Haar systems, KMS states on von Neumann algebras and C^*-algebras on dynamically defined groupoids and Noncommutative Integration

G. G. de Castro, A. O. Lopes and G. Mantovani

March 17, 2022

Abstract

We analyse Haar systems associated to groupoids obtained by certain equivalence relations of dynamical nature on sets like $\{1, 2, ..., d\}^{\mathbb{Z}}$, $\{1, 2, ..., d\}^{\mathbb{N}}$, $S^1 \times S^1$, or $(S^1)^{\mathbb{N}}$, where S^1 is the unitary circle. We also describe properties of transverse functions, quasi-invariant probabilities and KMS states for some examples of von Neumann algebras (and also C^*-Algebras) associated to these groupoids. We relate some of these KMS states with Gibbs states of Thermodynamic Formalism. We will show new results but we will also describe in detail several examples and basic results on the above topics. Some known results on non-commutative integration are presented, more precisely, the relation of transverse measures, cocycles and quasi-invariant probabilities.

We describe the results in a language which is more familiar to the people in Dynamical Systems. Our intention is to study Haar systems, quasi-invariant probabilities and von Neumann algebras as a topic on measure theory (intersected with ergodic theory) avoiding questions of algebraic nature (which, of course, are also extremely important).

1 The groupoid associated to a partition

We will analyze properties of Haar systems, quasi-invariant probabilities, transverse measures, C^*-algebras and KMS states related to Thermodynamic Formalism and Gibbs states. We will consider a specific particular setting where the groupoid will be defined by some natural equivalence relations on the sets of the form $\{1,2,...,d\}^{\mathbb{N}}$ or $\{1,2,...,d\}^{\mathbb{Z}}$, $S^1 \times S^1$, or $(S^1)^{\mathbb{N}}$. These equivalence relations will be of dynamic origin.

We will consider only the so called subgroupoids of pair groupoids (see [68]).

We will denote by X any one of the above sets.

The main point here is that we will use a notation which is more close to the one used on Ergodic Theory and Thermodynamic Formalism.

On section 2 we introduce the concept of transverse functions associated to groupoids and Haar systems.

On section 3 we consider modular functions and quasi-invariant probabilities on groupoids. In the end of this section we present a new result concerning a (non-)relation of the quasi-invariant probability with the SBR probability of the generalized Baker map.

On section 4 we consider a certain von Neumann algebra and the associated KMS states. On proposition 54 we present a new result concerning the relation between probabilities satisfying the KMS property (quasi-invariant) and Gibbs (DLR) probabilities of Thermodynamic Formalism on the symbolic space $\{1,2,...,d\}^{\mathbb{N}}$ for a certain groupoid. Proposition 55 shows that the KMS probability is not unique on this case.

[25], [23] and [24] are the classical references on measured groupoids and von Neumann algebras. KMS states and C^*-algebras are described on [52] and [46].

On section 5 we present a natural expression - based on quasi-invariant probabilities - for the integration of a transverse function by a transverse measure. Some basic results on non-commutative integration (see [12] for a detailed description of the topic) are briefly described.

On section 6 we present briefly the setting of C^*-algebras associated to groupoids on symbolic spaces. We present the well known and important concept of approximately proper equivalence relation and its relation with the direct inductive limit topology (see [17], [18], [19] and [56]).

On section 7 we present several examples of quasi-invariant probabilities for different kinds of groupoids and Haar systems.

Results on C^*-algebras and KMS states from the point of view of Thermodynamic Formalism are presented in [29], [56], [50], [61], [62], [1], [39], [21] and [22].

The paper [7] considers equivalence relations and DLR probabilities for certain interactions on the symbolic space $\{1,2,...d\}^{\mathbb{Z}}$ (not in $\{1,2,...d\}^{\mathbb{N}}$ like here).

Theorem 6.2.18 in Vol II of [9] and [4] describe the relation between KMS states and Gibbs probabilities for interactions on certain spin lattices (on the one-dimensional case corresponds to the space $\{1,2,..,d\}^{\mathbb{Z}}$).

We point out that Lecture 9 in [15] presents a brief introduction to C^*-Algebras and non-commutative integration.

In [34] and [35] are presented generalizations of some of the results (of sections 4 and section 5) described here.

We are indebted to Ruy Exel and Marcos Sebastiani for many helpful discussions and useful comments during the procedure of writing this paper. We also thanks Ali Tahzibi for some fruitful remarks.

We denote $\{1,2,...,d\}^{\mathbb{N}} = \Omega$ and consider the compact metric space with metric d where for $x = (x_0, x_1, x_2, ..) \in \Omega$ and $y = (y_0, y_1, y_2, ..) \in \Omega$

$$d(x,y) = 2^{-N},$$

where N is the smallest natural number $j \geq 0$, such that, $x_j \neq y_j$.

We also consider $\{1,2,...,d\}^{\mathbb{Z}} = \hat{\Omega}$ and elements in $\hat{\Omega}$ are denoted by $x = (..., x_{-n}, ..., x_{-1} \,|\, x_0, x_1, .., x_n, ..)$.

We will use the notation $\overleftarrow{\Omega} = \{1,2,...,d\}^{\mathbb{N}}$ and $\overrightarrow{\Omega} = \{1,2,...,d\}^{\mathbb{N}}$.

Given $x = (..., x_{-n}, ..., x_{-1} \,|\, x_0, x_1, .., x_n, ..) \in \hat{\Omega}$, we call $(..., x_{-n}, ..., x_{-1}) \in \overleftarrow{\Omega}$ the past of x and $(x_0, x_1, .., x_n, ..) \in \overrightarrow{\Omega}$ the future of x.

In this way we express $\hat{\Omega} = \overleftarrow{\Omega} \times | \overrightarrow{\Omega}$.

Sometimes we denote

$$(..., a_{-n}, ..., a_{-1} \,|\, b_0, b_1, .., b_n, ..) = < a \,|\, b >,$$

where $a = (..., a_{-n}, ..., a_{-1}) \in \overleftarrow{\Omega}$ and $b = (b_0, b_1, .., b_n, ..) \in \overrightarrow{\Omega}$.

On $\hat{\Omega}$ we consider the usual metric d, in such way that for $x, y \in \hat{\Omega}$ we set

$$d(x,y) = 2^{-N},$$

$N \geq 0$, where for

$$x = (..., x_{-n}, ..., x_{-1} \,|\, x_0, x_1, .., x_n, ..) \;\; , y = (..., y_{-n}, ..., y_{-1} \,|\, y_0, y_1, .., y_n, ..),$$

we have $x_j = y_j$, for all j, such that, $-N+1 \leq j \leq N-1$ and, moreover $x_N \neq y_N$, or $x_{-N} \neq y_{-N}$.

The shift $\hat{\sigma}$ on $\hat{\Omega} = \{1,2,...,d\}^{\mathbb{Z}}$ is such that

$$\hat{\sigma}(..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...) = (..., y_{-n}, ..., y_{-2}, y_{-1}, y_0 \,|\, y_1, ..., y_n, ...).$$

On the other hand the shift σ on $\Omega = \{1,2,...,d\}^{\mathbb{N}}$ is such that

$$\sigma(y_0, y_1, ..., y_n, ...) = (y_1, ..., y_n, ...).$$

A general equivalence relation R on a space X define classes and we will denote by $x \sim y$ when two elements x and y are on the same class. We denote by $[y]$ the class of $y \in X$.

Definition 1. *Given an equivalence relation $\sim$ on X, where X is any of the sets Ω, $\hat{\Omega}$, $(S^1)^{\mathbb{N}}$, or $S^1 \times S^1$, we denote by G the subset of $X \times X$, containing all pairs (x, y), where $x \sim y$. We call G the groupoid associated to the equivalence relation $\sim$.*

We also denote by G^0 the set $\{(x, x) \,|\, x \in X\} \sim X$, where X denote any of the sets Ω, $\hat{\Omega}$, $(S^1)^{\mathbb{N}}$, or $S^1 \times S^1$.

Remark: There is a general definition of groupoid (see [12]) which assumes more structure but we will not need this here. For all results we will consider there is no need for an additional algebraic structure (on the class of each point). In this way we can consider a simplified definition of groupoid as it is above. Our intention is to study C^*-algebras and Haar systems as a topic on measure theory (intersected with ergodic theory) avoiding questions of algebraic nature.

There is a future issue about the topology we will consider induced on G. One possibility is the product topology, which we call the standard structure, or, a more complex one which will be defined later on section 6 (specially appropriate for some C^*-algebras).

We will present several examples of dynamically defined groupoids. The equivalence relation of most of our examples is proper (see definition 81).

Example 2. *For example consider on $\{1, 2, ..., d\}^{\mathbb{N}}$ the equivalence relation R such that $x \sim y$, if $x_j = y_j$, for all $j \geq 2$, when $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$. This defines a groupoid G. In this case $G^0 = \Omega = \{1, 2, ..., d\}^{\mathbb{N}}$.*

For a fixed $x = (x_1, x_2, x_3, ...)$ the equivalence class associated to x is the set $\{\, (j, x_2, x_3, ...),\ j = 1, 2.., d \,\}$. We call this relation the **bigger than two relation**.

Example 3. *Consider an equivalence relation R which defines a partition η_0 of $\{1, 2, ..., d\}^{\mathbb{Z}-\{0\}} = \hat{\Omega}$ such its elements are of the form*

$$a \times |\, \overrightarrow{\Omega} = a \times \{1, 2, ..., d\}^{\mathbb{N}} = (..., a_{-n}, ..., a_{-2}, a_{-1}) \times |\, \{1, 2, ..., d\}^{\mathbb{N}},$$

where $a \in \{1, 2, ..., d\}^{\mathbb{N}} = \overleftarrow{\Omega}$. This defines an equivalence relation $\sim$.

In this way two elements x and y are related if they have the same past.

There exists a bijection of classes of η_0 and points in $\overleftarrow{\Omega}$.

Denote $\pi = \pi_2 : \hat{\Omega} \to \overleftarrow{\Omega}$ the transformation such that takes a point and gives as the result its class.

In this sense

$$\pi^{-1}(x) = \pi^{-1}((..., x_{-n}, ..., x_{-1} \,|\, x_1, .., x_n, ..))) =$$

$$(..., x_{-n}, ..., x_{-2}, x_{-1}) \times \; | \overrightarrow{\Omega} \cong \Omega.$$

The groupoid obtained by this equivalence relation can be expressed as $G = \{(x, y), \pi(x) = \pi(y)\}$. *In this way* $x \sim y$ *if they have the same past.*

In this case the number of elements in each fiber is not finite.

Using the notation of page 46 of [12] we have $Y \subset \hat{\Omega} \times \hat{\Omega}$ *and* $X = \overleftarrow{\Omega}$.

In this case each class is associated to certain $a = (a_{-1}, a_{-2}, ..) \in \overleftarrow{\Omega} = \{1, 2, ..., d\}^{\mathbb{N}}$..

We use the notation $(a \,|\, x)$ *for points on a class of the form*

$$(a \,|\, x) = (..., a_{-n}, ..., a_{-2}, a_{-1} \,|\, x_1, ..., x_n, ...) \; .$$

Example 4. *A particulary important equivalence relation* R *on* $\hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}}$ *is the following: we say* $x \sim y$ *if*

$$x = (..., x_{-n}, ..., x_{-2}, x_{-1} \,|\, x_0, x_1, ..., x_n, ...),$$

and,

$$y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...)$$

are such that there exists $k \in \mathbb{Z}$, *such that,* $x_j = y_j$, *for all* $j \leq k$.

The groupoid G_u *is defined by this relation* $x \sim y$.

By definition the unstable set of the point $x \in \hat{\Omega}$ *is the set*

$$W^u(x) = \{y \in \hat{\Omega}\,,\ \textit{such that}\ \lim_{n\to\infty} d(\,\hat{\sigma}^{-n}(x), \hat{\sigma}^{-n}(y)\,)\ = 0\,\}.$$

One can show that the unstable manifold of $x \in \hat{\Omega}$ *is the set*

$$W^u(x) = \{y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...) \,|\, \textit{there exists}$$

$$k \in \mathbb{Z},\ \textit{such that}\ x_j = y_j, \textit{for all}\ j \leq k\}.$$

If we denote by G_u *the groupoid defined by the above relation, then,* $x \sim y$, *if and only if,* $y \in W^u(x)$.

An equivalence relation of this sort - for hyperbolic diffeomorphism - was considered on [58] and [38].

Example 5. *An equivalence relation on* $\overrightarrow{\Omega} = \{1, 2, ..., d\}^{\mathbb{N}}$ *similar to the previous one is the following: we say* $x \sim y$ *if*

$$x = (\, x_0, x_1, ..., x_n, ...),$$

and,

$$y = (y_0, y_1, ..., y_n, ...)$$

are such that there exists $k \in \mathbb{N}$*, such that,* $x_j = y_j$*, for all* $j \geq k$*.*

Example 6. *Another equivalence relation on* $\overrightarrow{\Omega}$ *is the following: fix* $k \in \mathbb{N}$*, and we say* $x \sim_k y$*, if when*

$$x = (\, x_0, x_1, ..., x_n, ...),$$

and,

$$y = (y_0, y_1, ..., y_n, ...)$$

we have $x_j = y_j$*, for all* $j \geq k$*.*

In this case each class has d^k *elements.*

Example 7. *Given* $x, y \in \hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}}$*, we say that* $x \sim y$ *if*

$$\lim_{k \to +\infty} d(\hat{\sigma}^k x, \hat{\sigma}^k y) = 0$$

and

$$\lim_{k \to -\infty} d(\hat{\sigma}^k x, \hat{\sigma}^k y) = 0. \qquad (1)$$

This means there exists an $M \geq 0$*, such that,* $x_j = y_j$ *for* $j > M$*, and,* $j < -M$*. In other words, there are only a finite number of* i*'s such that* $x_i \neq y_i$*. This is the same to say that* x *and* y *are homoclinic.*

For example in $\hat{\Omega} = \{1, 2\}^{\mathbb{Z}}$ *take*

$$x = (..., x_{-n}, ..., x_{-7}, 1, 2, 2, 1, 2, 2 \,|\, 1, 2, 1, 2, 1, 1, x_7, ...x_n, ..)$$

and

$$y = (..., y_{-n}, ..., y_{-7}, 1, 2, 2, 1, 2, 2 \,|\, 1, 2, 1, 1, 1, 2, y_7, ...y_n, ..)$$

where $x_j = y_j$ *for* $|j| \geq 7$*.*

In this case $x \sim y$*.*

This relation is called the **homoclinic relation** *on* $\hat{\Omega}$*. It was considered for instance by D. Ruelle and N. Haydn in [57] and [26] for hyperbolic diffeomorphisms and also on more general contexts (see also [33], [43] and [7] for the symbolic case).*

Example 8. *Consider an expanding transformation $T : S^1 \to S^1$, of degree two, such that $\log T'$ is Holder and $\log T'(a) > \log \lambda > 0$, $a \in S^1$, for some $\lambda > 1$.*

Suppose $T(x_0) = 1$, where $0 < x_0 < 1$. We say that $(0, x_0)$ and $(x_0, 1)$ are the domains of injectivity of T.

Denote $\psi_1 : [0, 1) \to [0, x_0)$ the first inverse branch of T and $\psi_2 : [0, 1) \to [x_0, 1]$ the second inverse branch of T.

In this case for all y we have $T \circ \psi_1(y) = y$ and $T \circ \psi_2(y) = y$.

The **associated T-Baker map** *is the transformation $F : S^1 \times S^1$ such that satisfies for all a, b the following rule:*

1) if $0 \leq b < x_0$

$$F(a, b) = (\psi_1(a), T(b)),$$

and

2) if $x_0 \leq b < 1$

$$F(a, b) = (\psi_2(a)\,, T(b)).$$

In this case we take as partition the one associated to (local) unstable manifolds for F, that is, sets of the form $W_a = \{\,(a, b)\,|\, b \in S^1\}$, where $a \in S^1$.

Given two points $z_1, z_2 \in S^1 \times S^1$ we say that they are related if the the first coordinate is equal.

On $S^1 \times S^1$ we use the distance d which is the product of the usual arc length distance on S^1.

The bijection F expands vertical lines and contract horizontal lines.

As an example one can take $T(a) = 2a$ (mod 1) and we get (the inverse of) the classical Baker map (see [59]).

One can say that the dynamics of such F in some sense looks like the one of an Anosov diffeomorphism.

2 Kernels and transverse functions

A general reference for the material of this section is [12] (see also [27]).

We consider over $G \subset X \times X$ the Borel sigma-algebra $\mathcal{B}$ (on G) induced by the natural product topology on $X \times X$ and the metric d on X ([23] and [24] also consider this sigma algebra). This will be fine for the setting of von Neumann algebras. Later, another sigma-algebra will be considered for the setting C^*-algebras.

We point out that the only sets X which we are interested are of the form $\hat{\Omega}$, Ω, $(S^1)^{\mathbb{N}}$, or $S^1 \times S^1$.

We denote $\mathcal{F}^+(G)$ the space of Borel measurable functions $f : G \to [0, \infty)$ (a function of two variables (a, b)).

$\mathcal{F}(G)$ is the space of Borel measurable functions $f : G \to \mathbb{R}$. Note that $f(x, y)$ just make sense if $x \sim y$.

There is a natural involution on $\mathcal{F}(G)$ which is $f \to \tilde{f}$, where $\tilde{f}(x, y) = f(y, x)$.

We also denote $\mathcal{F}^+(G^0)$ the space of Borel measurable functions $f : G^0 \to [0, \infty)$ (a function of one variable a).

There is a natural identification of functions $f : G^0 \to \mathbb{R}$, of the form $f(x)$, with functions $g : G \to \mathbb{R}$ which depend only on the first coordinate, that is $g(x, y) = f(x)$. This will be used without mention, but if necessary we write $(f \circ P_1)(x, y) = f(x)$ and $(f \circ P_2)(x, y) = f(y)$.

Definition 9. *A* **measurable groupoid** *G is a groupoid with the topology induced by the product topology over $X \times X$, such that, the following functions are measurable for the Borel sigma-algebra:*

$P_1(x, y) = x,\ P_2(x, y) = y,\ h(x, y) = (y, x)$ *and* $Z(\,(x, s), (s, y)\,) = (x, y)$, *where* $Z : \{\,(\,(x, s), (r, y)\,) \mid r = s\,\} \subset G \times G \to G$.

Now, we will present the definition of kernel (see beginning of section 2 in [12]).

Definition 10. *A* **G-kernel** *ν on the measurable groupoid G is an application of G^0 in the space of measures over the sigma-algebra $\mathcal{B}$, such that,*

1) for any $y \in G^0$, we have that ν^y has support on $[y]$,

and

2) for any $A \in \mathcal{B}$, the function $y \to \nu^y(A)$ is measurable.

The set of all G- kernels is denoted by $\mathcal{K}^+$.

Example 11. *As an example consider for the case of the groupoid G associated to the bigger than two relation, the measure ν^y, for each $y = (y_1, y_2, y_3, ...)$, such that $\nu(j, y_2, y_3, ...) = 1,\ j = 1, 2, ..., d$. In other words we are using the counting measure on each class. We call this the standard G-kernel for the the bigger than two relation.*

More precisely, the **counting measure** *is such that $\nu^y(A) = \#(A \cap [y])$, for any $A \in \mathcal{B}$.*

Example 12. *Another possibility is to consider the G-kernel such that ν^y, for each $y = (y_1, y_2, y_3, ...)$, is such that $\nu(j, y_2, y_3, ...) = \frac{1}{d}$, We call this the normalized standard G-kernel for the the bigger than two relation.*

Example 13. *Given any groupoid G another example of kernel is the* **delta kernel** *ν which is the one such that for any $y \in G^0$ we have that $\nu^y(dx) = \delta_y(d\,x)$, where δ_y is the delta Dirac on y. We denote by $\mathfrak{d}$ such kernel.*

We denote by $\mathcal{F}_\nu(G)$ the set of ν-integrable functions.

Definition 14. *Given a G-kernel ν and an integrable function $f \in \mathcal{F}_\nu(G)$ we denote by $\nu(f)$ the function in $\mathcal{F}(G^0)$ defined by*

$$\nu(f)\ (y) = \int f\,(s, y)\,\nu^y(ds), \;\; y \in G^0.$$

A kernel ν is characterized by the law

$$f \in \mathcal{F}_\nu(G) \;\; \rightarrow \;\; \nu(f) \in \mathcal{F}(G^0).$$

In other words, for a kernel ν we get

$$\nu :\ \mathcal{F}_\nu(G) \rightarrow \mathcal{F}(G^0).$$

By notation given a kernel ν and a positive $f \in \mathcal{F}_\nu(G)$ then the kernel $f\,\nu$ is the one defined by $f(x, y)\nu^y(dx)$. In other words the action of the kernel $f\,\nu$ get rid of the first coordinate:

$$h(x, y) \;\; \rightarrow \;\; \int h(s, y) f\,(s, y)\,\nu^y(ds).$$

In this way if $f \in \mathcal{F}_\nu(G^0)$ we get $f(x)\nu^y(dx)$.

Note that $\nu(f)$ is a function and $f\,\nu$ is a kernel.

Definition 15. *A* **transverse function** *is a G-kernel ν, such that, if $x \sim y$, then, the finite measures ν^y and ν^x are the same. The set of transverse functions for G is denoted by $\mathcal{E}^+$. We call probabilistic transverse function any one such that for each $y \in G^0$ we get that ν^y is a probability on the class of y.*

The above means that

$$\int f(a)\nu^x(d\,a) = \int f(a)\nu^y(d\,a),$$

if x and y are related. In the above we have $x \sim y \sim a$.

Remark 16. *The above equality implies that a transverse function is left (and right) invariant.*

The concept of transverse function considered here is a particular version of the general definition presented in [12].

The standard G-kernel for the bigger than two relation (see example 11) is a transverse function.

The normalized standard G-kernel for the the bigger than two relation (see example 12) is a probabilistic transverse function.

If we consider the equivalence relation such that each point is related just to itself, then the transverse functions can be identified with the positive functions defined on X.

The difference between a function and a transverse function is that the former takes values on the set of real numbers and the later on the set of measures.

If ν is transverse, then $\nu^x = \nu^y$ when $x \sim y$, and we have from definition 59:

$$(\nu * f)(x,y) = \int f\,(x,s)\,\nu^x(ds) = \nu(\tilde{f})(x), \;\; \forall y \sim x \tag{2}$$

and,

$$(f * \nu)(x,y) = \int f\,(s,y)\,\nu^y(ds) = \nu(f)(y), \;\; \forall x \sim y. \tag{3}$$

Definition 17. *The pair (G,ν), where $\nu \in \mathcal{E}^+$, is called the* **measured groupoid for the transverse function** *ν or a* **Haar System**. *We assume any ν we consider is such that ν^y is not the zero measure for any y.*

In the case ν is such that, $\int \nu^y(ds) = 1$, for any $y \in G^0$, the Haar system will be called a probabilistic Haar system.

Note that the delta kernel $\mathfrak{d}$ is not a transverse function.

Given a measured groupoid (G,ν) and two measurable functions $f, g \in \mathcal{F}_\nu(G)$, we define $(f \underset{\nu}{*} g) = h$ in such way that for any $(x,y) \in G$

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\,f(s,y)\;\nu^y(ds) = h(x,y).$$

$(f \underset{\nu}{*} g)$ is called the **convolution** of the functions f, g for the measured groupoid (G,ν).

Example 18. *Consider the groupoid G of Example 2 and the family ν^y, $y \in \{1,2,...,d\}^{\mathbb{N}}$, of measures (where each measure ν^y has support on the equivalence class of y), such that, ν^y is the counting measure. This defines a transverse function (Haar system) called the* **standard Haar system**.

Example 19. *Consider the groupoid G of Example 2 and the normalized standard family ν^y, $y \in \{1,2,...,d\}^{\mathbb{N}}$. This defines a transverse function called the* **normalized standard Haar system**.

More precisely the family ν^y, $y \in \{1,2,...,d\}^{\mathbb{N}}$, $y = (y_1, y_2, y_3, ...)$, of probabilities on the set

$$\{\, (a, y_2, y_3, ...), a \in \{1,2,..,d\} \,\},$$

is such that, $\nu^y(\, \{(a, y_2, y_3, ...)\,\}) = \frac{1}{d}$, $a \in \{1,2,..,d\}$

Example 20. *In example 6 in which k is fixed consider the transverse function ν such that for each $y \in G^0$, we get that ν^y is the counting measure on the set of points $x \sim_k y$.*

Example 21. *Suppose $J : \{1,2,...,d\}^{\mathbb{N}} \to \mathbb{R}$ is continuous positive function such that for any $x \in \Omega$ we have that $\sum_{a=1}^{d} J(ax) = 1$. For the groupoid G of Example 2, the family ν^y, $y \in \{1,2,...,d\}^{\mathbb{N}}$, of probabilities on $\{\, (a, y_2, y_3, ...), a \in \{1,2,..,d\} \,\}$, such that, $\nu^y(a, y_2, y_3, ...) = J(a, y_2, y_3, ...)$, $a \in \{1,2,..,d\}$ defines a Haar system. We call it the* **probability Haar system associated to** J.

Example 19 is a particular case of the present example.

Example 22. *On the groupoid over $\{1,2,...,d\}^{\mathbb{Z}}$ described on example 3, where we consider the notation: for each class specified by $a \in \overleftarrow{\Omega}$ the general element in the class is given by*

$$(a \,|\, x) = (..., a_{-n}, ..., a_{-2}, a_{-1} \,|\, x_1, ..., x_n, ...),$$

where $x \in \overrightarrow{\Omega}$.

Consider a fixed probability μ on $\overrightarrow{\Omega}$. We define the transverse function $\nu^a(dx) = \mu(dx)$ independent of a.

Example 23. *We will show that the above defined concept of convolution generalize (in some sense) the product of matrices. Consider over the set $G^0 = \{1,2..,d\}$ the equivalence relation where all points are related. In this case $G = \{1,2..,d\} \times \{1,2..,d\}$. Take ν as the counting measure. A function $f : G \to \mathbb{R}$ is denoted by $f(i,j)$, where $i \in \{1,2..,d\}, j \in \{1,2..,d\}$.*

In this case the set of functions $f : G \to \mathbb{R}$ can be identified with the set of d by d matrices with real entries. A matrix A can be identified with $A = (f_{i,j})_{i\in\{1,2..,d\},j\in\{1,2..,d\}}$.

The convolution product is

$$(f \underset{\nu}{*} g)(i,j) = \sum_k g(i,k)\, f(k,j).$$

The convolution is just the product of matrices.

Example 24. *The so called generalized XY model consider space $(S^1)^{\mathbb{N}}$, where S^1 is the unitary circle and the shift acting on it (see).*

We can consider the equivalence relation R such that $x \sim y$, if $x_j = y_j$, for all $j \geq 2$, when $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$. This defines a groupoid G. In this case $G^0 = (S^1)^{\mathbb{N}}$.

For a fixed $x = (x_1, x_2, x_3, ...)$ the equivalence class associated to x is the set $\{\, (a, x_2, x_3, ...),\ a \in S^1 \,\}$. We call this relation the **bigger than two relation for the XY model** *and G the* **standard XY groupoid** *over $(S^1)^{\mathbb{N}}$.*

Given the class $\{(x, x_2, x_3, ..),\ x \in S^1\}$, where $(x_2, x_3, ..)$ is fixed, the transverse function could be dx for instance.

We refer the reader to [6], [36], [40] and [44] for general results on the Thermodynamic Formalism for the XY model. We point out that as in this example the cardinality of each set is not countable (and the transverse is dx in each class) it will be natural to consider a Ruelle operator with an a priori probability equal to dx. The dynamics is given by the shift. These papers are useful for the understanding of Haar systems in such type of groupoids. This claim is related to the future Theorem 54(see [35])

Example 25. *In the example 8 we consider the partition of $S^1 \times S^1$ given by the sets $W_a = \{\, (a,b) \,|\, b \in S^1\}$, where $a \in S^1$. For each $a \in S^1$, consider a probability $\nu^a(d\,b)$ over $\{\, (a,b) \,|\, b \in S^1\}$ such that for any Borel set $K \subset S^1 \times S^1$ we have that $a \to \nu^a(K)$ is measurable. This defines a probabilistic transverse function and a Haar system.*

Consider a continuous function $A : S^1 \times S^1 \to \mathbb{R}$. For each a consider the kernel ν^a such that $\int f(b)\nu^a(db) = \int f(b)\, e^{A(a,b)} db$, where db is the Lebesgue measure. This defines a transverse function.

We call the **standard Haar system on $\mathbf{S^1} \times \mathbf{S^1}$** *the case where for each a we consider as the probability $\nu^a(d\,b)$ over $\{\, (a,b) \,|\, b \in S^1\}$ the Lebesgue probability on S^1.*

We will present several properties of kernels and transverse functions on Section 5.

A question of notation: for a fixed groupoid G we will describe now for the reader the common terminology on the field (see [12], [27], [53] and [56]). It is usual to denote a general pair $(x, y) \in G$ by γ (of related elements x, y). The γ is called the directed arrow from x to y. In this case we call $s(\gamma) = x$ and $r(\gamma) = y$ (see [45] for a more detailed description of the arrow's setting).

Here, for each pair of related elements (x, y) there exist an unique directed arrow γ satisfying $s(\gamma) = x$ and $r(\gamma) = y$. Note that, since se are dealing with equivalence relations, (y, x) denotes another arrow. In category language: there is a unique morphism γ that takes $\{x\}$ to $\{y\}$, whenever x and y are related, and this morphism is associated in a unique way to the pair (x, y).

In this notation $r^{-1}(y)$ is the set of all arrows that end in y. This is in a bijection with all elements on the same class of equivalence of y. We call $r^{-1}(y)$ the fiber over y. If $x \sim y$, then $r^{-1}(y) = r^{-1}(x)$.

We adapt the notation in [12] and [27] to our notation. We use here the expression (s, y) instead of $\gamma\, \gamma'$. This makes sense considering that $\gamma = (x, y)$ and $\gamma' = (s, x)$. We use the expression (y, s) for $(\gamma')^{-1}\, \gamma$, where in this case, $\gamma = (x, y)$ and $\gamma' = (s, y)$, and, finally, $\nu^y(\gamma')$ means $\nu^y(ds)$ for $\gamma' = (s, y)$.

In the case of the groupoid G associated to the bigger than two relation we have for each $x = (x_1, x_2, x_3, ...)$ the property $r^{-1}(x) = \{\, (j, x_2, x_3, ...),$ $j = 1, 2.., d\,\}$.

The terminology of arrows will not be essentially used here. It was introduced just for the reader to make a parallel (a dictionary) with the one commonly used on papers on the topic.

Using the terminology of arrows Definition 15 is equivalent to say that: if, $\gamma = (x, y) = (s(\gamma), r(\gamma))$, then,

$$\nu^y = \gamma\, \nu^x.$$

Related results on Haar systems and transverse functions appear in [34], [35], [37].

3 Quasi-invariant probabilities

Definition 26. *A function $\delta : G \to \mathbb{R}$ such that*

$$\delta(x, z) = \delta(x, y)\, \delta(y, z),$$

for any (x, y), $(y, z) \in G$ is called a **modular function** *(also called a multiplicative* **cocycle***).*

In the arrow notation this is equivalent to say that

$$\delta(\gamma_1\gamma_2) = \delta(\gamma_1)\delta(\gamma_2).$$

Note that $\delta(x,y)\,\delta(y,y) = \delta(x,y)$ and it follows that for any y we have $\delta(y,y) = 1$. Moreover, $\delta(x,y)\,\delta(y,x) = \delta(x,x) = 1$ is true. Therefore, we get $\tilde{\delta} = \delta^{-1}$.

Example 27. *Given $W : G^0 \to \mathbb{R}$, $W(x) > 0, \forall x$, a natural way to get a modular function is to consider $\delta(x,y) = \frac{W(x)}{W(y)}$. In this case we say that the modular function is derived from W.*

Example 28. *In the case of example 8 the equivalence relation is: given two points $z_1, z_2 \in S^1 \times S^1$ they are related if the first coordinate is equal.*

Consider a expanding transformation T and the associated Baker map F. Note que $F^n(a,b) = (*, T^n(b))$ for some point $*$.

Given two points $z_1 \sim z_2$, for each n there exist z_1^n and z_2^n, such that, respectively, $F^n(z_1^n) = z_1$ and $F^n(z_2^n) = z_2$, and $z_1^n \sim z_2^n$.

For each pair $z_1 = (a, b_1)$ and $z_2 = (a,\, b_2)$, and $n \geq 0$, the elements z_1^n, z_2^n are of the form $z_1^n = (a^n, b_1^n)$, $z_2^n = (a^n, b_2^n)$.

In this case $T^n(b_1^n) = b_1$ and $T^n(b_2^n) = b_2$.

Note also that $T^n(a) = a^n$.

The distances between b_1^n and b_2^n are exponentially decreasing with n.

We denote

$$\delta(z_1, z_2) = \Pi_{j=1}^{\infty}\ \frac{T\,'(b_1^n)}{T\,'(b_2^n)} < \infty.$$

This product is well defined because

$$\sum_n \log \frac{T\,'(b_1^n)}{T\,'(b_2^n)} = \sum_n \left[\log T\,'(b_1^n)\ -\ \log T\,'(b_2^n)\right]$$

converges. This is so because $\log T\,'$ is Holder and for all n we have $|b_1^n - b_2^n| < \lambda^{-n}$, where $T'(x) > \lambda > 1$ for all x.

This δ is a cocycle.

In the case of example 25 considered a Holder continuous function $A(a,b)$, where $A : S^1 \times S^1 \to \mathbb{R}$.

Define for $z_1 = (a, b_1)$ and $z_2 = (a,\, b_2)$

$$\delta(z_1, z_2) = \Pi_{j=1}^{\infty}\ \frac{e^{A(z_1^n)}}{e^{A(z_2^n)}}.$$

The modular function $\delta(z_1, z_2)$ is well defined because A is Holder.

We will show that δ can be expressed in the form of example 27. Indeed, fix a certain $b_0 \in S^1$, then, taking $z_1 = (a, b_1)$ consider $z_0 = (a, b_0)$. We denote in an analogous way z_1^n and z_0^n the ones such that $F^n(z_1^n) = z_1$ and $F^n(z_0^n) = z_0$.

Define $V : G^0 \to \mathbb{R}$ by

$$V(z_1) = \Pi_{j=1}^{\infty} \frac{e^{A(z_1^n)}}{e^{A(z_0^n)}}. \tag{4}$$

V is well defined and if $z_1 \sim z_2$ we get that

$$\delta(z_1, z_2) = \frac{V(z_1)}{V(z_2)}.$$

We will show later (see Proposition 97) that $V(a, b)$ does not depend on a, and then we can write $V(b)$, and finally

$$\delta(z_1, z_2) = \frac{V(b_1)}{V(b_2)}.$$

Example 29. *Consider a fixed Holder function $\hat{A} : \{1, 2, ..., d\}^{\mathbb{Z}} \to \mathbb{R}$ and the groupoid given by the equivalence relation of example 4. Denote for any (x, y)*

$$\delta(x, y) = \Pi_{j=1}^{\infty} \frac{\hat{A}(\hat{\sigma}^{-j}(s(\gamma)))}{\hat{A}(\hat{\sigma}^{-j}(r(\gamma)))} = \Pi_{j=1}^{\infty} \frac{\hat{A}(\hat{\sigma}^{-j}(x))}{\hat{A}(\hat{\sigma}^{-j}(y))}.$$

The modular function δ is well defined because $\hat{A}$ is Holder. Indeed, this follows from the bounded distortion property.

In a similar way as in the last example one can show that such δ can be expressed on the form of example 27.

Definition 30. *Given a measured groupoid G for the transverse function ν we say that a probability M on G^0 is* **quasi-invariant** *for ν if there exist a modular function $\delta : G \to \mathbb{R}$, such that, for any integrable function $f : G \to \mathbb{R}$ we have*

$$\int \int f(s, x) \nu^x(ds) dM(x) = \int \int f(x, s)\, \delta^{-1}(x, s)\, \nu^x(ds) dM(x). \tag{5}$$

In a more accurate way we say that M is quasi-invariant for the transverse function ν and the modular function δ.

For the existence of quasi-invariant probabilities see [41] and [42].

Note that if $\delta(x,s) = \frac{B(x)}{B(s)}$ we get that the above condition (5) can be written as

$$\int \int f(s,x) B(s) \nu^x(ds) dM(x) = \int \int f(x,s)\, B(s)\, \nu^x(ds) dM(x). \qquad (6)$$

Indeed, in (5) replace $f(s,x)$ by $B(s) f(s,x)$.

Quasi-invariant probabilities will be also described as the ones which satisfies the so called the KMS condition (on the setting of von Neumann algebras, or C^*-algebras) as we will see later on section 4.

As an extreme example consider the equivalence relation such that each point is related to just itself. In this case a modular function δ takes only the value 1. Given any transverse function ν the condition

$$\int \int f(s,x) \nu^x(ds) dM(x) = \int \int f(x,s)\, \delta^{-1}(x,s)\, \nu^x(ds) dM(x) \qquad (7)$$

is satisfied by any probability M on X. In this case the set of probabilities is the set of quasi-invariant probabilities.

Example 31. Quasi invariant probability and the SBR probability for the Baker map

We will present a particular example where we will compare the probability M satisfying the quasi invariant condition with the so called SBR probability. We will consider a different setting of the case described on [58] (considering Anosov systems) which, as far as we know, was never published.

We will show that the **quasi invariant probability is not the SBR probability**.

We will address later on the end of this example the kind of questions discussed on [58] and [38].

We will consider the groupoid of example 8, that is, we consider the equivalence relation: given two points $z_1, z_2 \in S^1 \times S^1$ they are related if the first coordinate is equal.

In example 8 we consider an expanding transformation $T : S^1 \to S^1$ and F denotes the associated T-Baker map. The associated SBR probability is the only absolutely continuous F-invariant probability over $S^1 \times S^1$.

The dynamical action of F in some sense looks like the one of an Anosov diffeomorphism.

Consider the measured groupoid $(G\,, \nu)$ where in each vertical fiber over the point a we set ν^a as the Lebesgue probability db over the class (a,b), $0 \leq b \leq 1$.

This groupoid corresponds to the local unstable foliation for the transformation F.

We fix a certain point $b_0 \in (0,1)$. For each pair $x = (a,b)$ and $y = (a, b_0)$, where $a, b \in S^1$, and $n \geq 0$, the elements z_1^n, z_2^n, $n \in \mathbb{N}$, are such that $F^n(z_1^n) = x = (a,b)$ and $F^n(z_2^n) = y = (a, b_0)$. Note that they are of the form $z_1^n = (a^n, b^n)$, $z_2^n = (a^n, s^n)$. We use the notation $z_1^n(x)$, $b^n(x)$, $n \in \mathbb{N}$, to express the dependence on x.

We denote for $x \in S^1 \times S^1$

$$V(x) = V(a,b) = \Pi_{n=1}^{\infty} \, \frac{T'(b^n(x))}{T'(s^n)} = \Pi_{n=1}^{\infty} \, \frac{T'(b^n(a,b))}{T'(s^n)} < \infty.$$

This is finite because s^n and $b^n(x)$ are on the same domain of injectivity of T for all n and T' is of Holder class.

In a similar fashion as in [58] we define δ by the expression

$$\delta(\,(a,y_1)\,,\,(a,y_2)\,) = \frac{V(a,y_1)}{V(a,y_2)} = \frac{V(y_1)}{V(y_2)} = \Pi_{n=1}^{\infty} \, \frac{T'(b^n(a,y_1))}{T'(b^n(a,y_2))},$$

where $(a,y_1) \sim (a,y_2)$.

Consider the probability M on $S^1 \times S^1$ given by

$$dM(a,b) = \frac{V(a,b)}{\int V(a,c)dc} \, db\, da.$$

The density $\psi(a,b) = \frac{V(a,b)}{\int V(a,c)dc}$ satisfies the equation

$$\psi(a,b)\frac{1}{T'(b)} = \psi(F(a,b)). \tag{8}$$

Denote $F(a,b) = (\tilde{a}, \tilde{b})$, then, it is known that the density $\varphi(a,b)$ of the SBR probability for F satisfies

$$\varphi(a,b)\frac{T'(\tilde{a})}{T'(b)} = \varphi(F(a,b)). \tag{9}$$

This follows from the F-invariance of the SBR

Therefore, M is not the SBR probability - by uniqueness of the SBR.

We will show that M satisfies the quasi invariant condition.

Note that

$$\int \int f(\,(a,b),(a,s)\,)\; \nu^a(d\,s)\; dM(a,b) =$$

$$\int\int\int f(\,(a,b),(a,s)\,)\ \ \frac{V(a,b)}{\int V(a,c)dc}\,ds\,db\,da.$$

On the other hand

$$\int\int f(\,(a,s),(a,b)\,)\,\frac{V(a,s)}{V(a,b)}\ \nu^a(d\,s)\ dM(a,b) =$$

$$\int\int\int f(\,(a,s),(a,b)\,)\,\frac{V(a,s)}{V(a,b)}\ \ \frac{V(a,b)}{\int V(a,c)dc}\,ds\,db\,da =$$

$$\int\int\int f(\,(a,s),(a,b)\,)\ \ \frac{V(a,s)}{\int V(a,c)dc}\,ds\,db\,da.$$

If we exchange the variables b and s, and using Fubini's theorem, we get that M satisfies the quasi invariant condition.

The relation of quasi-invariant probabilities and transverse measures is described on section 5.

The result considered on Theorem 6.18 in [58] for an Anosov diffeomorphism concerns transverse measures and cocycles. [58] did not mention quasi-invariant probabilities.

Note that from equations (8) and (9) one can get that the conditional disintegration along unstable leaves of both the SRB and the quasi-invariant probability M are equal (see page 533 in [32]).

Using the relation of quasi-invariant probabilities, cocycles and transverse measures one can say that one of the main claims in [58] (see Theorem 6.18) and [38] (both considering the case of Anosov Systems) can be expressed in some sense via the above mentioned property about conditional disintegration along unstable leaves (using the analogy with the case of the above Baker map F).

In section 7 we will present more examples of quasi-stationary probabilities.

4 von Neumann Algebras derived from measured groupoid

We refer the reader to [2], [27] and [12] as general references for von Neumann algebras related to groupoids.

Here $X \sim G^0$ will be either $\hat{\Omega}$, Ω or $S^1 \times S^1$. We will denote by G a general groupoid obtained by an equivalence relation R.

Definition 32. *Given a measured groupoid G for the transverse function ν and two measurable functions $f, g \in \mathcal{F}_\nu(G)$, we define the convolution $(f \underset{\nu}{*} g) = h$, in such way that, for any $(x, y) \in G$*

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\, f(s,y)\; \nu^y(ds) = h(x,y).$$

In the case there exists a multiplicative neutral element for the operation $*$ we denote it by $\mathbb{1}$.

The above expression in some sense resembles the way we get a matrix from the product of two matrices.

For a fixed Haar system ν the product $\underset{\nu}{*}$ defines an algebra on the vector space of ν-integrable functions $\mathcal{F}_\nu(G)$.

As usual function of the form $f(x,x)$ are identified with functions $f : G^0 \to \mathbb{R}$ of the from $f(x)$.

Example 33. *In the particular case where ν^y is the counting measure on the fiber over y then*

$$(f \underset{\nu}{*} g)(x,y) = \sum_s g(x,s)\, f(s,y).$$

Denote by I_Δ the indicator function of the diagonal on $G^0 \times G^0$. In this case, I_Δ is the neutral element for the product $\underset{\nu}{*}$ operation.

In this case $\mathbb{1} = I_\Delta$.

Note that I_Δ is measurable but generally not continuous. This is fine for the von Neumann algebra setting. However, we will need a different topology (and σ-algebra) on $G^0 \times G^0$ - other than the product topology - when considering the unit $\mathbb{1} = I_\Delta$ for the C^*-algebra setting (see [14], [55], [56]).

Remark: The indicator function of the diagonal on $G^0 \times G^0$ is not always the multiplicative neutral element on the von Neumann algebra obtained from a general Haar system (G, ν).

Example 34. *Another example: consider the standard Haar system of example 18.*

In this case

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\, f(s,y) \nu^y(ds) =$$

$$\frac{1}{d} \sum_{a=1}^{d} g(x,\, (\,a, x_2, x_3, ...)\,)\; f(\,(a, x_2, x_3, ...)\,, y\,)\; = h(x,y).$$

The neutral element is $d\, I_\Delta = \mathbb{1}$.

Example 35. *Suppose* $J : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ *is a continuous positive function such that for any* $x \in \Omega$ *we have that* $\sum_{a=1}^{d} J(ax) = 1$. *The measured groupoid* (G, ν) *of Example 21, where* ν^y, $y \in \{1, 2, ..., d\}^{\mathbb{N}}$, *is such that given* $f, g : G \to \mathbb{R}$, *we have for any* $(x, y) \in G$, $x = (x_1, x_2, x_3, ...)$, $y = (y_1, x_2, x_3, ..)$ *that*

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y) \nu^y(ds) =$$

$$\sum_{a=1}^{d} g(x,\, (\, a, x_2, x_3, ...)\,)\; f(\,(a, x_2, x_3, ...)\,, y\,)\; J(a, x_2, x_3, ...) = h(x, y).$$

Note that $x_j = y_j$ *for* $j \geq 2$.

Suppose that f *is such that for any string* $(x_2, x_3, ...)$ *and* $a \in \{1, 2, ., d\}$ *we get*

$$f(\,(a, x_2, x_3, ...)\,, (a, x_2, x_3, ...)\,) \;=\; \frac{1}{J(a, x_2, x_3, ...)},$$

and, $a, b \in \{1, 2, ., d\}$, $a \neq b$

$$f(\,(a, x_2, x_3, ...)\,, (b, x_2, x_3, ...)\,) \;=\; 0.$$

In this case the neutral multiplicative element is $\mathbf{1}(x, y) = \frac{1}{J(x)} I_\Delta(x, y)$.

Consider a measured groupoid (G, ν), $\nu \in \mathcal{E}$, then, given two functions ν-integrable $f, g : G \to \mathbb{R}$, we had defined before an algebra structure on $\mathcal{F}_\nu(G)$ in such way that $(f \underset{\nu}{*} g) = h$, if

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y) \nu^y(ds) = h(x, y),$$

where $(x, y) \in G$ and $(s, y) \in G$.

To define the von Neumann algebra associated to (G, ν), we work with complex valued functions $f : G \to \mathbb{C}$. The product is again given by the formula

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y) \nu^y(ds).$$

The involution operation $*$ is the rule $f \to \tilde{f} = f^*$, where $\tilde{f}(x, y) = \overline{f(y, x)}$. The functions $f \in \mathcal{F}(G^0)$ are of the form $f(x) = f(x, x)$ are such that $\tilde{f} = f$.

Following Hanh [25], we define the I-norm

$$\|f\|_I = \max\left\{ \left\| y \mapsto \int |f(x,y)|\, \nu^y(dx) \right\|_\infty , \left\| y \mapsto \int |f(y,x)|\, \nu^y(dx) \right\|_\infty \right\},$$

and the algebra $I(G,\nu) = \{f \in L^1(G,\nu) : \|f\|_I < \infty\}$ with the product and involution as above. An element $f \in I(G,\nu)$ defines a bounded operator L_f of left convolution multiplication by a fixed f on $L^2(G,\nu)$. This gives the left regular representation of $I(G,\nu)$.

Definition 36. *Given a measured groupoid (G,ν), we define the* **von Neumann Algebra associated to** *(G,ν), denoted by $W^*(G,\nu)$, as the the von Neumann generated by the left regular representation of $I(G,\nu)$, that is, $W^*(G,\nu)$ is the closure of $\{L_f : f \in I(G,\nu)\}$ in the weak operator topology.*

The multiplicative unity is denoted by $\imath$.

In the case ν is such that such that, $\int \nu^y(ds) = 1$, for any $y \in G^0$, we say that the von Neumann algebra is normalized.

In the setting of von Neumann Algebras we do not require that $\imath$ is continuous.

Definition 37. *We say an element $h \in W^*(G\,,\nu)$ is positive if there exists a g such that $h = g * \tilde{g}$.*

This means

$$h(x,y) = (g \underset{\nu}{*} \tilde{g})(x,y) = \int g(x,s)\, \overline{g(y,s)} \nu^y(ds) = h(x,y).$$

Note que $h(x,x) = (g \underset{\nu}{*} \tilde{g})(x,x) \geq 0$.

The next example is related to Example 23.

Example 38. *Consider the equivalence relation where all points are related on the set $G^0 = \{1, 2.., d\}$. In this case $G = \{1, 2.., d\} \times \{1, 2.., d\}$. Take ν as the counting measure. A function $f : G \to \mathbb{C}$ is denoted by $f(i,j)$, where $i \in \{1, 2.., d\}, j \in \{1, 2.., d\}$.*

The convolution product is

$$(f \underset{\nu}{*} g)(i,j) = \sum_k g(i,k)\, f(k,j).$$

In this case the associated von Neumann algebra (the set of functions $f : G \to \mathbb{C}$) is identified with the set of matrices with complex entries. The convolution is the product of matrices and the identity matrix is the unit $\imath$. The involution operation is to take the Hermitian A^* of a matrix A.

Note that the diagonal elements of a positive matrix (A is of the form $A = B\,B^*$) are non negative real numbers.

Example 39. *For the groupoid G of Example 2 and the counting measure, given $f, g : G \to \mathbb{C}$, we have that*

$$(f \underset{\nu}{*} g)(x,y) =$$

$$\sum_{a\in\{1,2,..,d\}} g(\,(\,x_1,x_2,...\,)\,,\,(\,a,x_2,x_3,..)\,)\; f(\,(\,a,x_2,x_3,..)\,,\,(\,y_1,x_2,...\,)\,).$$

We call standard von Neumann algebra on the groupoid G (of Example 2) the associated von Neumann algebra. For this $W^*(G\,,\nu)$ the neutral element $\mathbb{1}$ (or, more formally $L_{\mathbb{1}}$) is the indicator function of the diagonal (a subset of G). In this case $\mathbb{1}$ is measurable but not continuous.

Example 40. *For the probabilistic Haar system (G, ν) of Example 21, given $f, g : G \to \mathbb{C}$, we get*

$$(f \underset{\nu}{*} g)(x,y) =$$

$$\sum_{a\in\{1,2,...,d\}} \varphi(a,x_2,x_3,...)\, g((\,x_1,x_2,...\,),(\,a,x_2,x_3..))\; f((\,a,x_2,x_3,...),(\,y_1,x_2,...\,)),$$

where φ is Holder and such that $\sum_{a\in\{1,2,...,d\}} \varphi(a,x_1,x_2,...) = 1$, for all $x = (x_1,x_2,...)$.

This φ is a Jacobian.

The neutral element is described in example 35.

Example 41. *In the case $\nu^y = \delta_{x_0}$ for a fixed x_0 independent of y, then*

$$(f \underset{\nu}{*} g)(x,y) = g(x,x_0)\, f(x_0,y).$$

Proposition 42. *If (G, ν) is a measured groupoid, then for $f, g \in I(G, \lambda)$.*

$$(f \underset{\nu}{*} g)^{\sim} = \tilde{g} \underset{\nu}{*} \tilde{f}.$$

Proof: Remember that for (x, y) in G

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\, f(s,y)\nu^y(ds) = h(x,y).$$

Then,

$$(f \underset{\nu}{*} g)^{\sim}(x,y) = \int \overline{g(y,s)\, f(s,x)}\nu^x(ds).$$

On the other hand

$$(\tilde{g} \underset{\nu}{*} \tilde{f})(y,x) = \int \overline{f(s,x)}\, \overline{g(y,s)} \nu^y(ds).$$

As $\nu^y = \nu^x$ we get that the two expressions are equal.

□

Then by proposition 42 we have for the involution $*$ it is valid the property

$$(f \underset{\lambda}{*} g)^* = g^* \underset{\lambda}{*} f^*.$$

For more details about properties related to this definition we refer the reader to chapter II in [53] and section 5 in [27].

We say that $c : G \to \mathbb{R}$ is a linear cocycle function if $c(x,y) + c(y,z) = c(x,z)$, for all x, y, z which are related. If c is a linear cocycle then e^δ is a modular function (or, a multiplicative cocycle).

Definition 43. *Consider the von Neumann algebra $W^*(G,\nu)$ associated to (G,ν).*

Given a continuous cocycle function $c : G \to \mathbb{R}$ we define the **group homomorphism** *$\alpha : \mathbb{R} \to Aut(W^*(G,\nu))$, where for each $t \in \mathbb{R}$ we have that $\alpha_t \in Aut(W^*(G,\nu))$ is defined by: for each fixed $t \in \mathbb{R}$ and $f : G \to \mathbb{R}$ we set $\alpha_t(f) = e^{t\,i\,c} f$.*

Remark: Observe that in the above definition that for each fixed $t \in \mathbb{R}$ and any $f : G^0 \to \mathbb{R}$, we have $\alpha_t(f) = f$, since $c(x,x) = 0$ for all $x \in G^0$.

We are particularly interested here in the case where $G^0 = \Omega$ or $G^0 = \hat{\Omega}$.

The value t above is related to temperature and not time. We are later going to consider complex numbers z in place of t. Of particular interest is $z = \beta i$ where β is related to the inverse of temperature in Thermodynamic Formalism (or, Statistical Mechanics).

Definition 44. *Consider the von Neumann Algebra $W^*(G,\nu)$ with unity $\mathbf{1}$ associated to (G,ν). A von Neumann* **dynamical state** *is a linear functional w (acting on the linear space $W^*(G,\nu)$) of the form $w : W^*(G,\nu) \to \mathbb{C}$, such that, $w(a) \geq 0$, if a is a positive element of $W^*(G,\nu)$, and $w(\mathbf{1}) = 1$.*

Example 45. *Consider over $\Omega = \{1,2,..,d\}^{\mathbb{N}}$ the equivalence relation R of Example 2 and the Haar system (G,ν) associated to the counting measure in each fiber $r^{-1}(x) = \{(a, x_2, x_3, ...)\,|a \in \{1,2,...,d\}\}$, where $x = (x_1, x_2, ...)$.*

Given a probability μ over Ω we can define a von Neumann dynamical state φ_μ in the following way: for $f : G \to \mathbb{C}$ define

$$\varphi_\mu(f) = \int f(x,x)\, d\mu(x) = \int \; f((\, x_1, x_2, x_3, ...\,), (\, x_1, x_2, x_3, ...))\; d\mu(x). \tag{10}$$

If h is positive, that is, of the form $h(x,y) = \int\; g(x,s)\, \overline{g(y,s)} \nu^y(ds)$, then

$$\varphi_\mu(h) = \int (\; \int \; \|g(x,s)\|^2 \nu^x(ds)\;)\; d\mu(x) \geq 0.$$

Note that $\varphi_\mu \mathbf{1} = 1$.

Then, φ_μ is indeed a von Neumann dynamical state.

In this case given $f, g : G \to \mathbb{C}$

$$\varphi_\mu(f \underset{\nu}{*} g) =$$

$$\int \sum_{a \in \{1,2,..,d\}} f((\, x_1, x_2, ...\,), (\, a, x_2, x_3..))\; g((\, a, x_2, x_3..), (\, x_1, x_2, ...\,))\, d\mu(x).$$

It seems natural to try to obtain dynamical states from probabilities M on G^0 (adapting the reasoning of the above example). Then, given a cocycle c it is also natural to ask: what we should assume on M in order to get a KMS state for c?

Example 46. *For the von Neumann algebra of complex matrices of example 38 taking $p_1, p_2, .., p_d \geq 0$, such that $p_1 + p_2 + .. + p_d = 1$, and $\mu = \sum_{j=1}^d \delta_j$, we consider φ_μ such that*

$$\varphi_\mu(A) = A_{11} p_1 + A_{22} p_2 + ... + A_{dd} p_d,$$

where A_{ij} are the entries of A.

Note first that $\varphi_\mu(I) = 1$.

If $B = A\, A^*$, then the entries $B_{jj} \geq 0$, for $j = 1, 2, ..., d$.

Therefore, φ_μ is a dynamical state on this von Neumann algebra.

Example 47. *Consider over $\Omega = \{1, 2, .., d\}^{\mathbb{N}}$ the equivalence relation R of Example 21 and the associated probability Haar system ν.*

Given a probability μ over Ω we can define a von Neumann dynamical state φ_μ in the following way: given $f : G \to \mathbb{C}$ we get $\varphi_\mu(f) = \int f(x,x)\, J(x)\, d\mu(x)$. In this way given f, g we have

$$\varphi_\mu(f \underset{\nu}{*} g) =$$

$$\int \sum_{a\in\{1,2,..,d\}} J(a,x_2,..)\, g((x_1,x_2,\dots),(a,x_2,..))\; f((a,x_2,..),(x_1,x_2\dots))\, J(x)d\mu(x).$$

For the neutral multiplicative element $\mathbf{1}(x,y) = \frac{1}{J(x)} I_\Delta(x,y)$ *we get*

$$\varphi_\mu(\mathbf{1}) \;=\; \int \frac{1}{J(x)} I_\Delta(x,x)\, J(x)d\mu(x) = 1.$$

Consider G a groupoid and a von Neumann Algebra $W^*(G,\nu)$, where ν is a transverse function, with the algebra product $f \underset{\nu}{*} g$ and involution $f \to \tilde{f}$.

Given a continuous cocycle $c : G \to \mathbb{R}$ we consider $\alpha : \mathbb{R} \to \mathrm{Aut}(W^*(G,\nu))$, $t \mapsto \alpha_t$, the associated homomorphism according to definition 43: for each fixed $t \in \mathbf{R}$ and $f : G \to \mathbb{R}$ we set $\alpha_t(f) = e^{t\,i\,c} f$.

Definition 48. *An element $a \in W^*(G,\nu)$ is said to be* **analytical** *with respect to α if the map $t \in \mathbb{R} \mapsto \alpha_t(a) \in W^*(G,\nu)$ has an analytic continuation to the complex numbers.*

More precisely, there is a map $\varphi : \mathbb{C} \to W^*(G,\nu)$, such that, $\varphi(t) = \alpha_t(a)$, for all $t \in \mathbb{R}$, and moreover, for every $z_0 \in \mathbb{C}$, there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $W^*(G,\nu)$, such that, $\varphi(z) = \sum_{n=0}^\infty (z-z_0)^n a_n$ in a neighborhood of z_0.

The analytical elements are dense on the von Neumann algebra (see [49]).

Definition 49. *We say that a von Neumann dynamical state w is a* **KMS state for β and c** *if*

$$w\,(\,b \underset{\nu}{*} (\alpha_{i\,\beta}(a)\,)\,) = w\,(a \underset{\nu}{*} b\,),$$

for any b and any analytical element a.

It follows from general results (see [49]) that it is enough to verify: for any $f, g \in I(G,\nu)$ and $\beta \in \mathbb{R}$ we get

$$w\,(\,g \underset{\nu}{*} \alpha_{\beta\,i}(\,f\,)\,) = w\,(\,g \underset{\nu}{*} (\,e^{-\beta\,c} f\,)\,) = w\,(f \underset{\nu}{*} g\,). \tag{11}$$

Consider the functions

$$u(x,y) = (f * g)(x,y) = \int g(x,s)\, f(s,y)\, \nu^y(ds),$$

and

$$v(x,y) = (\, g * (e^{-\beta\, c}\, f)\,)\,(x,y) = \int e^{-\beta c(x,s)} f(x,s)\, g(s,y)\, \nu^y(ds).$$

Equation (11) means

$$w(\, u(x,y)\,) = w(\, v(x,y)\,). \tag{12}$$

Note that equation (11) implies that a KMS von Neumann (or, C^*)-dynamical state w satisfies:

a) for any $f : G^0 \to \mathbb{C}$ and $g : G \to \mathbb{C}$:

$$w\,(\, g \underset{\nu}{*} f\,) = w\,(f \underset{\nu}{*} g\,). \tag{13}$$

This follows from the fact that for any $t \in \mathbb{R}$ and any $f : G^0 \to \mathbb{R}$, we have that $\alpha_t(f) = f$.

b) if the function $\mathbb{1}$ depends just on $x \in G^0$, then, for any β

$$\alpha_{i\,\beta}(\mathbb{1}) = \mathbb{1}.$$

c) w is invariant for the group α_t, $t \in \mathbb{R}$. Indeed,

$$w(\,\alpha_t(f)) = w(\mathbb{1} \underset{\nu}{*} \alpha^t(f)) = w(f \underset{\nu}{*} \mathbb{1}) \;=\; w(f).$$

Example 50. *For the von Neumann algebra (C^*-algebra) of complex matrices of examples 38 and 46 consider the dynamical evolution $\sigma_t = e^{i\,t\,H}$, $t \in \mathbb{R}$, where H is a diagonal matrix with entries the real numbers $H_{11} = U_1, H_{22} = U_2, ..., H_{dd} = U_d$. The KMS state ρ for β is*

$$\rho(A) = A_{1\,1}\rho_1 + A_{2\,2}\rho_2 + ... + A_{d\,d}\rho_d,$$

where $\rho_i = \frac{e^{-\beta U_i}}{\sum_{j=1}^d e^{-\beta U_j}}$, $i = 1, 2, ..., d$, and $A_{i,j}$, $i, j = 1, 2, ..., d$, are the entries of the matrix A (see [56]).

The probability μ of example 45 corresponds in some sense to the probability $\mu = (\rho_1, \rho_2, .., \rho_d)$ on $\{1, 2, ..., d\}$. That is, $\rho = \varphi_\mu$.

This is a clear indication that the μ associated to the KMS state has in some sense a relation with Gibbs probabilities. This property will appear more explicitly on Theorem 54 for the case of the bigger than two equivalence relation.

Remember that if c is a cocycle, then $c(x,z) = c(x,y) + c(y,z)$, $\forall x \sim y \sim z$, and, therefore,

$$\delta(x,y) = e^{\beta c(x,y)} = e^{-\beta c(y,x)}$$

is a modular function.

Definition 51. *Given a cocycle $c : G \to \mathbb{R}$ we say that a probability M over G^0 satisfies the (c,β)-KMS condition for the groupoid (G, ν), if for any $h \in I(G,\nu)$, we have*

$$\int \int h(s,x) \nu^x(ds) dM(x) = \int \int h(x,s) e^{-\beta\, c(x,s)}\, \nu^x(ds) dM(x), \qquad (14)$$

where $\beta \in \mathbb{R}$.

In this case we will say that M is a **KMS probability**.

The above means that M **is quasi-invariant for** ν **and** $\delta(x,s) = e^{-\beta c(s,x)}$.

When $\beta = 1$ and c is of the form $c(s,x) = V(x) - V(s)$ the above condition means

$$\int \int h(s,x) \nu^x(ds) e^{V(x)} dM(x) = \int \int h(x,s) e^{V(x)}\, \nu^x(ds) dM(x). \qquad (15)$$

Proposition 52. *(J. Renault - Proposition II.5.4 in [53]) Suppose that the state w is such that for a certain probability μ on G^0 we have that for any $h \in I(G,\nu)$ we get $w(h) = \int h(x,x) d\mu(x)$. Then, to say that μ satisfies the (c,β)-KMS condition for (G,ν) according to Definition 51 is equivalent to say that w is KMS for (G,ν), c and β, according to equation* (11).

Proof: Note that for any f, g

$$(\, f \underset{\nu}{*}\, g\,)(x,y) = \int g(x,s)\, f(s,y) d\nu^x(s)$$

and

$$(\, g \underset{\nu}{*}\, (\, e^{-\beta\, c}\, f\,)\,)(x,y) = \int f(x,s)\, g(s,y)\, e^{-\beta\, c(x,s)}\, d\nu^x(s).$$

We have to show that $\int u(x,x) d\mu(x) = \int v(x,x) d\mu(x)$ (see equation (12)).

Then, if the (c,β)-KMS condition for M is true, we take $h(s,x) = g(x,s)\, f(s,x)$ and we got equation (12) for such w.

By the other hand if (12) is true for such w and any f, g, then take $f(s,x) = h(s,x)$ and $g(s,x) = 1$.

□

Example 53. *In the case for each y we have that ν^y is the counting measure we get that to say that a probability M over $\hat{\Omega}$ satisfies the (c, β)-KMS condition means: for any $h : G \to \mathbb{C}$*

$$\sum_{y \sim x} \int h(x, y) e^{-\beta\, c(x,y)} dM(x) = \sum_{x \sim y} \int h(x, y)\, dM(y). \tag{16}$$

In the notation of [54] we can write the above in an equivalent way as

$$\int h\, e^{-\beta\, c} d(s^*(M)) = \int h\, d(r^*(M)).$$

Note that in [54] it is considered $r(x, y) = x$ and $s(x, y) = y$.

Suppose $c(x, y) = \varphi(x) - \varphi(y)$. Then, taking $h(x, y) = k(x, y)\, e^{\beta\, \varphi(x)}$ we get an equivalent expression for (16): for any $k(x, y)$

$$\sum_{y \sim x} \int k(x, y) e^{\beta\, \varphi(y)} dM(x) = \sum_{x \sim y} \int k(x, y)\, e^{\beta\, \varphi(x)} dM(y). \tag{17}$$

For a Holder continuous potential $A : \{1, 2.., d\}^{\mathbb{N}} \to \mathbb{R}$ the Ruelle operator $\mathcal{L}_A$ acts on continuous functions $v : \{1, 2.., d\}^{\mathbb{N}} \to \mathbb{R}$ by means of $\mathcal{L}_A(v) = w$, if

$$\mathcal{L}_A(v)(x_1, x_2, x_3, ...) = \sum_{a=1}^{d} e^{A(a, x_1, x_2, x_3, ...)} v(a, x_1, x_2, x_3, ...) = w(x).$$

For a Holder continuous potential $A : \{1, 2.., d\}^{\mathbb{N}} \to \mathbb{R}$ there exist a continuous positive eigenfunction f, such that, $\mathcal{L}_A(f) = \lambda\, f$, where λ is positive and also the spectral radius of $\mathcal{L}_A$ (see [47]).

The dual $\mathcal{L}_A^*$ of $\mathcal{L}_A$ acts on probabilities by Riesz Theorem (see [47]). We say that the probability m on $\{1, 2.., d\}^{\mathbb{N}}$ is **Gibbs for the potential** A, if $\mathcal{L}_A^*(m) = \lambda\, m$ (same λ as above). In this case we say that m is an eigenprobability for A.

Gibbs probabilities for Holder potentials A are also **DLR probabilities** on $\{1, 2, ...d\}^{\mathbb{N}}$ (see [11]).

Gibbs probabilities for Holder potentials A can be also obtained via Thermodynamic Limit from boundary conditions (see [11]).

We say that the **potential** A **is normalized** if $\mathcal{L}_A(1) = 1$. In this case a probability μ is Gibbs (equilibrium) for the normalized potential A if it is a fixed point for the dual of the Ruelle operator, that is, $\mathcal{L}_A^*(\mu) = \mu$.

Suppose $\Omega = \{-1, 1\}^{\mathbb{N}}$ and $A : \Omega \to \mathbb{R}$ is of the form

$$A(x_0, x_1, x_2, ...) = x_0 a_0 + x_1 a_1 + x_2 a_2 + x_3 a_3 + ... + x_n a_n + ...$$

where $\sum a_n$ is absolutely convergent.

In [10] the explicit expression of the eigenfunction for $\mathcal{L}_A$ and the eigenprobability for the dual $\mathcal{L}_A^*$ of the Ruelle operator $\mathcal{L}_A$ is presented. The eigenprobability is not invariant for the shift.

In example 39 consider $\Omega = \{1,2\}^{\mathbb{N}}$ and take ν^y the counting measure on the class of y. Consider the von Neumann algebra associated to this measured groupoid (G,ν) where G is given by the bigger than two relation.

In this case $\mathbf{1}(x,y) = I_\Delta(x,y)$.

Consider $c(x,y) = \varphi(x) - \varphi(y)$, where φ is Holder. We do not assume that φ is normalized.

A natural question is: the eigenprobability μ for such potential φ is such that $f \to \varphi_\mu(f) = \int f(x,x)\, d\mu(x)$ defines the associated KMS state? For each modular function c?

The purpose of the next results is to analyze this question when $c(x,y) = \varphi(x) - \varphi(y)$.

Consider the equivalence relation on $\Omega = \{1,2...,d\}^{\mathbb{N}}$ which is

$$x = (x_1, x_2, x_3, ..) \sim y = (y_1, y_2, y_3, ...)\ ,\text{ if an only if }\ ,\ x_j = y_j \quad \text{for all } j \geq 2.$$

In this case the class $[x]$ of $x = (x_1, x_2, x_3, ..)$ is

$$[x] = \{\, (1, x_2, x_3, ..),\ (2, x_2, x_3, ..), ..., (d, x_2, x_3, ..)\,\}.$$

The associated groupoid by $G \subset \Omega \times \Omega$, is

$$G = \{(x,y)\,|\, x \sim y\}.$$

G is a closed set on the compact set $\Omega\times\Omega$. We fix the measured groupoid (G,ν) where ν^x is the counting measure. The results we will get are the same if we take the Haar system as the one where each point y on the class of x has mass $1/d$.

In this case equation (14) means

$$\sum_j \int f(\,(j, x_2, x_3, .., x_n, ..)\,,\,(x_1, x_2, x_3, .., x_n, ..))dM(x) =$$

$$\sum_j \int f(\,(x_1, x_2, x_3, ..)\,,\,(j, x_2, x_3, ..))e^{-c((j,x_2,x_3,..)\,,\,(x_1,x_2,x_3,..))}dM(x). \tag{18}$$

The first question: given a cocycle c does there exist M as above?

Suppose $c(x,y) = \varphi(y) - \varphi(x)$.

In this case equation (18) means

$$\sum_j \int f(\,(j,x_2,x_3,..,x_n,..)\,,\,(x_1,x_2,x_3,..,x_n,..))dM(x) =$$

$$\sum_j \int f(\,(x_1,x_2,x_3,..)\,,\,(j,x_2,x_3,..))e^{-\varphi(j,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}dM(x). \quad (19)$$

Among other things we will show later that if we assume that φ depends just on the first coordinate then we can take M as the independent probability (that is, such independent M satisfies the KMS condition (19)).

In section 3.4 in [56] and in [29] the authors present a result concerning quasi-invariant probabilities and Gibbs probabilities on $\{1,2,..d\}^{\mathbb{N}}$ which has a different nature when compared to the next one. The groupoid is different from the one we will consider (there elements are of the form (x,n,y), $n \in \mathbb{Z}$). In [56] and [29] for just one value of β you get the existence of the quasi invariant probability. Moreover, the KMS state is unique (here this will not happen as we will show on Theorem 55)

In [57], [26], [43] and [33] the authors present results which have some similarities with the next theorem. They consider Gibbs (quasi-invariant) probabilities in the case of the symbolic space $\{1,2,..d\}^{\mathbb{Z}}$ and not $\{1,2,..d\}^{\mathbb{N}}$ like here. In all these papers the quasi-invariant probability is unique and invariant for the shift. In [7] the authors consider DLR probabilities for interactions in $\{1,2,..d\}^{\mathbb{Z}}$. The equivalence relation (the homoclinic relation of Example 7) in all these cases is quite different from the one we will consider.

The next result were generalized in [34] and [35].

Theorem 54. *Consider the Haar system with the counting measure ν for the bigger than two relation on $\{1,2,..d\}^{\mathbb{N}}$. Suppose that φ depends just on the first k coordinates, that is, and*

$$\varphi(x_1,x_2,...,x_k,x_{k+1},x_{k+2},..) = \varphi(x_1,x_2,...,x_k).$$

Then, the eigenprobability μ (a DLR probability) for the potential $-\varphi$ (that is, $\mathcal{L}^*_{-\varphi}(\mu) = \lambda\mu$, for some positive λ) satisfies the KMS condition (is quasi-invariant) for the associated modular function $c(x,y) = \varphi(y) - \varphi(x)$.

The same result is true, of course, for βc, where $\beta > 0$.

Proof: We are going to show that the Gibbs probability μ for the potential $-\varphi$ satisfies the KMS condition.

We point out that in general the eigenvalue $\lambda \neq 1$.

We have to show that (19) is true when $M=\mu$. That is, μ is a KMS probability for the Haar system and the modular function.

Denote for any finite string $a_1,a_2,...a_n$ and any n

$$p_{a_1,a_2,...a_n}=\frac{e^{-[\varphi(a_1,a_2,...a_n,1^\infty)+\varphi(a_2,...a_n,1^\infty)+...+\varphi(a_n,1^\infty)]}}{\sum_{b_1,b_2,...b_n}e^{-[\varphi(b_1,b_2,...b_n,1^\infty)+\varphi(b_2,...b_n,1^\infty)+...+\varphi(b_n,1^\infty)]}}.$$

Note that for $n>k$ we have that

$$e^{-[\varphi(a_1,a_2,...a_n,1^\infty)+\varphi(a_2,...a_n,1^\infty)+...+\varphi(a_n,1^\infty)]}=$$

$$e^{-[\varphi(a_1,a_2,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,1,..,1}_{k-1})+(n-k)\varphi(\underbrace{1,1,..,1}_{k-1})]}.$$

Therefore,

$$\sum_{b_1,b_2,...b_n}e^{-[\varphi(b_1,b_2,...b_n,1^\infty)+\varphi(b_2,...b_n,1^\infty)+...+\varphi(b_n,1^\infty)]}=$$

$$e^{-(n-k)\varphi(\underbrace{1,1,..,1}_{k-1})}\sum_{b_1,b_2,...b_n}e^{-[\varphi(b_1,b_2,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,1,..,1}_{k-1})]}.$$

Consider the probability μ_n, such that,

$$\mu_n=\sum_{a_1,a_2,...a_n}\delta_{(a_1,a_2,...a_n,1^\infty)}\,p_{a_1,a_2,...a_n}=$$

$$\sum_{a_1,...a_n}\delta_{(a_1,...a_n,1^\infty)}\frac{e^{-[\varphi(a_1,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n}e^{-[\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}}.$$

and μ such that $\mu=\lim_{n\to\infty}\mu_n$.

Note that

$$p_{a_1,a_2,..,a_n}=\frac{e^{-[\varphi(a_1,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n}e^{-[\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}}.$$

If φ is Holder it is known that the above probability μ is the eigenprobability for the dual of the Ruelle operator $\mathcal{L}_{-\varphi}$ (a DLR probability). That is, there exists $\lambda > 0$ such that $\mathcal{L}^*_{-\varphi}(\mu) = \lambda\mu$. This follows from the Thermodynamic Limit with boundary condition property as presented in [11].

We claim that the above probability μ satisfies the KMS condition.

Indeed, note that

$$\sum_j \int f(\,(j, x_2, x_3, .., x_n, ..)\,,\,(x_1, x_2, x_3, .., x_n, ..))d\mu(x) =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1,a_2,...a_n} f(\,(j, a_2, a_3, .., a_n,\, 1^\infty)\,,\,(a_1, a_2, a_3, .., a_n,\, 1^\infty)\,)p_{a_1,a_2,...,a_n} =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1} \sum_{a_2,...a_n} f((j, a_2, a_3, .., a_n,\, 1^\infty)(a_1, a_2, a_3, .., a_n, 1^\infty)p_{a_1,a_2,...,a_n}. \quad (20)$$

On the other hand

$$\sum_j \int f(\,(x_1, x_2, x_3, ..)\,,\,(j, x_2, x_3, ..))e^{-\varphi(j,x_2,x_3,...)\,+\,\varphi(x_1,x_2,x_3,..)}d\mu(x) =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1,a_2,...,a_n} f(\,(a_1, .., a_n,\, 1^\infty)\,,\,(j, a_2, .., a_n,\, 1^\infty)\,)e^{-\varphi(j,a_2,...,a_k)\,+\,\varphi(a_1,a_2,...a_k)}p_{a_1,a_2,...,a_n} =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1} \sum_{a_2,...a_n} f(\,(a_1, .., a_n,\, 1^\infty)\,,\,(j, a_2, .., a_n,\, 1^\infty)\,)e^{-\varphi(j,a_2,...,a_k)\,+\,\varphi(a_1,a_2,...a_k)}$$

$$\frac{e^{-\,[\,\varphi(a_1,a_2,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n} e^{-[\,\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}} =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1} \sum_{a_2,...a_n} f(\,(a_1, .., a_n,\, 1^\infty)\,,\,(j, a_2, .., a_n,\, 1^\infty)\,)$$

$$\frac{e^{-\,[\,\varphi(j,a_2,,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n} e^{-[\,\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}}. =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1} \sum_{a_2,...a_n} f(\,(a_1, .., a_n,\, 1^\infty)\,,\,(j, a_2, .., a_n,\, 1^\infty)\,)p_{j,a_2,...,a_n}.$$

On this last equation if we exchange coordinates j and a_1 we get expression (20).

Then, such μ satisfies the KMS condition.

$\square$

The above theorem can be extended to the case the potential φ is Holder. We refer the reader to [34] for more general results. This paper consider a more general relation defining the so called continuous groupoids. In the case the transverse function is not the counting measure (for instance when each class is not a countable set) a similar kind of results as above can be shown using Thermodynamic Formalism for the so called generalized XY model (see [35])

We will show now that under the above setting the KMS probability **is not unique**.

Proposition 55. *Suppose μ satisfies the KMS condition for the measured groupoid (G, ν) where $c(x, y) = \varphi(y) - \varphi(x)$. Suppose φ is normalized for the Ruelle operator, where $\varphi : G^0 = \Omega \to \mathbb{R}$. Consider $v(x_1, x_2, x_3, ..)$ which does not depend of the first coordinate. Then, $v(x)d\mu(x)$ also satisfies the KMS condition for the measured groupoid (G, ν).*

Proof: Suppose μ satisfies the (c, β)-KMS condition for the measured groupoid (G, ν). This means: for any $g \in I(G, \nu)$

$$\int \sum_{a \in \{1,2,..,d\}} g((\, a, y_2, y_3..)\,,\,(\, y_1, y_2, \dots\,))\, e^{\beta \varphi(\,a, y_2, y_3..)}\, d\mu(y) =$$

$$\int \sum_{a \in \{1,2,..,d\}} g((\, x_1, x_2, \dots\,), (\, a, x_2, x_3..))\, e^{\beta \varphi(\,a, x_2, x_3..)}\, d\mu(x). \tag{21}$$

Take

$$h(x_1, x_2, x_3, ..), (y_1, y_2, y_3, ..)) =$$
$$k(\,(x_1, x_2, x_3, ..), (y_1, y_2, y_3, ..))\, v(x_1, x_2, x_3, ..)\,) =$$
$$k(\,(x_1, x_2, x_3, ..), (y_1, y_2, y_3, ..))\, v(x_2, x_3, ..)\,).$$

From the hypothesis about μ we get that

$$\int \sum_{a \in \{1,2,..,d\}} h((\, a, y_2, y_3..)\,,\,(\, y_1, y_2, \dots\,))\, e^{\beta \varphi(\,a, y_2, y_3..)}\, d\mu(y) =$$

$$\int \sum_{a \in \{1,2,..,d\}} h((\, x_1, x_2, x_3, \dots\,), (\, a, x_2, x_3..))\, e^{\beta \varphi(\,a, x_2, x_3..)}\, d\mu(x).$$

This means, for any continuous k the equality

$$\int \sum_{a\in\{1,2,..,d\}} k((\,a,y_2,y_3..)\,,\,(\,y_1,y_2,...\,))\,e^{\beta\varphi(\,a,y_2,y_3..)}\;v(y_2,y_3,...)d\mu(y) =$$

$$\int \sum_{a\in\{1,2,..,d\}} k((\,x_1,x_2,x_3,...\,),(\,a,x_2,x_3..))\,e^{\beta\varphi(\,a,x_2,x_3..)}\;v(x_2,x_3,..)d\mu(x).$$

Therefore, $v(x)d\mu(x)$ also satisfies the (c,β)-KMS condition for the measured groupoid (G,ν).

□

It follows from the above result that the probability that satisfies the KMS condition for c and the measured groupoid (G,ν) is not always unique.

A probability ρ satisfies the Bowen condition for the potential $-\varphi$ if there exists constants $c_1,c_2>0$, and P, such that, for every

$$x=(x_1,...,x_m,...)\in\Omega=\{1,2,...d\}^{\mathbb{N}},$$

and all $m\geq 0$,

$$c_1 \leq \frac{\rho\{y: y_i=x_i, \quad \forall i=1,...,m\}}{\exp\left(-Pm-\sum_{k=1}^{m}\varphi(\sigma^k(x)\right)} \leq c_2. \tag{22}$$

Suppose φ is Holder, then, if ρ is the equilibrium probability (or, if ρ is the eigenprobability for the dual of Ruelle operator $\mathcal{L}_{-\varphi}$) one can show that it satisfies the Bowen condition for $-\varphi$.

In the case v is continuous and does not depend on the first coordinate then $v(x)d\mu(x)$ also satisfies the Bowen condition for φ. The same is true for the probability $\hat{\rho}$ of example 56 on the case $-\varphi=\log J$.

There is an analogous definition of the Bowen condition on the space $\{1,2,...d\}^{\mathbb{Z}}$ but it is a much more strong hypothesis on this case (see section 5 in [33]).

Example 56. *We will show an example where the probability μ of theorem 54 (the eigenprobability for the potential $-\varphi$) is such that if f is a function that depends just on the first coordinate, then, $f\,\mu$ does not necessarily satisfies the KMS condition.*

Suppose $\varphi=-\log J$, where $J(x_1,x_2,x_3,..)=J(x_1,x_2)>0$, and $\sum_i P_{i,j}=1$, for all i. In other words the matrix P, with entries $P_{i,j}$, $i,j\in\{1,2..,d\}$, is a column stochastic matrix. The Ruelle operator for $-\varphi$ is the Ruelle operator for $\log J$. The potential $\log J$ is normalized for the Ruelle operator.

We point out that in Stochastic Process it is usual to consider line stochastic matrices which is different from our setting.

There exists a unique right eigenvalue probability vector π for P (acting on vectors on the right). The Markov chain determined by the matrix P and the initial vector of probability $\pi = (\pi_1, \pi_2, .., \pi_d)$ determines an stationary process, that is, a probability ρ on the Bernoulli space $\{1, 2, ..., d\}^{\mathbb{N}}$, which is invariant for the shift acting on $\{1, 2, ..., d\}^{\mathbb{N}}$.

For example, we have that $\rho(\overline{21}) = P_{21}\pi_1$.

We point out that such ρ is the eigenprobability for the $\mathcal{L}^*_{\log J}$ (associated to the eigenvalue 1). Therefore, ρ satisfies the KMS condition from the above results.

The Markov Process determined by the matrix P and the initial vector of probability $\pi = (1/d, 1/d, ..., 1/d)$ defines a probability $\hat{\rho}$ on the Bernoulli space $\{1, 2, ..., d\}^{\mathbb{N}}$, which is not invariant for the shift acting on $\{1, 2, ..., d\}^{\mathbb{N}}$.

In this case, for example, $\hat{\rho}(\overline{21}) = P_{21}\, 1/d$.

Note that the probability ρ satisfies $\rho = u\, \hat{\rho}$ where u depends just on the first coordinate.

Note that unless P is double stochastic is not true that for any j_0 we have that $\sum_k P_{j_0,k} = 1$.

Assume that there exists j_0 such that $\sum_k P_{j_0,k} \neq 1$.

We will check that, in this case $\hat{\rho}$ does not satisfies the KMS condition for the function $f(x, y) = I_{X_1=i_0}(x)\, I_{X_1=j_0}(y)$.

Indeed, equation (19) means

$$\sum_j \int f(\,(j, x_2, x_3, .., x_n, ..)\,,\,(x_1, x_2, x_3, .., x_n, ..))d\hat{\rho}(x) =$$

$$\sum_j \int I_{X_1=i_0}(j, x_2, x_3, ..)\, I_{X_1=j_0}(x_1, x_2, x_3...)d\hat{\rho}(x) =$$

$$\int I_{X_1=i_0}(i_0, x_2, x_3, ..)\, I_{X_1=j_0}(x_1, x_2, x_3...)d\hat{\rho}(x) =$$

$$\int I_{X_1=j_0}(x_1, x_2, x_3...)d\hat{\rho}(x) = \hat{\rho}(\overline{j_0}) = 1/d =$$

$$\sum_j \int f(\,(x_1, x_2, x_3, ..)\,,\,(j, x_2, x_3, ..))e^{-\varphi(j,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\sum_j \int I_{X_1=i_0}(x_1, x_2, x_3, ..)\, I_{X_1=j_0}(j, x_2, x_3...)e^{-\varphi(j,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\int I_{X_1=i_0}(x_1, x_2, x_3, ..)\, I_{X_1=j_0}(j_0, x_2, x_3...)e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\int I_{X_1=i_0}(x_1,x_2,x_3,..)\, e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)} d\hat{\rho}(x) =$$

$$\int_{X_1=i_0} e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(i_0,x_2,x_3,..)} d\hat{\rho}(x) =$$

$$\sum_k \int_{X_1=i_0\,,\,X_2=k} e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(i_0,x_2,x_3,..)} d\hat{\rho}(x) =$$

$$\sum_k \int_{X_1=i_0\,,\,X_2=k} P_{j_0,k}\ P_{i_0,k}^{-1} d\hat{\rho}(x) =$$

$$\sum_k P_{j_0,k}\ P_{i_0,k}^{-1}\, P_{i_0,k} 1/d =$$

$$\sum_k P_{j_0,k}\ 1/d\ \neq 1/d = \hat{\rho}(\overline{j_0}).$$

Therefore, $\hat{\rho}$ does not satisfies the KMS condition.

Example 57. *Consider $\Omega = \{1,2\}^{\mathbb{N}}$, a Jacobian J and take ν^y the probability on each class y given by $\sum_a J(a,y_2,y_3,..)\delta_{(a,y_2,y_3,..)}$.*

Note first that $\varphi = \log J$ is a normalized potential. Does the equilibrium probability for $\log J$ satisfies the KMS condition? We will show that this in not always true.

The question means: is it true that for any function k is valid

$$\int \sum_{a\in\{1,2\}} k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\, e^{\varphi(a,y_2,...)}\, d\mu(y) =$$

$$\int \sum_{a\in\{1,2\}} k(\,(a,x_2,..),\,(x_1,x_2,..)\,)\, e^{\varphi(a,x_2,...)}\, d\mu(x) =$$

$$\int \sum_{a\in\{1,2\}} k(\,(x_1,x_2,..),\,(a,x_2,..)\,)\, e^{\varphi(a,x_2,...)}\, d\mu(x)? \tag{23}$$

Consider the example: take $c(x,y) = \varphi(x) - \varphi(y)$, for $\varphi : \{1,2\}^{\mathbb{N}} \to \mathbb{R}$, such that,

$$\varphi(a,\,.\,,\,.\,,...) = \log p$$

where $p = p_a$, for $a \in \{1,2\}$, and $p_1 + p_2 = 1$, $p_1, p_2 > 0$.

The Gibbs probability μ for such φ is the independent probability associated to p_1, p_2.

Given such probability μ over Ω we can define a dynamical state φ_μ in the following way: given $f : G \to \mathbb{R}$ we get $\varphi_\mu(f) = \int f(x,x)\, d\mu(x)$.

Take $\beta = 1$. *We will show that* φ_μ *is not KMS for* c.
The equation (23) for such μ *means for any* $k(x,y)$

$$\int \sum_{a\in\{1,2\}} p_a\, k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\; d\mu(y) =$$

$$\int \sum_{a\in\{1,2\}} k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\, p(a,y_2,...)\, d\mu(y) =$$

$$\int \sum_{b\in\{1,2\}} k(\,(x_1,x_2,..),\,(b,x_2,..)\,)\, p(b,x_2,...)\, d\mu(x) =$$

$$\int \sum_{b\in\{1,2\}} p_b\, k(\,(x_1,x_2,..),\,(b,x_2,..)\,)\; d\mu(x).$$

It is not true that μ *is Gibbs for the potential* $\log p$.
Indeed, given k *consider the function*

$$g(y_1,y_2,y_3,y_4,...) = k(\,(y_1,y_3,...)\,,\,(y_2,y_3,...)\,).$$

Note that

$$\mathcal{L}_{\log p}(g)(y_1,y_2,y_3,..) = \sum_{a\in\{1,2\}} p(a,y_1,y_2,y_3,...)\, g(a,y_1,y_2,...)$$

$$\sum_{a\in\{1,2\}} p_a\, k(\,(a,y_2,y_3,..),\,(y_1,y_2,..)\,).$$

Then,

$$\int \sum_{a\in\{1,2\}} p_a\, k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\; d\mu(y) =$$

$$\int \mathcal{L}_{\log p}(g)(y_1,y_2,y_3,..)\, d\mu(y) = \int k(\,(y_1,y_3,...)\,,\,(y_2,y_3,...)\,)d\mu(y).$$

Now, given k *consider the function*

$$h(x_1,x_2,x_3,x_4,...) = k(\,(x_2,x_3,...)\,,\,(x_1,x_3,x_4,...)\,).$$

Then,

$$\int \mathcal{L}_{\log p}(h)(x_1,x_2,x_3,..)\, d\mu(y) = \int k(\,(x_2,x_3,...)\,,\,(x_1,x_3,...)\,)\, d\mu(x).$$

For the Gibbs probability μ *for* $\log p$ *is not true that for all* k

$$\int k(\,(x_2,x_3,...)\,,\,(x_1,x_3,...)\,)\, d\mu(x) = \int k(\,(x_1,x_3,...)\,,\,(x_2,x_3,...)\,)d\mu(x).$$

5 Noncommutative integration and quasi-invariant probabilities

In non-commutative integration the transverse measures are designed to integrate transverse functions (see [12] or [27]).

In the same way we can say that a function can be integrated by a measure resulting in a real number we can say that the role of a transverse measure is to integrate transverse functions (producing a real number).

The main result here is Theorem 66 which describes a natural way to define a transverse measure from a modular function δ and a Haar system $(G\,, \hat{\nu})$.

We refer the reader to [35] for new results on the topic (for instance related to the entropy of transverse measures, etc...)

As a motivation for the topic of this section consider a foliation of the two dimensional torus where we denote each leave by l. This partition defines a grupoid with a quite complex structure. Each leave is a class on the associated equivalence relation. This motivation is explained with much more details in [13]

We consider in each leave l the intrinsic Lebesgue measure on the leave which will be denoted by ρ_l.

A random operator q is the association of a bounded operator $q(l)$ on $\mathcal{L}^2(\rho_l)$ for each leave l. We will avoid to describe several technical assumptions which are necessary on the theory (see page 51 in [13]).

The set of all random operators defines a von Neumann algebra under some natural definitions of the product, etc... (see Proposition 2 in page 52 in [13]) (*).

This setting is the formalism which is natural on **noncommutative geometry** (see [13]).

Important results on the topic are for instance the characterization of when such von Neumann algebra is of type I, etc... (see page 53 in [13]). There is a natural trace defined on this von Neumann algebra.

A more abstract formalism is the following: consider a fixed grupoid G. Given a transverse function λ one can consider a natural operator L_λ : $\mathcal{F}^+(G) \to \mathcal{F}^+(G)$, which satisfies

$$f \to \lambda * f = L_\lambda(f)\,.$$

L_λ acts on $\mathcal{F}^+(G)$ and can be extended to a linear action on the von Neumann algebra $\mathcal{F}(G)$. This defines a Hilbert module structure (see section 3.2 in [31] or [27]).

Given λ we can also define the operator $R_\lambda : \mathcal{F}^+(G) \to \mathcal{F}^+(G)$ by

$$h(x,y) = R_\lambda(f)(\gamma) = \int f(s,y) d\lambda^x(d\,s),$$

for any (x,y).

Definition 58. *Given two G-kernels λ_1 and λ_2 we get a new G-kernel $\lambda_1 * \lambda_2$, called the convolution of λ_1 and λ_2, where given the function $f(x,y)$, we get the rule*

$$(\lambda_1 * \lambda_2)\,(f) = g \in \mathcal{F}(G^0),$$

given by

$$g(y) = \int \Big(\int f(s,y)\ \lambda_2^x(d\,s) \Big)\ \ \lambda_1^y(dx).$$

In the above $x \sim y \sim s$.

In other words $(\lambda_1 * \lambda_2)$ is such that for any y we have

$$(\lambda_1 * \lambda_2)^y(dx) = \int \lambda_2^x(d\,s)\ \lambda_1^y(dx). \tag{24}$$

Note that

$$R_{\lambda_1 * \lambda_2} = R_{\lambda_1} \circ R_{\lambda_2}.$$

For a given fixed transverse function λ, for each class $[y]$ on the grupoid G, we get that R_λ defines an operator acting on functions $f(r,s)$, where $f : [y] \times [y] \to \mathbb{C}$, and where $R_\lambda(f) = h$.

In this way, each transverse function λ defines a random operator q, where $q([y])$ acts on $\mathcal{L}^2(\lambda^y)$ via R_λ.

A transverse measure can be seen as an integrator of transverse functions or as an integrator of random operators (which are elements on the von Neumann algebra (*) we mention before).

First we will present the basic definitions and results that we will need later on this section.

Remember that $\mathcal{E}^+$ is the set of transverse functions for the grupoid $G \subset X \times X$ associated to a certain equivalence relation $\sim$.

$\mathcal{F}^+(G)$ denotes the space of Borel measurable functions $f : G \to [0,\infty)$ (a real function of two variables (a,b)).

Definition 59. *Given a G kernel ν and an integrable function $f \in \mathcal{F}_\nu(G)$ we can define two functions on G:*

$$(x,y) \to (\nu * f)(x,y) = \int f\,(x,s)\,\nu^y(ds)$$

and

$$(x,y) \to (f * \nu)(x,y) = \int f\,(s,y)\,\nu^x(ds).$$

Note that $\nu * 1 = 1$ if ν^y is a probability for all y. Also note that $(f * \nu)(y,y) = \nu(f)\;(y)$ (see definition 14).

About (24) we observe that

$$(\lambda_1 * \lambda_2)\,(f) = \lambda_1(f * \lambda_2).$$

A kind of analogy of the above concept of convolution (of kernels) with integral kernels is the following: given the kernels $K_1(s,x)$ and $K_2(x,y)$ we define the kernel

$$\hat{K}(s,y) = \int K_2(s,x)\,K_1(x,y)\,dx.$$

This is a kind of convolution of integral kernels.

This defines the operator

$$f(x,y) \to g(y) = \int f(s,y)\hat{K}(s,y)ds = \int (\int f(s,y)\,K_2(s,x)\,d\,s\,)\;K_1(x,y)\,dx.$$

Example 60. *Given any kernel ν we have that $\mathfrak{d} * \nu = \nu$, where $\mathfrak{d}$ is the delta kernel of Example 13.*

Indeed, for any $f \in \mathcal{F}(G)$

$$\int f\,(\,\mathfrak{d} * \nu\,)^y = \int\int f(s,y)\,\nu^x(d\,s)\,\mathfrak{d}^{\,y}(d\,x) = \int f(s,y)\,\nu^y(d\,s)\; = \int f\,\nu^y$$

In the same way for any ν we have that $\nu * \mathfrak{d} = \nu$.

Example 61. *Given a fixed positive function $h(x,y)$ and a fixed kernel ν, we get that the kernel $\nu * (\,h\,\mathfrak{d}\,)$, where $\mathfrak{d}$ is the Dirac kernel, is such that given any $f(x,y)$,*

$$(\nu * (\,h\,\mathfrak{d}\,))(f)(y) = \int (\int f(s,y)\,h(s,x)\,\mathfrak{d}^x(d\,s)\,)\quad \nu^y(dx) =$$

$$\int f(x,y)\,h(x,x)\quad \nu^y(dx).$$

Particularly, taking $h = 1$, we get $\nu * \mathfrak{d} = \nu$.

Example 62. *For the bigger than two equivalence relation of example 24 on* $(S^1)^{\mathbb{N}}$*, where* S^1 *is the unitary circle, the equivalence classes are of the form* $\{ (a, x_2, x_3, ...), \, a \in S^1 \}$*, where* $x_j \in S^1$*,* $j \geq 2$*, is fixed.*

Given $x = (x_1, x_2, x_3, ...)$ *we define* $\nu^x(da)$ *the Lebesgue probability on* S^1*, which can be identified with* $S^1 \times (x_2, x_3, .., x_n, ...)$*. This defines a transverse function where* $G^0 = (S^1)^{\mathbb{N}}$*. We call it the* **standard XY Haar system***.*

In this case given a function $f(x, y) = f(\, (x_1, x_2, x_3, ..), (y_1, y_2, y_3, ..)\,)$

$$(\nu * f)(x, y) = \int f(x, s)\, \nu^x(d\, s) = \int f\,(\, (x_1, x_2, x_3, ..), (s, x_2, x_3, ..)\,)\, ds,$$

where $s \in S^1$*. Note that in the present example the information on* y *was lost after convolution.*

Such ν *is called in [6] the a priori probability for the Ruelle operator. Results about Ruelle operators and Gibbs probabilities for such kind of XY models appear in [6] and [36].*

After Proposition 72 we will present several properties of convolution of transverse function (we will need soon some of them).

Note that if ν is transverse and λ is a kernel, then $\nu * \lambda$ is transverse.

Remember that given a kernel λ and a fixed y the property $\lambda^y(1) = 1$ means $\int \lambda^y(dx) = 1$.

Definition 63. *A transverse measure* Λ *over the modular function* $\delta(x, y)$*,* $\delta : G \to \mathbb{R}$*, is a linear function* $\Lambda : \mathcal{E}^+ \to \mathbb{R}^+$*, such that, for each kernel* λ *which satisfies the property* $\lambda^y(1) = 1$*, for any* y*, if* ν_1 *and* ν_2 *are transverse functions such that* $\nu_1 * (\delta\lambda) = \nu_2$*, then,*

$$\Lambda(\nu_1) = \Lambda(\nu_2). \tag{25}$$

A measure produces a real number from the integration of a classical function (which takes values on the real numbers), and, on the other hand, the transverse measure produces a real number from a transverse function ν (which takes values on measures).

The assumptions on the above definition are necessary (for technical reasons) when considering the abstract concept of integral of a transverse function by Λ (as is developed in [12]). We will show later that there is a more simple expression providing the real values of such process of integration by Λ which is related to quasi-invariant probabilities.

If one consider the equivalence relation such that each point is related just to itself, any cocycle is constant equal 1 and the only kernel satisfying $\lambda^y(1) = 1$, for any y, is the delta Dirac kernel $\mathfrak{d}$. In this case if ν_1 and ν_2 are such that $\nu_1 * (\delta\lambda) = \nu_2$, then, $\nu_1 = \nu_2$ (see Example 60). Moreover, $\mathcal{E}^+$ is just the set of positive functions on X. Finally, we get that the associated transverse measure Λ is just a linear function $\Lambda : \mathcal{E}^+ \to \mathbb{R}^+$

Example 64. *Given a probability μ over G^0 we can define*

$$\Lambda(\nu) = \int \int \nu^y(dz) d\mu(y).$$

Suppose that λ satisfies $\lambda^x(1) = 1$, for any x, and

$$\nu_1 \, * \, \lambda = \nu_2.$$

Then, $\Lambda(\nu_1) = \Lambda(\nu_2)$. This means that Λ is invariant by translation on the right side.

Indeed, note that,

$$\Lambda(\nu_1) = \int \int \nu_1^y(dz) d\mu(y),$$

and, moreover

$$\Lambda(\nu_2) = \int \int \nu_2^y(dz) d\mu(y) =$$

$$\int [\int (\int \lambda^x(d\, s)) \,) \, \nu_1^y(dx))] \, d\mu(y) =$$

$$\int (\int \nu_1^y(dx)) \, d\mu(y).$$

Therefore, Λ is a transverse measure of modulus $\delta = 1$.

In this way for each measure μ on G^0 we can associate a transverse measure of modulus 1 by the rule $\nu \to \Lambda(\nu) = \int \int \nu^y(dz) d\mu(y) \in \mathbb{R}$.

The condition

$$\nu_1 \, * \, (\delta\lambda) = \nu_2$$

means for any f we get

$$\int f(x,y) \, (\, \nu_1 * (\delta\lambda\,))^y \, (d\,x) \; = \int (\int f(s,y) \, [\, \delta(s,x) \lambda^x(ds) \,] \,) \quad \nu_1^y(dx) =$$

$$\int f(x,y)\ \nu_2^y(dx). \tag{26}$$

We define before (see (5)) the concept of quasi-invariant probability for a given modular function δ, a grupoid G and a fixed transverse function ν.

For reasons of notation we use a little bit variation of that definition. In this section we say that M is quasi invariant probability for δ and ν if for any $f(x,y)$

$$\int\int f(y,x)\,\nu^y(x)dM(y) = \int\int f(x,y)\delta(x,y)^{-1}\nu^y(x)dM(y). \tag{27}$$

Proposition 65. *Given a modular function δ, a grupoid G and a fixed transverse function $\hat{\nu}$ denote by M the quasi invariant probability for δ.*

Assume that $\int \hat{\nu}^y(dr) \neq 0$ for all y.

If $\hat{\nu} * \lambda_1 = \hat{\nu} * \lambda_2$, where λ_1, λ_2 are kernels, then,

$$\int \delta^{-1}\lambda_1(1)\ d\,M = \int \delta^{-1}\lambda_2(1)\ d\,M.$$

This is equivalent to say that

$$\int\int \delta^{-1}(s,y)\lambda_1^y(ds)dM(y) = \int\int \delta^{-1}(s,y)\lambda_2^y(ds)dM(y).$$

Proof: By hypothesis $g(y) = (\hat{\nu} * \lambda_1)(\delta^{-1})\,(y) = (\hat{\nu} * \lambda_2)(\delta^{-1})(y)$.

Then, we assume that (see (24))

$$\int g(y)\frac{1}{\int \hat{\nu}^y(dr)}dM(y) = \int\int\int \frac{1}{\int \hat{\nu}^y(dr)}\delta^{-1}(s,y)\lambda_1^x(ds)\hat{\nu}^y(dx)\,dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}\delta^{-1}(s,y)\lambda_2^x(ds)\hat{\nu}^y(dx)\,dM(y). \tag{28}$$

Therefore,

$$\int\int \delta^{-1}(s,y)\lambda_1^y(ds)dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^x(dr)}\delta(y,s)\lambda_1^y(ds)\hat{\nu}^y(dx)\,dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}[\,\delta(y,x)\,\delta(x,s)\,\lambda_1^y(ds)\,]\hat{\nu}^y(dx)\,dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}[\,\delta(x,y)^{-1}\,\delta(x,s)\,\lambda_1^y(ds)\,]\hat{\nu}^y(dx)\,dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^x(dr)} [\, \delta(y,x)^{-1}\, \delta(y,s)\, \lambda_1^x(ds)\,]\, \delta^{-1}(x,y) \hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^x(dr)} \delta(y,s) \lambda_1^x(ds) \hat{\nu}^y(dx)\, dM(y) \tag{29}$$

On the above from the fourth to the fifth line we use the quasi-invariant expression (27) for M taking

$$f(y,x) = \int \frac{1}{\int \hat{\nu}^y(dr)}\, \delta(x,y)^{-1}\, \delta(x,s)\, \lambda_1^y(ds).$$

Note that if $\hat{\nu}$ is transverse $\int \hat{\nu}^x(dr)$ does not depends on x on the class $[y]$.

Finally, from the above equality (29) (and replacing λ_1^x by λ_2^x) it follows that

$$\int\int \delta^{-1}(s,y) \lambda_1^y(ds)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)} \delta(y,s) \lambda_1^y(ds) \hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)} \delta(y,s) \lambda_2^y(ds) \hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int \delta^{-1}(s,y) \lambda_2^y(ds)\, dM(y).$$

□

From now on we assume that $\int \hat{\nu}^y(dr) \neq 0$ for all y.

Theorem 66. *Given a modular function δ and a Haar system $(G\,, \hat{\nu})$, suppose M is quasi invariant for δ.*

We define Λ on the following way: given a transverse function ν there exists a kernel ρ such that $\nu = \hat{\nu} * \rho$ by proposition 74. We set

$$\Lambda(\nu) = \int\int \delta(x,y)^{-1} \rho^y(dx) dM(y). \tag{30}$$

Then, Λ is well defined and it is a transverse measure.

Proof: Λ is well defined by proposition 65.

We have to show that if $\lambda^x(1) = 1$, for any x, and ν_1 and ν_2 are such that $\nu_1 * (\delta\lambda) = \nu_2$, then, $\Lambda(\nu_1) = \Lambda(\nu_2)$.

Suppose $\nu_1 = \hat{\nu} * \lambda_1$, then, $\nu_2 = \hat{\nu} * (\, \lambda_1 * (\delta\, \lambda)\,)$.

Note that

$$\Lambda(\nu_1) = \int\int \delta(x,y)^{-1}\,\lambda_1^y(dx) dM(y).$$

On the other hand from (24)

$$\Lambda(\nu_2) = \int\int \delta(s,y)^{-1}\,(\,\lambda_1 * (\delta\,\lambda\,)\,)^y(ds) dM(y) =$$

$$\int\int\int \delta(s,y)^{-1}\,\delta(s,x)\,\lambda^x(ds)\,\lambda_1^y(dx)\,dM(y) =$$

$$\int\int\int \delta(x,y)^{-1}\,\lambda^x(ds)\,\lambda_1^y(dx)\,dM(y) =$$

$$\int\int \delta(x,y)^{-1}\ \left(\int \lambda^x(ds)\,\right)\ \lambda_1^y(dx)\,dM(y) =$$

$$\int\int \delta(x,y)^{-1}\ \ \lambda_1^y(dx)\,dM(y) = \Lambda(\nu_1).$$

□

Remark: The last proposition shows that given a quasi invariant probability M - for a transverse function $\hat{\nu}$ and a cocycle δ - there is a natural way to define a transverse measure Λ (associated to a grupoid G and a modular function δ).

One can ask the question: given transverse measure Λ (associated to a grupoid G and a modular function δ) is it possible to associate a probability on G_0? In the affirmative case, is this probability quasi invariant? We will elaborate on that.

Definition 67. *Given a transverse measure Λ for δ we can associate by Riesz Theorem to a transverse function $\hat{\nu}$ a measure M on G^0 by the rule: given a non-negative continuous function $h : G^0 \to \mathbb{R}$ we will consider the transverse function $h(x)\,\hat{\nu}^y(dx)$ and set*

$$h \to \Lambda(h\,\hat{\nu}) = \int h(x) dM(x).$$

Such M is a well defined measure (a bounded linear functional acting on continuous functions) and we denote such M by $\Lambda_{\hat{\nu}}$.

$\Lambda_{\hat{\nu}}$ means the rule $h \to \Lambda(h\,\hat{\nu}) = \Lambda_{\hat{\nu}}(h)$.

Proposition 68. *Given any transverse measure Λ associated to the modular function δ and any transverse functions ν and ν' we have for any continuous f that*

$$\Lambda_{\nu'}(\nu(\tilde{\delta} f)) = \Lambda(\nu(\tilde{\delta} f)\nu') = \Lambda(\nu'(\tilde{f})\nu) = \Lambda_{\nu}(\nu'(\tilde{f})).$$

Proof: If $\lambda^y(1) = 1\ \forall y$, that is, $\int 1\lambda^y(ds) = 1\ \ \forall y$, then $\Lambda(\nu * \delta\lambda) = \Lambda(\nu)$. If $g(x) = \lambda^x(1) = \int 1\lambda^x(ds) \neq 1$, then we can write $\lambda'^x(ds) = \frac{1}{g(x)}\lambda^x(ds)$, where λ and λ' are just kernels. In this way $(\nu * \delta\lambda) = (g\nu) * \delta\lambda'$. Indeed, for $h(x, y)$,

$$\int h(x,y)\,(\nu * \delta\lambda)^y(dx) = \int h(s,y)\delta(s,x)\lambda^x(ds)\nu^y(dx)$$

$$= \int h(s,y)\delta(s,x)\lambda'^x(ds)g(x)\nu^y(dx) = \int h(x,y)((g\nu) * \delta\lambda')^y(dx).$$

Denoting $\lambda(1)(x) = g(x) = \lambda^x(1) = \int 1\lambda^x(ds)$, it follows that

$$\Lambda(\nu*\delta\lambda) = \Lambda(g\nu*\delta\lambda') = \Lambda(g\nu) = \Lambda_\nu(g) = \Lambda_\nu(\lambda(1)) = \Lambda_\nu(\int 1\lambda^x(ds)). \quad (31)$$

From, (2) if ν is a kernel and $f = f(x, y)$

$$(\nu * f)(x,y) = \nu(\tilde{f})(x),$$

and, from Lemma 75, if λ is a kernel and ν is a transverse function, then, for any $f = f(x, y)$,

$$\lambda * (f\nu) = (\lambda * f)\nu.$$

It follows that, for transverse functions ν and ν', we get

$$\nu * [(\delta\tilde{f})\nu'] = [\nu * (\delta\tilde{f})]\nu' = [\nu(\tilde{\delta} f)]\nu'.$$

As a consequence

$$\Lambda_{\nu'}(\nu(\tilde{\delta} f)) = \Lambda([\nu(\tilde{\delta} f)]\nu') = \Lambda(\nu * [(\delta\tilde{f})\nu']) = \Lambda(\nu * \delta(\tilde{f}\nu')) =$$

$$\Lambda_\nu((\tilde{f}\nu')(1)) = \Lambda_\nu(\int 1 \cdot \tilde{f}(s,y)\nu'^y(ds)) = \Lambda_\nu(\nu'(\tilde{f})).$$

Above we use equation (31) with $\lambda = \tilde{f}\nu'$. □

Corollary 69. *If $\nu \in \mathcal{E}^+$, then for any f*

$$\Lambda(\nu(\tilde{f})\,\nu\,) \;=\; \Lambda(\nu(\tilde{\delta}\,f)\;\nu) \quad (32)$$

Proof: Just take $\nu = \nu'$ on last Proposition.

□

Among other things we are interested on a modular function δ, a transverse function $\hat{\nu}$ and a transverse measure Λ (of modulo δ) such that $M_{\Lambda,\hat{\nu}} = M$ is Gibbs for a Jacobian J. What conditions are required from M?

The main condition of the next theorem is related to the KMS condition of definition 51

Proposition 70. *Given a transverse measure* Λ *associated to the modular function* δ, *and a* **transverse function** $\hat{\nu}$, *consider the associated* $M = \Lambda_{\hat{\nu}}$. *Then,* M *is quasi invariant for* δ. *That is,* M *satisfies for all* g

$$\int \int g(s,x)\hat{\nu}^x(ds)dM(x) = \int \int g(x,s)\delta(x,s)\,\hat{\nu}^x(ds)dM(x). \qquad (33)$$

Proof: First we point out that (37) is consistent with (27) (we are just using different variables).

A transverse function $\hat{\nu}$ defines a function of $f \in \mathcal{F}(G) \to \mathcal{F}(G^0)$.

The probability M associated to $\hat{\nu}$ satisfies for any continuous function $h(x)$, where $h : G^0 \to \mathbb{R}$ the rule

$$h \to \Lambda(h\,\hat{\nu}) = \int h(x)dM(x),$$

where $h(x)\,\hat{\nu}^y(dx) \in \mathcal{E}^+$.

From proposition 69 we have that for the continuous function $f(s,x) = \tilde{g}(s,x)$, where $f : G \to \mathbb{R}$, the expression

$$\Lambda(\hat{\nu}\ (g)\,\hat{\nu}\,) = \Lambda(\hat{\nu}\ (\tilde{f})\,\hat{\nu}\,) = \Lambda(\hat{\nu}(\delta^{-1}\,f)\ \hat{\nu}\ ) = \Lambda(\hat{\nu}(\delta^{-1}\,\tilde{g})\ \hat{\nu}\ )$$

For a given function $g(s,x)$ it follows from the above that

$$\Lambda(\hat{\nu}\ (g)\,\hat{\nu}\,) = \int \hat{\nu}\ (g)(x)\,dM(x) = \int \int g(s,x)\hat{\nu}^x(ds)\,dM(x).$$

On the other hand

$$\Lambda(\hat{\nu}(\delta^{-1}\,\tilde{g})\ \hat{\nu}\ ) = \int \hat{\nu}(\delta^{-1}\,\tilde{g})(x)d\,M(x) = \int g(x,s)\delta^{-1}(s,x)\,\hat{\nu}^x(ds)\,dM(x).$$

□

Proposition 71. *Given a modular function* δ, *a grupoid* G, *a transverse measure* Λ *and a transverse function* $\hat{\nu}$, *suppose for any* ν, *such that* $\nu = \hat{\nu} * \rho$, *we have that*

$$\Lambda(\nu) = \Lambda(\hat{\nu} * \rho) = \int \int \delta(s,x)^{-1}\rho^x(ds)d\mu(x) = \int \delta^{-1}\rho(1)\,d\,\mu.$$

Then, $\mu = \Lambda_{\hat{\nu}}$.

Proof: Given $f \in \mathcal{F}(G_0)$ consider λ the kernel such that $\lambda^x(ds) = f(x)\delta_x(ds)$, where δ_x is the Delta Dirac on x.

Then, using the fact that $\delta(x,x) = 0$ we get that the kernel $f(x)\hat{\nu}^y(dx)$ is equal to $\hat{\nu} * \delta\, \lambda$.

Then, taking $\rho = \delta\, \lambda$ on the above expression we get

$$\Lambda(\, f\hat{\nu}) = \Lambda(\hat{\nu} * \, (\delta\, \lambda)\,) =$$

$$\int \delta^{-1} \rho(1) d\mu = \int \delta^{-1}\, (\delta\lambda)(1) d\mu = \int \lambda(1) d\mu = \int f(x) d\mu(x).$$

Therefore, $\Lambda_{\hat{\nu}} = \mu$.

□

Now we present a general procedure to get transverse measures.

Proposition 72. *For a fixed modular function δ we can associate to any given probability μ over G^0 a transverse measure Λ by the rule*

$$\nu \to \Lambda(\nu) = \int \int \delta(s,x)^{-1}\, \nu^x(ds)\;\; d\mu(x). \tag{34}$$

Proof:

Consider $\nu\,' \in \mathcal{E}^+$ and λ, such that, $\int \lambda^r(ds) = 1$, for all r, and moreover that $\nu\,' = \nu * (\delta\lambda)$.

We will write

$$(\nu *\, \delta\lambda)\, (\delta^{-1}) = \int \int \delta^{-1}(s,x)\, \delta(s,r)\, \lambda^r(ds) \nu^x(dr)$$

which is a function of x

Then

$$\Lambda(\nu\,') = \int \int \delta(s,x)^{-1}\, \nu\,'^x(ds)\;\; d\mu(x) = \int \nu\,'(\delta^{-1})(x) d\mu(x) =$$

$$= \int \Big(\nu * (\delta\lambda)\Big)(\delta^{-1})(x) d\mu(x) = \int \int \int \delta(s,x)^{-1} \delta(s,r) \lambda^r(ds) \nu^x(dr) d\mu(x) =$$

$$= \int \int \int \delta(r,x)^{-1} \lambda^r(ds) \nu^x(dr) d\mu(x) = \int \int \delta(r,x)^{-1} \nu^x(dr) d\mu(x) = \Lambda(\nu).$$

□

This last transverse measure is defined in a quite different way that the one described on Theorem 66.

Now we will present some general properties of convolution of transverse functions.

Lemma 73. *Suppose* $\nu \in \mathcal{E}^+$ *is a transverse function,* ν_0 *a kernel, and* $g \in \mathcal{F}^+(G)$ *is such that* $\int g(s,x)\nu_0^y(dx) = 1$*, for all* s, y*. Then,* $\nu_0 * (g\,\nu) = \nu$ *, where* $g\,\nu$ *is a kernel.*

Remark The condition $\int g(s,x)\nu_0^y(dx) = 1$, for all s, y means $(\nu_0 * g)(s,y) = 1$ for all s, y, that is $\nu_0 * g \equiv 1$ (See lemma 3 below).

Proof:

$$z(y) = \int f(s,y)\ \nu^y(ds) =$$

$$\int f(s,y)\ [\int g(s,x)\nu_0^y(dx)]\,\nu^x(ds) \qquad =$$

$$\int \int f(s,y)\ [\,g(s,x)\nu^x(ds)\,] \quad \nu_0^y(dx) =$$

$$\int f(s,y)\,(g\,\nu)^x(ds)\nu_0^y(dx) = \int f(s,y)\,(\,\nu_0 *\ (g\,\nu))^y(ds).$$

□

We say that the kernel ν is fidel if $\int \nu_0^y(ds) \neq 0$ for all y.

Proposition 74. *For a fixed transverse function* ν_0 *we have that for each given transverse function* ν *there exists a kernel* λ*, such that,* $\nu_0 * \lambda = \nu$*.*

Proof: Given the kernel ν_0 take $g_0(s) = \frac{1}{\int 1\nu_0^s(dr)} \geq 0$. Note that $g_0(v)$ is constant for $v \in [s]$. Then $\nu_0(g) = 1$, that is, for each s we get that $\int g_0(s)\nu_0^s(dx) = 1$.

We can take $\lambda = g_0\,\nu$ as a solution. Indeed, in a similar way as last lemma we get

$$z(y) = \int f(s,y)\ \nu^y(ds) =$$

$$\int f(s,y)\ [\int g_0(s)\nu_0^s(dx)]\,\nu^x(ds) \qquad =$$

$$\int \int f(s,y)\ [\,g_0(s)\nu^x(ds)\,] \quad \nu_0^y(dx) =$$

$$\int f(s,y)\,(g_0\,\nu)^x(ds)\nu_0^y(dx) = \int f(s,y)\,(\,\nu_0 *\ (g_0\,\nu))^y(ds) =$$

$$\int f(s,y)\,(\,\nu_0 *\ \lambda)^y(ds).$$

□

The next Lemma is just a more general form of Lemma 73.

Lemma 75. *Suppose* $\nu \in \mathcal{E}^+$, $g \in \mathcal{F}^+(G)$ *and* λ *a kernel, then* $\lambda * (g\,\nu) = (\lambda * g)\ \nu$, *where* $g\,\nu$ *is a kernel and* $\lambda * g$ *is a function.*

Proof:

Given $f \in \mathcal{F}(G)$ we get

$$(\lambda * (g\,\nu))(f)(y) = \int f(x,y)\,(\lambda * (g\,\nu))^y(dx)$$

$$= \int\int f(s,y)[(g\nu)^x(ds)]\,\lambda^y(dx) = \int\int f(s,y)[g(s,x)\nu^x(ds)]\,\lambda^y(dx).$$

On the other hand

$$[(\lambda * g)\ \nu](f)(y) = \int f(s,y)[(\lambda * g)\,\nu]^y(ds) = \int f(s,y)[(\lambda * g)(s,y)]\nu^y(ds)$$

$$= \int f(s,y)[\int g(s,x)\lambda^y(dx)]\nu^y(ds) = \int\int f(s,y)g(s,x)\nu^x(ds)\lambda^y(dx).$$

□

Proposition 76. *Suppose* ν *and* λ *are transverse. Given* $f \in \mathcal{F}^+(G)$, *we have that*

$$\lambda\,(\nu * f) = \nu\,(\lambda * \tilde{f}).$$

Proof:

Indeed, by definition 59, $(\nu * f)(x,y) = g(x,y) = \int f(x,s)\nu^y(ds)$, and by definition 14

$$\lambda\,(\nu * f)(y) = \lambda(g)(y) = \int g(x,y)\lambda^y(dx) = \int\int f(x,s)\nu^y(ds)\lambda^y(dx).$$

By the same arguments $(\lambda * \tilde{f})(x,y) = h(x,y) = \int \tilde{f}(x,s)\lambda^y(ds)$, and

$$\nu\,(\lambda * \tilde{f})(y) = \nu(h)(y) = \int h(x,y)\nu^y(dx) = \int\int \tilde{f}(x,s)\lambda^y(ds)\nu^y(dx) =$$

$$= \int\int f(s,x)\lambda^y(ds)\nu^y(dx) = \lambda\,(\nu * f)(y),$$

if we exchange the coordinates x and s.

Note that in the case $f \in \mathcal{F}(G^0)$ we denote $f(x,s) = f(x)$. In the same way

$$\lambda\,(\nu * f) = \nu\,(\lambda * \tilde{f})$$

in the following sense:

$$\int\int f(x)\nu^y(ds)\lambda^y(dx) = \int\int f(s)\lambda^y(ds)\nu^y(dx).$$

□

6 C^*-Algebras derived from Haar Systems

In this section the functions $f : G \to \mathbb{R}$ will be required to be continuous (not just measurable).

An important issue here is that we need suitable hypotheses in such way that the indicator of the diagonal $\imath$ belongs to the underlying space we consider. On von Neumann algebras setting the unit is just measurable and not continuous (this is good enough). We want to consider another setting (certain C^*-algebras associated to Haar Systems) where the unit will be required to be a continuous function. In general terms, given a groupoid $G \subset \Omega \times \Omega$, as we will see, we will need another topology (not the product topology) on the set G for the C^*-Algebra formalism and for defining KMS states.

We will begin with some more examples. The issue here is to set a certain appropriate topology.

Example 77. *For $n \in \mathbb{N}$ we define the partition η_n over $\overrightarrow{\Omega} = \{1, 2, ..., d\}^{\mathbb{N}}$, $d \geq 2$, such that two elements $x \in \overrightarrow{\Omega}$ and $y \in \overrightarrow{\Omega}$ are on the same element of the partition, if and only if, $x_j = y_j$, for all $j > n$. This defines an equivalence relation denoted by R_n.*

Example 78. *We define a partition η over $\overrightarrow{\Omega}$, such that two elements $x \in \overrightarrow{\Omega}$ and $y \in \overrightarrow{\Omega}$ are on the same element of the partition, if and only if, there exists an n such that $x_j = y_j$, for all $j > n$. This defines an equivalence relation denoted by R_∞.*

Example 79. *For each fixed $n \in \mathbb{Z}$ consider the equivalence relation on $\hat{\Omega}$: $x \sim y$ if*

$$y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...)$$

is such that $x_j = y_j$ for all $j \leq n$, where $\hat{\Omega} = \overleftarrow{\Omega} \times \overrightarrow{\Omega}$.

This defines a groupoid.

Example 80. *Recall that by definition the unstable set of the point $x \in \hat{\Omega}$ is the set*

$$W^u(x) = \{y \in \hat{\Omega}\,, \text{ such that } \lim_{n\to\infty} d(\hat{\sigma}^{-n}(x), d(\hat{\sigma}^{-n}(y)\,) \,= 0\,\}$$

One can show that the unstable manifold of $x \in \hat{\Omega}$ is the set

$$W^u(x) = \{y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...) \,|\, \text{there exists}$$
$$k \in \mathbb{Z}, \text{ such that } x_j = y_j, \text{for all } j \leq k\}.$$

If we denote by G_u the groupoid defined by the above relation, then, $x \sim y$, if and only if $y \in W^u(x)$.

Definition 81. *Given the equivalence relation R, when the quotient* $\hat{\Omega}/R$ *(or,* $\overrightarrow{\Omega}/R$*) is Hausdorff and locally compact we say that R is a proper equivalence.*

For more details about proper equivalence see section 2.6 in [56].

On the set $X = \overrightarrow{\Omega}$, if we denote $x = (x_1, x_2, ..x_n, ..)$, the family $U_x(m) = \{y \in \overrightarrow{\Omega}$, such that, $y_1 = x_1, y_2 = x_2, ..., y_m = x_m\}$, $m = 1, 2, ...$, is a fundamental set of open neighbourhoods on Ω.

Considering the relations R_m and R_∞ we get the corresponding groupoids

$$G_1 \subset G_2 \subset ... \subset G_m \subset ... \subset G_\infty \subset \overrightarrow{\Omega} \times \overrightarrow{\Omega} = X \times X.$$

The equivalence relation described in example 78 (and also 4) is not proper if we consider the product topology on $\overrightarrow{\Omega}$ (respectively on $\hat{\Omega}$). The equivalence relation described in example 77 (and also 79) is proper if we consider the product topology on $\overrightarrow{\Omega}$ (respectively on $\hat{\Omega}$) (see [21]).

We consider over G_n the quotient topology.

Lemma 82. *Given* $X = \overrightarrow{\Omega}$*, for each n the map defined by the canonical projection* $X \to G_n$ *is open.*

Proof: Given an open set $U \subset X$ take $V = \{y \in X \mid$ there exists $x \in X$, satisfying $y \sim x$ for the relation $R_n\}$. We will show that V is open.

Consider $y \in V$, $y \in U$, such that, $y \sim x$ for the relation R_n. There exists $m > n$, such that, $U_x(m) \subset U$. Then, $U_y(m) \subset V$. Indeed, if $z \in U_y(m)$, take $z' \in X$, such that $z'_j = x_j$, when $1 \leq j \leq m$, and $z'_j = z_j$, when $j > m$.

Then, $z' \sim z$ for the relation R_n. But, as $z \in U_y(m)$, this implies that $z_j = y_j$, when $1 \leq j \leq m$, and $y \sim x$, for R_n, implies that $y_j = x_j$, when $j > n$. Then, $z'_j = x_j$, if $1 \leq j \leq m$. Therefore, $z' \in U_x(m) \subset U$.

□

Lemma 83. *Given* $X = \overrightarrow{\Omega}$*, for each n the map defined by the canonical projection* $X \to G_\infty$ *is open.*

Proof: Given an open set $U \subset X$ take $V = \{y \in X \mid$ there exists $x \in X$, satisfying $y \sim x$ for the relation $R_n\}$ and $V_\infty = \{y \in X \mid$ there exists $x \in X$, satisfying $y \sim x$ for the relation $R_\infty\}$. Then, $V_\infty = \cup_{n=1}^\infty V_n$ is open.

□

Lemma 84. *Given* $X = \overrightarrow{\Omega}$*, for each* $n = 1, 2..., n, ...$*, the set* G_n *is Hausdorff.*

Proof: Given a fixed n, and $x, y \in X$, such that x and y are not related by R_n, then, there exists $m > n$ such that $x_m \neq y_m$. From this follows that no element of $U_x(m)$ is equivalent by R_n to an element of $U_y(m)$. By lemma 82 it follows that G_n is Hausdorff.

□

Lemma 85. *Given $X = \overrightarrow{\Omega}$ the set G_∞ is not Hausdorff.*

Proof: If $x_m = (\underbrace{1, 1, ..., 1}_{m}, d, d, d...)$, then $\lim_{n\to\infty} x_n = (1, 1, 1..., 1, ...)$ and $(\underbrace{1, 1, ..., 1}_{m}, d, d, d...) \sim (d, d, d, ..., d, ...)$, for the relation R_∞. Note, however, that $(1, 1, 1..., 1, ...)$ is not in the class $(d, d, d, ..., d, ...)$ for the relation R_∞.

□

Lemma 86. *Given $X = \overrightarrow{\Omega}$ denote by D the diagonal set on $X \times X$. Then, D is open on G_n for any n, where we consider on D the topology induced by $X \times X$.*

Proof: Given $x \in X$, we have that $U_x(n) \times U_x(n)$ is an open set of $X \times X$ which contains (x, x).

Consider $y, z \in U_x(n)$ such that y and z are related by R_n. Then, $y_j = x_j = z_j$, when $1 \leq j \leq n$, and $y_j = z_j$, when $j > n$. Therefore, $y = z$.

From this we get that

$$U_x(n) \times U_x(n) \cap G_n \subset D$$

□

Definition 87. *An equivalence relation R on a compact Hausdorff space X is said to be approximately proper if there exists an increasing sequence of proper equivalence relations R_n, $n \in \mathbb{R}$, such that $R = \cup_n R_n$, $n \in \mathbb{N}$. This in the sense that if $x \sim_R y$, then there exists an n such that $x \sim_{R_n} y$.*

Example 88. *Consider the equivalence relation R_∞ of example 78 and R_n the one of example 77. For each n the equivalence relation R^n is proper.*

Then, $R_\infty = \cup_n R_n$, $n \in \mathbb{N}$ is approximately proper (see [21]).

Definition 89. *Consider a fixed set K, a sequence of subsets $W_0 \subset W_1 \subset W_2 \subset ... \subset W_n \subset ... \subset K$ and a topology $\mathcal{W}_n$ for each set $W_n \subset K$.*

By the direct inductive limit

$$t - \lim_{n\to\infty} \mathcal{W}_n = \mathcal{K}$$

we understand the set K endowed with the largest topology $\mathcal{K}$ turning the identity inclusions $W_n \to K$ into continuous maps.

The topology of $t - \lim_{n\to\infty} \mathcal{W}_n = \mathcal{K}$ can be easily described: it consists of all subsets $U \subset K$ whose intersection $U \cap W_n$ is in $\mathcal{W}_n$ for all n.

For more details about the inductive limit (see section 2.6 in [56]).

In the case $W_n = G_n$ we consider as $\mathcal{W}_n$ the product topology.

Lemma 90. *Given $X = \overrightarrow{\Omega}$ if we consider over $K = G_\infty$ the inductive limit topology defined by the sequence of the $G_n \subset X \times X$, then, the indicator function $\mathbb{1}$ on the diagonal is continuous.*

Proof: By lemma 86 the diagonal D is an open set.

Moreover, $(G_\infty - D) \cap G_n = (\,(X \times X) - D) \cap G_\infty) \cap G_n = (\,(X \times X\,) - D) \cap G_n$ is open on G_n for all n. Then, $(G_\infty - D)$ is open on G_∞.

$\square$

Remark: Note that on G_∞ we have that D is not open on the induced topology by $X \times X$. Indeed, consider $a = (1, 1, 1, .., 1, ..)$ and $b_m = (\underbrace{1, 1, ..., 1}_{m-1}, d, 1, 1, 1, ..1, ...)$. Then, $\lim_{m\to\infty}(a, b_m) = (a, a) \in D$, and $(a, b_m) \in G_\infty$ but (a, b_m) is not on D, for all m.

Example 91. *In the above definition 89 consider $W_n = G_n \subset \overleftarrow{\Omega} \times \overrightarrow{\Omega}$, $n \in \mathbb{N}$, which is the groupoid associated to the equivalence relation R_n (see example 77). Then, $\cup_n W_n = K = G \subset \overleftarrow{\Omega} \times \overrightarrow{\Omega}$, where G is the groupoid associated to the equivalence relation R^∞. Consider on W_n the topology $\mathcal{W}_n$ induced by the product topology on $\overrightarrow{\Omega} \times \overrightarrow{\Omega}$.*

For a fixed x the set $U = \{y \,|\, x_j = y_j \text{ for all } j \leq n\} \cap G_n$ is open on G_n, that is, an element on $\mathcal{W}_n$.

Note that $G_n \cap (U \times U)$ is a subset of the diagonal.

Points of the form

$$(\,(x_1, x_2, ..., x_n, z_{n+1}, z_{n+2}, ...), (x_1, x_2, ..., x_n, z_{n+1}, z_{n+2}, ...)\,)$$

are on this intersection.

Then, the diagonal $\{(y, y)\,,\, y \in \overrightarrow{\Omega}\}$ is an open set in the inductive limit topology $\mathcal{K}$ over G

From this follows that the indicator function of the diagonal, that is, I_Δ, where $\Delta = \{(x, x) |\, x \in \overrightarrow{\Omega}\}$, is a continuous function.

Example 92. *Consider the partition* η_n, $n \in \mathbb{Z}$, *over* $\hat{\Omega}$ *of Example 79,* $W_n = G_n$, *for all* n, *and* $K = G_u$.

We consider the topology $\mathcal{W}_n$ *over* G_n *induced by the product topology. In this way* $A \in \mathcal{W}_n$ *if*

$$A = B \cap G_n,$$

where B *is an open set on the product topology for* $\hat{\Omega} \times \hat{\Omega}$.

In this way A *is open on* $t - \lim_{n \to -\infty} \mathcal{W}_n = \mathcal{K}$ *if for all* n *we have that*

$$A \cap X_n \in \mathcal{X}_n.$$

Denote by D *the diagonal on* $\hat{\Omega} \times \hat{\Omega}$ *and consider the indicator function* $I_D : \hat{\Omega} \times \hat{\Omega} \to \mathbb{R}$.

The function I_D *is continuous over the inductive limit topology* $\mathcal{K}$ *over* $K = G_u$.

Here G^0 will be the set $\hat{\Omega} = \overleftarrow{\Omega} \times \overrightarrow{\Omega}$. We will denote by G a general groupoid obtained by an equivalence relation R.

The measures we consider on this section are defined over the sigma-algebra generated by the inductive limit topology.

Definition 93. *Given a Haar system* (G, ν), *where* $G^0 = \hat{\Omega} = \overleftarrow{\Omega} \times \overrightarrow{\Omega}$ *is equipped with the inductive limit topology, considering two continuous functions with compact support* $f, g \in C_C(G)$, *we define* $(f \underset{\nu}{*} g) = h$ *in such way that for any* $(x, y) \in G$

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y)\ \nu^y(ds) = h(x, y).$$

The closure of the operators of left multiplication by elements of $C_C(G)$, $\{L_f : f \in C_C(G)\} \subseteq B(L^2(G, \nu))$, *with respect to the norm topology is called the reduced C*-algebra associated to* (G, ν) *and denoted by* $C_r^*(G, \nu)$.

Remark: There is another definition of a C*-algebra associated to (G, ν) called the full C*-algebra. For a certain class of groupoid, namely the amenable groupoids, the full and reduced C*-algebras coincide. See [3] for more details.

As usual function of the form $f(x, x)$ are identified with functions $f : G^0 \to \mathbb{C}$ of the form $f(x)$.

The collection of these functions is commutative sub-algebra of the C^*-algebra $C_r^*(G, \nu)$.

We denote by $\imath$ the indicator function of the diagonal on $G^0 \times G^0$. Then, $\imath$ is the neutral element for the product $\underset{\nu}{*}$ operation. Note that $\imath$ is continuous according to example 92.

In the case there exist a neutral multiplicative element we say the C^*-Algebra is unital.

Similar properties to the von Neumann setting can also be obtained.

We can define in analogous way to definition 44 the concept of C^*-dynamical state (which requires an unit $\imath$) and the concept of KMS state for a continuous modular function δ.

General references on the C^*-algebra setting are [53], [56], [17], [18], [19], [21], [22], [28], [51], [29] and [1] .

7 Examples of quasi-stationary probabilities

On this section we will present several examples of measured groupoids, modular functions and the associated quasi-stationary probability (KMS probability).

Example 94. *Considering the example 3 we get that each* $a \in \{1,2,...,d\}^{\mathbb{N}} = \overleftarrow{\Omega}$ *defines a class of equivalence*

$$a \times | \overrightarrow{\Omega} = a \times \{1,2,...,d\}^{\mathbb{N}} = (..., a_{-n}, ..., a_{-2}, a_{-1}) \times | \{1,2,...,d\}^{\mathbb{N}}.$$

On next theorem we will denote by G *such groupoid.*

Given a Haar system ν *over such* $G \subset \hat{\Omega} \times \hat{\Omega}$*, note that if* $z_1 =< a|b_1 >$ *and* $z_2 =< a|b_2 >$*, then* $\nu^{z_1} = \nu^{z_2}$*. In this way it is natural do index the Haar system by* ν^a*, where* $a \in \overleftarrow{\Omega}$*. In other words, we have*

$$\nu^{<a|b>}(d\ <a|\tilde{b}>) = \nu^a(d\,\tilde{b}). \tag{35}$$

Consider $V : G \to \mathbb{R}$*,* m *a probability over* $\overleftarrow{\Omega}$ *and the modular function* $\delta(x,y) = \frac{e^{V(x)}}{e^{V(y)}}$*, where* $(x,y) \in G$*.*

Finally we denote by $\mu_{m,\nu,V}$ *the probability on* $G^0 = \hat{\Omega}$*, such that, for any function* $g : \hat{\Omega} \to \mathbb{R}$ *and* $y =< a\,|\,b >$

$$\int g(y) d\mu_{m,\nu,V}(y) = \int_{\overleftarrow{\Omega}} \Big(\int_{\overrightarrow{\Omega}} g(<a|b>)\, e^{V(<a|b>)}\, d\nu^a(d\,b) \Big)\, dm(da).$$

Note that $\hat{\nu} = e^V \nu$ *is a* G*-kernel but maybe not transverse. The next theorem will provide a large class of examples of quasi-invariant probabilities for such groupoid* G*.*

Theorem 95. *Consider a Haar System (G, ν) for the groupoid of example 94. Then, given m, V, using the notation above we get that $M = \mu_{m,\nu,V}$ is quasi-invariant for the modular function $\delta(x, y) = \frac{e^{V(x)}}{e^{V(y)}}$.*

Proof:

From (6) we just have to prove that for any $f : G \to \mathbb{R}$

$$\int \int f(x, y)e^{V(x)}\nu^y(dx)d\mu_{m,\nu,V}(dy) = \int \int f(y, x)e^{V(x)}\nu^y(dx)d\mu_{m,\nu,V}(dy). \tag{36}$$

We denote $y =< a|b >$ and $x =< \tilde{a}|\tilde{b} >$. Note that if $y \sim x$, then $a = \tilde{a}$. Note that, from (35)

$$\int (\int f\,(x, y)\,e^{V(x)}\nu^y(dx)\ )\ d\mu_{m,\nu,V}(dy) =$$

$$\int (\int f\,(< \tilde{a}|\tilde{b} >, < a|b >)\,e^{V(<\tilde{a}|\tilde{b}>)}\nu^{<a|b>}(d < \tilde{a}|\tilde{b} >)\ d\mu_{m,\nu,V}(d\ < a|b >) =$$

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int \int f\,(< \tilde{a}|\tilde{b} >, < a|b >)\ e^{V(<\tilde{a}|\tilde{b}>)}\nu^{<a|b>}(d < \tilde{a}|\tilde{b} >)\ \ e^{V(<a|b>)}\,d\nu^a(d\,b)\,)\ ]\,d\,m(da) =$$

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int \int f\,(< a|\tilde{b} >, < a|b >)\ e^{V(<a|\tilde{b}>)}\ e^{V(<a|b>)}\,\nu^a(d\,\tilde{b})\,d\nu^a(d\,b)\,)\ ]\,d\,m(da).$$

In the above expression we can exchange the variables b and $\tilde{b}$, and, finally, as $a = \tilde{a}$, we get

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int \int f\,(< a|b >, < a|\tilde{b} >)\ e^{V(<a|b>)}\ e^{V(<a|\tilde{b}>)}\,\nu^a(d\,b)\,d\nu^a(d\,\tilde{b})\,)\ ]\,d\,m(da) =$$

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int \int f\,(< a|b >, < a|\tilde{b} >)\,e^{V(<a|\tilde{b}>)}\,d\nu^a(d\,\tilde{b})\ \ e^{V(<a|b>)}\,\nu^a(d\,b)\ )\ ]\,d\,m(da) =$$

$$(\int \int f\,(y,\ x)\,e^{V(x)}\ d\nu^y(dx)\ d\mu_{m,\nu,V}(dy).$$

This shows the claim.

□

Example 96. *Consider G associated to the equivalence relation given by the unstable manifolds for $\hat{\sigma}$ acting on $\hat{\Omega}$ (see example 4). Let's fix for good a certain $x_0 \in \overrightarrow{\Omega}$. Note that in the case $x =< a^1|b^1 >$ and $y =< a^2|b^2 >$ are on the same unstable manifold, then there exists an $N > 0$ such that $a^1_j = a^2_j$, for any $j < -N$. Therefore, when $\hat{A} : \hat{\Omega} \to \mathbb{R}$ is Holder and $(x, y) \in G$ then it is well defined*

$$\delta(x, y) = \Pi_{i=1}^{\infty} \frac{e^{\hat{A}(\hat{\sigma}^{-i}(x))}}{e^{\hat{A}(\hat{\sigma}^{-i}(y))}} = \Pi_{i=1}^{\infty} \frac{e^{\hat{A}(\hat{\sigma}^{-i}(<a^1|b^1>))}}{e^{\hat{A}(\hat{\sigma}^{-i}(<a^2|b^2>))}}.$$

Fix a certain $x_0 =< a^0, b^0 >$, then the above can also be written as

$$\delta(x,y) = \frac{e^{V(x)}}{e^{V(y)}} = \frac{e^{V(<a^1|b^1>)}}{e^{V(<a^2|b^2>)}},$$

where

$$e^{V(<a|b>)} = \Pi_{i=1}^{\infty} \frac{e^{\hat{A}(\hat{\sigma}^{-i}(<a|b>))}}{e^{\hat{A}(\hat{\sigma}^{-i}(<a^0|b^0>))}}.$$

Then, in this case δ is also of the form of example 27.

In this case, given any Haar system ν and any probability m, Theorem 95 can be applied and we get examples of quasi-invariant probabilities.

The next result has a strong similarity with the reasoning of [38] and [58].

Proposition 97. *Given the modular function δ of example 28 consider the probability $M(d\,a, d\,b) = W(b)\,d\,b\,d\,a$ on $S^1 \times S^1$. Assume ν^y, $y = (a_0, b_0)$, is the Lebesgue probability db on the fiber $(a_0, b), 0 \leq b < 1$, then, M satisfies for all f*

$$\int\int f(s,y)\nu^y(ds)dM(y) = \int\int f(y,s)\delta^{-1}(y,s)\,\nu^y(ds)dM(y). \qquad (37)$$

Proof:

We consider the equivalence relation: given two points $z_1, z_2 \in S^1 \times S^1$ they are related if the first coordinate is equal.

In the case of example 28 we take the a priori transverse function $\nu^{z_1}(d\,b) = \nu^a(d\,b)$, $z_1 = (a, \tilde{b})$, constant equal to $d\,b$ in each fiber. This corresponds to the Lebesgue probability on the fiber.

For each pair $z_1 = (a, b)$ and $z_2 = (a,\, s)$, and $n \geq 0$, the elements z_1^n, z_2^n, $n \in \mathbb{N}$, such that $F^n(z_1^n) = z_1 = (a, b)$ and $F^n(z_2^n) = z_2 = (a, s)$, are of the form $z_1^n = (a^n, b^n)$, $z_2^n = (a^n, s^n)$.

We define the cocycle

$$\delta(z_1, z_2) = \Pi_{j=1}^{\infty}\ \frac{A(z_1^n)}{A(z_2^n)}.$$

Fix a certain point $z_0 = (a, c)$ and define V by

$$V(z_1) = \Pi_{j=1}^{\infty}\ \frac{A(z_1^n)}{A(z_0^n)}.$$

Note that we can write

$$\delta(z_1, z_2) = \frac{V(z_1)}{V(z_2)},$$

for such function V.

Remember that by notation x_0 is a point where $(0, x_0)$ and $(x_0, 1)$ are intervals which are domains of injectivity of T.

Remark: Note the important point that if $x = (a, b)$ and $x' = (a', b)$, with $x_0 \leq a \leq a'$, we get that $b_n(x) = b_n(x')$. In the same way if $0 \leq a \leq x_0$ we get that $b_n(x) = b_n$. In this way the b_n does not depends on a.

This means, there exists W such that we can write

$$\delta(z_1, z_2) = \delta^{-1}(\,(a, b), (a, s)\,) = Q(s, b) = \frac{W(s)}{W(b)},$$

where $b, s \in S^1$.

Condition (37) for y of the form $y = (a, b)$ means for any f:

$$\int\int f(\,(a, b), (a, s)\,)\ \nu^a(d\,s)\ dM(a, b) =$$

$$\int\int f(\,(a, s), (a, b)\,)\ \delta^{-1}(\,(a, b), (a, s)\,)\ \nu^a(d\,s)\ dM(a, b) =$$

$$\int\int f(\,(a, s), (a, b)\,)\ Q(s, b)\ \nu^a(d\,s)\ dM(a, b).$$

Now, considering above $f(\,(a, b), (a, s)\,)V(s)$ instead of $f(\,(a, b), (a, s)\,)$, we get the equivalent condition: for any f:

$$\int\int f(\,(a, b), (a, s)\,)\, W(s)\, d\,s\ dM(a, b) =$$

$$\int\int f(\,(a, s), (a, b)\,)\ W(s)\ d\,s\ dM(a, b).$$

As $dM = W(b) d\,b\, da$ we get the alternative condition

$$\int\int f(\,(a, b), (a, s)\,)\, W(s)\, d\,s\ W(b) d\,b\, d\,a\ =$$

$$\int\int f(\,(a, s), (a, b)\,)\ W(s)\ d\,s\ W(b) d\,b\, d\,a\,, \tag{38}$$

which is true because we can exchange the variables b and s on the first term above.

□

Example 98. *Consider the groupoid G associated to the equivalence relation of example 5. In this case x and y are on the same class when there exists an $N > 0$ such that $x_j = y_j$, for any $j \geq N$. Each class has a countable number of elements.*

Consider a Holder potential $A : \overrightarrow{\Omega} \to \mathbb{R}$.

For $(x, y) \in G$ it is well defined

$$\delta(x, y) = \Pi_{i=0}^{\infty} \frac{e^{A(\sigma^i(x))}}{e^{A(\sigma^i(y))}}.$$

Consider the counting Haar system ν on each class.

We say $f : G \to \mathbb{R}$ is admissible if for each class there exist a finite number of non zero elements.

The quasi-invariant condition (5) for the probability M on $\overrightarrow{\Omega}$ means: for any admissible integrable function $f : G \to \mathbb{R}$ we have

$$\sum_s \int f(s, x) dM(x) = \sum_s \int f(x, s)\, \Pi_{i=0}^{\infty} \frac{e^{A(\sigma^i(s))}}{e^{A(\sigma^i(x))}}\, dM(x). \qquad (39)$$

Suppose B is such that $B = A + \log h - \log(g \circ \sigma) - c$. This expression is called a coboundary equation for A and B. Under this assumption, as $x \sim s$, we get

$$\sum_s \int f(x, s)\, \Pi_{i=0}^{\infty} \frac{e^{B(\sigma^i(x))}}{e^{B(\sigma^i(s))}}\, dM(x) =$$

$$\sum_s \int f(x, s)\, \Pi_{i=0}^{\infty} \frac{e^{A(\sigma^i(x))}}{e^{A(\sigma^i(s))}} \frac{h(x)}{h(s)}\, dM(x).$$

Take $f(s, x) = g(s, x)\, h(x)$, then, as M is quasi-invariant for A, we get that

$$\sum_s \int g(x, s)\, \Pi_{i=0}^{\infty} \frac{e^{B(\sigma^i(x))}}{e^{B(\sigma^i(s))}}\, h(x)\, dM(x) = \sum_s \int g(s, x)\, h(x)\, dM(x). \qquad (40)$$

As $g(x, s)$ is a general function we get that $h(x)\, dM(x)$ is quasi-invariant for B.

Any Holder function A is coboundary to a normalized Holder potential. In this way, if we characterize the quasi-invariant probability M for any given normalized potential A, then, we will be able to determine, via the corresponding coboundary equation, the quasi-invariant probability for any Holder potential.

References

[1] Z. Afsar, A. Huef and I. Raeburn, KMS states on C^*-algebras associated to local homeomorphisms, Internat. J. Math. 25 (2014), no. 8, 1450066, 28 pp

[2] C. Anatharaman-Delaroche, Ergodic Theory and Von Neumann algebras: an introduction, preprint Univ d'Orleans (France)

[3] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, Monographs of L'Enseignement Mathmatique, 36. L'Enseignement Mathmatique, Geneva, 2000. 196 pp.

[4] H. Araki, On the Equivalence of KMS and Gibbs Conditions for States of Quantum Lattice Systems, Commun. Math. Phys, 35, 1–12 (1974)

[5] T. Banakh, Direct Limit topologies and a characterization of LF-spaces

[6] A. Baraviera, L. M. Cioletti, A. O. Lopes, J. Mohr and R. R. Souza, On the general XY Model: positive and zero temperature, selection and non-selection, *Rev. in Math Phys*, Vol. 23, N. 10, pp 1063-1113 (2011).

[7] R. Bissacot and B. Kimura, Gibbs Measures on Multidimensional Subshifts, preprint (2016)

[8] B. Blackadar, Operator algebras. Theory of C^*-algebras and von Neumann algebras. Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.

[9] O. Bratteli and D. Robinson, Operator Algebras and Quantum Statistical Mechanics I and II, Springer Verlag.

[10] L. Cioletti, M. Denker, A. O. Lopes and M. Stadlbauer, Spectral Properties of the Ruelle Operator for Product Type Potentials on Shift Spaces, Journal of the London Mathematical Society, Volume 95, Issue 2, 684–704 (2017)

[11] L. Cioletti and A. O. Lopes. Interactions, specifications, DLR probabilities and the Ruelle operator in the one-dimensional lattice, Discrete and Cont. Dyn. Syst. - Series A, Vol 37, Number 12, 6139 – 6152 (2017)

[12] A. Connes, Sur la Theorie non commutative de l'integration, Lect. Notes in Math. 725, Semminaire sur les Algebres d'Operateurs, Editor P. de la Harpe, 19–143 (1979)

[13] A. Connes, Noncommutative Geometry, Academic Press (1994)

[14] V. Deaconu. Groupoids associated with endomorphisms. Trans. Amer. Math. Soc., 347(5):1779-1786, (1995).

[15] G. DellAntonio, Lectures on the Mathematics of Quantum Mechanics II, Atlantis Press (2016)

[16] J. Diximier, Von Neumann Algebras, North Holland Publishing (1981)

[17] R. Exel, A new look at the crossed-product of a C^*-algebra by an endomorphism. Ergod. Th. and Dynam. Sys. 23(6) (2003), 1733-1750.

[18] R. Exel, Crossed-products by finite index endomorphisms and KMS states. J. Funct. Anal. 199(1) (2003), 153-188.

[19] R. Exel, KMS states for generalized gauge actions on Cuntz-Krieger algebras (An application of the Ruelle-Perron-Frobenius Theorem), Bol. Soc. Brasil. Mat. (2004)

[20] R. Exel, Inverse semigroups and combinatorial C^*-algebras, Bull. Braz. Math. Soc. (N.S.), 39 (2008), 191–313.

[21] R. Exel and A. Lopes, C^*-algebras, approximately proper equivalence relations and thermodynamic formalism, Ergod. Theo. and Dynam. Sys., 24, 1051-1082 (2004)

[22] R. Exel and A. Lopes, C^*- Algebras and Thermodynamic Formalism, Sao Paulo Journal of Mathematical Sciences - Vol. 2, 1, 285–307 (2008)

[23] J. Feldman and C. Moore, Ergodic equivalence relations, cohomologies, von Neumann algebras I, TAMS 234 (1977) 289-359.

[24] J. Feldman and C. Moore, Ergodic equivalence relations, cohomologies, von Neumann algebras II, TAMS 234 (1977).

[25] P. Hahn, The regular representations of measure groupoids. Trans. Amer. Math. Soc. 242 (1978), 35–72

[26] N. T. A. Haydn and D. Ruelle, Equivalence of Gibbs and Equilibrium states for homeomorphisms satisfying expansiveness and specification, Comm. in Math. Phys., 148, 155-167, 1992

[27] D. Kastler, On Connes' Noncommutative Integration Theory, Comm. in Math. Physics, v 85, 99–120 (1982)

[28] M. Kessebohmer, M. Stadlbauer and B. Stratmann, Lyapunov spectra for KMS states on Cuntz-Krieger algebras. Math. Z. 256 (2007), no. 4, 871–893.

[29] A. Kumjian and J. Renault, KMS states on C^*-Algebras associated to expansive maps, Proc. AMS Vol. 134, No. 7, 2067–2078 (2006)

[30] A. Kishimoto and A. Kumjian, Simple stably projectionless C^*-algebras arising as crossed products. Canad. J. Math. 48 (1996), no. 5, 980–996.

[31] N. Landsman, Lecture Notes on C^*-Algebras and Quantum Mechanics, Univ. of Amsterdam (1998)

[32] F. Ledrappier and L.-S. Young, The Metric Entropy of Diffeomorphisms: Part I: Characterization of measures satisfying Pesin's Entropy Formula, Annals of Mathematics, Vol. 122, No. 3, 509–539 (1985)

[33] A. O. Lopes and G. Mantovani, The KMS Condition for the homoclinic equivalence relation and Gibbs probabilities, Sao Paulo Journ. of Math. Sciences, Volume 13, Issue 1, pp 248-282 (2019)

[34] A. O. Lopes and E. Oliveira, Continuous groupoids on the symbolic space, quasi-invariant probabilities for Haar systems and the Haar-Ruelle operator, Bull of the Braz. Math. Soc. Volume 50, Issue 3, pp 663-683 (2019)

[35] A. O. Lopes and J. K. Mengue, Thermodynamic Formalism for Haar systems in Noncommutative Integration: transverse functions and entropy of transverse measures, Erg. Theory and Dyn Systems, 41, 1835-1863 (2021)

[36] A. Lopes, J. K. Mengue, J. Mohr and R. R. Souza, Entropy and variational principle for one-dimensional Lattice systems with a general a-priori probability: positive and zero temperature, Erg. Theory and Dyn Systems, 35 (6), 1925–1961 (2015)

[37] A. Lopes and J. Mengue, On information gain, Kullback-Leibler divergence, entropy production and the involution kernel, to appear in Disc. Cont. Dyn. Systm. Series A

[38] G. Mantovani, Teoria não comutativa de integração e dinâmica hiperbólica, Dissertação de Mestrado, ICMC-USP - São Carlos - Brasil (2013)

[39] K. Matsumoto, Continuous orbit equivalence of topological Markov shifts and KMS states on Cuntz–Krieger algebras, arXiv (2019)

[40] S. Muir and M. Urbanski. Thermodynamic formalism for a modified shift map, *Stochastics and Dynamics*, Vol. 14, No. 02, 1350020 (2014)

[41] B. Miller, The existence of measures of a given cocycle. I. Atomless, ergodic -finite measures. Ergodic Theory Dynam. Systems 28 (2008), no. 5, 1599-1613.

[42] B. Miller, The existence of measures of a given cocycle. II. Probability measures. Ergodic Theory Dynam. Systems 28 (2008), no. 5, 1615-1633

[43] T. Meyerovitch, Gibbs and equilibrium measures for some families of subshifts, Ergod. Th. and Dynam. Sys., 33, 934-953, (2013)

[44] J. Mohr, Product type potential on the XY model: selection of maximizing probability and a large deviation principle, arXiv (2019)

[45] M. O'Uchi, Measurable groupoids and associated von Neumann algebras, Ehime University Notes (1984)

[46] J. Panangaden, Energy Full Counting Statistics in Return-to-Equilibrium Jane Panangaden, thesis McGill Univ (2016)

[47] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque* Vol 187-188 (1990).

[48] M. Pollicott and M. Yuri, Dynamical systems and Ergodic Theory, Cambrige Press, 1998

[49] G. K. Pedersen, C^*-Algebras and and their automorphism groups, Acad Press, (1979)

[50] I. Putnam, Lecture Notes on Smale Spaces, Univ. of Victoria - Canada (2015)

[51] I. Putnam and J. Spielberg, The Structure of C^*-Algebras associated with hyperbolic dynamical systems, J. Funct. Anal. 163 (1999), no. 2, 279–299

[52] I. Putnam, Lecture Notes on C^*-algebras (2019)

[53] J. Renault, A Groupoid approach to C^*-algebras, Lecture Notes in Mathematics 793, Springer-Verlag, (1980)

[54] J. Renault, AF equivalence relations and their cocycles, Operator algebras and mathematical physics (Constanca, 2001), 365–377, Theta, Bucharest, (2003).

[55] J. Renault, The Radon-Nikodym problem for appoximately proper equivalence relations. Ergodic Theory Dynam. Systems 25 (2005), no. 5, 1643–1672.

[56] J. Renault, C^*-Algebras and Dynamical Systems, XXVII Coloquio Bras. de Matematica - IMPA (2009)

[57] D. Ruelle, Noncommutative algebras for hyperbolic diffeomorphisms, Inv. Math. 93, 1-13 (1988)

[58] J. Segert, Hyperbolic Dynamical Systems and the Noncommutative Theory of Connes, PhD Thesis, Department of Physics, Princeton University (1987)

[59] L. Smith, Chaos: A Very Short Introduction, Cambridge Univ. Press

[60] M. Takesaki, Tomita's Theory of Modular Hilbert Algebras and its Applications, Lecture Notes in Mathematics 128, Springer-Verlag, (1970)

[61] K. Thomsen, The homoclinic and heteroclinic C^*-algebras of a generalized one-dimensional solenoid, Ergodic Theory Dynam. Systems 30, no. 1, 263–308 (2010).

[62] K. Thomsen, C^*-algebras of homoclinic and heteroclinic structure in expansive dynamics. Mem. Amer. Math. Soc. 206, no. 970 (2010)

[63] K. Thomsen, On the C^*-algebra of a locally injective surjection and its KMS states. Comm. Math. Phys. 302, no. 2, 403–423 (2011)

[64] K. Thomsen, Phase transitions on O_2, Comm. Math. Phys, 349 (2), 481-492 (2017)

[65] K. Thomsen, KMS-states and conformal measures, Comm. Math. Phys. 316 (2012), 615-640

[66] K. Thomsen, KMS weights on groupoid and graph C^*-algebras, J. Func. Analysis 266 (2014), 2959-2998.

[67] K. Thomsen, KMS weights on graph C^*-algebras II. Factor types and ground states, Arxiv

[68] A. Weinstein, Groupoids: unifying internal and external symmetry. A tour through some examples. Notices Amer. Math. Soc. 43 (1996), no. 7, 744-752