$

Theorem 34. *Given $n \geq 1$*

$$\| \; [\mathcal{D},\pi(\mathcal{K}^n\mathcal{L}^n)] \; \| = 1. \tag{83}$$

Proof. We know that $\mathcal{K}$ preserves inner products, so that for any bounded operator A:

$$\begin{aligned}\| \mathcal{K}A \| &= \sup_{|g|=1} |\mathcal{K}Ag|\\ &= \sup_{|g|=1} \sqrt{\langle \mathcal{K}Ag,\mathcal{K}Ag\rangle}\\ &= \sup_{|g|=1} \sqrt{\langle Ag,Ag\rangle}\\ &= \sup_{|g|=1} |Ag|\\ &= \| A \|.\end{aligned}$$

Substituting A for $\mathcal{K}^n\mathcal{L}^n-\mathcal{K}^{n-1}\mathcal{L}^{n-1}$ we get:

$$\begin{aligned}\| \; [\mathcal{D},\pi(\mathcal{K}^n\mathcal{L}^n)] \; \| &= \| \; \left(\mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^n\mathcal{L}^n\right)\mathcal{L} \, \|\\ &= \| \, \mathcal{K}\left(\mathcal{K}^n\mathcal{L}^n-\mathcal{K}^{n-1}\mathcal{L}^{n-1}\right) \, \|\\ &= \| \, \mathcal{K}^n\mathcal{L}^n-\mathcal{K}^{n-1}\mathcal{L}^{n-1} \, \|\\ &= \| \, \mathcal{K}^{n-1}\mathcal{L}^{n-1}-\mathcal{K}^n\mathcal{L}^n \, \|\\ &= 1.\end{aligned}$$

□

3 A generalized boson formalism for the maximal entropy probability

In the space $L^2(\mu)$ consider the action of the Koopman operator $\mathcal{K}=\mathcal{L}^*$.

In this section we elaborate on the meaning of a dynamical version of generalized boson systems which we introduced before. We present the computations that are required for the justification of several claims presented in the Introduction section 1. We will describe in detail different estimates that will corroborate our claims for this setting, and the differences and similarities with respect to the non-dynamical point of view. The main results are summarized in Propositions 36 and 37.

Denote $\hat{f} = \frac{1}{\sqrt{2}}\mathcal{L}$ and $\hat{f}^\dagger = \frac{1}{\sqrt{2}}\mathcal{K}$. The main CAR relation should be $\{\hat{f}, \hat{f}^\dagger\} = I$ (it will not be true here).

Assume that J is continuous, positive, and satisfies for any x

$$\sum_{\sigma(y)=x} J(y) = 1.$$

We call J of Jacobian.

The Ruelle operator $\mathcal{L}_{\log J}$ acts on continuous functions ϕ and is defined by

$$\mathcal{L}_{\log J}(\phi)(x) = \sum_{\sigma(y)=x} J(y)\phi(y).$$

$\phi \to (\mathcal{L}_{\log J} \circ \mathcal{K})(\phi)$ is the identity.

We want to obtain results similar to the ones in [6].

Note that $\mathcal{K}\mathcal{L}(f)$ is the conditional expectation of f given the sigma-algebra $\sigma^{-1}(\mathcal{B})$, where $\mathcal{B}$ is the Borel sigma-algebra in $\{0,1\}^{\mathbb{N}}$. Therefore, if f does not depend of the first coordinate we get

$$\mathcal{K}\mathcal{L}(f) = f. \tag{84}$$

However, as we mentioned before, the CAR relations are not true: indeed, for any x and any ϕ

$$\{\mathcal{L}_{\log J}, \mathcal{K}\}(\phi)(x) = (\mathcal{L}_{\log J}\mathcal{K} + \mathcal{K}\mathcal{L}_{\log J})(\phi)(x) = \phi(x) + \sum_{\sigma(y)=\sigma(x)} J(y)\phi(y).$$

This means

$$\{\hat{f}, \hat{f}^\dagger\}(\phi)(x) = \frac{1}{2}(\phi(x) + \sum_{\sigma(y)=\sigma(x)} J(y)\phi(y)) \neq \phi(x).$$

Note that in the case ϕ does not depends on the first coordinate we get that

$$\{\hat{f}, \hat{f}^\dagger\}(\phi) = \phi. \tag{85}$$

Is it natural to consider bounded operators acting on the $L^2(\mu)$ space $\mathcal{F}$ of the functions ϕ that do not depend on the first coordinate. Then, CAR

$$\{\hat{f}, \hat{f}^\dagger\} = I \tag{86}$$

is true on $\mathcal{F}$.

Nonetheless, for the general case of ϕ

$$\int \{\mathcal{L}, \mathcal{K}\}(\phi)\, d\mu = 2 \int \phi\, d\mu.$$

Then, we get

$$\int \{\hat{f}, \hat{f}^\dagger\}(\phi)\, d\mu = \int \phi\, d\mu. \tag{87}$$

In this way $\{\hat{f}, \hat{f}^\dagger\} \neq I$, but anyway, when we consider the action of integrating functions in the $L^2(\mu)$ space, we get something similar to CAR.

Note that

$$[\mathcal{L}_{\log J}, \mathcal{K}]\,(\phi)(x) = (\mathcal{L}_{\log J}\mathcal{K} - \mathcal{K}\mathcal{L}_{\log J})(\phi)(x) = \phi(x) - \sum_{\sigma(y)=\sigma(x)} J(y)\phi(y). \tag{88}$$

For the boson formalism the Number operator $\mathcal{K}\mathcal{L}$ acts on the $L^2(\mu)$ space. In this direction, the CCR $\left[\hat{f}, \hat{f}^\dagger\right] = 1$ is not true, however, a generalized boson form of CCR, as in [19], will be considered in expression (127).

Here we take $f(n) = 2^{-n/2}$ for the dynamical generalized boson system we consider.

From now on $\Omega = \{0,1\}^{\mathbb{N}}$, $J = \frac{1}{2}$ and μ is the measure of maximal entropy for the shift σ.

Denote by $w = w_1 w_2 w_3 \cdots w_l$ a finite word on the symbols $\{0,1\}$.

Given a finite word $w = w_1 w_2 w_3 \cdots w_l$, we denote by $l(w) = l$ the size of the word w. We denote by $[w] = [w_1 w_2 w_3 \cdots w_l]$ the associated cylinder set.

We say that $\tilde{w} = a_1 a_2 \cdots a_r$ is a prefix of $w = b_1 b_2 \cdots b_s$ if $r < s$ and $w = a_1 a_2 \cdots a_r b_{r+1} \cdots b_s$. We use the notation $\tilde{w} < w$. Note that if w is not a prefix of $\tilde{w}$ and also $\tilde{w}$ is not a prefix of w, then the product of the functions $e_{\tilde{w}}\, e_w = 0$.

If $x < y$ we get that

$$e_x\, e_y = \sqrt{2^{l(x)}} e_y = -(-1)^{y_{l(x)+1}} \sqrt{2^{l(x)}} e_y. \tag{89}$$

Moreover, if $x = y$, then

$$e_x\, e_x = 2^{l(x)}\ \left(\mathbf{1}_{[x0]} + \mathbf{1}_{[x1]}\right) = 2^{l(x)} \mathbf{1}_{[x]}. \tag{90}$$

From [6] (see also [16]), given w, denote

$$e_w \frac{1}{\sqrt{\mu([w])}} \left(\sqrt{\tfrac{\mu([w0])}{\mu([w1])}} \mathbf{1}_{[w1]} - \sqrt{\tfrac{\mu([w1])}{\mu([w0])}} \mathbf{1}_{[w0]} \right) \tag{91}$$

W^* denotes the set of finite words w with size $l(w) \geq 1$. By abuse of notation we will say that $\sigma(w_1 w_2 w_3 \cdots w_l) = w_2 w_3 \cdots w_l$ when $l \geq 2$.

In Appendix B in [6], adapting Theorem 3.5 in [16] to our case, it was shown that

$$\mathbb{B}\,\{e_w;\ w \in W^*\} \cup \left\{ -\mu([0])^{-\frac{1}{2}} \mathbf{1}_{[0]}\ ,\ \mu([1])^{-\frac{1}{2}} \mathbf{1}_{[1]} \right\}$$

is a Haar basis of $L^2(\mu)$.

In the case $\log J = \log 1/2$ we get a more simple expression: for each finite word w

$$e_w\ =\ \frac{1}{\sqrt{\mu([w])}} \left(\mathbf{1}_{[w1]} - \mathbf{1}_{[w0]}\right). \tag{92}$$

Note that $\mu([w]) = 2^{-l}$, where $l = l(w) \geq 1$ is the size of w.

In order to get a (Haar) basis we should add to the different e_w the functions

$$e_\emptyset^0 = -\frac{1}{\sqrt{\mu([0])}} \mathbb{1}_{[0]} = -\sqrt{2}\, \mathbb{1}_{[0]}, \quad \text{and } e_\emptyset^1 = \frac{1}{\sqrt{\mu([1])}} \mathbb{1}_{[1]} = \sqrt{2}\, \mathbb{1}_{[1]}. \tag{93}$$

The relations (89) and (90) will play an important role in this section. From [22] (see also [23]) we get:

Proposition 35. *Given $x = x_1 x_2 \cdots x_n$ with a size larger than 1. When $\log J = \log 1/2$, for $n > 1$*

$$\sqrt{1/2}\, \mathcal{L}\,(e_{x_1 x_2 \cdots x_n}) = \frac{1}{2}\, e_{x_2 x_3 \cdots x_n}. \tag{94}$$

that is,

$$\mathcal{L}\,(e_{x_1 x_2 \cdots x_n}) = \frac{1}{\sqrt{2}}\, e_{x_2 x_3 \cdots x_n}. \tag{95}$$

Moreover,

$$\mathcal{L}\,(e_{[1]}) = \mathcal{L}\,(e_{[0]}) = \frac{\sqrt{2}}{2}\,(\mathbb{1}_{[1]} - \mathbb{1}_{[0]}) = \frac{1}{2}(e_\emptyset^1 + e_\emptyset^0), \tag{96}$$

$$\mathcal{L}\,(\frac{1}{\sqrt{2}}(e_\emptyset^1 + e_\emptyset^0)) = 0. \tag{97}$$

(95) *and* (2) *justifies to call* $\mathfrak{a} = \mathcal{L}$ *a generalized annihilation operator.*

Other relations will be required for our reasoning. Note that for $k < n$

$$\mathcal{L}^k\,(e_{x_1 x_2 \cdots x_n}) = (\frac{1}{\sqrt{2}})^k\, e_{x_{k+1} x_{k+2} \cdots x_n}. \tag{98}$$

Moreover,

$$\hat{f}(e_\emptyset^0) = \frac{1}{\sqrt{2}} \mathcal{L}(e_\emptyset^0) = -\frac{1}{2} \tag{99}$$

and

$$\hat{f}(e_\emptyset^1) = \frac{1}{2} = \frac{1}{\sqrt{2}} \mathcal{L}(e_\emptyset^1). \tag{100}$$

Note that in this case for $l(w) \geq 1$

$$\hat{f}^\dagger(e_w) = \frac{1}{\sqrt{2}} \mathcal{K}(e_w) = \frac{1}{2}(e_{0w} + e_{1w}). \tag{101}$$

that is,

$$\mathcal{K}(e_w) = \frac{1}{\sqrt{2}}(e_{0w} + e_{1w}), \tag{102}$$

and

$$\sqrt{2}\, \mathcal{K}(e_w) = (e_{0w} + e_{1w}). \tag{103}$$

Therefore, from (2) it is natural to call $\mathfrak{c} = \mathcal{K}$ a generalized creation operator.

Moreover,

$$\mathcal{K}(e^0_\emptyset) = -\sqrt{2}\,(\mathbb{1}_{00} + \mathbb{1}_{10}), \tag{104}$$

and

$$\mathcal{K}(e^1_\emptyset) = \sqrt{2}\,(\mathbb{1}_{01} + \mathbb{1}_{11}). \tag{105}$$

As

$$\mathcal{L}(e^1_\emptyset + e^0_\emptyset) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0, \tag{106}$$

we get

$$\mathcal{K}\mathcal{L}(e^1_\emptyset + e^0_\emptyset) = 0. \tag{107}$$

Therefore, it is natural to call $e^1_\emptyset + e^0_\emptyset$ the vacuum. Note that from (104) and (105)

$$\mathcal{K}(e^1_\emptyset + e^0_\emptyset) = \mathfrak{c}(e^1_\emptyset + e^0_\emptyset) = e_0 + e_1, \tag{108}$$

which means

$$\hat{f}^\dagger(e^1_\emptyset + e^0_\emptyset) = \frac{1}{\sqrt{2}}(e_0 + e_1) = f(1)(e_0 + e_1). \tag{109}$$

One can show that

$$(\hat{f}^\dagger)^n(e^1_\emptyset + e^0_\emptyset) = f(n) \sum_{a_1, a_2, \cdots, a_n = 0}^{1} e_{a_1 a_2 \cdots a_n}. \tag{110}$$

The above expression corresponds to (A3) in [19], when $f(n) = 2^{-n/2}$.

Note for the word $w = w_1 w_2 \cdots w_l$, $l > 1$

$$\hat{f}^\dagger\, \hat{f}(e_w) = \frac{1}{\sqrt{2}}\mathcal{K}\,\frac{1}{\sqrt{2}}\mathcal{L}(e_w) = \frac{1}{4}(e_{0 w_2 \cdots w_l} + e_{1 w_2 \cdots w_l}), \tag{111}$$

that is

$$\mathfrak{c}\mathfrak{a}(e_w) = \mathcal{K}\ \mathcal{L}(e_w) = \frac{1}{2}(e_{0 w_2 \cdots w_l} + e_{1 w_2 \cdots w_l}). \tag{112}$$

Therefore, from (2) and (112) it is natural to call $\mathfrak{c}\mathfrak{a}$ a generalized number operator.

Note that given $w = w_1 w_2 \cdots w_l$

$$[\mathfrak{a}, \mathfrak{c}]\,(e_w) = [\mathcal{L}\,,\,\mathcal{K}]\,(e_w) =$$

$$(\mathcal{L}\ \mathcal{K} - \mathcal{K}\ \mathcal{L})(e_w) = e_w - \frac{1}{2}(e_{0 w_2 \cdots w_l} + e_{1 w_2 \cdots w_l}). \tag{113}$$

The above expression will help to get (127).

Moreover, from (96)

$$\hat{f}^\dagger\, \hat{f}(e_1) = \frac{1}{2}\mathcal{K}\,\mathcal{L}(e_1) =$$

$$\frac{1}{2\sqrt{2}}(\mathbb{1}_{w_2=1} - \mathbb{1}_{w_2=0}) = \frac{1}{4}(e_1 + e_0) = \hat{f}^\dagger\, \hat{f}(e_0). \tag{114}$$

Finally,
$$\hat{f}^\dagger\,\hat{f}(e_\emptyset^0) = \frac{1}{\sqrt{2}}\mathcal{K}(\frac{1}{\sqrt{2}}\mathcal{L}(e_\emptyset^0)) = \frac{1}{\sqrt{2}}\mathcal{K}(-\frac{1}{2}) = -\frac{1}{2\sqrt{2}}, \tag{115}$$
and
$$\hat{f}^\dagger\,\hat{f}(e_\emptyset^1) = \frac{1}{\sqrt{2}}\mathcal{K}(\frac{1}{\sqrt{2}}\mathcal{L}(e_\emptyset^1)) = \frac{1}{\sqrt{2}}\mathcal{K}(\frac{1}{2}) = \frac{1}{2\sqrt{2}}. \tag{116}$$

Moreover, for w such that $l(w) > 1$
$$\{\hat{f}, \hat{f}^\dagger\}(e_w) = \{\frac{1}{\sqrt{2}}\mathcal{L}, \frac{1}{\sqrt{2}}\mathcal{K}\}(e_w) =$$
$$\frac{1}{2}e_w + \frac{1}{\sqrt{2}}\mathcal{K}(\sqrt{1/2}\,e_{w_2\cdots w_l}) = \frac{1}{2}e_w + \frac{1}{4}(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}). \tag{117}$$

It follows that
$$\{\hat{f}, \hat{f}^\dagger\}(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}) =$$
$$[\frac{1}{2}e_{0w_2\cdots w_l} + \frac{1}{4}(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l})] +$$
$$[\frac{1}{2}e_{1w_2\cdots w_l} + \frac{1}{4}(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l})] = e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}. \tag{118}$$

Given any word $w = w_1 w_2 \cdots w_l)$, $l > 1$, consider $\phi_w = \phi_{w_1 w_2\cdots w_l}$, of the form
$$\phi_{w_1 w_2\cdots w_l} = e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}. \tag{119}$$

It follows from (118) that ϕ_w, $l(w) > 1$, is an eigenfunction for $\{\hat{f}, \hat{f}^\dagger\}$ associated to the value 1 (see Remark 3).

Given two words w and $\tilde{w}$, if $\tilde{w}_1 \neq w_2$, but $w_2 \cdots w_l = \tilde{w}_2 \cdots \tilde{w}_l$, then $\phi_w = \phi_{\tilde{w}}$. In any other case, $\langle \phi_w, \phi_{\tilde{w}} \rangle = 0$.

Moreover, from (112) we get for $w = w_1 w_2 \cdots w_l$
$$\mathcal{K}^m \mathcal{L}^m\,(e_w) = 2^{-m}\left(\sum_{\ell(y)=m} e_{y w_{m+1} w_{m+2}\cdots w_l}\right) \neq (\mathcal{K}\mathcal{L})^m\,(e_w). \tag{120}$$

Note that
$$(\mathcal{K}\mathcal{L})\phi_{w_1 w_2\cdots w_l} = (\mathcal{K}\mathcal{L})(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}) =$$
$$[\frac{1}{2}(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}) + \frac{1}{2}(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l})] =$$
$$(e_{0w_2\cdots w_l} + e_{1w_2\cdots w_l}) = \phi_{w_1 w_2\cdots w_l}. \tag{121}$$

Then, $\phi_{w_1 w_2\cdots w_l}$ is an eigenfunction for $\mathcal{K}\mathcal{L}$ associated to the eigenvalue 1.

From (121) and (107), we can say that $\mathcal{K}\mathcal{L}$ is a version of the generalized fermion Number operator (see (2) and Remark 3).

The C^*-algebra generated by $\mathcal{K}^m \mathcal{L}^m$, $m \geq 0$, is the topic of [11] and [10]. The C^*-algebra generated by $\mathcal{K}^m \mathcal{L}^n$, $m, n \geq 0$, is the topic of [13].

The next proposition summarizes the results mentioned above.

Proposition 36. *Consider the operators* $\mathfrak{a} = \mathcal{L}$ *and* $\mathfrak{a}^* = \mathfrak{c} = \mathcal{K}$ *(and also* $\hat{f} = \frac{1}{\sqrt{2}}\mathcal{L}$ *and* $\hat{f}^\dagger = \frac{1}{\sqrt{2}}\mathcal{K}$*) .*

From (98) *and* (102) *we get that when* $l(w) = l(w_1 w_2 \cdots w_k) = k > m > n$

$$\mathcal{K}^m\, \mathcal{L}^n(e_w) = \mathcal{K}^m\, \mathcal{L}^n(e_{w_1 w_2 \cdots w_k})$$

$$(\frac{1}{\sqrt{2}})^{m+n} \sum_{l(y)=m} e_{y\,\sigma^n(w)} = (\frac{1}{\sqrt{2}})^{m+n} \sum_{l(y)=m} e_{y w_{n+1} \cdots w_k}. \tag{122}$$

Note that $l(y w_{n+1} \cdots w_k) = m + k - n$ *and* $\mathfrak{c}^m\ \mathfrak{a}^n$ *is not hermitian for* $n \neq m$*.*
Given $w_{n+1} \cdots w_k$

$$\mathfrak{c}^m\ \mathfrak{a}^n \sum_{l(y)=m} (e_{y w_{n+1} \cdots w_k}) =$$

$$2^m\, (\frac{1}{\sqrt{2}})^{m+n} \sum_{l(y)=m} e_{y w_{n+1} \cdots w_k} = 2^{m/2 - n/2} \sum_{l(y)=m} e_{y w_{n+1} \cdots w_k}. \tag{123}$$

The above generalizes (121). The proof follows from the estimates we presented before on this section.

Therefore, from the above, there exists at least a subspace of dimension $k-n$ of eigenfunctions $\sum_{l(z)=m} (e_{z_1 z_2 \cdots z_m w_{n+1} \cdots w_k})$ for $\mathfrak{c}^m\ \mathfrak{a}^n$, associated to the eigenvalue $2^{m/2-n/2}$.

Finally, note that from (110) we get

$$\hat{f}^\dagger\, \hat{f}(\sum_{a_1, a_2, \cdots, a_n = 0}^{1} e_{a_1 a_2 \cdots a_n}) =$$

$$\frac{1}{2} \sum_{a_1, a_2, \cdots, a_n = 0}^{1} e_{a_1 a_2 \cdots a_n} = (\frac{f(n)}{f(n-1)})^2 \sum_{a_1, a_2, \cdots, a_n = 0}^{1} e_{a_1 a_2 \cdots a_n}, \tag{124}$$

which is consistent with (4), when taking into account (110).

From (113), given w

$$[\mathfrak{a}, \mathfrak{c}]\, (e_w) = e_w - \frac{1}{2}(e_{0 w_2 \cdots w_l} + e_{1 w_2 \cdots w_l}) \neq e_w, \tag{125}$$

therefore,

$$\langle e_w, [\mathfrak{a}, \mathfrak{c}]\, (e_w) \rangle = \langle e_w, e_w - \frac{1}{2}(e_{0 w_2 \cdots w_l} + e_{1 w_2 \cdots w_l}) \rangle = 1 - \frac{1}{2} = \frac{1}{2}.$$

Given $e_{w_1, w_2 \cdots w_l}$ denote $\hat{e}_{w_1, w_2 \cdots w_l}$ the element $e_{v_1, w_2 \cdots w_l}$, where $v_1 = 1$ if $w_1 = 0$, and $v_1 = 0$ if $w_1 = 1$.

Then, given w

$$[\mathfrak{a}, \mathfrak{c}]\, (e_w) = \frac{1}{2} e_w - \frac{1}{2} \hat{e}_w \neq e_w. \tag{126}$$

Now we will show a CCR version for our generalized boson setting.

Taking into account expressions (A2), and the equality on the left-hand side of (A5) in [19], and its notation, we introduce a function F: given a word w take $F(w) = 2\,\langle e_w, [\mathfrak{a},\mathfrak{c}]\,(e_w)\rangle = 1$; and a new form of commutator $\lceil\cdot,\cdot\rceil$. In this way we get a similar (but not exactly equal) expression to (A5): the generalized CCR relation.

Proposition 37.

$$\lceil \mathfrak{a},\mathfrak{c}\rceil =$$
$$\sum_w F(w)|e_w\rangle\langle e_w| = \sum_w 2\langle e_w, [\mathfrak{a},\mathfrak{c}]\,(e_w)\rangle|e_w\langle\rangle e_w| = I. \tag{127}$$

Note that $[\mathfrak{a},\mathfrak{c}]\,(e_w) \neq e_w$.

The proof follows from (125) and the expressions we obtained above.

Example 38. *It is possible to consider an analogy between our study and what is observed in the harmonic oscillator. Denote by* $|n\rangle$ *the* n*th eigenfunction of the quantized operator* $\mathbf{H}$ *associated to the classical Harmonic oscillator Hamiltonian* $H(x,p) = \frac{\hat{p}^2}{2m} + \frac{m\,w^2\,\hat{x}^2}{2}$. *Denote by* $\hat{x}$ *and* $\hat{p}$*, respectively, the position and momentum operator acting on* $L^2(dx)$*, for the Lebesgue measure* dx *on* $\mathbb{R}$.

The creation operator $\mathcal{C}$ *and the annihilation operator* $\mathcal{A}$ *(see Section 11 in [14]) are given by*

$$\mathcal{C} = \sqrt{\frac{m\,w}{2\,\hbar}}\;\hat{x} + i\sqrt{\frac{1}{2mw\,\hbar}}\;\hat{p} = \sqrt{\frac{m\,w}{2\,\hbar}}\;\hat{x} + \sqrt{\frac{1}{2mw\,\hbar}}\,\frac{d}{dx} \tag{128}$$

and

$$\mathcal{A} = \sqrt{\frac{m\,w}{2\,\hbar}}\;\hat{x} - i\sqrt{\frac{1}{2mw\,\hbar}}\;\hat{p} = \sqrt{\frac{m\,w}{2\,\hbar}}\;\hat{x} - \sqrt{\frac{1}{2mw\,\hbar}}\,\frac{d}{dx}. \tag{129}$$

From the property $[\hat{p},\hat{x}] = -i\,\hbar\,I$*, we can get the Canonical Commutation Relation*

$$[\mathcal{C},\mathcal{A}] = I.$$

In this case for $\mathcal{C}$ *and* $\mathcal{A} = \mathcal{C}^*$*, respectively, the creation and annihilation operators, we get*

$$\mathcal{C}(|n\rangle) = \sqrt{n+1}\;\;|(n+1)\rangle$$

and

$$\mathcal{A}(|n\rangle) = \sqrt{n}\;\;|(n-1)\rangle.$$

From (2)*, this corresponds to the case where* $f(n) = \sqrt{n!}$.

In this case, $|0\rangle$ *corresponds to the eigenfunction associated with the smallest eigenvalue of the Hamiltonian operator* $\mathbf{H}$*, and*

$$\mathcal{A}(|0\rangle) = 0. \tag{130}$$

For this reason $|0\rangle$ *is called the vacuum (eigenfunction) for the quantized harmonic oscillator (see* (3)*).*

Moreover, in this case for $k > m > n$

$$\mathcal{C}^m \mathcal{A}^n(|k\rangle) = (k-n)\cdots(k-1)\sqrt{k}\cdots\sqrt{k+m-n-1}\,|(k+m-n)\rangle; \tag{131}$$

in particular

$$\mathcal{C}^m \mathcal{A}^m(|k\rangle) = (k-m)\cdots(k-1)\,|k\rangle.$$

(131) *should be compared with the generalized case* (123) *(the functions $n \to f(n)$ are, of course, different in each case).*

4 Compact Hermitian Operators

In this section we are interested in estimates of $\|\, [\mathcal{D}, \pi(H)] \,\|$ for a given compact hermitian operator $H : L^2(\mu) \longrightarrow L^2(\mu)$. Every such operator can be put in a diagonal form with vanishing eigenvalues which we are going to write as:

$$H = \sum_i \lambda_i \hat{\psi}_i \; = \sum_i \lambda_i \,|\psi_i\rangle\langle\psi_i|\,. \tag{132}$$

The vectors $\{\psi_i\}$ form an orthonormal basis for $L^2(\mu)$. The eigenvalues $\{\lambda_i\}$ are such that $\lim_{i\to+\infty} \lambda_i = 0$, and if only finitely many of them are non-zero, then we can also rearrange them, so their sequence is non-increasing. The basis $\{\psi_i\}$ is of course not necessarily equal to the basis $\mathbb{B}$ we have been using. Still, for any given $f \in L^2(\mu)$ we may write:

$$\begin{aligned} f &= \sum_i \langle f, \psi_i\rangle \psi_i \\ &= \sum_i f_i \psi_i, \end{aligned}$$

and then follows that:

$$\|\, f \,\| = 1 \iff \sum_i |f_i|^2 = 1.$$

Considering the Dirac operator $\mathcal{D}$ we introduced earlier (19) and the diagonal representation $\pi : L^2(\mu) \longrightarrow L^2(\mu) \times L^2(\mu)$, we observe that:

$$[\mathcal{D}, \pi(H)] = \begin{pmatrix} 0 & \mathcal{K}H - H\mathcal{K} \\ \mathcal{L}H - H\mathcal{L} & 0 \end{pmatrix},$$

and since H is hermitian, we have:

$$\|\, [\mathcal{D}, \pi(H)] \,\| = \|\, \mathcal{K}H - H\mathcal{K} \,\| = \|\, \mathcal{L}H - H\mathcal{L} \,\|.$$

This already allows us to prove that:

Proposition 39. *For any hermitian operator H:*

$$\|\, H \,\| \leq \frac{1}{2} \implies \|\, [\mathcal{D}, \pi(H)] \,\| \leq 1.$$

Proof. This happens because:

$$\begin{aligned}\| \, [\mathcal{D}, \pi(H)] \, \| &= \| \, \mathcal{K}H - H\mathcal{K} \, \| \\ &\leq 2\| \, K \, \| \| \, H \, \| \\ &= 1.\end{aligned}$$

□

Remark 40. *We stress the fact that any "non-$\mathcal{D}$-constant" hermitian operator H (see Definition 41) may be normalized by its "Lipschitz constant" ($\| \, [\mathcal{D}, \pi(H)] \, \|$) so as to produce a new operator ($\tilde{H} = \frac{1}{\| \, [\mathcal{D},\pi(H)] \, \|} H$) that satisfies $\| \left[\mathcal{D}, \pi(\tilde{H})\right] \| = 1$. This corresponds with the fact that any non-constant function with a finite Lipschitz constant may be normalized by it, effectively setting the constant to 1.*

Definition 41. *We say an hermitian operator H is $\mathcal{D}$-constant if $\| \, [\mathcal{D}, \pi(H)] \, \| = 0$.*

Proposition 42. *The following are equivalent:*

(i) H is $\mathcal{D}$-constant.

(ii) H commutes with $\mathcal{K}$.

(iii) H commutes with $\mathcal{L}$.

Proof. Follows directly from identity (79). □

If H commutes with both $\mathcal{K}$ and $\mathcal{L}$, then H commutes with their products, like $\mathcal{KL}$, and therefore H leaves both $\ker \mathcal{KL}$ and $\operatorname{im} \mathcal{KL}$ invariant. Also, because $\mathcal{KL}$ is a projection, we know H takes the form $H_0 \oplus H_1$ with $H_0 \in \mathcal{B}(\ker \mathcal{KL})$ and $H_1 \in \mathcal{B}(\operatorname{im} \mathcal{KL})$. In other words, writing $L^2(\mu) = \ker \mathcal{KL} \oplus \operatorname{im} \mathcal{KL}$, we get:

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}.$$

Since commuting with $\mathcal{K}$ and $\mathcal{L}$ is a stronger requirement than commuting with $\mathcal{KL}$, we note that not every H that takes the above form satisfies $\| \, [\mathcal{D}, \pi(H)] \, \| = 0$.

We are now going to estimate expression $\| \, [\mathcal{D}, \pi(H)] \, \|$ for a compact self-adjoint operator H.

Proposition 43. *Consider the compact hermitian operator $H : L^2(\mu) \longrightarrow L^2(\mu)$ in the form*

$$H = \sum_i \lambda_i \hat{\psi}_i = \sum_i \lambda_i \, |\psi_i\rangle\langle\psi_i| \, . \tag{133}$$

where the vectors $\{\psi_i\}$ form an orthonormal basis for $L^2(\mu)$, and λ_i are the real eigenvalues.

Then,

$$\| \, [\mathcal{D}, \pi(H)] \, \|^2 \leq 2 \left(\| \, H \, \|^2 + \sum_{i,j} |\lambda_i \lambda_j| \right). \tag{134}$$

Proof. Note that

$$\begin{aligned}
\| \, [\mathcal{D}, \pi(H)] \, \|^2 &= \| \, \mathcal{K}H - H\mathcal{K} \, \|^2 \\
&= \sup_f \langle (\mathcal{K}H - H\mathcal{K})\, f, (\mathcal{K}H - H\mathcal{K})\, f \rangle \\
&= \sup_f \langle \mathcal{K}Hf, \mathcal{K}Hf \rangle + \langle H\mathcal{K}f, H\mathcal{K}f \rangle - 2\langle \mathcal{K}Hf, H\mathcal{K}f \rangle \\
&= \sup_f \langle Hf, Hf \rangle + \langle H\mathcal{K}f, H\mathcal{K}f \rangle - 2\langle \mathcal{K}Hf, H\mathcal{K}f \rangle .
\end{aligned}$$

After a huge calculation we finally get:

$$\| \, [\mathcal{D}, \pi(H)] \, \|^2 \leq 2 \left(\| \, H \, \|^2 - \inf_{|f|=1} \sum_{i,j} \lambda_i \lambda_j \langle \mathcal{K}f, \mathcal{K}\psi_i \rangle \langle \mathcal{K}f, \psi_j \rangle \langle \mathcal{K}\psi_i, \psi_j \rangle \right) .$$

Considering that f, $\mathcal{K}f$, ψ_i and $\mathcal{K}\psi_i$ all are unit vectors, we can apply the Cauchy-Schwarz inequality, to get:

$$\| \, [\mathcal{D}, \pi(H)] \, \|^2 \leq 2 \left(\| \, H \, \|^2 + \sum_{i,j} |\lambda_i \lambda_j| \right) .$$

□

5 Appendix

In the Appendix we will present the proofs of several claims we mentioned before.

Proposition 44. *Suppose* $e_w = e_{w_1 w_2 \cdots w_n}$, $l(w) > 1$, *then*

$$\begin{aligned}
(\mathcal{K}\hat{e}_w - \hat{e}_w \mathcal{K})(\phi) &= \sqrt{\frac{1}{2}} [(\int e_w \, \phi d\mu)(e_{0w} + e_{1w}) - \int e_{w_2 w_3 \cdots w_n} \, \phi \, d\mu \;\; e_w \,] \\
&= \frac{1}{\sqrt{2}} \left[\langle e_w, \phi \rangle \, (e_{0w} + e_{1w}) - \langle e_{\sigma(w)}, \phi \rangle e_w \right] ,
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{L}\hat{e}_w - \hat{e}_w \mathcal{L})(\phi) &= \sqrt{\frac{1}{2}} [\int e_w \, \phi \, d\mu \; e_{w_2 \cdots w_n} - \int (e_{0w} + e_{1w}) \, \phi \; d\mu \;\; e_w \,] \\
&= \frac{1}{\sqrt{2}} \left[\langle e_w, \phi \rangle e_{\sigma(w)} - \langle e_{0w} + e_{1w}, \phi \rangle e_w \right] .
\end{aligned}$$

Moreover,

$$(\mathcal{K}\hat{e}_{e^0_\emptyset} - \hat{e}_{e^0_\emptyset} \mathcal{K})(\phi) = -\sqrt{2} \, \langle e^0_\emptyset, \phi \rangle (\mathbf{1}_{00} + \mathbf{1}_{10}) + \langle e^0_\emptyset, (\phi \circ \sigma) \rangle e^0_\emptyset ,$$

$$(\mathcal{K}\hat{e}_{e^1_\emptyset} - \hat{e}_{e^1_\emptyset}\mathcal{K})(\phi) = \sqrt{2}\,\langle e^1_\emptyset, \phi\rangle(\mathbf{1}_{01} + \mathbf{1}_{11}) - \langle e^1_\emptyset, (\phi\circ\sigma)\rangle e^1_\emptyset,$$

$$(\mathcal{L}\hat{e}_{e^0_\emptyset} - \hat{e}_{e^0_\emptyset}\mathcal{L})(\phi) = -\sqrt{\frac{1}{2}}\,\langle e^0_\emptyset, \phi\rangle + \sqrt{2}\langle(\mathbf{1}_{00} + \mathbf{1}_{10}), \phi\rangle\, e^0_\emptyset,$$

and

$$(\mathcal{L}\hat{e}_{e^1_\emptyset} - \hat{e}_{e^1_\emptyset}\mathcal{L})(\phi) = \sqrt{\frac{1}{2}}\,\langle e^1_\emptyset, \phi\rangle - \sqrt{2}\langle(\mathbf{1}_{01} + \mathbf{1}_{11}), \phi\rangle\, e^1_\emptyset.$$

Proof. Note that $\forall x$

$$\begin{aligned}(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(\phi)(x) &= \mathcal{K}\left(\langle e_w, \phi\rangle e_w\right) - \left(\langle e_w, \mathcal{K}\phi\rangle e_w\right)\\ &= \left(\langle e_w, \phi\rangle \mathcal{K}e_w\right) - \left(\langle \mathcal{L}e_w, \phi\rangle e_w\right)\\ &= \langle e_w, \phi\rangle\left(\frac{1}{\sqrt{2}}\left(e_{0w} + e_{1w}\right)\right) - \left(\langle \mathcal{L}e_w, \phi\rangle e_w\right)\\ &= \langle e_w, \phi\rangle\left(\frac{1}{\sqrt{2}}\left(e_{0w} + e_{1w}\right)\right) - \left(\langle \frac{1}{\sqrt{2}} e_{w_2w_3\cdots w_n}, \phi\rangle e_w\right)\\ &= \sqrt{\frac{1}{2}}\left[\,\left(\int e_w\,\phi d\mu\right)(e_{0w} + e_{1w}) - \int e_{w_2w_3\cdots w_n}\,\phi\, d\mu\ \ e_w\ \right].\end{aligned}$$

Moreover, $\forall x$

$$\begin{aligned}(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(\phi)(x) &= \mathcal{L}\left(\langle e_w, \phi\rangle e_w\right) - \left(\langle e_w, \mathcal{L}\phi\rangle e_w\right)\\ &= \left(\langle e_w, \phi\rangle \mathcal{L}e_w\right) - \left(\langle \mathcal{K}e_w, \phi\rangle e_w\right)\\ &= \langle e_w, \phi\rangle\left(\frac{1}{\sqrt{2}} e_{w_2w_3\cdots w_n}\right) - \left(\langle \frac{1}{\sqrt{2}}\left(e_{0w} + e_{1w}\right), \phi\rangle e_w\right)\\ &= \sqrt{\frac{1}{2}}\left[\int e_w\,\phi\, d\mu\ e_{w_2\cdots w_n} - \int (e_{0w} + e_{1w})\,\phi\ d\mu\ \ e_w\,\right].\end{aligned}$$

□

Proposition 45. *For a word w satisfying $l(w) > 1$, given a generic word $\tilde{w}$, and the corresponding element $e_{\tilde{w}}$, we get that*

$$|(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}})|^2 = \int [\,(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}})\,]^2\, d\mu \tag{135}$$

is equal to 0, if $w \neq \tilde{w} \neq w_2w_3\cdots w_n$, is equal to $1/2$ if $w \neq \tilde{w} = w_2w_3\cdots w_n$, and is equal to 1 if $\tilde{w} = w$. Moreover,

$$|(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(e_{\tilde{w}})|^2 = \int [\,(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(e_{\tilde{w}})\,]^2\, d\mu \tag{136}$$

is equal to 0, if $\tilde{w} \neq w \neq \sigma\tilde{w}$, and is equal to $1/2$ if either $\tilde{w} \neq w = \sigma\tilde{w}$ or $\tilde{w} = w$.

Proof. For a fixed word w such that $l(w) > 1$, given a generic word $\tilde{w} \neq w = w_1 w_2 w_3 \cdots w_n$, and the corresponding element $e_{\tilde{w}}$, we get

$$\begin{aligned}(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}}) &= \frac{1}{\sqrt{2}}\left[\langle e_w, e_{\tilde{w}}\rangle\,(e_{0w} + e_{1w}) - \langle e_{\sigma(w)}, e_{\tilde{w}}\rangle e_w\right] \\ &= 0 - \frac{1}{\sqrt{2}}\langle e_{\sigma(w)}, e_{\tilde{w}}\rangle e_w \\ &= 0 - \sqrt{\frac{1}{2}}\int e_{w_2 w_3 \cdots w_n}\, e_{\tilde{w}}\, d\mu\;\; e_w.\end{aligned}$$

When, $w = \tilde{w}$ we get

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_w) = \sqrt{\frac{1}{2}}[\,(\,e_{0w1} + e_{1w1}) - (e_{0w0} + e_{1w0})\,]. \tag{137}$$

When, $w \neq \tilde{w} \neq w_2 w_3 \cdots w_n$ we get

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}}) = 0. \tag{138}$$

When, $w \neq \tilde{w} = w_2 w_3 \cdots w_n$ we get

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_{\tilde{w}}) = -\sqrt{\frac{1}{2}}\,2^{n-1}\int 1_{\tilde{w}} d\mu\, e_w = -\sqrt{\frac{1}{2}}\,e_w. \tag{139}$$

If $l(w) > 2$ we get that

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_1) = 0 = (\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_0). \tag{140}$$

It is also true that for $w = 01$ and $w = 11$

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_1) = -\sqrt{\frac{1}{2}}\,e_w, \tag{141}$$

and, for $w = 10$ and $w = 00$

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_1) = 0. \tag{142}$$

Moreover, for $w = 10$ and $w = 00$

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_0) = -\sqrt{\frac{1}{2}}\,e_w, \tag{143}$$

and, for $w = 01$ and $w = 11$

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_0) = 0. \tag{144}$$

Note that for a finite word $w = w_1 w_2 w_3 \cdots w_n$, where $l(w) > 1$

$$(\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_\emptyset^0) = 0 = (\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(e_\emptyset^1). \tag{145}$$

With regards to $\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L}$, $l(w) > 1$ note that:

1. if $\tilde{w} = w$, then:

$$(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})\, e_{\tilde{w}} = \frac{1}{\sqrt{2}} e_{\sigma(w)},$$

2. if $\tilde{w} = 0w$ or $\tilde{w} = 1w$ (in other words, if $\sigma(\tilde{w}) = w$), then:

$$(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})\, e_{\tilde{w}} = -\frac{1}{\sqrt{2}} e_w,$$

3. else:

$$(\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})\, e_{\tilde{w}} = 0.$$

□

Theorem 46. *Given a fixed word $w = w_1 w_2 w_3 \cdots w_n$, $l(w) \geq 2$, we get for the operator norm*

$$\| \, (\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K}) \, \| = 1, \tag{146}$$

and also:

$$\| \, (\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L}) \, \| = 1; \tag{147}$$

therefore:

$$\| \, [\mathcal{D}, \pi(\hat{e}_w)] \, \| = 1. \tag{148}$$

Proof. For getting (148) we will use (31). Consider $\phi = \sum_u a_u e_u + \alpha_0 e^0_\emptyset + \alpha_1 e^1_\emptyset$ in such way that $\sum_u |a_u|^2 + |\alpha_0|^2 + |\alpha_1|^2 = 1$. Take $\tilde{w} = w_2 w_3 \cdots w_n$.

$$\begin{aligned}
\int ((\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(\phi))^2 \, d\mu &= \int ((\mathcal{K}\hat{e}_w - \hat{e}_w\mathcal{K})(a_w e_w + a_{\tilde{w}} e_{\tilde{w}}))^2 \, d\mu \\
&= \int \left(a_w \frac{1}{\sqrt{2}} (e_{0w} + e_{1w}) - a_{\tilde{w}} \frac{1}{\sqrt{2}} e_w \right)^2 \, d\mu \\
&= \frac{1}{2} \int a_w^2 e_{0w}^2 + a_w^2 e_{1w}^2 + a_{\tilde{w}}^2 e_w^2 \, d\mu \\
&= a_w^2 + \frac{1}{2} a_{\tilde{w}}^2 \\
&\leq \left(a_w^2 + a_{\tilde{w}}^2 \right) \\
&\leq 1
\end{aligned}$$

If we take $\phi = e_w$ we get the maximal value which is 1.
Besides,

$$\begin{aligned}
\int ((\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(\phi))^2 \, d\mu &= \int ((\mathcal{L}\hat{e}_w - \hat{e}_w\mathcal{L})(a_w e_w + a_{0w} e_{0w} + a_{1w} e_{1w}))^2 \, d\mu \\
&= \int \left(a_w \frac{1}{\sqrt{2}} e_{\sigma(w)} - \frac{1}{\sqrt{2}} (a_{0w} + a_{1w})\, e_w \right)^2 \, d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \int a_w^2 e_{\sigma(w)}^2 + (a_{0w} + a_{1w})^2 e_w^2 \, d\mu \\
&= \frac{1}{2} \left(a_w^2 + (a_{0w} + a_{1w})^2 \right) \\
&\leq \frac{1}{2} \left(a_w^2 + 2 \left(a_{0w}^2 + a_{1w}^2 \right) \right) \\
&\leq \left(a_w^2 + a_{0w}^2 + a_{1w}^2 \right) \\
&\leq 1
\end{aligned}
$$

If we take $\phi = \frac{1}{\sqrt{2}} (e_{0w} + e_{1w})$ we get the maximal value which is 1.

Now consider $\phi = \sum_u a_u e_u$ and $\psi = \sum_u b_u e_u$ such that:

$$
\begin{aligned}
\| (\phi, \psi) \|^2 &= \| \phi \|^2 + \| \psi \|^2 \\
&= \sum_u |a_u|^2 + \sum_u |b_u|^2 \\
&= \sum_u |a_u|^2 + |b_u|^2 \\
&\leq 1.
\end{aligned}
$$

and compute the squared norm of $[D, \pi(\hat{e}_w)]$ applied to it:

$$
\begin{aligned}
\| \, [D, \pi(\hat{e}_w)] \, (\phi, \psi) \, \|^2 &= \int \left((\mathcal{K}\hat{e}_w - \hat{e}_w \mathcal{K})(\psi) \right)^2 \, d\mu \\
&\quad + \int \left((\mathcal{L}\hat{e}_w - \hat{e}_w \mathcal{L})(\phi) \right)^2 \, d\mu \\
&= b_w^2 + \frac{1}{2} b_{\tilde{w}}^2 + \frac{1}{2} \left(a_w^2 + (a_{0w} + a_{1w})^2 \right) \\
&\leq b_w^2 + b_{\tilde{w}}^2 + a_w^2 + a_{0w}^2 + a_{1w}^2 \\
&\leq 1.
\end{aligned}
$$

Now, if we take either the pair $(0, e_w)$ or the pair $\left(\frac{1}{\sqrt{2}} (e_{0w} + e_{1w}) \, , 0 \right)$, we get the maximal value, which is 1. □

Theorem 47. *Using the notation of (43) and (44), take a non-constant $\psi = \sum_u b_u e_u + \beta_0 e_\emptyset^0 + \beta_1 e_\emptyset^1 \in L^2(\mu)$, such that $|\psi| = 1$, and denote by B_ψ the value given by (46).*

Then, if $\hat{\psi}$ denotes the projection operator, we get the operator norm:

$$
1 \leq \| \, \left[D, \pi(\hat{\psi}) \right] \, \| = \max \left\{ \| \, (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \, \|, \| \, (\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L}) \, \| \right\} \leq \frac{3}{2\sqrt{2}}.
$$

This is so because

$$
\frac{3}{2\sqrt{2}} \geq \| \, (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \, \| = \tag{149}
$$

$$
\sup_{|\phi|=1} \sqrt{\langle \phi, \psi \rangle^2 - 2 \langle \phi, \psi \rangle \langle \mathcal{K}\phi, \psi \rangle \langle \mathcal{K}\psi, \psi \rangle + \langle \mathcal{K}\phi, \psi \rangle^2} \tag{150}
$$

$$\geq \sup_{w} \sqrt{b_w^2 + \frac{1}{2}\left(b_{0w} + b_{1w}\right)^2 - 2b_w \frac{1}{\sqrt{2}}\left(b_{0w} + b_{1w}\right) B_\psi}, \tag{151}$$

and

$$\frac{3}{2\sqrt{2}} \geq \| \left(\mathcal{L}\hat{\psi} - \hat{\psi}\mathcal{L}\right) \| = \tag{152}$$

$$\sup_{|\phi|=1} \sqrt{\langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{L}\phi, \psi \rangle \langle \mathcal{L}\psi, \psi \rangle + \langle \mathcal{L}\phi, \psi \rangle^2} \tag{153}$$

$$\geq \sup_{w} \sqrt{b_w^2 + \frac{1}{2} b_{\sigma(w)}^2 - 2b_w \frac{1}{\sqrt{2}} b_{\sigma(w)} B_\psi}. \tag{154}$$

The inequality comes from Proposition 20.

In terms of the elements of the basis, we get for ϕ of the form (44)

$$|\mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi)|^2 = \langle \mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi), \mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi) \rangle =$$

$$\langle \phi, \psi \rangle^2 + \Big(\frac{1}{\sqrt{2}} \sum_v a_v \left(b_{0v} + b_{1v}\right) + \frac{1}{2}(\alpha_1 + \alpha_0)(b_1 + b_0) + \frac{1}{2}(\alpha_1 - \alpha_0)(\beta_1 - \beta_0)\Big)^2$$

$$-2 \langle \phi, \psi \rangle \Big\{\Big[\frac{1}{\sqrt{2}} \sum_v a_v \left(b_{0v} + b_{1v}\right) + \frac{1}{2}(\alpha_1 + \alpha_0)(b_1 + b_0) + \frac{1}{2}(\alpha_1 - \alpha_0)(\beta_1 - \beta_0)\Big]$$

$$\Big[\sum_{l(u)>1} \frac{b_u}{\sqrt{2}}(b_{0u} + b_{1u}) + \frac{1}{2}(b_0 + b_1)(\beta_0 + \beta_1)\Big]\Big\}. \tag{155}$$

Proof. Given a fixed ψ, the equality to $\frac{3}{2\sqrt{2}}$ in (56) is true because the Koopman operator satisfies the hypothesis of Proposition 20. Equality (59) follows from duality. The expressions (57) and (60) will follow from (156) and (157).

First note that

$$\langle \mathcal{K}(\phi), \psi) \rangle =$$

$$\langle \Big(\sum_v a_v \frac{1}{\sqrt{2}}(e_{0v} + e_{1v}) - \alpha_0 \sqrt{2}(\mathbb{1}_{00} + \mathbb{1}_{10})\Big) + \sqrt{2}\alpha_1(\mathbb{1}_{01} + \mathbb{1}_{11}),$$

$$\Big(\sum_u b_u e_u + \beta_0 e_\emptyset^0 + \beta_1 e_\emptyset^1\Big) \rangle =$$

$$\frac{1}{\sqrt{2}} \sum_v a_v \left(b_{0v} + b_{1v}\right) + \frac{1}{2}(\alpha_0 b_0 + \alpha_0 b_1 + \alpha_1 b_0 + \alpha_1 b_1\,) + \frac{1}{2}(\alpha_0\beta_0 - \alpha_0\beta_1 + \alpha_1\beta_1 - \alpha_1\beta_0).$$

From (102), (104), (105)

$$\mathcal{K}\hat{\psi}(\phi) = \mathcal{K}(\langle \phi, \psi \rangle \sum_u b_u e_u + \beta_0 e_\emptyset^0 + \beta_1 e_\emptyset^1) =$$

$$\langle \phi, \psi \rangle \Big[\sum_u b_u \frac{1}{\sqrt{2}}(e_{0u} + e_{1u}) - \sqrt{2}\beta_0(\mathbb{1}_{00} + \mathbb{1}_{10}) + \sqrt{2}\beta_1(\mathbb{1}_{01} + \mathbb{1}_{11})\Big],$$

and

$$\hat{\psi}\mathcal{K}(\phi) = \langle \mathcal{K}(\phi), \psi \rangle (\sum_u b_u e_u + \beta_0 e^0_\emptyset + \beta_1 e^1_\emptyset) =$$

$$[\,\frac{1}{\sqrt{2}} \sum_v a_v\,(b_{0v} + b_{1v}) + \frac{1}{2}(\alpha_1 + \alpha_0)(b_1 + b_0) + \frac{1}{2}(\alpha_1 - \alpha_0)(\beta_1 - \beta_0)\,]$$

$$(\sum_u b_u e_u + \beta_0 e^0_\emptyset + \beta_1 e^1_\emptyset).$$

Then, given ϕ such that $|\phi| = 1$

$$\mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi) =$$

$$\langle \phi, \psi \rangle\,[\sum_u b_u \frac{1}{\sqrt{2}}(e_{0u} + e_{1u}) - \sqrt{2}\beta_0(\imath_{00} + \imath_{10}) + \sqrt{2}\beta_1(\imath_{01} + \imath_{11})]$$

$$-[\,\frac{1}{\sqrt{2}} \sum_v a_v\,(b_{0v} + b_{1v}) + \frac{1}{2}(\alpha_1 + \alpha_0)(b_1 + b_0) + \frac{1}{2}(\alpha_1 - \alpha_0)(\beta_1 - \beta_0)\,]$$

$$(\sum_u b_u e_u + \beta_0 e^0_\emptyset + \beta_1 e^1_\emptyset).$$

We leave it for the reader to calculate expression (62).
Alternatively, note that.

$$\langle \langle \phi, \psi \rangle \mathcal{K}\psi, \langle \phi, \psi \rangle \mathcal{K}\psi \rangle = \langle \phi, \psi \rangle^2 \langle \mathcal{K}\psi, \mathcal{K}\psi \rangle = \langle \phi, \psi \rangle^2,$$

and

$$\langle \langle \mathcal{K}\phi, \psi \rangle \psi, \langle \mathcal{K}\phi, \psi \rangle \psi \rangle = \langle \mathcal{K}\phi, \psi \rangle^2 \langle \psi, \psi \rangle \langle \mathcal{K}\phi, \psi \rangle^2.$$

Moreover,

$$2\langle \langle \phi, \psi \rangle \mathcal{K}\psi, -\langle \mathcal{K}\phi, \psi \rangle \psi \rangle = -2\langle \phi, \psi \rangle \langle \mathcal{K}\phi, \psi \rangle \langle \mathcal{K}\psi, \psi \rangle.$$

Finally,

$$\langle \mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi), \mathcal{K}\hat{\psi}(\phi) - \hat{\psi}\mathcal{K}(\phi) \rangle =$$

$$\langle \mathcal{K}\hat{\psi}(\phi), \mathcal{K}\hat{\psi}(\phi) \rangle - 2\langle \mathcal{K}\hat{\psi}(\phi), \hat{\psi}\mathcal{K}(\phi) \rangle + \langle \hat{\psi}\mathcal{K}(\phi), \hat{\psi}\mathcal{K}(\phi) \rangle$$

$$\langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{K}\phi, \psi \rangle \langle \mathcal{K}\psi, \psi \rangle + \langle \mathcal{K}\phi, \psi \rangle^2. \tag{156}$$

Similarly, note that:

$$\langle \mathcal{L}\hat{\psi}(\phi) - \hat{\psi}\mathcal{L}(\phi), \mathcal{L}\hat{\psi}(\phi) - \hat{\psi}\mathcal{L}(\phi) \rangle =$$

$$\langle \mathcal{L}\hat{\psi}(\phi), \mathcal{L}\hat{\psi}(\phi) \rangle - 2\langle \mathcal{L}\hat{\psi}(\phi), \hat{\psi}\mathcal{L}(\phi) \rangle + \langle \hat{\psi}\mathcal{L}(\phi), \hat{\psi}\mathcal{L}(\phi) \rangle =$$

$$\langle \phi, \psi \rangle^2 - 2\langle \phi, \psi \rangle \langle \mathcal{L}\phi, \psi \rangle \langle \mathcal{L}\psi, \psi \rangle + \langle \mathcal{L}\phi, \psi \rangle^2. \tag{157}$$

Applying the right hand-side of (62) to $\phi = e_w$, we get

$$b_w^2 + \frac{1}{2}\,(b_{0w} + b_{1w})^2 - 2b_w \frac{1}{\sqrt{2}}\,(b_{0w} + b_{1w})\,B_\psi. \tag{158}$$

Expression

$$\| (\mathcal{K}\hat{\psi} - \hat{\psi}\mathcal{K}) \| \geq$$

$$\sup_w \sqrt{b_w^2 + \frac{1}{2}\left(b_{0w} + b_{1w}\right)^2 - 2b_w \frac{1}{\sqrt{2}}\left(b_{0w} + b_{1w}\right) B_\psi} \tag{159}$$

follows from (158). □

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References

[1] A. Arai, Analysis on Fock Spaces and Mathematical Theory of Quantum Fields, World Scient (2017)

[2] S. Bezuglyi and P. E. T. Jorgensen, Transfer Operators, Endomorphisms, and Measurable Partitions, Lect. Notes in Mat. 2217, Springer Verlag

[3] J-B Bru and W. de S Pedra, C^*-Algebras and Mathematical Foundations of Quantum Statistical Mechanics, Springer Verlag (2023)

[4] M. de Carvalho, Mean, What do you Mean?, The American Statistician Volume 70, Issue 3, 270-274 (2016)

[5] Z-P Cian, G. Zhu, S-K Chu, A. Seif, W. DeGottardi, L. Jiang, and M. Hafezi, Photon Pair Condensation by Engineered Dissipation, Phys. Rev. Lett. 123, 063602 -2019

[6] L. Cioletti, L. Hataishi. A. O. Lopes and M. Stadlbauert, Spectral Triples on Thermodynamic Formalism and Dixmier Trace Representations of Gibbs Measures: theory and examples, Nonlinearity, Volume 37, Number 10, 105010 (56pp) (2024)

[7] A. Connes, Compact metric spaces, Fredholm modules, and hyperfiniteness. Ergodic Theory Dynam. Systems, 9(2):207-220, 1989.

[8] A. Connes, Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994

[9] J. Conway, A Course in Functional Analysis, Springer Verlag, 1990

[10] R. Exel and A. O. Lopes, C^*-Algebras, approximately proper equivalence relations and Thermodynamic Formalism, Vol 24, pp 1051–1082 Erg Theo and Dyn Syst (2004).

[11] R. Exel and A. O. Lopes, C^*-Algebras and Thermodynamic Formalism, Sao Paulo Journal of Mathematical Sciences 2, 1 (2008), 285-307

[12] R. Exel, KMS states for generalized gauge actions on CuntzKrieger algebras (An application of the Ruell-Perron-Frobenius Theorem). Bol. Soc. Brasil. Mat. 35(2004), no. 1, 1-12.

[13] R. Exel and A. Vershik, C^*-Algebras of Irreversible Dynamical Systems, Canad. J. Math. Vol. 58 (1), 2006 pp. 39-63

[14] B. C. Hall, Quantum theory for mathematicians, Springer Verlag (2013)

[15] A. Julien and I. Putnam. Spectral triples for subshifts. J. Funct. Anal., 270(3):10311063, 2016

[16] M. Kessebohmer and T. Samuel, Spectral metric spaces for Gibbs measures, Journal of Functional Analysis, 265, 1801-1828 (2013)

[17] M. Kessebohmer, M. Stadlbauer and B. O. Stratmann, Lyapunov spectra for KMS states on Cuntz-Krieger algebras. Math. Z. 256 (4): 871-893 (2007).

[18] A. Kolmogorov, Sur la Notion de la Moyenne, Atti della Academia Nazionale dei Lincei, 12, 323-343 (1930)

[19] E-J Kuo,Y Xu, D.Hangleiter, A. Grankin and M. Hafezi, Boson sampling for generalized bosons, Physical Review Research, 4, 043096 (2022)

[20] B-S Lin, H-L Chen and T-H Heng, Note on Connes spectral distance of qubits, arXiv (2022)

[21] A. O. Lopes, Uma breve Introducao a Matematica da Mecanica Quantica, XXXI Coloquio Brasileiro de Matematica (2017)

[22] A. O. Lopes and R. Ruggiero, The sectional curvature of the infinite dimensional manifold of Hölder equilibrium probabilities, to appear in Proc. of the Edinburg Matematical Soc.

[23] A. O. Lopes and R. Ruggiero, Geodesics and dynamical information projections on the manifold of Hölder equilibrium probabilities, preprint

[24] D. S. Mitrinovic and P. M. Vasic, Analytic inequalities. Vol. 1. Berlin: Springer, 1970.

[25] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* Vol 187-188 1990

[26] M. Pollicott and R. Sharp, A Weil-Petersson type metric on spaces of metric graphs. Geom. Dedicata 172, 229-244 (2014)

[27] P. P. Potts, Quantum Thermodynamics, arxiv (2024)

[28] R. Sharp. Spectral triples and Gibbs measures for expanding maps on Cantor sets. J. Noncommut. Geom., 6(4):801-817 2012.

[29] R. Sharp. Conformal Markov systems, Patterson-Sullivan measure on limit sets and spectral triples. Discrete Contin. Dyn. Syst., 36(5):2711-2727, 2016

[30] F. Schwabl, Advanced Quantum Mechanics, Springer Verlag (2005)