A GENERAL RENORMALIZATION PROCEDURE ON THE ONE-DIMENSIONAL LATTICE AND DECAY OF CORRELATIONS

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ABSTRACT. We present a general form of Renormalization operator $\mathscr{R}$ acting on potentials $V : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$. We exhibit the analytical expression of the fixed point potential V for such operator $\mathscr{R}$. This potential can be expressed in a naturally way in terms of a certain integral over the Hausdorff probability on a Cantor type set on the interval $[0,1]$. This result generalizes a previous one by A. Baraviera, R. Leplaideur and A. Lopes where the fixed point potential V was of Hofbauer type.

For the potentials of Hofbauer type (a well known case of phase transition) the decay is like $n^{-\gamma}$, $\gamma > 0$

Among other things we present the estimation of the decay of correlation of the equilibrium probability associated to the fixed potential V of our general renormalization procedure. In some cases we get polynomial decay like $n^{-\gamma}$, $\gamma > 0$, and in others a decay faster than $n e^{-\sqrt{n}}$, when $n \to \infty$.

The potentials g we consider here are elements of the so called family of Walters' potentials on $\{0,1\}^{\mathbb{N}}$ which generalizes a family of potentials considered initially by F. Hofbauer. For these potentials some explicit expressions for the eigenfunctions are known.

In a final section we also show that given any choice $d_n \to 0$ of real numbers varying with $n \in \mathbb{N}$ there exist a potential g on the class defined by Walters which has a invariant probability with such numbers as the coefficients of correlation (for a certain explicit observable function).

1. INTRODUCTION

It is a classical problem in Statistical Mechanics the study of phase transitions (see [12], [13], [1] , [15], [28] and [29]) for general potentials. In several examples it is known that on the point of phase transition one gets polynomial decay of correlation for the associated equilibrium probability. As far as we know, there is no unified way for understanding all kinds of phase transitions.

One important technique on the analysis of such class of problems is the use of the renormalization operator (see for instance [5], [18], [4] and [8]). The potential which is fixed for the renormalization operator in most of the cases is quite relevant for the understanding of the problem.

Phase transitions in the setting of Thermodynamic Formalism has been considered from different points of view (see for instance [2], [14], [19], [20], [21], [7] , [6], [22], [23], [9], [10], [11] and [17]).

In [2] it was introduced a renormalization operator (acting on potentials) on the one-dimensional lattice $\{0,1\}^{\mathbb{N}}$ and the fixed point one gets is the well known Hofbauer potential. The equilibrium probability for such potential exhibit polynomial decay of correlation for a certain class of observables.

Here we consider a more general procedure of renormalization of potentials defined on the lattice $\{0,1\}^{\mathbb{N}}$. We want to exhibit a fixed point potential for such procedure. The fixed point potential will be on a class of functions previously considered by P. Walters in [27] (which includes the Hofbauer potential). We are particulary interested in the decay of correlation (for a certain natural function) associated to the equilibrium probability for such potential.

In some cases (first type) the fixed point will produce polynomial decay of correlation for some natural observable. We point out that in this case in section 5 we will show the exact speed of decay of correlations and not just an upper bound.

In others cases (second type) not of polynomial type (some of them faster than $n e^{-\sqrt{n}}$, when $n \to \infty$). In this case is an upper bound. This shows that fixed point for the renormalization operator (on

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the sense described here) does not necessarily present polynomial decay of correlations (for some natural observable).

In order to give an idea to the reader of the kind of results we will get here we present an example: in a particular case the renormalization operator $\mathscr{R}$ acts on continuous potentials U, such that U is constant on cylinder sets of the form $\overline{\underbrace{0\ldots0}_{n}1}$ and $\overline{\underbrace{1\ldots1}_{n}0}$, for all $n\in\mathbb{N}$. For each fixed n the values are the same on both cylinder sets

We define $U_2=\mathscr{R}(U_1)$ in the following way: suppose x is of the form $x=(\underbrace{0\ldots0}_{n},1,\ldots)$, then

$$U_2(x)\,=\mathscr{R}(U_1)\,(x)=U_1(\underbrace{0\ldots0}_{3n},1)+U_1(\underbrace{0\ldots0}_{3n-2},1).$$

In this case the fixed point potential V for the renormalization operator $\mathscr{R}$ is the function

$$V(x)=-\int_K\frac{1}{(n-t)^{\alpha}}d\nu(t),$$

when $x=(\underbrace{0\ldots0}_{n},1,\ldots)$, $n\in\mathbb{N}$, and where ν is the Hausdorff probability on the classical Cantor set (a subset of the interval $[0,1]$) and which has Hausdorff dimension $\alpha=\frac{\log 2}{\log 3}$.

In several distinct classes of problems the fixed point potential for the renormalization operator can be expressed on an integral form associated to a fractal probability (see for instance [5], [3], [18], [4] and [24]). Here we will also get a similar kind of expression.

We will present now the main definitions giving more details on the specific problems we want to analyze.

We will consider a subclass of potentials $g:\{0,1\}^{\mathbb{N}}\to\mathbb{R}$ which were described in [27]. In [27] it is considered four sequence of real numbers a_n,b_n,c_n,d_n and constants a,b,c,d. It is introduced the potential g which satisfies

$$g(0^n1z)=a_n,g(0^\infty)=a,g(10^n1z)=d_n,g(10^\infty)=d,$$
$$g(01^n0z)=b_n,g(01^\infty)=b,g(1^n0z)=c_n,g(1^\infty)=c,$$

and is assumed that $a_n\to a$, $b_n\to b$, $c_n\to c$ and $d_n\to d$.

A potential of this kind is continuous and called a Walters' potential. Depending of the choice of the sequences some potentials are of Hölder class and others are not. Theorem 1.1 in [27] characterize when a potential of this form is on the Bowen's class (which contains the Hölder's - or, Lichitz's - potentials).

In our model $a=0=c$ and the sequences $b_n=b$ and $d_n=d$ are constant. Moreover, we are taking $c_n=a_n$, for all n.

Given $a_2,a_3,\ldots,a_n,\ldots<0$ define η_j, by $\eta_j:=e^{(a_2+a_3+\ldots+a_{j+1})}$, $j\geq 1$.

For the sequence $1\geq\eta_n>0$, $n\in\mathbb{N}$, $n\geq 1$, $\eta_1=1$, we assume that, $\eta_n\to 0$, and $\sum_{n=1}^{\infty}\eta_n^{\beta}=W(\beta)>1$, for $\beta>1$.

Particular cases which we are interested are when $\eta_n=e^{-n^{1/2}}$, $\eta_n=e^{-n^{1-\frac{\log 2}{\log 5}}}$ or $\eta_n=n^{-\gamma}$, $\gamma>2$.

Consider the two infinite collections of cylinder sets given by

$$L_n=\overline{0^n1}\quad\text{and}\quad R_n=\overline{1^n0},\ \text{ for all } n\geq 1.$$

Using these parameters we can define a continuous potential $g=g_\beta:\Omega\to\mathbb{R}$, which is simply denote by g, in the following way: for any $x\in\Omega$ and $\beta\geq 0$

$$g(x)=\begin{cases}\beta a_n, & \text{if } x\in L_n,\ \text{for some } n\geq 2;\\ \beta a_n, & \text{if } x\in R_n,\ \text{for some } n\geq 2;\\ -\log W(\beta), & \text{if } x\in L_1;\\ -\log W(\beta), & \text{if } x\in R_1;\\ 0, & \text{if } x\in\{1^\infty,0^\infty\}.\end{cases}$$

In Statistical Mechanics the above parameter β is considered the inverse of temperature.

We are mainly interested in potentials g which are not of Hölder type.

A particular example where phase transition occurs is presented on section 2 of [10].

Definition 1. *Given $A : \Omega = \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, the Ruelle operator $\mathscr{L}_A$ acts on functions $\psi : \Omega \to \mathbb{R}$ in the following way*

$$\varphi(x) = \mathscr{L}_A(\psi)(x) = \sum_{a=0}^{1} e^{A(ax)} \psi(ax).$$

By this we mean $\mathscr{L}_A(\psi) = \varphi$.

From the general result described in page 1341 in [27] applied to our particular case, we get that the eigenfunction φ_β of the Ruelle operator for $A = \beta\, g$, for $\beta \leq 1$, is such that, for any $n \geq 1$

$$\varphi_\beta(0^n 1...) = \alpha \left(1 + \frac{1}{\eta_n^\beta} \sum_{j=1}^{\infty} \frac{\eta_{n+j}^\beta}{\lambda(\beta)^j}\right), \tag{1}$$

$$\varphi_\beta(1^n 0...) = b(\beta) \left(1 + \frac{1}{\eta_n^\beta} \sum_{j=1}^{\infty} \frac{\eta_{n+j}^\beta}{\lambda(\beta)^j}\right), \tag{2}$$

$$\varphi_\beta(0^\infty) = \alpha \qquad \text{and} \qquad \varphi_\beta(1^\infty) = b(\beta).$$

We claim that there exists a unique positive eigenfunction for $\beta > 1$. This follows from an argument similar to the one used on page 24 of [25].

We assume that for $\beta = 1$ we get $\lambda(\beta) = 1 = \alpha = b(\beta)$

When $\beta = 1$, it is easy to see that in the case it is well defined the probability ρ on the Borel sets of Ω, in such way that, for any natural number $q \geq 1$,

$$\rho\left(\overline{0^q 1}\right) = \eta_q \qquad \text{and} \qquad \rho\left(\overline{1^q 0}\right) = \eta_q, \tag{3}$$

then, ρ is an eigenprobability for the dual of the Ruelle operator for g. This claim can be obtained by generalizing the reasoning of section 5 in [10].

Therefore, from a general procedure (see [25]) one gets that the probability μ, such that,

$$\mu\left(\overline{0^q 1}\right) = \eta_q\, \varphi(0^n 1...) = \sum_{j=0}^{\infty} \eta_{n+j} \qquad \text{and} \qquad \mu\left(\overline{1^q 0}\right) = \eta_q\, \varphi(1^n 0...) = \sum_{j=0}^{\infty} \eta_{n+j} \tag{4}$$

is the equilibrium probability for g (maximizes pressure).

We will show in Theorem 6 that decay of correlation d_q, as a function of $q \in \mathbb{N}$, of the equilibrium probability μ for the potential g and for the indicator of the cylinder $\overline{0}$ is of order

$$\sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \eta_{(k+q+s)}.$$

Given a certain sequence $d_q \to 0$, there exists (under mild conditions) a choice of η_n such that $d_q = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \eta_{(k+q+s)}$. Indeed, first for the given sequence d_q, $q \in \mathbb{N}$, take inductively c_n such that $c_n = d_n - d_{n+1}$, $n \in \mathbb{N}$. In this way $\sum_{j=q}^{\infty} c_j = d_q$.

Now, given such sequence c_n, $n \in \mathbb{N}$, take a new sequence η_r, such that, $\eta_r = c_r - c_{r+1}$, $r \in \mathbb{N}$. In this case $\sum_{i=p}^{\infty} \eta_i = c_p$.

It follows that for all q we get $d_q = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \eta_{(k+q+s)}$.

Part of the results presented here were a consequence of fruitful discussions with A. Baraviera and R. Leplaideur which contributed in a non trivial way to our reasoning. Our thanks to them.

In sections 2 and 3 we present results about the fixed point potential for a certain renormalization operator. On sections 4 and 5 we estimate the decay of correlations for the class o potentials of Walters type and we show that a large class of types of decay of correlations are attained by such family. On the appendix we use results of the previous sections to show that for some of these fixed point potentials the decay is not of purely polynomial type.

2. A GENERAL RENORMALIZATION - FIRST TYPE

The potentials f which we are interested are on the Walters' class

$$f(0^n 1z) = a_n, \quad f(0^\infty) = a, \quad f(10^n 1z) = d_n, \quad f(10^\infty) = d,$$
$$f(01^n 0z) = b_n, \quad f(01^\infty) = b, \quad f(1^n 0z) = c_n, \quad f(1^\infty) = c,$$

and is assumed that $a_n \to a$, $b_n \to b$, $c_n \to c$ and $d_n \to d$.

We will assume that $b_n = d_n = b = d$.

The renormalization operator will act on such functions f.

Fix a natural number $k \geq 2$. We will generalize the kind of renormalization described in [2] where $k = 2$.

Now take for $n > 1$

$$H_k(0^n 1x) = 0^{kn-(k-2)} 1x,$$
$$H_k(1^n 0y) = 1^{kn-(k-2)} 0y,$$

and

$$H_k(01^m x) = 0\,1^m x,$$
$$H_k(1\,0^m y) = 1\,0^m y,$$

and, also $H_k(0^\infty) = 0^\infty$, $H_k(1^\infty) = 1^\infty$.

Now define the renormalization operator as

$$R_k(V)(x) = V(H_k(x)) + V(\sigma H_k(x)) + V(\sigma^2 H_k(x)) + ... + V(\sigma^{k-1} H_k(x)).$$

We will show that it is possible to exhibit a fixed point V for the renormalization operator R_k.

Note that the renormalization operator is globally defined.

The class of functions we consider here in the section 1 is large enough to be able to get fixed points V. This means we will be able to find a sequence a_n, $n \in \mathbb{N}$, which will define such V

An interesting question is to estimate the decay of correlation of the equilibrium probability for the fixed point potential V. One can show (see section 5) that for the case of renormalization of first type the potential V is such that we get polynomial decay of correlation.

If V is fixed for the renormalization operator R_k, then $V(1^\infty) = V(0^\infty) = 0 = a = c$.

Moreover,

$$V(01^\infty) = V(01^\infty) + V(1^\infty) + V(1^\infty) + .. + V(1^\infty) = b,$$

and

$$a_2 = V(00\,1^\infty) = V(\underbrace{00...000}_{k+2} 1^\infty) + V(\underbrace{00...000}_{k+1} 1^\infty) + ... + V(0001^\infty) = a_3 + a_4 + ... + a_{k+2},$$

$$a_3 = V(000\,1^\infty) = V(\underbrace{00...000}_{2k+2} 1^\infty) + V(\underbrace{00...000}_{2k+1} 1^\infty) + ... + V(\underbrace{00...000}_{k+3} 1^\infty) = a_{k+3} + a_{k+4} + ... + a_{2k+2}.$$

As we will see once we fix the value of a_2 and b the potential V will be determined.

The Walters potential V which is the fixed point for R_k should satisfy for $n \geq 2$:

$$a_n = a_{k(n-1)+2} + a_{k(n-1)+1} + ... + a_{k(n-2)+3},$$
$$a_n = c_n = c_{k(n-1)+2} + c_{k(n-1)+1} + ... + c_{k(n-2)+3},$$

and

$$d_n = b_n = b = d$$

are constant.

Now we solve $\alpha(2)$ as the solution of

$$a_2 = -\log \frac{2 + \alpha(2)}{2 + \alpha(2) - 1}.$$

Now, $\alpha(n)$ is defined by induction for $n \geq 2$,

$$\alpha_{k(n-1)+2} = \alpha_{k(n-1)+1} = ... = \alpha_{k(n-2)+3},$$

and

$$\alpha_{k(n-1)+2} = k\,\alpha(n) + (k-2).$$

Taking $a_n = d_n$, $n \geq 2$, of the form

$$a_n = -\log \frac{n+\alpha(n)}{n+\alpha(n)-1} < 0$$

one can show that we get the fixed point solution of the Renormalization operator.

We point out that a_2 and b are free.

One can show that in this case for such V there exists γ such that $\log \eta_n \sim -\gamma \log n$, $n \geq 1$. If $\gamma > 2$, one gets polynomial decay of correlation of the form $n^{2-\gamma}$ similar to the Hofbauer case. This kind of question was previously consider in [22] [23] [14] [10]. This can be also obtained from the general proof of section 5.

3. A GENERAL RENORMALIZATION - SECOND TYPE

We consider in this section a far more general generalization of the renormalization operator and we exhibit the fixed point potential V.

We point out that in many examples the fixed point potential of a renormalization operator is described by an expression obtained by the integral of a kernel with respect to a certain fractal probability (see for instance [5], [3], [18], [4] and [24]). Here we will get a similar kind of result.

In order to define the second type renormalization operator, we first introduce the map H (which is on Walters' class). Let k be an integer larger or equal than 2; take $h(0) = \underbrace{00\ldots0}_{k}$ and $h(1) = \underbrace{11\ldots1}_{k}$; hence we define

$$H(x_1, x_2, x_3, \ldots) = (h(x_1)h(x_2)h(x_3)\ldots).$$

It is easy to see that $\sigma^k H = H\sigma$. Hence

$$H(\underbrace{00\ldots0}_{n}1\ldots) = \underbrace{00\ldots0}_{kn}1\ldots \qquad \text{say, if } x \in L_n, \text{ then } H(x) \in L_{kn}$$

and

$$H(\underbrace{11\ldots1}_{n}0\ldots) = \underbrace{11\ldots1}_{kn}0\ldots \qquad \text{say, if } x \in R_n, \text{ then } H(x) \in R_{kn}$$

We now define the renormalization operator as the map on functions defined as follows: fix l such that $2 \leq l \leq k$ and the constants $0 \leq c_1 < c_2 < \ldots < c_l \leq k$. Then

$$\mathscr{R}\,V(x) := V(\sigma^{c_1}H(x)) + V(\sigma^{c_2}H(x)) + \cdots + V(\sigma^{c_l}H(x)).$$

One can see that the Renormalization operator of last section (the case $l = k$) is a particular case of this new one .

The equation above for the potential Vcan be rewritten, analogously to the expressions in the previous section, as

$$R\,a_n = a_{kn-c_1} + a_{kn-c_2} + \cdots + a_{kn-c_l}$$

(and a similar expression holds to c_n).

We are interested on finding a fixed potential V for such renormalization operator. For a potential on the class of Walters the fixed point equation is

$$a_n = a_{kn-c_1} + a_{kn-c_2} + \cdots + a_{kn-c_l}.$$

Then, we get the following result:

Lemma 2. *Given $N \geq 1$ then*

$$\mathscr{R}^N V(x) = \sum_j V(\sigma^j H^N(x),$$

where j is of the form

$$j = b_0 k^0 + b_1 k^1 + b_2 k^2 + \cdots + b_{N-1} k^{N-1},$$

for b_i chosen on the set $\{c_1, c_2, \ldots, c_l\}$.

Proof. Indeed, the expression it obviously valid for $N = 1$ (by definition of the operator $\mathscr{R}$). Now, admiting that it holds for N; we need to show that this implies the expression for $\mathscr{R}^{N+1} V$. But,

$$\mathscr{R}^{N+1} V = \mathscr{R}(\mathscr{R}^N V) = \mathscr{R}^N \sigma^{c_1} H + \mathscr{R}^N \sigma^{c_2} H + \cdots + \mathscr{R}^N \sigma^{c_l} H =$$

$$\sum_j V(\sigma^j H^N \sigma^{c_1} H) + \ldots$$

Now, it is easy to see (also by induction) that

$$H \sigma^P H = \sigma^{Pk} H^2$$

and

$$H^P \sigma H = \sigma^{k^P} H^2 \ = \ \sigma^{k^P} H^{P+1}.$$

With this two expressions we get

$$H^N \sigma^c H = \sigma^{ck^N} H^{N+1}.$$

Then, we can show that $\mathscr{R}^{N+1} V$ is

$$\sum_j V(\sigma^j H^N \sigma^{c_1} H) + \ldots =$$

$$\sum_j V(\sigma^j \sigma^{c_1 k^N} H^{N+1}) + \ldots = \sum_j V(\sigma^{j + c_1 k^N} H^{N+1}) + \ldots,$$

where $j = b_0 k^0 + b_1 k^1 + b_2 k^2 + \cdots + b_{N-1} k^{N-1}$; hence the expression above is indeed

$$\sum_J V(\sigma^J H^{N+1}) + \ldots,$$

where $J = d_0 k^0 + d_1 k^1 + d_2 k^2 + \cdots + d_{N-1} k^{N-1} + d_N k^N$, showing that $\mathscr{R}^{N+1}$ satisfies what we claim. □

Now we want to show that we have a fixed point of $\mathscr{R}$ that in the neighborhood of 0^∞ behaves like

$$V(\underbrace{00\ldots0}_{n}1\ldots) \sim -\frac{1}{n^{\frac{\log l}{\log k}}}.$$

As we will see in a moment it will be natural to consider the (Cantor like) set $K = K(l,k)$ which is the closure of the set

$$\{x = \sum_{n \geq 1} a_i k^{-i} \ \text{ for } a_i \in \{c_1, \ldots, c_l\}, n \in \mathbb{Z} \ \}.$$

Note that when $k = l$ the set K will be the interval $[0,1]$.

Example 3. *Consider the following particular case.*

Define $H_3 : \Omega = \{0,1\}^{\mathbb{N}} \to \Omega$ by:

$$H_3((\underbrace{0,...,0}_{c_1},\underbrace{1,...,1}_{c_2}\underbrace{0,...,0}_{c_3},1,...)) = (\underbrace{0,...,0}_{3c_1},\underbrace{1,...,1}_{c_2}\underbrace{0,...,0}_{c_3},1,\ldots),$$

and

$$H_3((\underbrace{1,\ldots,1}_{c_1},\underbrace{0,\ldots,0}_{c_2}\underbrace{1,\ldots,1}_{c_3},1,\ldots)) = (\underbrace{1,\ldots,1}_{3c_1},\underbrace{0,\ldots,0}_{c_2}\underbrace{1,\ldots,1}_{c_3},1,\ldots).$$

In this case the renormalization operator is

$$R_3(V)(x) = V(H_3(x)) + V(\sigma H_3(x)) + V(\sigma^2 H_3(x)).$$

If $x = (\underbrace{0,...,0}_{c_1},1,..)$ *or* $x = (\underbrace{1,...,1}_{c_1},0,..)$ *denote* $V(x) = \log \frac{c_1}{c_1-1}$. *Note that for each* c_1 *we get that*

$$\log \frac{3c_1}{3c_1-1} + \log \frac{3c_1-1}{3c_1-2} + \log \frac{3c_1-2}{3c_1-3} = \log \frac{c_1}{c_1-1}.$$

In this case V *is a fixed point for the renormalization operator.*
We can choose the signal of V *and then, we take* V *in the form*

$$V(\underbrace{1,...,1}_{c_1},0,..) = V(\underbrace{0,...,0}_{c_1},1,..) = -\int_0^1 \frac{1}{c_1-t}\,dt = -\log \frac{c_1}{c_1-1}.$$

In order to find a good guess (in the general case) for the fixed point potential consider

$$U(x) = \frac{1}{n^\alpha}, \qquad \text{for any } n,$$

when, $x = \underbrace{0,...,0}_{n},1,..)$, or, $x = \underbrace{1,...,1}_{n},0,..)$.

Then, by iteration of U we get:

$$\mathscr{R}^N U(x) = \sum_j \frac{1}{(nk^N - j)^\alpha} = \sum_j \frac{1}{k^{N\alpha}} \frac{1}{(n - \frac{j}{k^N})^\alpha},$$

for $j = b_0k^0 + b_1k^1 + b_2k^2 + \cdots + b_{N-1}k^{N-1}$ (and, $b_i \in \{c_1,\ldots,c_l\}$).

From the above seems natural to try to find a fixed point V of the form

$$V(x) = -\int_K \frac{1}{(n-t)^\alpha} d\nu(t),$$

when $(\underbrace{0\ldots0}_{n},1,..)$, for some probability ν on the interval.

Example 4. *Let us fix* $k = 3$, $l = 2$ *and* $c_1 = 0, c_2 = 2$. *In this case the Cantor set* K *we generate is indeed the classical Cantor set. We can write* $K = K_0 \cup K_2$ *where* $K_0 = K \cap [0,1/3]$ *and* $K_2 = K \cap [2/3,1]$.

Note that the transformation $T(x) = 3x$ *(mod* 1*) is two to one when restricted to* K.

Moreover, $T[0,1/3] = [0,1] = T[2/3,1]$.

Let ν *be the measure supported on* K *and consider* $\alpha = \log 2/\log 3$, *that is the Hausdorff dimension of* K.

Note that the Radon-Nykodin derivative of ν *in each injective branch of* T *restricted to* $K_0 \subset [0,1/3]$ *and* $K_1 \subset [2/3,1]$ *is equal to* 2.

In this case the function

$$V(\underbrace{0\ldots0}_{n}1) = -\int_K \frac{1}{(n-t)^\alpha} d\nu(t)$$

is a fixed point of the renormalization operator, i.e.,

$$V(x) = V(H(x)) + V(\sigma^2 H(x)).$$

Theorem 5. *Let* x *of the form:* $x = \underbrace{00\ldots0}_{n}1\ldots$, *or* $x = \underbrace{11\ldots1}_{n}0\ldots$, *then the operator*

$$\mathscr{R}V(x) := V(\sigma^{c_1}H(x)) + V(\sigma^{c_2}H(x)) + \cdots + V(\sigma^{c_l}H(x))$$

has a fixed point V *of the form*

$$V(x) = -\int_K \frac{1}{(n-t)^\alpha} d\nu(t),$$

where $\alpha = \frac{\log l}{\log k}$, *and where* ν *is the measure over* K *that maximizes the entropy for the transformation* $T(x) = kx$ *(mod* 1*) acting on* K .

Proof. The reasoning is similar to the last example. Consider $\alpha = \frac{\log l}{\log k}$.

The transformation $T(x) = kx \pmod 1$ acting on K is l to one. The maximal entropy probability ν has entropy $\log l$.

There exists l sets $K^1, K^2, ..., K^l \subset [0,1]$ such that $T(K^j \cap K) = K$, and by considering T restricted to $K^j = [c_j/k, (c_j+1)/k]$, $j = 1, 2, ..., l$, we get an injective map. Consider the induced probability $\nu_j = T^*(\nu)$ on K_j. The Radon-Nykodin derivative of ν_j with respect to ν in each set K^j is equal to l.

If $x = (\underbrace{0\ldots0}_{n}, 1, ...)$ (or, $x = (\underbrace{1\ldots1}_{n}, 0, ...)$) we want to show that

$$V(x) = -\int_K \frac{1}{(n-t)^\alpha} d\nu(t) = -\sum_{j=1}^{l} \int_K \frac{1}{((kn-c_j)-t)^\alpha} d\nu(t) = \mathscr{R}V(x).$$

For each j we have that

$$\int_K \frac{1}{((kn-c_j)-t)^\alpha} d\nu(t) = \int_K \frac{1}{k^\alpha (n - \frac{c_j+t}{k})^\alpha} d\nu(t) = \int_K \frac{1}{l\,(n - \frac{c_j}{k} - \frac{t}{k})^\alpha} d\nu(t).$$

For each j we consider the change of coordinate $t \to t/k$ on the set K^j.

Then, we get that

$$\int_K \frac{1}{(n - \frac{c_j+t}{k})^\alpha} d\nu(t) = \int_{K^j} \frac{1}{(n-y)^\alpha}\, l\, d\nu(y).$$

As

$$\int_K \frac{1}{(n-t)^\alpha} d\nu(t) = \sum_{j=1}^{l} \int_{K^j} \frac{1}{(n-t)^\alpha} d\nu(t)$$

we get the claim.

□

Note that

$$-\int_K \frac{1}{(n-t)^\alpha} d\nu(t) \sim -n^{-\alpha},$$

for large n. In this case η_n will be of order $e^{-n^{1-\alpha}}$. When $k = 5$ and $l = 2$ we will get that the equilibrium state for fixed point V has a decay faster than $n\,e^{-\sqrt{n}}$ (see Appendix).

4. THE ASSOCIATED JACOBIAN

Now, we are interested on the estimation of the decay of correlation associated to equilibrium probabilities associated to the fixed point potentials we get via the renormalization operator. We will show that it is not always of polynomial type. In fact, we will present the decay of correlation for a general potential of Walters type.

Suppose g is of Walter type. If we add a constant to a potential its equilibrium state will not change. Then, we can assume without lost of generality that the potential g has pressure zero.

The associated Jacobian J for the potential defined by g at the inverse temperature $\beta = 1$ is such that $\log J = g + \log\varphi - \log(\varphi \circ \sigma)$, where φ is the eigenfunction of the Ruelle operator for g (see (2).

We will define $r(q)$ for $q \geq 1$.

We split the expression of $\log J(x)$ in five cases:

a) for $q \geq 2$ and $x \in L_q$ or $x \in R_q$ we have

$$\log J(x) = \beta a_q + \log\left(1 + \eta_q \sum_{n=2}^{\infty} \eta_{(n+q-1)}\right) - \log\left(1 + \eta_{(q-1)} \sum_{n=2}^{\infty} \eta_{(n+q-2)}\right)$$
$$:= \beta a_q + \log r(q) - \log r(q-1),$$

b) for $x = 0\overbrace{111...1}^{q}0... \in L_1$ we have

$$\log J(x) = -\log\left(1+\eta_q \sum_{n=2}^{\infty} \eta_{(n+q-1)}\right) = -\log(r(q)),$$

c) for $x = 1\overbrace{000...0}^{q}1... \in R_1$ we have

$$\log J(x) = -\log\left(1+\eta_q \sum_{n=2}^{\infty} \eta_{(n+q-1)}\right) = -\log(r(q)),$$

d) for $x = 0^\infty$ or $x = 1^\infty$ we have $J(x) = 1$,
e) for $x = 10^\infty$ or $x = 01^\infty$ we have $J(x) = 0$.

Note that $r(1)\eta_1 = W$.

It follows from [27] that when J is well defined the equilibrium probability μ for g has Jacobian J and

$$\mathscr{L}^*_{\log J}(\mu) = \mu.$$

5. DECAY OF CORRELATION

In this section we want to estimate the decay of correlation of the observable $I_{[0]}$ for the equilibrium probability μ at the critical inverse temperature T where $\beta = 1 = \frac{1}{T}$.

When the potential on the Walters' class is Hölder the decay is exponential (there exists an spectral gap) according to [25].

We assume from now on that $\beta = 1$.

That is, we will estimate the asymptotic behavior with k of the decay of correlation of the observable $I_{\overline{0}}$

$$\int_\Omega (I_{[0]} \circ \sigma^q)\,[I_{[0]} - \mu_1[0]]\,d\mu \sim \alpha(q),$$

where μ is the equilibrium probability for g, when $\beta = 1$.

We will show, as a particular case, that if $\eta_q \sim q^{-\gamma}$, with $\gamma > 2$, then, $\alpha(q)$ goes to zero like $q^{2-\gamma}$. The next result is quite general.

Theorem 6. *The decay of correlation as a function of q for the equilibrium probability μ (for the potential g) for the indicator of the cylinder $\overline{0}$ is*

$$\sum_{s=1}^{\infty}\sum_{k=0}^{\infty} \eta_{(k+q+s)}$$

in the case is well defined.

Proof: The technique is similar to the one employed in [14] and [10].

First we need to estimate the asymptotic limit

$$\mu_1[0] - \mathscr{L}^q_{\log J}(I_{[0]})(01..).$$

We claim that

$$\mu_1[0] - \mathscr{L}^q_{\log J}(I_{[0]})(01..) \sim \sum_{n=1}^{q}\sum_{j=q+1}^{\infty} \eta_j. \tag{5}$$

We will present the proof of this fact later.

Now, for a fixed q and any $s \geq 2$ we need to evaluate the difference

$$\mu_1[0] - \mathscr{L}^q_{\log J}(I_{[0]})(\underbrace{00...0}_{s}1..).$$

For fixed q and variable s, let $B_q^s = \mathscr{L}_{\log J}^q(I_{[0]})(\overbrace{00...0}^{s}1..)$, $A(t) = \mathscr{L}_{\log J}^t(I_{[0]})(01...)$ and finally denote $p_n^s = \frac{r(s+n)\eta_{s+n}}{r(s)\eta_s}$, $n = 1,..,q-2$, $s \in \mathbb{N}$.

We denote $\alpha(q,s) = \frac{\eta_{(s+q)}\, r(s+q)}{\eta_s\, r(s)}$.

One can show (see figure 3 page 1092 and expression in the bottom of page 1090 [14] where a similar case is described) that for any $s, q \geq 2$

$$B_q^s = A_{q-1} + e^{a_{s+1}}\,\frac{r(s+1)}{r(s)}A_{q-2} + e^{a_{s+1}+a_{s+2}}\,\frac{r(s+2)}{r(s)}A_{q-3} + ... + e^{a_{s+1}+a_{s+2}+...+a_{q+s-2}}\,\frac{r(s+q-2)}{r(s)}A_1 +$$

$$e^{a_{s+1}+a_{s+2}+...+a_{q+s-1}+a_{q+s}}\,\frac{r(s+q)}{r(s)} =$$

$$A_{q-1} + \frac{r(s+1)\,\eta_{(s+1)}}{r(s)\,\eta_s}A_{q-2} + \frac{r(s+2)\eta_{s+2}}{r(s)\eta_s}A_{q-3} + ... + \frac{r(s+q-2)\,\eta_{(s+q-2)}}{r(s)\,\eta_s}A_1 + \frac{r(s+q)\,\eta_{(s+q)}}{r(s)\,\eta_s} =$$

$$A_{q-1} + \frac{r(s+1)\,\eta_{(s+1)}}{r(s)\,\eta_s}A_{q-2} + \frac{r(s+2)\eta_{s+2}}{r(s)\eta_s}A_{q-3} + ... + \frac{r(s+q-2)\,\eta_{(s+q-2)}}{r(s)\,\eta_s}A_1 + \alpha(n,s). \tag{6}$$

This is not a kind of renewal equation.

Let us now consider for $n \geq 1$ fixed, and $s \geq 2$ the sequence $V_n^s = \mu_1[0] - B_n^s$.

We also introduce, for $n \geq 1$, the sequences $V_n = \mu_1[0] - A_n$, and

$$U_n^s = -\mu[0](\frac{r(s+1)\,\eta_{(s+1)}}{r(s)\,\eta_s} + \frac{r(s+2)\eta_{s+2}}{r(s)\eta_s} + ... + \frac{r(s+q-2)\,\eta_{(s+q-2)}}{r(s)\,\eta_s}) - \alpha(n,s).$$

From the equation (6) we deduce that for q fixed

$$\mu_1[0] - \mathscr{L}_{\log J}^q I_{[0]}(\underbrace{00...0}_{s}1..) = \mu[0] - B_q^s = V_q^s = V_{q-1} + V_{q-2}\,p_1^s + V_{q-3}\,p_2^s + ... + V_1\,p_{q-2}^s + U_q^s. \tag{7}$$

Remember that (4) claims

$$\mu\left(\overline{0^s\,1}\right) = \eta_s\,\varphi_1(0^s 1...) = \eta_s\,r(s). \tag{8}$$

For each fixed q we will have to estimate later for each s the value

$$\mu(\overline{0^s\,1})\left[\mathscr{L}_{\log J}^q I_{[0]}(\underbrace{00...0}_{s}1..) - \mu[0]\right] = \eta_s\,r(s)\left[\mathscr{L}_{\log J}^q I_{[0]}(\underbrace{00...0}_{s}1..) - \mu[0]\right] =$$

$$V_{q-1}\,\eta_s\,r(s) + V_{q-2}\,r(1+s)\eta_{1+s} + V_{q-3}\,r(2+s)\eta_{2+s} + ... + V_1\,r(s+q-2)\eta_{s+q-2} - r(s+q)\eta_{s+q}. \tag{9}$$

As mentioned before the Ruelle operator $\mathscr{L}_{\log J}$ is the dual (in the $\mathscr{L}^2(\Omega, \mathscr{B}, \mu_1)$ sense) of the Koopman operator $\mathscr{K}(\varphi) = \varphi \circ \sigma$, using this duality, expressions (9) and (15) we get that for fixed q that

$$\begin{aligned}
\int_\Omega (I_{[0]} \circ \sigma^q)\,[I_{[0]} - \mu_1[0]]\,d\mu &= \int I_{[0]}\,\mathscr{L}_{\log J}^q[I_{[0]} - \mu[0]]\,d\mu \\
&= \int_\Omega I_{[0]}(x)\left[\mathscr{L}_{\log J}^q I_{[0]}(x) - \mu[0]\right]d\mu(x) \\
&= \sum_{s=1}^{\infty}\int_\Omega I_{[0]}(\underbrace{00...0}_{s}1..)\left[\mathscr{L}_{\log J}^q I_{[0]}(\underbrace{00...0}_{s}1..) - \mu[0]\right]d\mu \\
&= \sum_{s=1}^{\infty}\mu[\underbrace{00...0}_{s}1]\left[\mathscr{L}_{\log J}^q I_{[0]}(\underbrace{00...0}_{s}1..) - \mu[0]\right] \\
&= -\sum_{s=1}^{\infty}\mu[0]\,(\,r(s+1)\,\eta_{(s+1)} + r(s+2)\eta_{s+2} + ... + r(s+q-2)\,\eta_{(s+q-2)}\,) \\
&+ \sum_{s=1}^{\infty}V_{q-1}\,\eta_s\,r(s) + ... + V_1\,r(s+q-2)\eta_{s+q-2} - r(s+q)\eta_{s+q} \sim
\end{aligned}$$

$$\sum_{j=1}^{q-1} V_{q-j} \sum_{k=0}^{\infty} \eta_{(k+s+j-1)} - \sum_{s=1}^{\infty}\sum_{k=0}^{\infty} \eta_{(k+q+s)} \sim -\sum_{s=1}^{\infty}\sum_{k=0}^{\infty} \eta_{(k+q+s)}. \tag{10}$$

Note that in the case $\eta_k = e^{-k^{1/2}}$ we get that $r(s)\eta_s$ is of order $\sqrt{s}\,e^{-\sqrt{s}}$ (see Appendix) which goes to zero when $s \to \infty$. Moreover,

$$\sum_{s=1}^{\infty}\sum_{k=0}^{\infty} \eta_{(k+q+s)} \sim q\,e^{-\sqrt{q}}.$$

In the case $\log \eta_n \sim -\gamma \log n$, $n \geq 1$, one easily get from the above that the decay of correlation is of polynomial type $n^{2-\gamma}$.

Now we prove that

$$\mu[0] - \mathscr{L}^q_{\log J}(I_{[0]})(01..) = V_q \sim \sum_{n=1}^{q} \sum_{j=q+1}^{\infty} \eta_j.$$

Using the scheme of smaller trees on the right side we get that the general term of $\mathscr{L}^q_{\log J}(I_{[0]})(01...)$ for $1 \leq j \leq q-1$ is given by the following expression

$$\mathscr{L}^q_{\log J}(I_{[0]})(01\ldots) = \frac{\eta_{(q-1)}}{W(\gamma)} \mathscr{L}^1_{\log J}(I_{[0]})(01\ldots) + \ldots$$

$$+\frac{\eta_2}{W(\gamma)} \mathscr{L}^{q-2}_{\log J}(I_{[0]})(01\ldots) + \frac{1}{W(\gamma)} \mathscr{L}^{q-1}_{\log J}(I_{[0]})(01\ldots) + \frac{\eta_{(q+1)} r(q+1)}{W(\gamma)} \tag{11}$$

Denote $p_q = \frac{\eta_q}{W(\beta)}$, $q \geq 1$, and $\alpha(q) = \frac{\eta_{(q+1)} r(q+1)}{W(\beta)}$ and $K = \mu[0] = 1/2 = \frac{\sum_q \alpha(q)}{\sum_q q\, p_q}$.

Define $V_q = \mu[0] - A(q) = \mu[0] - \mathscr{L}^q_{\log J}(I_{[0]})(01..)$, for $q \geq 1$.

For example, $V_2 = p_1 V_1 + \mu_1[0]\,(p_2 + p_3 + ...) - \alpha(3)$.

We want to obtain the behavior of $V(q)$, when $q \to \infty$. From the renewal equation (11) we get another renewal equation: for $q \geq 3$

$$V_q = \sum_{j=1}^{q-1} V_j p_{q-j} + \left[\mu[0] \sum_{j=q}^{\infty} p_j - \alpha(q)\right]. \tag{12}$$

We denote by K_q the last term on the above equality, that is,

$$K_q = \mu[0] \sum_{j=q}^{\infty} p_j - \alpha(q), \tag{13}$$

$q \geq 1$.

Note that $K_1 = \mu[0] - \alpha(1) = \mu[0] - \mathscr{L}^1_{\log J}(I_{[0]})(01..) = V_1$.

Note also that

$$\alpha(q) = \frac{\sum_{j=q+1}^{\infty} \eta_j}{W}$$

and

$$K_q = 1/2 \sum_{j=q}^{\infty} \frac{\eta_j}{W(\beta)} - \frac{\sum_{j=q+1}^{\infty} \eta_j}{W(\beta)} \sim -\sum_{j=q+1}^{\infty} \eta_j \to 0,$$

when $q \to \infty$.

Now consider the following formal power series

$$f(z) = \sum_{j=1}^{\infty} p_j z^{j-1}, \quad V(z) = \sum_{j=1}^{\infty} V_j z^j, \quad \text{and} \quad K(z) = \sum_{j=1}^{\infty} K_j z^{j-1}.$$

From the renewal equation (12) we get that $V(z)\,f(z) + K(z) = V(z)$. Therefore

$$V(z) = \frac{K(z)}{1-f(z)} = \frac{K(z)}{1-z}\,\frac{1-z}{1-f(z)}.$$

We assume that p_n, $n \in \mathbb{N}$, is such that $f(z)$ is differentiable on $z = 1$ (this is an assumption on η_n, $n \in \mathbb{N}$) and the derivative is not zero.

Up to a bounded multiplicative constant we get

$$V(z) \sim \frac{K(z)}{1-z},$$

from where we obtain (asymptotically)

$$V(z) \sim \sum_{j=1}^{\infty} K_j z^j \ \ (1+z+z^2+z^3+...) =$$

$$K_1 \ + (K_1+K_2)z + (K_1+K_2+K_3)z^2 + (K_1+K_2+K_3+K_4)z^3....$$

In this way we get the following recurrence relation: $V_n \sim K_1+K_2+K_3+...+K_n$.

This argument complete the proof of

$$\mu[0] - \mathscr{L}^q_{\log J}(I_{[0]})(01..) = V_q \sim \sum_{n=1}^{q} \sum_{j=q+1}^{\infty} \eta_j. \tag{14}$$

This goes to zero because $\sum_q q\,\eta_q < \infty$.
When $\eta_k = e^{-k^{1/2}}$, we get that

$$V_k \sim k^2 e^{-k^{1/2}}. \tag{15}$$

In the same way

$$\mu[0] - \mathscr{L}^q_{\log J}(I_{[0]})(10..) \sim \sum_{n=1}^{q} \sum_{j=q+1}^{\infty} \eta_j.. \tag{16}$$

□

6. Appendix

First we consider the case $\eta_k = e^{-k^{1/2}}$ and we will take the estimation (10).
The estimate of $\sum_{k=0}^{\infty} e^{-\sqrt{n+j+k}}$, for n, j fixed, can be done via the integral

$$\int_1^{\infty} e^{-\sqrt{n+j+x}}\, dx.$$

Taking the change of coordinates $y = \sqrt{n+j+x}$, we get the equivalent expression

$$2\int_{\sqrt{n+j+1}}^{\infty} e^{-y}\, y\, dy.$$

As

$$(e^{-y}\, y)\,|_{\sqrt{n+j+1}}^{\infty} = \int_{\sqrt{n+j+1}} \frac{d\,(e^{-y}\, y)}{dy}\, dy = \int e^{-y}\, dy - \int y\, e^{-y}\, dy,$$

we get that $\int_1^{\infty} e^{-\sqrt{n+j+x}}\, dx$ is of order $\sqrt{n+j+1}\, e^{-\sqrt{n+j+1}}$.
Now for n fixed, we want to estimate the expression $\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} e^{-\sqrt{n+j+k}} = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \eta_{n+j+k}$.
In order to to that we consider the integral

$$\int_1^{\infty} \sqrt{n+x}\, e^{-\sqrt{n+x}}\, dx.$$

Considering the change of coordinates $y = \sqrt{n+x}$ we get that

$$(e^{-y}\, y^2)\,|_{\sqrt{n+1}}^{\infty} = \int_{\sqrt{n+1}} \frac{d\,(e^{-y}\, y^2)}{dy}\, dy = 2\int_{\sqrt{n+1}} e^{-y}\, y\, dy - \int y^2\, e^{-y}\, dy,$$

and finally

$$\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \eta_{n+j+k} \sim \int_1^{\infty} \sqrt{n+x}\, e^{-\sqrt{n+x}}\, dx \sim n\, e^{-\sqrt{n}}.$$

In a similar way, if $\eta_k = e^{-k^{1-\frac{\log 2}{\log 5}}}$ we get a similar result, that is, the decay is faster than polynomial.

Indeed, $\frac{\log 2}{\log 5} = 0.4306$, and then $1 - \frac{\log 2}{\log 5} > 0.5$. From this follows that $e^{-k^{1-\frac{\log 2}{\log 5}}} \leq e^{-k^{1/2}}$. Therefore, for the case of the fixed point V for the renormalization operator when $k = 5$ and $l = 2$, we get that the associated equilibrium state has a decay faster than $n\,e^{-\sqrt{n}}$.

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