Idempotent approach to level-2 variational principles in Thermodynamical Formalism

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Abstract

In this work, we adapt the idempotent formalism to the level-2 setting, introducing the concept of idempotent pressure and its associated density entropy at level-2. In a broad sense, level-2 corresponds to the study of potentials or probabilities defined on the set of probabilities. The idempotent pressure is a natural concept that corresponds to the meaning of measure in the level-2 Max-Plus context. Given an idempotent pressure and a potential g, we investigate the equilibrium states which maximize an associated variational principle akin to the topological pressure; they are level-1 probabilities and not necessarily unique. Our results can be seen as part of Tropical Geometry. We also study a level-2 mpIFS operator, which acts on idempotent pressures and is defined from a family of dual Ruelle operators as the maps of the IFS acting on probabilities; requiring the control of a uniform bound of contraction for an uncountable family of classical dual-Ruelle operators. We prove the existence of a unique idempotent pressure which is fixed under its action. This idempotent pressure is also fixed by the action of the push-forward map.

1 Introduction

In this paper we aim to define an idempotent analysis approach to level-2 variational principles in Thermodynamical Formalism. Among other issues, we investigate the resemblances and dissimilarities of our setting with the general convex pressure framework introduced recently for level-1 functions in [BCMV22].Some of our results consider an Iterated Function System defined by a family of Ruelle (also known as transfer) operators in symbolic dynamics.

The use of Idempotent Analysis is a reasonable choice, because as we will see it encompasses many of the classical constructions of thermodynamic formalism. Also, entropy and pressure are variational concepts which can be modeled in an idempotent framework, in such a way that its properties will derive from the fundamental theorems of Idempotent Analysis (the Max-Plus setting). Idempotent analysis is also known as Tropical Geometry (see [Mas16]). A. Connes and

C. Consani in their seminal paper [Con11] propose to contrast two branches of science: Real Analysis $\to$ Idempotent Analysis on the one hand, and Geometry over $\mathbb{C} \to$ Tropical Geometry on the other; both topics of great importance in applications.

Basic properties of the Max-Plus algebra can be found in chapter 6 in [BLL13] and [Kol01] (see also Section 5.2 in [Gar17]). The idempotent mathematics utilizes the idempotent semiring $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ endowed with the operations $\oplus := \max$ and $\odot := +$, where we assume $-\infty \oplus a = a$ and $-\infty \odot a = -\infty \ \forall a \in \mathbb{R}_{\max}$. Note that $r \oplus r = \max\{r, r\} = r, \forall r \in \mathbb{R}_{\max}$ and the neutral elements for $\oplus$ and $\odot$ are, respectively, $-\infty$ and 0.

The study of optimization problems was the motivation for the introduction of the Idempotent Analysis by Kolokol'tsov and Maslov in [KM89]. The idempotent formalism allows one to handle in a linear way, non-linear problems occurring in mathematical physics. The use of properties obtained in the Max-Plus setting in several areas of applications is described in [Litvi02], [Litvi05], and [Litvi07]. For example, as mentioned in Section 6 in [Litvi07], the representation of a solution of the Schrödinger Schrödinger equation in terms of the Feynman integral corresponds to the Lax-Olenik solution formula for the Hamilton-Jacobi equation (see [Fey65]). Given a pair of quantum actions S_1 and S_2, obtained from a quantum system associated to a certain Planck constant $\hbar$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, in the semiclassical limit, when $\hbar \to 0$, one gets that $\lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ is a solution of the Hamilton-Jacobi equation and this equation can be treated as linear in the Max-Plus sense (see [Roublev05], [Baca01], [Chou87] and [Flem00]). Maslov Dequantization is associated to a natural passage from standard quantum mechanics (standard algebra) to the Max-Plus algebra. Moreover, the study of the statistical mechanics properties of cyclic and polyhedral water clusters via the transfer-matrix and the Max-Plus algebra method was the topic of [Kirov09]. In the study of ground states (temperature zero), it is natural to introduce the Max-Plus formalism (see chapter 6 in [BLL13], Section 5.2 in [Gar17], [Chou86] and [Kol01]). Recent results appear in [MO25], [Li25] and [LM26]. In Appendix 8 we will elaborate on the relation of our work with Physics and other areas of applications.

In [MO24], via the idempotent framework, the authors addressed the issue of idempotent measures for max-plus iterated function systems (mpIFS). As a tool, a representation for idempotent probabilities on compact metric spaces was proved there and we will use it here.

Among other things we will show that adapting the idempotent formalism of [MO24] to the level-2 setting, we can use idempotent probabilities to model variational principles associated to the non-linear thermodynamic formalism framework. Level-1 properties are related to points in a compact metric space X and the variational principles are defined for continuous functions $\varphi : X \to \mathbb{R}$. On the other hand, a level-2 property is related to points in $\mathcal{P}(X)$ (the space of Borel probabilities on X equipped with the weak* topology) and the variational principles are defined for continuous functions $g : \mathcal{P}(X) \to \mathbb{R}$. A simple example of a continuous function $g : \mathcal{P}(X) \to \mathbb{R}$ is

$$g_A(\mu) = \int_X A\, d\mu, \text{ where } A : X \to \mathbb{R} \text{ is continuous and fixed.}$$

In ergodic theory, questions at level-2 refer to properties related to the global study of the set of different probabilities on a given compact metric space X. For example, in [Lop90] the author studies large deviations in the set of probabilities over the symbolic space with a finite number of symbols, and minus entropy plays the role of a deviation function, while in [LO24], the authors study thermodynamic formalism, when the dynamics is given by the push-forward map acting on $\mathcal{P}(X)$; a form of Ruelle operator is introduced and a kind of entropy was defined.

Given a compact metric space (X, d), the space $\mathcal{P}(X)$ of probabilities on the Borel sigma-algebra of X is a compact space with respect to the weak* topology. One can metrize such a topology as in [Vil03], by choosing the 1-Wasserstein (or Monge-Kantorovich) metric

$$W_1(\mu, \nu) = \sup_{\mathrm{Lip}(f) \leq 1} \mu(f) - \nu(f), \tag{1}$$

where $\mathrm{Lip}(f) = \sup_{x \neq y} \frac{f(x)-f(y)}{d(x,y)}$. Indeed, it is widely known that $(\mathcal{P}(X), W_1)$ is a compact metric space, as a consequence of the Banach-Alaoglu theorem.

We denote by $C(X, \mathbb{R})$ the space of continuous functions from X to $\mathbb{R}$ and by $C(\mathcal{P}(X), \mathbb{R})$ the space of continuous functions on $\mathcal{P}(X)$ taking values in $\mathbb{R}$. For $g, g' \in C(\mathcal{P}(X), \mathbb{R})$ we associate $g \oplus g' \in C(\mathcal{P}(X), \mathbb{R})$ defined by $(g \oplus g')(\mu) = g(\mu) \oplus g'(\mu)$ and $g \odot g' \in C(\mathcal{P}(X), \mathbb{R})$ defined by $(g \odot g')(\mu) = g(\mu) \odot g'(\mu)$.

Definition 1. *A (level-2)* ***idempotent pressure function****, is a max-plus linear functional $\ell : C(\mathcal{P}(X), \mathbb{R}) \to \mathbb{R}$, that is, the following axioms are fulfilled for any $c \in \mathbb{R}$ and $g, g' \in C(\mathcal{P}(X), \mathbb{R})$*

- *Axiom A1*
$$\ell(c \odot g) = c \odot \ell(g) \tag{2}$$
- *Axiom A2*
$$\ell(g \oplus g') = \ell(g) \oplus \ell(g'). \tag{3}$$

From Riesz Theorem, measures can be seen as classical linear operators acting on the set of continuous functions. In the idempotent formalism, ℓ acts as a Max-Plus linear functional on the set $C(\mathcal{P}(X), \mathbb{R})$; so it is fair to say that idempotent pressure functions play the role of measures in the level-2 Max-Plus context.

Any idempotent pressure $\ell : C(\mathcal{P}(X), \mathbb{R}) \to \mathbb{R}$ is particularly monotonic (from Axiom A2), translation invariant (from Axiom A1) and therefore Lipschitz continuous. Indeed, if $f - \epsilon \leq g \leq f + \epsilon$, then

$$\ell(f) - \epsilon = \ell(f - \epsilon) \leq \ell(g) \leq \ell(f + \epsilon) = \ell(f) + \epsilon.$$

The terminology *pressure* used above in Definition 1 is in line with the next theorem which is a consequence of Theorem 1.2 in [MO24]. It presents a characterization of the idempotent pressure function as a variational principle.

Theorem 2. *Let (X,d) be a compact metric space and $\mathcal{P}(X)$ be the set of probabilities over the Borel sigma algebra. Consider $\mathcal{P}(X)$ as a metric space with any metric equivalent to the weak-* topology.*

If $\ell : C(\mathcal{P}(X),\mathbb{R}) \to \mathbb{R}$ is an idempotent pressure function, then there exists a unique upper semi-continuous (u.s.c.) function $h : \mathcal{P}(X) \to \mathbb{R}_{\max}$ such that

$$\ell(g) = \sup_{\mu\in\mathcal{P}(X)} [g(\mu) + h(\mu)], \tag{4}$$

for any $g \in C(\mathcal{P}(X),\mathbb{R})$. Reciprocally, if $h : \mathcal{P}(X) \to \mathbb{R}_{\max}$ is bounded above and it is not identically $-\infty$ then equation (4) defines an idempotent pressure function.

With the introduction of the above result, the following definition is natural.

Definition 3. *Let (X,d) be a compact metric space, $\ell : C(\mathcal{P}(X),\mathbb{R}) \to \mathbb{R}$ be an idempotent pressure and $h_\ell : \mathcal{P}(X) \to \mathbb{R}$ be the unique u.s.c. function satisfying (4). We say that h_ℓ is the* ***density entropy*** *associated to the idempotent pressure function ℓ. Moreover, given ℓ, we call any probability $\nu \in \mathcal{P}(X)$ attaining the supremum, that is,*

$$\ell(g) = h_\ell(\nu) + g(\nu), \tag{5}$$

an ***equilibrium state*** *associated to the idempotent pressure function ℓ.*

We remark that the existence of equilibrium states associated to the idempotent pressure function ℓ is a consequence of the fact that $\mathcal{P}(X)$ is compact and the map $\mu \mapsto [g(\mu) + h_\ell(\mu)]$ to be u.s.c. In this way the supremum in (4) is in fact a maximum. Given ℓ, in the broad framework portrayed by Definition 3, the concept of density entropy h_ℓ does not necessarily have to be of a dynamical nature.

In section 2 we present a short exposition of idempotent measures and in Section 3 we present examples. Some examples illustrate the connections with classical constructions in ergodic theory and thermodynamic formalism. It is simple to exhibit in our setting examples where the equilibrium state is not unique and moreover the set of equilibrium states is not convex (see Example 11). We exhibit also how the idempotent pressure and its variational characterization can be used to model non-linear dependencies of the potential.

Our main results are presented in Sections 4 and 5. In [BCMV22] the authors consider a generalization of thermodynamic formalism via convex analysis; not primarily aimed to analyze level-2 questions.

In accordance with [BCMV22], we call a function $\Gamma : C(X,\mathbb{R}) \to \mathbb{R}$ of a (level-1) convex pressure function if it satisfies:

i. $\varphi \leqslant \psi \Rightarrow \Gamma(\varphi) \leqslant \Gamma(\psi) \quad \forall \varphi,\psi \in C(X,\mathbb{R})$;

ii. $\Gamma(\varphi + c) = \Gamma(\varphi) + c \quad \forall \varphi \in C(X,\mathbb{R}) \quad \forall c \in \mathbb{R}$;

iii. $\Gamma(t\varphi + (1-t)\psi) \leqslant t\Gamma(\varphi) + (1-t)\Gamma(\psi) \quad \forall \varphi, \psi \in C(X,\mathbb{R}) \quad \forall t \in [0,1]$.

In this case, following [BCMV22] and [BCM+23], the function $\mathfrak{h} : \mathcal{P}(X) \to \mathbb{R}_{\max}$ defined by

$$\mathfrak{h}(\mu) = \inf_{\varphi \in C(X,\mathbb{R})} \left[\Gamma(\varphi) - \int \varphi d\mu \right]$$

is a concave and upper semi-continuous function satisfying

$$\Gamma(\varphi) = \sup_{\mu \in \mathcal{P}(X)} \left[\int \varphi d\mu + \mathfrak{h}(\mu) \right] \quad \forall \varphi \in C(X,\mathbb{R}).$$

We will exhibit the connection between a level-2 idempotent pressure and a level-1 convex pressure by proving the following theorem.

Theorem 4. *Consider an idempotent pressure function ℓ with density entropy h_ℓ and the canonical inclusion $j : C(X,\mathbb{R}) \to C(\mathcal{P}(X),\mathbb{R})$ given by $j(\varphi)(\mu) = \int_X \varphi(x) d\mu(x)$, $\mu \in \mathcal{P}(X)$. If we define $\Gamma_\ell : C(X,\mathbb{R}) \to \mathbb{R}$ by*

$$\Gamma_\ell(\varphi) := \ell(j(\varphi)),$$

then Γ_ℓ is a convex pressure function. Furthermore $h_\ell \leq \mathfrak{h}$. Reciprocally, each convex pressure function is of the form Γ_ℓ where ℓ is an idempotent pressure function. If X is not a singleton, such map $\ell \mapsto \Gamma_\ell$ is not injective.

Suppose that an idempotent pressure ℓ has a concave density entropy h_ℓ. Then for all $\mu \in \mathcal{P}(X)$ we have

$$h_\ell(\mu) = \inf_{g \in C(\mathcal{P}(X),\mathbb{R})} [\ell(g) - g(\mu)]. \tag{6}$$

Finally, considering Γ_ℓ and its entropy $\mathfrak{h}$ (given in Theorem 19) we have $\mathfrak{h} = h_\ell$.

We will prove this theorem and explain the connection between the setting in [BCMV22] and the present level-2 setting in Section 4. We point out that the results we derive are independent and distinct from those in [BCMV22]. For example, our setting covers, including, among others, the non-linear thermodynamic formalism (see Examples 12 and 13). The setting in [BCMV22] could also be applied to level-2 functions in $C(\mathcal{P}(X),\mathbb{R})$, introducing in this case the variational principle

$$\sup_{\pi \in \mathcal{P}(\mathcal{P}(X))} \int g(\mu)\, d\pi(\mu) \; + \mathfrak{h}(\pi), \; g \in C(\mathcal{P}(X),\mathbb{R}), \tag{7}$$

where $\mathfrak{h}$ is a u.s.c and concave function defined on $\mathcal{P}(\mathcal{P}(X))$. However, Theorem 3 yields equilibrium states in $\mathcal{P}(X)$, whereas the equilibrium states of (7) belong to $\mathcal{P}(\mathcal{P}(X))$. This difference is mathematically pertinent since, in general, the space $\mathcal{P}(X)$ is much simpler than $\mathcal{P}(\mathcal{P}(X))$ as well as is the variational principle (4) instead of (7). Nevertheless, we prove in Proposition 25 that an idempotent pressure is a particular case of level-2 convex pressure. We do not consider probabilities on the set of probabilities.

In Section 5 we present characterizations of the entropy in order to get an idempotent pressure invariant for a dynamical system or a transfer operator. In [MO24], mpIFS and max-plus transfer operators were studied. In [MO25] such theory was applied to the study of zero temperature limits and large deviations in temperature for Gibbs probabilities of IFS. We will apply the results of [MO24] for a level-2 max-plus transfer operator (see equation (8)), defined by an IFS of level-2 maps $\{L_J^* : \mathcal{P}(X) \to \mathcal{P}(X) \,|\, J \in \mathcal{D}\}$ (see definitions in Theorem 5) where each L_J^* is a dual Ruelle operator and X is a full shift. Our main result of Section 5 is the following.

Theorem 5. *Consider the shift map σ over the compact metric space (Ω, d_γ), where $\Omega = \{1, ..., d\}^{\mathbb{N}}$, $0 < \gamma < \frac{1}{d+1}$ and (denoting $x = (x_0, x_1, x_2, ...)$ and $y = (y_0, y_1, y_2, ...)$) the metric d_γ satisfies $d_\gamma(x, y) = \gamma^{\min\{j \in \mathbb{N} | x_j \neq y_j\}}$, if $x \neq y$. Let*

$$\mathcal{D} = \{J : \Omega \to [0, 1] \,|\, \mathrm{Lip}(J) \leq 1 \text{ and } \sum_{a=1}^{d} J(ax) = 1 \,\forall x \in \Omega\}.$$

Consider, for each $J \in \mathcal{D}$, the dual of the Ruelle Operator $L_J^* : \mathcal{P}(X) \to \mathcal{P}(X)$, that is, $L_J^*(\mu) = \nu$ means that $\int f(x)\, d\nu(x) = \int \sum_{a=1}^{d} J(ax) f(ax)\, d\mu(x)$.
1. For each sequence $(J_1, J_2, J_3 ...)$ of elements of $\mathcal{D}$ there exists a unique probability measure $\mu \in \mathcal{P}(X)$, such that

$$\mu = \lim_{n\to\infty} L_{J_1}^* \circ \cdots \circ L_{J_n}^*(\nu),$$

for any probability $\nu \in \mathcal{P}(X)$.
2. Consider $\mathcal{D}$ as a metric space with the supremum norm and for a fixed non-empty closed subset $D \subseteq \mathcal{D}$ let $q : D \to \mathbb{R}$ be a continuous function, such that, $\sup_{J\in D} q_J = 0$. Consider the operator $\mathcal{M} : I(P(X)) \to I(P(X))$, acting over idempotent pressures, which is given by

$$\mathcal{M}(\ell)(f) := \bigoplus_{J\in\mathcal{D}} q_J \odot \ell(f \circ L_J^*). \tag{8}$$

There exists a unique idempotent pressure function ℓ satisfying $\ell(0) = 0$ and invariant for $\mathcal{M}$. Its density entropy is given by

$$h_\ell(\mu) = \bigoplus_{\substack{(J_1, J_2, J_3, ...) \in D^{\mathbb{N}} \text{ such that} \\ L_{J_1}^* \circ \cdots \circ L_{J_n}^* \to \mu}} [q_{J_1} + q_{J_2} + q_{J_3} + \cdots]$$

and $h_\ell(\mu) = -\infty$ if no such sequence $(J_1, J_2, J_3, ...)$ exists.

This level-2 mpIFS has an infinite number of maps $\{L_J^* : \mathcal{P}(X) \to \mathcal{P}(X) \,|\, J \in \mathcal{D}\}$, requiring a uniform bound of contraction for an uncountable family of operators.

We point out that in the literature, to the best of our knowledge, there are not many examples of probabilities that are invariant for the push-forward map (which acts on the set of probabilities); a dynamics at level-2. In [LO24] the authors provide some new trivial examples, within the framework of classical algebra. Results related to the topological properties of the dynamics of the push-forward map appear in [Rod12], [BS75] and [BV16]. In this work, with different methods, we get idempotent measures, which are invariant for the push-forward map, which are produced via a formalism akin to the use of transfer operators of Statistical Mechanics (see Section 5). Our setting generalizes the classical one which is well-known at level-1. The idempotent formalism is well suited to describe the main protagonists of this more general point of view.

In systems with long-range interactions, every particle interacts with the aggregate state of the system, requiring a non-linear approach. The idempotent formalism encompasses the non-linear thermodynamic formalism which is closely related to the study of the Curie-Weiss model, a topic of great importance in Statistical Mechanics, see [FV18], [Ku24], [BH22], [BKL23], [LW19], [LW20] and [BdSPL25]. With the goal of clarifying this topic for the reader, in Example 12, obtaining probabilities on $\mathcal{P}(X)$, we develop this topic (for a non-dynamical framework) in a slightly simpler formulation; we produce explicit solutions. In [Kos18] and [BdSPL25] results for non-linear equilibrium probabilities are obtained via the use of large deviation properties for the classical pressure. For a dynamical setting, in [BdSPL25] and [BPL26], functionals which are convex or concave, when defined in the set of shift invariant probabilities, are considered (see the idempotent pressure function (16) for a simple example fitting this formalism), and the use of the Bogoliubov's approximation method (see [Bogo66]) helps to determine non-linear equilibrium probabilities (maybe more than one). Our Example 13 does not fit into the linear (or, in the non-linear, as described in Definition 16) Thermodynamic Formalism.

In Section 6 we analyze the inverse problem of finding a max-plus IFS for which a given idempotent pressure function is invariant. We also study in Section 7, of independent interest, some max-plus dynamical aspects involving the max-plus averages of a dynamical system. The main result of Section 7 is Proposition 45. In Section 8 we describe some applications of the Max-Plus setting (and the corresponding algebra) to mathematical physics.

2 Idempotent measures

In this section we present a short discussion concerning idempotent measures and also explain how the Level-2 approach and Theorem 2 can be considered as applications of idempotent mathematics to the space of continuous functions on probabilities.

Given a compact metric space (Z, d), consider the set $C(Z, \mathbb{R})$ of continuous functions on Z.

Definition 6. *A function $m : C(Z, \mathbb{R}) \to \mathbb{R}$ is an idempotent (or Maslov) measure over Z if*

- $m(c \odot f) = c \odot m(f)$, *$c \in \mathbb{R}$ and $f \in C(Z, \mathbb{R})$;*

- $m(f \oplus f') = m(f) \oplus m(f'),\ f, f' \in C(Z, \mathbb{R})$.

The set of idempotent probabilities over Z, denoted $I(Z)$, is the set of all idempotent measures m satisfying $m(0) = 0$.

The set of all idempotent measures is the max-plus dual of $C(Z, \mathbb{R})$. The idempotent pressure in Definition 1 corresponds to a Maslov measure when $Z = \mathcal{P}(X)$.

The next result was proved in [BRZ10].

Theorem 7. *[BRZ10] $I(Z)$ endowed with the pointwise convergence topology is compact.*

Definition 8. *We denote by $U(Z)$ the set of all u.s.c. functions λ taking values in $\mathbb{R}_{\max}$ such that $\lambda(z_0) > -\infty$ for some $z_0 \in Z$. In other words* $\mathrm{supp}(\lambda) = \{z | \lambda(z) > -\infty\} \neq \varnothing$.

A representation theorem for idempotent measures, similar to [KM89] was proved in [MO24].

Theorem 9. *[MO24, Theorem 1.2] Let (Z, d) be a compact metric space. A function $m : C(Z, \mathbb{R}) \to \mathbb{R}$ is an idempotent measure if, and only if, there is $\lambda \in U(Z)$ such that*

$$m(f) = \sup_{z \in Z} [\lambda(z) + f(z)], \tag{9}$$

for any $f \in C(Z, \mathbb{R})$. Moreover, such function λ is unique in $U(Z)$ and $m \in I(Z)$ if, and only if, $\sup_{z \in Z} \lambda(z) = 0$.

It is worth noticing that an analogous result was previously stated for separable locally compact topological spaces; the original work [KM89], considered functionals acting on continuous functions, tending to zero at infinity and with compact support, taking image in a metric semiring. The setting in [KM89] is not exactly the same as in [MO24].

We can now prove Theorem 2.

Proof of Theorem 2. In our case, the hypothesis ensures that $Z = \mathcal{P}(X)$ is a compact metric space, thus the idempotent pressure function ℓ is actually an idempotent measure on $\mathcal{P}(X)$. By Theorem 9 we obtain the representation in Equation (4). Reciprocally, if $h : \mathcal{P}(X) \to \mathbb{R}_{\max}$ is bounded above and it is not identically $-\infty$, then defining $\ell : C(\mathcal{P}(X), \mathbb{R}) \to \mathbb{R}$ by $\ell(g) = \sup_{\mu \in \mathcal{P}(X)} [g(\mu) + h(\mu)]$, we get an idempotent pressure functional. Indeed,

$$\ell(g_1 \oplus g_2) = \sup_{\mu \in \mathcal{P}(X)} [(g_1 \oplus g_2)(\mu) + h(\mu)] = \sup_{\mu \in \mathcal{P}(X)} [g_1(\mu) \oplus g_2(\mu) + h(\mu)]$$

$$= \sup_{\mu \in \mathcal{P}(X)} [[\max_{i \in \{1,2\}} g_i(\mu)] + h(\mu)] = \max_{i \in \{1,2\}} \sup_{\mu \in \mathcal{P}(X)} [g_i(\mu) + h(\mu)] = \ell(g_1) \oplus \ell(g_2)$$

and

$$\ell(c \odot g) = \sup_{\mu \in \mathcal{P}(X)} [c + g(\mu) + h(\mu)] = c + \sup_{\mu \in \mathcal{P}(X)} [g(\mu) + h(\mu)] = c \odot \ell(g).$$

□

Remark 10. *A version of Theorem 2 for idempotent pressure functions on topological spaces can be constructed (see [KM89, Theorem 1]). In this case, the original work [KM89], considered bounded maps on separable locally compact topological spaces taking value in some semiring, tending to infinity at infinity and with compact support. In that work they considered the semimodule of linear (max-plus) functionals acting on those functions with image in a metric semiring.*

3 Some examples

Given a fixed density entropy h and a fixed $g \in C(\mathcal{P}(X), \mathbb{R})$, one interesting problem is to find a $\mu \in \mathcal{P}(X)$ attaining the value $\ell_h(g)$ (an equilibrium state). However, the next example shows that it is possible for infinitely many equilibrium states to exist, and that a characterization of them seems to be as difficult as trying to optimize any possible u.s.c. function over a compact and convex set.

Example 11. *Take $X = \{1, ..., d\}$ and let $\mathcal{P}(X)$ be the simplex*

$$\mathcal{P}(X) = \{p = (p_1, p_2, ..., p_d) \mid \sum_{j=1}^{d} p_j = 1;\, p_j \geq 0 \,\forall j \in \{1, 2, .., d\}\} \subset \mathbb{R}^d. \quad (10)$$

Consider the Shannon entropy

$$h(p_1, p_2, ..., p_d) = -\sum_{j=1}^{d} p_j \log p_j, \quad (11)$$

where $0 \cdot \log(0) = 0$ by convention. This function h is continuous and concave. Consider the functional ℓ_h as defined by (4). Then, ℓ_h is an idempotent pressure function.

Let us consider the level-1 case. Fix a function $g : X \to \mathbb{R}$ and denote $g(j) = g_j$. For a probability $p = (p_1, ..., p_d)$ on X we denote $\int g\, dp = \sum_j g_j p_j$. In this way we consider

$$\ell_h(g) = \bigoplus_{p \in \mathcal{P}(X)} h(p) \odot \int g\, dp = \sup_{p \in \mathcal{P}(X)} [(-\sum_j p_j \log p_j) + \sum_j g_j p_j].$$

Question: given such g, what is the probability $p \in \mathcal{P}(X)$ that attains the value $\ell_h(g)$? It is well known that the Gibbs probability, $p_j = \frac{e^{g_j}}{\sum_k e^{g_k}}$, $j \in \{1, 2, ..., d\}$, is the solution (see Lemma 9.9 in [Wal82]).

We consider now the level-2 case. In this way, for a continuous function $g : \mathcal{P}(X) \to \mathbb{R}$, the idempotent pressure of g is given by

$$\ell_h(g) = \bigoplus_{p \in \mathcal{P}(X)} h(p) \odot g(p) = \sup_{p \in \mathcal{P}(X)} [(-\sum_j p_j \log p_j) + g(p_1, ..., p_d)]. \quad (12)$$

For any continuous function $f : \mathbb{R}^d \to \mathbb{R}$, defining

$$g(p_1, ..., p_d) := f(p_1, ..., p_d) + \sum_j p_j \log p_j,$$

we get $\ell_h(g) = \bigoplus_{p \in \mathcal{P}(X)} f(p)$. Consequently, we are just maximizing this continuous function f over the compact set $\mathcal{P}(X) \subset \mathbb{R}^d$. There is at least one solution attaining the maximal value, but clearly, we cannot claim that there exists a unique equilibrium measure. Furthermore, depending on the nature of g (or f) we can have a different subset of $\mathcal{P}(X)$ as the set of equilibrium measures. It is a closed subset of $\mathcal{P}(X)$, but it does not need to be a convex set (for example, $X = \{1, 2\}$ and $f(p_1, p_2) = -p_1 \cdot p_2$). This example, which is not of dynamical nature, does not fit within the framework of [BCMV22] due to the possible generality of g.

When, for a given g, there are multiple maximizing solutions for (12), we say that a phase transition occurs. In the next example, we present cases where such phenomena take place.

Example 12. *As a particular case of above example, we take $d = 2$ and consider a form of non-linear pressure problem along the lines of the papers [LW19] and [BdSPL25]; it is related to the Curie-Weiss model.*

Let $A : \{1, 2\} \to \mathbb{R}$ be a function and denote $a_1 := A(1)$ and $a_2 := A(2)$. Initially let us consider the function $g(p) = \frac{1}{2} (\int A \, dp)^2$ and the variational problem

$$\ell_h(g) = \sup_{p_1 + p_2 = 1} [- \sum_{j=1}^{2} p_j \log p_j + \frac{1}{2} (a_1 p_1 + a_2 p_2)^2]. \tag{13}$$

For a fixed A, we are interested in finding a probability $p = (p_1, p_2)$ maximizing (13).

If $a_1 = a_2$, for example, then we get $(p_1, p_2) = (1/2, 1/2)$ as the unique equilibrium state (measure) maximizing (13). If $a_1 = 10$ and $a_2 = -10$, for example, then $(p_1, p_2) = (1, 0)$ and $(p_1, p_2) = (0, 1)$ are the equilibrium states (measures) of (13) and a phase transition occurs. If $a_1 = 1.2$ and $a_2 = -1.2$ then a phase transition also occurs and the maximal value of g is attained for $p_1 \approx 0.083$ and $p_1 \approx 0.917$.

Consider the function

$$f(p_1) = -p_1 \log p_1 - (1 - p_1) \log(1 - p_1) + \frac{1}{2} (a_1 p_1 + a_2 (1 - p_1))^2.$$

We remark that

$$f''(p_1) = (a_1 - a_2)^2 + \frac{1}{p_1 (p_1 - 1)}. \tag{14}$$

If $(a_1 - a_2)^2 \leq 4$, then $f'' \leq 0$, and it follows that f' is monotone decreasing. In such a case, there is a unique point attaining the maximal value of f on $[0, 1]$.

If $(a_1 - a_2)^2 > 4$, then there are two points $a < b$ in $(0,1)$, such that f'' is positive in (a,b), but negative in $(0,a)$ and $(b,1)$. In this way, f' is increasing in (a,b), but decreasing in $(0,a)$ and $(b,1)$. The analysis of a function $f : [0,1] \to \mathbb{R}$ with such properties shows that it has at most two points in $[0,1]$ attaining their maximal values. This conclusion concerning the existence of at most two equilibrium probabilities aligns with what is observed in the classical Curie-Weiss model (see [Kos18]).

Now, let us consider the case $g(p) = \sin(\int A d\, p)$ and the following variational problem:

$$\ell_h(g) = \sup_{p_1+p_2=1} [-\sum_{j=1}^{2} p_j \log p_j + \sin(a_1 p_1 + a_2 p_2)]. \tag{15}$$

In this case, when $a_1 = 5, a_2 = -8.2$, we get two solutions p maximizing (15) which correspond to $p_1 \approx 0.73$ and $p_1 \approx 0.27$; in this case $\ell_h(g) \approx 1.58$; and we also get a phase transition for such parameters.

A consequence of considering non-linear potentials is the phenomena of ensemble inequivalence [Touch09, Campa14]. In the idempotent pressure, $\mu \mapsto g(\mu)$ can be non-linear and the density entropy can be non-concave. In such cases, the Canonical ensemble (fixed temperature) and Microcanonical ensemble (fixed energy) can yield different physical predictions, a phenomenon unique to long-range systems and non-linear potentials.

Suppose $F : \mathbb{R} \to \mathbb{R}$ is a continuous function. Using the above notation, take $g(p) = F(a_1 p_1 + a_2 p_2)$, for fixed $a_1, a_2 > 0$. Now, consider the idempotent pressure of g as

$$\ell_h(g) = \sup_{p_1+p_2=1} [-\sum_{j=1}^{2} p_j \log p_j + F(a_1 p_1 + a_2 p_2)]. \tag{16}$$

The study of this variational problem is included in the idempotent formalism described above (see Example 15 and Definition 16 for a more general formulation).

Given a potential A (minus the Hamiltonian) and a fixed entropy function, in most ofthe examples in Statistical Mechanics, the role of the function g we considered here is played by functions of the form $g(\mu) = \int A d\mu$ (classical pressure problem), or $g(\mu) = (\frac{1}{2} \int A d\mu)^2$ (Curie-Weiss type model). The study of this last case was one of the main goals of [LW19] and [LW20]. Our formalism contemplates the study of a more general class of examples of functions g, as mentioned above (not contemplated by the setting of [BCMV22]). For example, in our setting it is natural to consider integrals with respect to two distinct potentials. We present now an example to illustrate this broader generality.

Example 13. *Consider $X = \{1, 2\}$, and denote a probability p in X by (p_1, p_2). Fix the real values a_1, a_2, b_1, b_2, and denote by $A, B : X \to \mathbb{R}$, the functions that respectively satisfy $A(1) = a_1, A(2) = a_2$ and $B(1) = b_1, B(2) = b_2$.*

Consider the function $g : \mathcal{P}(X) \to \mathbb{R}$, given by

$$g(p) = \inf\{\int Adp, \int Bdp\}.$$

We want to analyze the following variational problem:

$$l(g) = \sup_{p \in \mathcal{P}(X)} \{-p_1 \log(p_1) - p_2 \log(p_2) + g(p)\}.$$

It is a kind of mixed-pressure problem.

It is natural to consider the following function $G : [0,1] \to \mathbb{R}$:

$$G(p_1) = -p_1 \log(p_1) - (1-p_1)\log(1-p_1) + \inf\{a_1 p_1 + a_2(1-p_1), b_1 p_1 + b_2(1-p_1)\},$$

and then to try to find the points p_1 where it reaches its maximum.

If there is a point $p_1 \in [0,1]$ satisfying $a_1 p_1 + a_2(1-p_1) = b_1 p_1 + b_2(1-p_1)$ then this point is equal to $\bar{p}_1 = \frac{a_2 - b_2}{-a_1 + a_2 + b_1 - b_2}$. The function G is concave and differentiable up to the point $\bar{p}_1$.

The differentiable function

$$\mathcal{A}(p_1) = -p_1 \log(p_1) - (1-p_1)\log(1-p_1)) + a_1 p_1 + a_2(1-p_1)$$

has derivative zero just at the point $r_1 = \frac{1}{1+e^{a_2 - a_1}}$.

The differentiable function

$$\mathcal{B}(p_1) = -p_1 \log(p_1) - (1-p_1)\log(1-p_1)) + b_1 p_1 + b_2(1-p_1)$$

has derivative zero just at the point $s_1 = \frac{1}{1+e^{b_2 - b_1}}$.

Therefore, $l(g)$ can be computed as

$$l(g) = \max\{G(1), G(0), G(r_1), G(s_1), G(\bar{p}_1)\}$$

(or $l(g) = \max\{G(1), G(0), G(r_1), G(s_1)\}$ if $\bar{p}_1$ does not exist).

For instance, when $a_1 = 0.1, a_2 = 0.7, b_1 = 0.4, b_2 = 0.2$, we get that the maximal value is 0.9981, attained at the point 0.549.

Our formulation encompasses several classical constructions, as the following examples show. Examples 14, 15 and 17 have a dynamical nature in contrast with Examples 11, 12 and 13 where there is no dynamics.

Example 14. *If $T : X \to X$ is a measurable map and $\mathcal{M}(T)$ is the set of invariant probabilities, by taking $h = 0$ over $\mathcal{M}(T)$ and $-\infty$ else, the idempotent pressure becomes $\ell(g) = \sup_{\mu \in \mathcal{M}(T)} g(\mu)$, which extends the standard level-1 ergodic optimization problem $\sup_{\mu \in \mathcal{M}(T)} \int_X A(x) d\mu(x)$.*

In another direction, if h is the Kolmogorov-Sinai entropy over $\mathcal{M}(T)$ and $-\infty$ elsewhere, the idempotent pressure associated with (4) is given by $\ell(g) = \sup_{\mu \in \mathcal{M}(T)}[g(\mu) + h(\mu)]$, which in level-1 is given by

$$\sup_{\mu \in \mathcal{M}(T)} \left[\int_X A(x) d\mu(x) + h(\mu) \right], \tag{17}$$

usually known as the variational principle for pressure.

Now, we consider more general level-2 variational principles where the dependence on a potential function, acting on measures, is non-linear. This formalism includes the study of the Curie-Weiss model, which is of great importance in Statistical Mechanics (see Section 2 in [FV18] and also [BH22], [BKL23], [LW19], [LW20], [BdSPL25] and [Campa14]).

The purpose of the next example is to show the wide range of cases that can be considered in the present context.

Example 15. *We consider on $\Omega = \{1, 2, ..., d\}^{\mathbb{N}}$ the distance d_γ defined by*

$$d_\gamma(x, y) = \begin{cases} 0, & x = y \\ \gamma^{i(x,y)}, & x \neq y \end{cases} , \tag{18}$$

*where $0 < \gamma < 1$, $x = (x_0, x_1, x_2, ...)$, $y = (y_0, y_1, y_2, ...)$ and $i(x, y) = min\ \{j \in \mathbb{N}, x_j \neq y_j\}$. Consider also the shift map $\sigma : \Omega \to \Omega$, given by $\sigma(x_0, x_1, x_2, ...) = (x_1, x_2, x_3, ...)$. We denote by $\mathcal{M}(\sigma)$ the set of σ-invariant probabilities on the Borel sigma-algebra of Ω. Consider a continuous function (a potential) $A : \Omega \to \mathbb{R}$. Given a continuous function $g : \mathcal{P}(\Omega) \to \mathbb{R}$, consider the functional $\ell_h(g) := \sup_{\mu \in \mathcal{P}(\Omega)} h(\mu) + g(\mu)$, where h is the **extended entropy** (it coincides with the Kolmogorov-Sinai entropy acting on the set of invariant probabilities and it is $-\infty$ for non-invariant probabilities). Then, we can define the "quadratic" pressure for A as*

$$P_2(A) = \sup_{\mu \in \mathcal{P}(\Omega)} \left[h(\mu) + \left(\int_X A d\mu \right)^2 \right] = \sup_{\mu \in \mathcal{M}(\sigma)} \left[h(\mu) + \left(\int_X A d\mu \right)^2 \right].$$

We call $P_2(A)$ the quadratic non-linear pressure for A (in the sense of [BKL23]). There exists at least one equilibrium state $\mathfrak{m} = \mathfrak{m}_{A,2}$ attaining the supremum value $P_2(A)$, where $\mathfrak{m} \in \mathcal{M}(\sigma) \subset \mathcal{P}(\Omega)$. Observe that, $\mu \to (\int A d\mu)^2$ is not affine. The probability maximizing $P_2(A)$ may not be unique, and some of them may not be ergodic (see [LW19] and an explicit example in Section 4 of [BKL23]).

Definition 16. *Given a continuous function $F : \mathbb{R} \to \mathbb{R}$, and a continuous potential $A : \Omega \to \mathbb{R}$, a probability $\mu = \mu_{F,A}$ is called the (F, A)-equilibrium probability, if it maximizes the pressure*

$$P_F(A) = \sup_{\mu \in \mathcal{P}(\Omega)} \left[h(\mu) + F \left(\int_X A d\mu \right) \right] = \sup_{\mu \in \mathcal{M}(\sigma)} \left[h(\mu) + F \left(\int A d\mu \right) \right],$$

where h is the extended entropy.

The classical pressure is obtained when $F(x) = x$ is the identity. In the other cases we will say that P_F is a non-linear pressure. The classical Curie-Weiss model is generally associated with the case where F is convex (or even quadratic); [Kos18] addresses this problem for a non-dynamical setting using properties of large deviations. The study of properties of the variational problem described in Definition 16, when F is concave or convex, is included in the idempotent formalism described above (the analysis of this specific problem is the main goal of [BdSPL25] and [BPL26]).

Example 17. *Consider the map $T : S^1 \to S^1$ given by $T(x) = 2\,x$ (mod1), where $S^1 = \mathbb{R}/\mathbb{Z}$ is topologically a circle. The usual addition + in $\mathbb{R}$ induces an operation + on the circle $\mathbb{R}/\mathbb{Z}$. Denote by $\mathcal{M}(T)$ the set of T-invariant probabilities and set $h(\mu)$ as the Kolmogorov-Sinai entropy of μ, for $\mu \in \mathcal{M}(T)$, and $h = -\infty$ on the set of non-invariant probabilities.*

Given two probabilities η and μ on S^1, the convolution $\eta * \mu$ is the probability on S^1 satisfying

$$\int \phi(z) d(\eta * \mu)(z) = \iint \phi(y + x)\, d\mu(y)\, d\,\eta(x), \tag{19}$$

for any continuous function $\phi : S^1 \to \mathbb{R}$. From Fubini's theorem we get $\eta * \mu = \mu * \eta$ for all η, μ. Moreover, for a fixed μ, the function $\eta \to \eta * \mu$ is affine and continuous in the weak topology.*

For a fixed probability μ and a fixed Hölder potential $B : S^1 \to \mathbb{R}$, let us define $g : \mathcal{P}(S^1) \to \mathbb{R}$ by $g(\eta) = \int B\, d\,(\eta * \mu)$. For such g, let us consider the variational problem

$$\ell(g) = \sup_{\eta \in \mathcal{M}(T)} [g(\eta) + h(\eta)]. \tag{20}$$

We remark that $g(\eta) = \int (\int B(y + x)\, d\mu(y))\, d\,\eta(x) = \int A(x)\, d\eta(x)$, where $A(x) = \int B(y + x)\, d\mu(y)$ is Hölder continuous. Therefore, there is a unique probability maximizing (17) (see [PP90]). Furthermore, as

$$\ell(g) = \sup_{\eta \in \mathcal{M}(T)} [g(\eta) + h(\eta)] = \sup_{\eta \in \mathcal{M}(T)} [\int A\, d\,\eta + h(\eta)], \tag{21}$$

there exists a unique equilibrium state for such ℓ.

Results about the convolution of equilibrium probabilities for Holder potentials appear in [L18].

4 Comparison with level-1 convex pressure

For the sake of comparison we recall that [BCMV22] (and its correction [BCM$^+$23]) defines a (convex, level-1) pressure function. We assume in the present work that X is just a compact metric space.

Definition 18. *[BCMV22, Definition 2.1] A function $\Gamma : C(X, \mathbb{R}) \to \mathbb{R}$ is called a pressure function if it satisfies the following conditions:*
(C1) Monotonicity: $\varphi \leqslant \psi \Rightarrow \Gamma(\varphi) \leqslant \Gamma(\psi) \quad \forall \varphi, \psi \in C(X, \mathbb{R})$.
(C2) Translation invariance: $\Gamma(\varphi + c) = \Gamma(\varphi) + c \quad \forall \varphi \in C(X, \mathbb{R}) \quad \forall c \in \mathbb{R}$.
(C3) Convexity: $\Gamma(t\varphi + (1-t)\psi) \leqslant t\Gamma(\varphi) + (1-t)\Gamma(\psi) \quad \forall \varphi, \psi \in C(X, \mathbb{R}) \quad \forall t \in [0, 1]$.

The broad framework portrayed by Definition 18 does not have to be necessarily of dynamical nature, but covers some important cases like the classical pressure in thermodynamic formalism (see [PP90]).

In [BCMV22] and [BCM+23] the authors show the following characterization (here applied for compact spaces):

Theorem 19. *[BCM+23, Theorem 1] Let $\Gamma : C(X,\mathbb{R}) \to \mathbb{R}$ be a pressure function in the sense of Definition 18. Then*

$$\Gamma(\varphi) = \sup_{\mu \in \mathcal{P}(X)} \left[\int \varphi d\mu + \mathfrak{h}(\mu) \right] \quad \forall \varphi \in C(X,\mathbb{R}) \tag{22}$$

where

$$\mathfrak{h}(\mu) = \inf_{\varphi \in \mathcal{A}_\Gamma} \int \varphi d\mu \quad \text{and} \quad \mathcal{A}_\Gamma = \{\varphi \in C(X,\mathbb{R}) : \Gamma(-\varphi) \leqslant 0\}.$$

Moreover, $\mathfrak{h}(\mu)$ is concave and upper semi-continuous. Furthermore, if $\alpha : \mathcal{P}(X) \to \mathbb{R} \cup \{-\infty\}$ is another function taking the role of $\mathfrak{h}$ in (22), then $\alpha \leqslant \mathfrak{h}$. In addition, one has $\mathfrak{h}(\mu) = \inf_{\varphi \in C(X,\mathbb{R})} \left[\Gamma(\varphi) - \int \varphi d\mu\right]$, $\forall \mu \in \mathcal{P}(X)$. Finally, the maximum in (22) is attained in $\mathcal{P}(X)$.

Remark 20. *We notice that there exists a canonical inclusion $j : C(X,\mathbb{R}) \to C(\mathcal{P}(X),\mathbb{R})$ given by*

$$j(\varphi)(\mu) = \int_X \varphi(x) d\mu(x), \ \mu \in \mathcal{P}(X).$$

In this particular case we have

$$\ell(j(\varphi)) = \sup_{\mu \in \mathcal{P}(X)} \left[h_\ell(\mu) + \int_X \varphi(x) d\mu(x) \right]. \tag{23}$$

However it is important to observe that in $C(X,\mathbb{R})$ there exists a natural max-operation, that is: $\sup(\varphi,\psi)(x) := \sup\{\varphi(x),\psi(x)\} = \varphi(x) \oplus \psi(x)$. In general, this max-operation structure on $C(X,\mathbb{R})$ does not agree with the $\oplus$ of $C(\mathcal{P}(X),\mathbb{R})$, that is, $j(\sup\{\varphi,\psi\}) \neq j(\varphi) \oplus j(\psi)$. As an example, if $X = \{1,2\}$ $\mu = (1/2,1/2)$, $\varphi(1) = 0$, $\varphi(2) = 1$, $\psi(1) = 1$, $\psi(2) = 0$ we get $j(\varphi)(\mu) = j(\psi)(\mu) = 1/2$ and so $[j(\varphi) \oplus j(\psi)](\mu) = 1/2$. On the other hand $j(\sup\{\varphi,\psi\})(\mu) = j(1)(\mu) = 1$. In general,

$$j(\sup\{\varphi,\psi\})(\mu) = \int \sup\{\varphi,\psi\} d\mu \geq \sup\{\int \varphi d\mu, \int \psi d\mu\} = [j(\varphi) \oplus j(\psi)](\mu).$$

Consequently,

$$\ell(j(\sup\{\varphi,\psi\})) \neq \sup\{\ell(j(\varphi)), \ell(j(\psi))\}.$$

We highlight the fact that, when considering the inclusion map j, the correct $\oplus$ operation to be used is the level-2 max-operation,

$$[j(\varphi) \oplus j(\psi)](\mu) = \sup\left\{ \int \varphi d\mu, \int \psi d\mu \right\}.$$

Example 21. *Suppose $X = \{1, 2\}$. Any probability $p = (p_1, p_2)$ can be identified with a number $p_1 \in [0, 1]$ and the metric space $\mathcal{P}(X)$ can be identified with the metric space $[0, 1]$. Let h be the density entropy defined by $h(p) = h(p_1, p_2) = \cos(4\pi p_1)$. It is continuous but not concave. Let us consider the idempotent pressure ℓ_h, associated to h from equation (4). For any level-2 continuous function $g : \mathcal{P}(X) \to \mathbb{R}$ we can write $\ell_h(g) = \sup_{x\in[0,1]}[g(x) + \cos(4\pi x)]$, where $x \in [0, 1]$ represents p_1. On the other hand, for any level-1 function $\varphi : \{1, 2\} \to \mathbb{R}$ we can consider $\Gamma(\varphi) = \sup_{x\in[0,1]}[(\varphi(1)x + \varphi(2)(1 - x)) + \cos(4\pi x)]$. Such map $\varphi \mapsto \Gamma(\varphi)$ defines a convex pressure and from Theorem 19 there is a u.s.c. and concave function $\mathfrak{h}$ such that for any $\varphi : \{1, 2\} \to \mathbb{R}$,*

$$\sup_{x\in[0,1]}[(\varphi(1)x + \varphi(2)(1 - x)) + \cos(4\pi x)] = \sup_{x\in[0,1]}[(\varphi(1)x + \varphi(2)(1 - x)) + \mathfrak{h}(x)].$$

Observe however that the points attaining the supremum on the left hand side do not need to coincide with the points for the right hand side, that is, the equilibrium probabilities may be different. Furthermore, $\ell_h \neq \ell_{\mathfrak{h}}$ because in the level-2 setting, the u.s.c. density entropy of an idempotent pressure is unique.

The proof of the Theorem 4 will be divided in two propositions (22 and 23). The next proposition shows that the restriction of a level-2 idempotent pressure function to functions of $C(X, \mathbb{R})$ is actually a level-1 convex pressure function in the sense of Definition 18. Precisely, the canonical inclusion $j : C(X, \mathbb{R}) \to C(\mathcal{P}(X), \mathbb{R})$ defines a projection of level-2 idempotent pressure functions to level-1 pressure functions.

Proposition 22. *Consider an idempotent pressure function ℓ with density entropy h_ℓ and the canonical inclusion $j : C(X, \mathbb{R}) \to C(\mathcal{P}(X), \mathbb{R})$ given by $j(\varphi)(\mu) = \int_X \varphi(x)d\mu(x)$, $\mu \in \mathcal{P}(X)$. If we define $\Gamma_\ell : C(X, \mathbb{R}) \to \mathbb{R}$ by*

$$\Gamma_\ell(\varphi) := \ell(j(\varphi))$$

then Γ_ℓ is a convex pressure function (in the sense of Definition 18). Let $\mathfrak{h}$ be the concave and upper semi-continuous (entropy) function given in Theorem 19. Then $h_\ell \leq \mathfrak{h}$. Reciprocally, each pressure function (in the sense of Definition 18) is of the form Γ_ℓ where ℓ is an idempotent pressure function. If X is not a singleton, such map $\ell \mapsto \Gamma_\ell$ is not injective.

Proof. Let ℓ be an idempotent pressure function. From Theorem 2 there exists a unique u.s.c. function $h_\ell : \mathcal{P}(X) \to \mathbb{R}_{\max}$ such that

$$\ell(g) = \sup_{\mu\in\mathcal{P}(X)}[g(\mu) + h_\ell(\mu)],$$

for any $g \in C(\mathcal{P}(X), \mathbb{R})$. Define $\Gamma_\ell(\varphi) := \ell(j(\varphi))$, $\varphi \in C(X, \mathbb{R})$. In this way,

$$\Gamma_\ell(\varphi) = \sup_{\mu\in\mathcal{P}(X)}[\int \varphi d\mu + h_\ell(\mu)].$$

Consequently, the hypotheses in Definition 18 are immediately satisfied by Γ_ℓ. Furthermore, by Theorem 19, $h_\ell \leq \mathfrak{h}$.

Reciprocally, consider $\Gamma(\varphi) = \sup_{\mu\in\mathcal{P}(X)} \left[\mathfrak{h}(\mu) + \int \varphi d\mu\right]$ a convex pressure function. Note that $\mathfrak{h}(\mu)$ is u.s.c., so the formula $\ell(g) = \sup_{\mu\in\mathcal{P}(X)}[\mathfrak{h}(\mu) + g(\mu)]$, $g \in C(\mathcal{P}(X), \mathbb{R})$ defines an idempotent pressure function which extends Γ. Clearly $\Gamma_\ell = \Gamma$.

If X is not a singleton and $\nu \in \mathcal{P}(X)$, the function $h_0 : \mathcal{P}(X) \to \mathbb{R}$ defined by $h_0(\mu) = 0$ if $\nu \neq \mu$ and $h_0(\nu) = 1$ is u.s.c. and not concave. The unique u.s.c. density entropy of ℓ_{h_0} is h_0. The convex pressure $\Gamma_{\ell_{h_0}}$ admits an u.s.c. and concave entropy $\mathfrak{h}_0$ and $\ell_{\mathfrak{h}_0}$ is an idempotent pressure with unique u.s.c. density entropy $\mathfrak{h}_0$. As $\mathfrak{h}_0 \neq h_0$ we get $\ell_{h_0} \neq \ell_{\mathfrak{h}_0}$. This shows that the map $\ell \mapsto \Gamma_\ell$ is not injective. □

We notice that the classical Kolmogorov-Sinai entropy is not always a u.s.c. function. In fact, the non existence of maximal measures shows that the entropy map is not u.s.c. (see [Rue02]). Also, in the IFS setting, the entropy is always concave (see Proposition 2.2 in [MO17]). Finally, from Example 2.4 of [MO17] we conclude that it may not be affine.

Proposition 23. *Suppose that an idempotent pressure ℓ has a concave density entropy h_ℓ. Then for all $\mu \in \mathcal{P}(X)$ we have*

$$h_\ell(\mu) = \inf_{g\in C(\mathcal{P}(X),\mathbb{R})} [\ell(g) - g(\mu)]. \tag{24}$$

Finally, considering Γ_ℓ and its entropy $\mathfrak{h}$ (given in Theorem 19) we have $\mathfrak{h} = h_\ell$.

Proof. We will adapt the proof of Theorem 9.12 in [Wal82] to the present case. Equation

$$\ell(g) = \sup_{\mu\in\mathcal{P}(X)} [h_\ell(\mu) + g(\mu)]$$

is a consequence of h_ℓ to be the density entropy of ℓ and its concavity is not necessary.

Given $\mu \in \mathcal{P}(X)$ and $g \in C(\mathcal{P}(X), \mathbb{R})$, from above equation we get

$$\ell(g) \geq h_\ell(\mu) + g(\mu).$$

Then

$$\ell(g) - g(\mu) \geq h_\ell(\mu)$$

and consequently

$$\inf_{g\in C(\mathcal{P}(X),\mathbb{R})} [\ell(g) - g(\mu)] \geq h_\ell(\mu). \tag{25}$$

On the other hand, considering $\Gamma_\ell : C(X, \mathbb{R}) \to \mathbb{R}$, defined by $\Gamma_\ell(\varphi) := \ell(j(\varphi))$ we get a convex pressure function. From, Theorem 19

$$\mathfrak{h}(\mu) = \inf_{\varphi\in C(X,\mathbb{R})} \left[\Gamma_\ell(\varphi) - \int \varphi d\mu\right].$$

Then

$$\mathfrak{h}(\mu) = \inf_{\varphi\in C(X,\mathbb{R})} [\ell(j(\varphi)) - j(\varphi)(\mu)] \geq \inf_{g\in C(\mathcal{P}(X),\mathbb{R})} [\ell(g) - g(\mu)] \geq h_\ell(\mu).$$

Now we will show that

$$h_\ell \geq \inf_{\varphi \in C(X,\mathbb{R})} \left[\Gamma_\ell(\varphi) - \int \varphi d\mu \right]. \tag{26}$$

Denote

$$C = \{(\mu, t) \in \mathcal{P}(X) \times \mathbb{R} \,|\, t \leq h_\ell(\mu)\}.$$

Note that for any μ in the support of h_ℓ, there exists a t such that $(\mu, t) \in C$. As h_ℓ is u.s.c. and concave by hypothesis we get that C is a convex closed set. Fix a probability μ_0 and take $b > h_\ell(\mu_0)$. As h_ℓ is u.s.c., we get that $(\mu_0, b) \notin C$. The set $\{(\mu_0, b)\}$ is compact and convex. It follows that there exists a continuous function $A : X \to \mathbb{R}$ and a real number α (see Separation Theorem in page 417 in [DS58] or Section 2 in [MN22]) such that

$$\int A d\mu + \alpha t > \int A \, d\mu_0 + \alpha b,$$

for all $(\mu, t) \in C \subset \mathcal{P}(X) \times \mathbb{R}$.

It follows that for any probability $\mu \in \mathcal{P}(X)$, taking $t = h_\ell(\mu)$,

$$\int A \, d\mu + \alpha h_\ell(\mu) > \int A \, d\mu_0 + \alpha b.$$

Now, taking $\mu = \mu_0$ in the above expression we get that $\alpha h_\ell(\mu_0) > \alpha b$. It follows that $\alpha < 0$. Therefore, for any $\mu \in \mathcal{P}(X)$

$$h_\ell(\mu) + \frac{1}{\alpha} \int A d\mu < b + \frac{1}{\alpha} \int A d\mu_0. \tag{27}$$

Taking the supremum on the left hand side with respect to $\mu \in \mathcal{P}(X)$ we get

$$\Gamma_\ell \left(\frac{A}{\alpha} \right) \leq b + \frac{1}{\alpha} \int A d\mu_0.$$

We conclude that, for any $b > h_\ell(\mu_0)$,

$$b \geq \Gamma_\ell \left(\frac{A}{\alpha} \right) - \frac{1}{\alpha} \int A d\mu_0 \geq \inf\{\Gamma_\ell(\varphi) - \int \varphi d\mu_0 \,|\, \varphi \in C(X, \mathbb{R})\}.$$

Finally, we get that

$$h_\ell(\mu_0) \geq \inf\{\Gamma_\ell(\varphi) - \int \varphi d\mu_0 \,|\, \varphi \in C(X, \mathbb{R})\}.$$

This shows that (26) is true and finishes the proof.

□

As a consequence of this proposition we get the following corollary.

Corollary 24. *Suppose that $\alpha : \mathcal{P}(X) \to \mathbb{R}_{\max}$ is a function playing the role of $\mathfrak{h}$ in (22). If α is u.s.c and concave then $\alpha = \mathfrak{h}$.*

Proof. Let Γ be a convex pressure function and suppose that $\alpha : \mathcal{P}(X) \to \mathbb{R}_{\max}$ is u.s.c, concave and satisfies

$$\Gamma(\varphi) = \sup_{\mu \in \mathcal{P}(X)} \left[\int \varphi d\mu + \alpha(\mu) \right] \quad \forall \varphi \in C(X, \mathbb{R}).$$

Let ℓ be the idempotent pressure with density entropy α. We have $\Gamma = \Gamma_\ell$ and applying Proposition 23 we get $\mathfrak{h} = \alpha$. $\square$

We finish this section considering the setting of [BCMV22] and [BCM+23] applied to level-2 functions. In this way, we consider the space $\mathcal{P}(X)$ and functions in $C(\mathcal{P}(X), \mathbb{R})$. From Definition 18, we will call a function $\Gamma : C(\mathcal{P}(X), \mathbb{R}) \to \mathbb{R}$ a level-2 convex pressure function if it satisfies the following conditions:
(Axiom B1) $\varphi \leqslant \psi \Rightarrow \Gamma(\varphi) \leqslant \Gamma(\psi) \quad \forall \varphi, \psi \in C(\mathcal{P}(X), \mathbb{R})$.
(Axiom B2) $\Gamma(\varphi + c) = \Gamma(\varphi) + c \quad \forall \varphi \in C(\mathcal{P}(X), \mathbb{R}) \quad \forall c \in \mathbb{R}$.
(Axiom B3) $\Gamma(t\varphi + (1-t)\psi) \leqslant t\Gamma(\varphi) + (1-t)\Gamma(\psi)\, \forall \varphi, \psi \in C(\mathcal{P}(X), \mathbb{R})\, \forall t \in [0, 1]$.

From Theorem 19 and Corollary 24 there is a unique u.s.c and concave function $\mathfrak{h} : \mathcal{P}(\mathcal{P}(X)) \to \mathbb{R}_{\max}$ such that

$$\Gamma(\varphi) = \sup_{\pi \in \mathcal{P}(\mathcal{P}(X))} \left[\int \varphi(\mu) d\pi(\mu) + \mathfrak{h}(\pi) \right] \quad \forall \varphi \in C(\mathcal{P}(X), \mathbb{R}). \tag{28}$$

Finally, the supremum is attained in $\mathcal{P}(\mathcal{P}(X))$.

Proposition 25. *Any idempotent pressure is a level-2 convex pressure.*

Proof. Clearly Axiom B1 is a consequence of Axiom A2 in Definition 1 and Axiom B2 is equivalent to Axiom A1 in Definition 1. Let ℓ be an idempotent pressure. It remains to prove that it satisfies Axiom B3, that is, it is convex. From Theorem 2 there is an u.s.c function h such that

$$\ell(g) = \sup_{\mu \in \mathcal{P}(X)} [g(\mu) + h(\mu)],\, \forall g \in C(\mathcal{P}(X), \mathbb{R}).$$

The convexity follows by standard arguments, that is, if $\lambda \in [0, 1]$ and $f, g \in C(\mathcal{P}(X), \mathbb{R})$ we have

$$\begin{aligned}
\ell(\lambda f + (1-\lambda)g) &= \sup_{\mu \in \mathcal{P}(X)} [\lambda f(\mu) + (1-\lambda)g(\mu) + h(\mu)] \\
&= \sup_{\mu \in \mathcal{P}(X)} [\lambda f(\mu) + (1-\lambda)g(\mu) + \lambda h(\mu) + (1-\lambda)h(\mu)] \\
&\leq \sup_{\nu \in \mathcal{P}(X)} [\lambda f(\nu) + \lambda h(\nu)] + \sup_{\rho \in \mathcal{P}(X)} [(1-\lambda)g(\rho) + (1-\lambda)h(\rho)] \\
&= \lambda \ell(f) + (1-\lambda)\ell(g).
\end{aligned}$$

$\square$

Using the canonical inclusion $j : C(X, \mathbb{R}) \to C(\mathcal{P}(X), \mathbb{R})$, Proposition 22 and above proposition we get the following inclusions

$$\begin{array}{c}\text{Set of level-1}\\ \text{convex pressures}\end{array} \subseteq \begin{array}{c}\text{Set of level-2}\\ \text{idempotent pressures}\end{array} \subseteq \begin{array}{c}\text{Set of level-2}\\ \text{convex pressures}\end{array}$$

with correspondent (and conceptually different) variational principles

$$\sup_{\mu\in\mathcal{P}(X)} \int \varphi d\mu + \mathfrak{h}(\mu), \qquad \sup_{\mu\in\mathcal{P}(X)} g(\mu) + h(\mu), \qquad \sup_{\pi\in\mathcal{P}(\mathcal{P}(X))} \int g(\mu) d\pi(\mu) + \mathfrak{h}(\pi).$$

5 Invariant idempotent pressures

We denote by $I(\mathcal{P}(X))$ the set of idempotent pressure functions (even using this notation we do not assume $\ell(0) = 0$). Given a map $S : I(\mathcal{P}(X)) \to I(\mathcal{P}(X))$, we will say that an idempotent pressure function $\ell \in I(\mathcal{P}(X))$ is S-invariant if $S(\ell) = \ell$. When $T : X \to X$ is a continuous dynamical system we can use the functorial action of T in $C(X, \mathbb{R})$ to define, by duality, a map in $\mathcal{P}(X)$, which is continuous for the weak* topology. It is the push-forward map $T^\sharp : \mathcal{P}(X) \to \mathcal{P}(X)$, where $T^\sharp(\mu) = \nu$ means that $\int f\, d\nu = \int f \circ T\, d\mu$, $\forall f \in C(X, \mathbb{R})$.

The usual pressure in thermodynamic formalism for level-1 functions has the property $P(f \circ T) = P(f)$, for any $f \in C(X, \mathbb{R})$. In this way it is natural to ask what are the idempotent pressures functions for level-2 functions satisfying $\ell(g) = \ell(g \circ T^\sharp)$ for any $g \in C(\mathcal{P}(X), \mathbb{R})$.

Our first result is the following

Proposition 26. *Let (X, d) be a compact metric space and $T : X \to X$ be a continuous map. Let $S : I(\mathcal{P}(X)) \to I(\mathcal{P}(X))$ be the map given by*

$$S(\ell)(g) := \ell(g \circ T^\sharp),\ \ell \in I(\mathcal{P}(X)).$$

If $h \in U(\mathcal{P}(X))$ satisfies $h(\mu) = -\infty$ when μ is not T-invariant, then the associated idempotent pressure is invariant for S. Furthermore, $\ell \in I(\mathcal{P}(X))$ is invariant for S if, and only if, its density entropy satisfies

$$h_\ell(\nu) = \begin{cases} \sup_{[\mu\in\mathcal{P}(X)\ |\ T^\sharp(\mu)=\nu]} h_\ell(\mu), & \text{if } \nu \in T^\sharp(\mathcal{P}(X)) \\ -\infty & \text{if } \nu \notin T^\sharp(\mathcal{P}(X)) \end{cases}.$$

Proof. As T is continuous and X is compact, there exists a T-invariant probability. Suppose initially that h is a u.s.c. function satisfying $h(\mu) = -\infty$ if μ is not T-invariant. Define $\ell(g) := \sup_{\mu\in P(X)} g(\mu) + h(\mu)$. We have then

$$S(\ell)(g) = \ell(g \circ T^\sharp) = \sup_{\mu\in P(X)} [g(T^\sharp(\mu)) + h(\mu)].$$

If $T^\sharp(\mu) \neq \mu$ then $h(\mu) = -\infty$ and consequently such μ does not attain the supremum. It follows that the above supremum is equal to $\sup_{\mu\in P(X)}[g(\mu) + h(\mu)]$ which is also equal to $\ell(g)$. This proves that ℓ is invariant under S.

Now we consider an idempotent pressure ℓ. Let us denote by h_ℓ its unique u.s.c. entropy density. So we have $\ell(g) = \sup_{\mu\in\mathcal{P}(X)}[h_\ell(\mu) + g(\mu)]$. We start by studying $S(\ell)$.

$$S(\ell)(g) = \ell(g \circ T^\sharp) = \sup_{\mu\in\mathcal{P}(X)} [h_\ell(\mu) + g(T^\sharp(\mu))] = \sup_{\nu\in\mathcal{P}(X)} [h'(\nu) + g(\nu)]$$

where

$$h'(\nu) = \begin{cases} \sup_{[\mu\in\mathcal{P}(X)\,\mid\, T^\sharp(\mu)=\nu]} h_\ell(\mu), & \text{if } \nu \in T^\sharp(\mathcal{P}(X)) \\ -\infty & \text{if } \nu \notin T^\sharp(\mathcal{P}(X)) \end{cases}$$

We claim that $h' \in U(\mathcal{P}(X))$. Indeed, if $h_\ell(\mu) \neq -\infty$ and $T^\sharp(\mu) = \nu$ we have $h'(\nu) \neq -\infty$ and hence $\operatorname{supp}(h') \neq \emptyset$. In order to show that h' is u.s.c. consider that $\nu_n \to \nu$ in the weak* topology. We need to show that $\limsup h'(\nu_n) \leq h'(\nu)$. So we can suppose $h'(\nu_n) \neq -\infty$ for all n. By definition of h' we conclude that $\nu_n \in T^\sharp(\mathcal{P}(X))$ and as $T^\sharp$ is continuous, the set $[\mu \in \mathcal{P}(X) \mid T^\sharp(\mu) = \nu_n]$ is compact. Therefore there exists $\mu_n \in (T^\sharp)^{-1}(\nu_n)$ such that $h_\ell(\mu_n) = h'(\nu_n)$. As the set $\mathcal{P}(X)$ is compact, by taking a subsequence, we can suppose there is μ such that $\mu_n \to \mu$. By continuity of $T^\sharp$ we get $T^\sharp(\mu) = \nu$. Using that h_ℓ is u.s.c, we have

$$h'(\nu) \geq h_\ell(\mu) \geq \lim_n h_\ell(\mu_n) = \lim_n h'(\nu_n),$$

as claimed.

As $S(\ell)$ has density $h' \in U(\mathcal{P}(X))$ and ℓ has density $h_\ell \in U(\mathcal{P}(X))$ from the uniqueness of the density, see Theorem 2, we get $S(\ell) = \ell$ iff

$$h_\ell(\nu) = \begin{cases} \sup_{[\mu\in\mathcal{P}(X)\,\mid\, T^\sharp(\mu)=\nu]} h_\ell(\mu), & \text{if } \nu \in T^\sharp(\mathcal{P}(X)) \\ -\infty, & \text{if } \nu \notin T^\sharp(\mathcal{P}(X)) \end{cases}.$$

□

Example 27. *In the above proposition, h does not need to be $-\infty$ at non-invariant probabilities in order to get $S(\ell) = \ell$. For example, consider $X = \{1, 2\}$. Then $\mathcal{P}(X) = \{(p, 1-p),\, p \in [0, 1]\}$. Consider the Shannon entropy $h(p, 1-p) = -p\log(p) - (1-p)\log(1-p)$, which satisfies $0 \leq h(p, 1-p) \leq \log(2)$ for any $p \in [0, 1]$. Suppose that $T : \{1, 2\} \to \{1, 2\}$ satisfies $T(1) = 2$ and $T(2) = 1$. Then $T^\sharp(p, 1-p) = (1-p, p)$ and the unique $T-$invariant probability is $(\frac{1}{2}, \frac{1}{2})$. Observe, however, that $h(T^\sharp(\mu)) = h(\mu)$ for any $\mu \in \mathcal{P}(X)$, that is $h(p, 1-p) = h(1-p, p)$. Consequently if ℓ has density entropy h we have*

$$\ell(g \circ T^\sharp) = \sup_{\mu\in\mathcal{P}(X)} [g(T^\sharp(\mu)) + h(\mu)] = \sup_{\mu\in\mathcal{P}(X)} [g(T^\sharp(\mu)) + h(T^\sharp(\mu))] = \ell(g),$$

where the last equality is due to the fact that $T^\sharp(\mathcal{P}(X)) = \mathcal{P}(X)$.

In this example

$$h(\nu) = \sup_{[\mu\in\mathcal{P}(X)\,\mid\, T^\sharp(\mu)=\nu]} h(\mu)$$

corresponds to $h(p, 1-p) = h(1-p, p)$.

Now we study an idempotent pressure functions which is invariant for the action of the dual Ruelle operator.

Example 28. *Let (X,d) be a compact metric space and $T : X \to X$ be a surjective and continuous map such that any point has a finite number of pre-images. Suppose that $L_J : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ is a transfer operator associated to a continuous Jacobian $J : X \to \mathbb{R}$ and consider the dual map $L_J^* : \mathcal{P}(X) \to \mathcal{P}(X)$. This means that*

$$L_J(f)(x) = \sum_{T(y)=x} J(y)f(y),\ f \in C(X,\mathbb{R}),$$

where $\sum_{T(y)=x} J(y) = 1$, $\forall x \in X$. Furthermore, $L_J^*(\mu) = \nu$ means that $\int f\, d\nu = \int L_J(f)\, d\mu$. By the Tychonoff-Schauder theorem, there is a probability μ_J satisfying $L_J^*(\mu_J) = \mu_J$.

We claim that $T^\sharp \circ L_J^*(\mu) = \mu$, $\forall \mu \in \mathcal{P}(X)$ (see Lemma 2.4 in [LO24]). Indeed, if $L_J^*(\mu) = \nu$ and $T^\sharp(\nu) = \omega$ then we have, for any $f \in C(X,\mathbb{R})$,

$$\int f\, d\omega = \int f \circ T\, d\nu = \int \sum_{T(y)=x} J(y)f(T(y))\, d\mu(x)$$

$$= \int \sum_{T(y)=x} J(y)f(x)\, d\mu(x) = \int f(x)\, d\mu(x).$$

Proposition 29. *Under the setting of the above example, consider the map $S : I(\mathcal{P}(X)) \to I(\mathcal{P}(X))$, given by*

$$S(\ell)(g) := \ell(g \circ L_J^*).$$

If h is a u.s.c function satisfying $h = -\infty$ at any probability which is not invariant for L_J^*, then its associated idempotent pressure function is invariant for S. More generally, the following statements concerning an idempotent pressure ℓ and its density entropy h_ℓ are equivalent:
i. ℓ is invariant for S;
ii. h_ℓ satisfies $h_\ell = h_\ell \circ T^\sharp$ and $h_\ell(\nu) = -\infty$ if $\nu \notin L_J^*(\mathcal{P}(X))$;
iii. h_ℓ satisfies $h_\ell = h_\ell \circ L_J^*$ and $h_\ell(\nu) = -\infty$ if $\nu \notin L_J^*(\mathcal{P}(X))$.

Proof. The first part of the proof follows the same lines as the proof of Proposition 26, replacing $T^\sharp$ by L_J^* in that proof.

Let us write
$$\ell(g) = \sup_{\mu \in \mathcal{P}(X)} [h_\ell(\mu) + g(\mu)]$$
where h_ℓ is u.s.c.

(iii. $\Rightarrow$ i.) Suppose that $h_\ell(\mu) = h_\ell(L_J^*(\mu))$ for any $\mu \in \mathcal{P}(X)$ and $h_\ell(\nu) = -\infty$ if $\nu \notin L_J^*(\mathcal{P}(X))$. We have

$$S(\ell)(g) = \ell(g \circ L_J^*) = \sup_{\mu \in \mathcal{P}(X)} [h_\ell(\mu) + g(L_J^*(\mu))] = \sup_{\mu \in \mathcal{P}(X)} [h_\ell(L_J^*(\mu)) + g(L_J^*(\mu))]$$

$$= \sup_{\nu \in L_J^*(\mathcal{P}(X))} [h_\ell(\nu) + g(\nu)] = \sup_{\nu \in \mathcal{P}(X)} [h_\ell(\nu) + g(\nu)],$$

where in the last equality we use that $h_\ell(\nu) = -\infty$ if $\nu \notin L_J^*(\mathcal{P}(X))$ and so such ν does not attain the maximum. This proves that ℓ is S-invariant.

(i. $\Rightarrow$ ii.) Suppose that $\ell(g) = \ell(g \circ L_J^*)$, for any $g \in C(\mathcal{P}(X), \mathbb{R})$. Observe that

$$\ell(g \circ L_J^*) = \sup_{\mu \in \mathcal{P}(X)} [h_\ell(\mu) + g(L_J^*(\mu))] = \sup_{\mu \in \mathcal{P}(X)} [h_\ell(T^\sharp(L_J^*(\mu)) + g(L_J^*(\mu))]$$
$$= \sup_{\nu \in L_J^*(\mathcal{P}(X))} [h_\ell(T^\sharp(\nu) + g(\nu)].$$

Writing

$$h'(\nu) := \begin{cases} h_\ell(T^\sharp(\nu)), & \nu \in L_J^*(\mathcal{P}(X)) \\ -\infty, & \text{otherwise} \end{cases},$$

we also get

$$\ell(g \circ L_J^*) = \sup_{\nu \in \mathcal{P}(X)} [h'(\nu) + g(\nu)].$$

Since $L_J^*(\mathcal{P}(X))$ is closed we obtain that h' is u.s.c. Thus h' is the density of $S(\ell)$. Since $S(\ell) = \ell$, from the uniqueness of the density h_ℓ, we conclude that $h' = h_\ell$.

(ii. $\Rightarrow$ iii.) For any μ we have $h_\ell(L_J^*(\mu)) = h_\ell(T^\sharp(L_J^*(\mu))) = h_\ell(\mu)$. □

From the above proposition, for a single Lipschitz Jacobian J, if an idempotent pressure ℓ is invariant for the action of L_J^* then its density entropy satisfies $h_\ell = h_\ell \circ L_J^*$ and $h_\ell(\nu) = -\infty$ if $\nu \notin L_J^*(\mathcal{P}(X))$. We claim that, if $\nu \notin (L_J^*)^2(\mathcal{P}(X))$ then $h_\ell(\nu) = -\infty$. Indeed, if $\nu = L_J^*(\nu_2)$ with $\nu_2 \notin (L_J^*)(\mathcal{P}(X))$ then

$$-\infty = h_\ell(\nu_2) = h_\ell \circ L_J^*(\nu_2) = h_\ell(\nu).$$

If we iterate this process, we get that $h_\ell(\mu) \neq -\infty$ only if $\mu \in (L_J^*)^n(\mathcal{P}(X))$ for any n. In symbolic dynamics, for a Lipschitz J, there exists a unique such μ and it is also the unique invariant probability for L_J^* (eigenprobablity). The next theorem considers a generalization of the above proposition for the case of multiple Jacobians in the context of symbolic dynamics.

Let us fix $d \in \{2, 3, 4, ...\}$ and consider $\Omega = \{1, 2, ..., d\}^{\mathbb{N}}$. A cylinder is a subset of Ω on the form

$$[x_1, ..., x_n] = \{(y_1, y_2, y_3, ...) \in \Omega \,|, y_1 = x_1, y_2 = x_2, ..., y_n = x_n\}.$$

Let us remember the claim in Theorem 5.

Theorem 5. *Consider the shift map σ over the compact metric space (Ω, d_γ), where $\Omega = \{1, ..., d\}^{\mathbb{N}}$, $0 < \gamma < \frac{1}{d+1}$ and (denoting $x = (x_0, x_1, x_2, ...)$ and $y = (y_0, y_1, y_2, ...)$) the metric d_γ satisfies $d_\gamma(x, y) = \gamma^{\min\{j \in \mathbb{N} | x_j \neq y_j\}}$, if $x \neq y$. Let*

$$\mathcal{D} = \{J : \Omega \to [0, 1] \,|\, \mathrm{Lip}(J) \leq 1 \text{ and } \sum_{a=1}^{d} J(ax) = 1 \,\forall x \in \Omega\}.$$

Consider, for each $J \in \mathcal{D}$, the dual of the Ruelle Operator $L_J^* : \mathcal{P}(X) \to \mathcal{P}(X)$, that is, $L_J^*(\mu) = \nu$ means that $\int f(x)\, d\nu(x) = \int \sum_{a=1}^d J(ax) f(ax)\, d\mu(x)$.
1. For each sequence $(J_1, J_2, J_3...)$ of elements of $\mathcal{D}$ there exists a unique probability $\mu \in \mathcal{P}(X)$ such that

$$\mu = \lim_{n\to\infty} L_{J_1}^* \circ \cdots \circ L_{J_n}^*(\nu),$$

for any probability $\nu \in \mathcal{P}(X)$.
2. Consider $\mathcal{D}$ as a metric space with the supremum norm and for a fixed non-empty closed subset $D \subseteq \mathcal{D}$ let $q : D \to \mathbb{R}$ be a continuous function, such that, $\sup_{J\in D} q_J = 0$. *Consider the operator $\mathcal{M} : I(P(X)) \to I(P(X))$, acting over idempotent pressures, which is given by*

$$\mathcal{M}(\ell)(f) := \bigoplus_{J\in\mathcal{D}} q_J \odot \ell(f \circ L_J^*). \tag{29}$$

There exists a unique idempotent pressure function ℓ satisfying $\ell(0) = 0$ and invariant for $\mathcal{M}$. Its density entropy is given by

$$h_\ell(\mu) = \bigoplus_{\substack{(J_1, J_2, J_3, ...) \in D^{\mathbb{N}} \text{ such that} \\ L_{J_1}^* \circ \cdots \circ L_{J_n}^* \to \mu}} [q_{J_1} + q_{J_2} + q_{J_3} + \cdots]$$

and $h_\ell(\mu) = -\infty$ if there is no such sequence $(J_1, J_2, J_3, ...)$.

Example 30. *Consider $D = \{J_1, J_2\}$ where J_1 and J_2 are two Jacobians in the set $\mathcal{D}$. Suppose that $q_{J_1} = q_{J_2} = 0$. Then we are considering the operator $\mathcal{M}$, acting over idempotent pressures, which satisfies*

$$\mathcal{M}(\ell)(f) := \ell(f \circ L_{J_1}^*) \oplus \ell(f \circ L_{J_2}^*).$$

There exists a unique idempotent pressure function ℓ satisfying $\ell(0) = 0$ and invariant under $\mathcal{M}$. Its density entropy is given by $h_\ell(\mu) = 0$ if there is a sequence $(i_1, i_2, i_3, ...) \in \{1, 2\}^{\mathbb{N}}$ such that $\mu = \lim_{n\to\infty} L_{J_{i_1}}^* \circ \cdots \circ L_{J_{i_n}}^*(\cdot)$ and $h_\ell(\mu) = -\infty$ if there is not such a sequence.

Example 31. *Suppose $\Omega = \{1, 2\}^{\mathbb{N}}$ and for each $p \in [0, 1]$ let J_p be defined by $J_p(1, x_1, x_2, ...) = p$ and $J_p(2, x_1, x_2, ...) = 1 - p$. Let $D = \{J_p \,|\, p \in [0, 1]\} \subset \mathcal{D}$. We denote also $P_p(1) = J_p(1) = p$ and $P_p(2) = J_p(2) = 1 - p$. Fix a sequence $(J_{p_1}, J_{p_2}, J_{p_3}, ...)$ and let μ be such that*

$$L_{J_{p_1}}^* \circ \cdots \circ L_{J_{p_n}}^*(\cdot) \to \mu.$$

We want to characterize such μ. We claim that μ satisfies

$$\mu[x_1, ..., x_n] = P_{p_1}(x_1) \cdots P_{p_n}(x_n), \tag{30}$$

for any cylinder set $[x_1, ..., x_n]$ (this is a generalization of the concept of Bernoulli probability, because the weight $(p_i, 1-p_i)$ is not the same for each coordinate i). We denote by I_A the indicator map of the set A. As the set of functions $\{I_A | A \, is \, cylinder \, set\}$ is dense in the set of continuous functions (with the supremum norm), we get that (30) determines μ. In order to verify that μ satisfies (30) we compute as below.
- if $f_1 = I_{[x_1]}$, we have, for any ν:

$$\int f_1 dL^*_{J_{p_1}}(\nu) = \int L_{J_{p_1}}(f_1)\, d\nu = \int J_{p_1}(1x) f_1(1x) + J_{p_1}(2x) f_1(2x)\, d\nu(x)$$

$$= p_1 I_{[x_1]}(1) + (1-p_1) I_{[x_1]}(2) = P_{p_1}(x_1).$$

It follows that $\mu[x_1] = P_{p_1}(x_1)$.
- if $f_2 = I_{[x_1,x_2]}$ we have, for any ν:

$$\int f_2\, d(L^*_{J_{p_1}} \circ L^*_{J_{p_2}}(\nu)) = \int L_{J_{p_1}}(f_2)\, dL^*_{J_{p_2}}\, d\nu = \int L_{J_{p_2}} \circ L_{J_{p_1}}(f_2)\, d\nu$$

$$= \int \sum_{i_2=1}^{2} J_{p_2}(i_2)(L_{J_{p_1}} f_2)(i_2 x)\, d\nu(x) = \int \sum_{i_2=1}^{2} J_{p_2}(i_2)(\sum_{i_1=1}^{2} J_{p_1}(i_1) f_2(i_1 i_2 x))\, d\nu(x)$$

$$= \sum_{i_2=1}^{2} J_{p_2}(i_2)(\sum_{i_1=1}^{2} J_{p_1}(i_1) f_2(i_1 i_2)) = \sum_{i_1,i_2} J_{p_1}(i_1) J_{p_2}(i_2) I_{[x_1,x_2]}(i_1,i_2) = P_{p_1}(x_1) \cdot P_{p_2}(x_2).$$

In general, if $f_n = I_{[x_1,x_2,...,x_n]}$ we have, for any ν:

$$\int f_n\, dL^*_{J_{p_1}} \circ \cdots \circ L^*_{J_{p_n}}(\nu) = \int L_{J_{p_n}} \circ \cdots \circ L_{J_{p_1}}(f_n)\, d\nu$$

$$= \int \sum_{i_1,...,i_n} J_{p_n}(i_n) \cdots J_{p_1}(i_1) I_{[x_1,...,x_n]}(i_1,...,i_n) = P_{p_1}(x_1) \cdots P_{p_n}(x_n).$$

Such μ usually is not σ-invariant. Indeed, a probability ν is invariant if it satisfies, for any cylinder set $[x_1, ..., x_n]$,

$$\sum_{i=1}^{2} \nu[i, x_1, x_2, ..., x_n] = \nu[x_1, x_2, ..., x_n].$$

For the above defined μ we have

$$\sum_{i=1}^{2} \mu[i, x_1, x_2, ..., x_n] = \sum_{i=1}^{2} P_{p_1}(i) P_{p_2}(x_1) \cdots P_{p_{n+1}}(x_n) = P_{p_2}(x_1) \cdots P_{p_{n+1}}(x_n)$$

which is different from $\mu[x_1, x_2, ..., x_n] = P_{p_1}(x_1) \cdots P_{p_n}(x_n)$. We get the equality for all cylinders just in the case $p_1 = p_2 = p_3 = ...$, that is μ is the Bernoulli probability.

5.1 Proof of Theorem 5

In order to prove Theorem 5 we need to consider several concepts and prove some technical preparatory results. In particular, we need some stability result showing how the image of a transfer operator acting on probabilities changes under small variations of the parameter. The proof follows by adapting results described in [Hut81] and [Mic14] to our level-2 context.

For a Lipschitz function $w : \Omega \to \mathbb{R}$, we denote $|w|_\gamma = \sup_{x \neq y} \frac{|w(x)-w(y)|}{d_\gamma(x,y)}$, which is called the Lipschitz constant of w, and $||w||_\infty = \sup_{x\in\Omega} |w(x)|$, which is the supremum norm of w.

Consider the set

$$\mathcal{D} = \{J : \Omega \to [0,1] \,|\, |J|_\gamma \leq 1 \text{ and } \sum_{a=1}^{d} J(ax) = 1 \,\forall x \in \Omega\}. \tag{31}$$

It is a compact metric space for the supremum norm $||\cdot||_\infty$. Indeed, as the potentials $J \in \mathcal{D}$ are uniformly bounded and have Lipschitz constant less than or equal to 1, any sequence has a convergent subsequence by the Arzela-Ascoli Theorem. Furthermore the limit function is in $\mathcal{D}$.

We remember that the 1-Wasserstein distance on $\mathcal{P}(\Omega)$, as defined in (1), is given by

$$W_1(\mu,\nu) = \sup_{|f|_\gamma \leq 1} [\mu(f) - \nu(f)]. \tag{32}$$

Observe that for any constant c we get $|f+c|_\gamma = |f|_\gamma$ and $\mu(f+c) - \nu(f+c) = \mu(f) - \nu(f)$. Then we can suppose that $\inf\{f(x) \,|\, x \in \Omega\} = 0$. Consequently, as $diam(\Omega) = 1$, we get also $\sup\{f(x) \,|\, x \in \Omega\} \leq 1$. In this way we can suppose $0 \leq f \leq 1$ for the computation of W_1.

The next proposition provides a uniform rate of contraction for an uncountable family of duals of Ruelle operators, extending the results of [KLS15].

Proposition 32. *For any $J \in \mathcal{D}$ and any $\mu,\nu \in \mathcal{P}(\Omega)$,*

$$W_1(L_J^*(\mu), L_J^*(\nu)) \leq r \cdot W_1(\mu,\nu), \tag{33}$$

where $r = (d+1)\gamma < 1$.

This shows that $\mu \to L_J^*(\mu)$ is a contraction on $\mathcal{P}(\Omega)$ for the metric W_1, with a constant r which is independent of $J \in \mathcal{D}$.

Proof. We first claim that $|L_J(f)|_\gamma \leq (d+1)\cdot\gamma\cdot|f|_\gamma$, for any Lipschitz function f satisfying $\inf\{f(x) \,|\, x \in \Omega\} = 0$ (and consequently $\sup\{f(x) \,|\, x \in \Omega\} \leq |f|_\gamma$). Indeed, for $x \neq y$ we have

$$\frac{|L_J(f)(x) - L_J(f)(y)|}{d_\gamma(x,y)} = \frac{|\sum_{a=1}^d J(ax)f(ax) - \sum_{a=1}^d J(ay)f(ay)|}{d_\gamma(x,y)}$$

$$\leq \frac{|\sum_{a=1}^d J(ax)f(ax) - \sum_{a=1}^d J(ay)f(ax)| + |\sum_{a=1}^d J(ay)f(ax) - \sum_{a=1}^d J(ay)f(ay)|}{d_\gamma(x,y)}$$

$$\leq \frac{|\sum_{a=1}^{d}[J(ax)-J(ay)]f(ax)|+|\sum_{a=1}^{d}J(ay)[f(ax)-f(ay)]|}{d_\gamma(x,y)}$$

$$\leq d\gamma|J|_\gamma||f||_\infty+\gamma|f|_\gamma \leq d\gamma|f|_\gamma+\gamma|f|_\gamma=(d+1)\gamma|f|_\gamma.$$

Now for any Lipschitz function f satisfying $|f|_\gamma \leq 1$ and $0 \leq f \leq 1$, as $|L_J(f)|_\gamma \leq r$ where $r=(d+1)\cdot\gamma$, we get

$$L_J^*(\mu)(f)-L_J^*(\nu)(f)=\mu(L_J(f))-\nu(L_J(f)) \leq \sup_{|g|_\gamma\leq r}[\mu(g)-\nu(g)]$$

$$=\sup_{|h|_\gamma\leq 1}[\mu(r\cdot h)-\nu(r\cdot h)]=r\cdot W_1(\mu,\nu).$$

Taking $\sup_{|f|_\gamma\leq 1}$ in the left hand side we conclude the proof.

□

Proposition 33. *For any probability $\mu\in\mathcal{P}(\Omega)$ and $J_1,J_2\in\mathcal{D}$,*

$$W_1(L_{J_1}^*(\mu),L_{J_2}^*(\mu)) \leq d\,||J_1-J_2||_\infty. \tag{34}$$

Proof. Consider a function $f:\Omega\to[0,1]$ satisfying $|f|_\gamma\leq 1$. We have

$$|L_{J_1}(f)(x)-L_{J_2}(f)(x)| \leq \sum_{a=1}^{d}|f(ax)J_1(ax)-f(ax)J_2(ax)|$$

$$\leq \sum_{a=1}^{d}|J_1(ax)-J_2(ax)| \leq d\,||J_1-J_2||_\infty.$$

Then,

$$|\int f d\,L_{J_1}^*(\mu)-\int f d\,L_{J_2}^*(\mu)|=|\int L_{J_1}(f)(x)d\mu(x)-\int L_{J_2}(f)(x)d\mu(x)| \leq$$

$$\int |\,L_{J_1}(f)(x)-L_{J_2}(f)(x)\,|d\mu(x) \leq d\,||J_1-J_2||_\infty.$$

□

Proposition 34. *Let $r=(d+1)\gamma<1$ and consider on $\mathcal{D}$ the metric defined by $\tilde{d}(J_1,J_2)=\frac{d}{r}||J_1-J_2||_\infty$. Then, for any $J_1,J_2\in\mathcal{D}$ and any $\mu_1,\mu_2\in\mathcal{P}(\Omega)$ we have*

$$W_1(L_{J_1}^*(\mu_1),L_{J_2}^*(\mu_2)) \leq r[W_1(\mu_1,\mu_2)+\tilde{d}(J_1,J_2)] \tag{35}$$

Proof. Note that from Propositions 32 and 33

$$W_1(L_{J_1}^*(\mu_1),L_{J_2}^*(\mu_2)) \leq W_1(L_{J_1}^*(\mu_1),L_{J_1}^*(\mu_2))+W_1(L_{J_1}^*(\mu_2),L_{J_2}^*(\mu_2)) \leq$$

$$r\;W_1(\mu_1,\mu_2)\;+\;d\,||J_1-J_2||_\infty=r[W_1(\mu_1,\mu_2)+\tilde{d}(J_1,J_2)].$$

□

We remark that the metric $\tilde{d}$ on $\mathcal{D}$ is equivalent to the supremum norm.

Proposition 35. *Given a closed (therefore compact) subset $D \subseteq \mathcal{D}$ with respect to the metric $\tilde{d}$ and the compact metric space $(\mathcal{P}(X), W_1)$, consider the iterated function system $(\mathcal{P}(X), (\phi_J)_{J\in D})$ where $\phi_J : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies $\phi_J(\mu) = L_J^*(\mu)$. Such an IFS is uniformly contractible, that is, for any $J_1, J_2 \in D$ and $\mu_1, \mu_2 \in \mathcal{P}(X)$,*

$$W_1(\phi_{J_1}(\mu_1), \phi_{J_2}(\mu_2)) \leq r[W_1(\mu_1, \mu_2) + \tilde{d}(J_1, J_2)],$$

where $r = (d+1)\gamma < 1$.

Applying the above proposition, the proof of Theorem 5 is a consequence of Lemma 4.1 and Theorem 4.7 in [MO24].

Remark 36. *Once it is proved that $(\mathcal{P}(X), (\phi_J)_{J\in D})$ is uniformly contractible, Theorem 3.6 in [MO24] can be also applied in order to get a characterization of invariant idempotent pressures for the non-place dependent case (where $q_J(\mu)$ depends on μ). Furthermore, all theory concerning transfer operators for uniformly contractible IFS (see [MO25]) can be applied for such IFS.*

6 The inverse problem

We start investigating the inverse problem, that is: given an idempotent pressure function ℓ, is there some uniformly contractible mpIFS in $\mathcal{P}(X)$ for which ℓ is invariant? Recall from Section 2, Definition 6, that $I(\mathcal{P}(X))$ is the set of idempotent probabilities over $\mathcal{P}(X)$.

An attempt to solve our inverse problem is to consider an IFS as general as possible. Let (X, d_X) be a compact metric space and consider the compact metric space (J, d_J) where $J := \mathcal{P}(X)$ and $d_J := \frac{1}{\gamma} d_{\mathcal{P}(X)}$, for a fixed γ satisfying $0 < \gamma < 1$. Consider the iterated function system given by a family of maps $\{\phi_\nu : \mathcal{P}(X) \to \mathcal{P}(X) \,|\, \nu \in \mathcal{P}(X)\}$ which are defined by

$$\phi_\nu(\mu) = \nu, \ \forall \nu \in \mathcal{P}(X), \mu \in \mathcal{P}(X).$$

We notice that such IFS is uniformly contractible, that is,

$$d_{\mathcal{P}(X)}(\phi_{\nu_1}(\mu_1), \phi_{\nu_2}(\mu_2)) = d_{\mathcal{P}(X)}(\nu_1, \nu_2) \leq \gamma \cdot [d_J(\nu_1, \nu_2) + d_{\mathcal{P}(X)}(\mu_1, \mu_2)].$$

A continuous function $q : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$, $q(\nu, \mu) = q_\nu(\mu)$, is a normalized family of weights if it satisfies $|q_\nu(\mu_1) - q_\nu(\mu_2)| < Cd(\mu_1, \mu_2) \,\forall \nu, \mu_1, \mu_2 \in \mathcal{P}(X)$ and $\oplus_{\nu\in\mathcal{P}(X)} q_\nu(\mu) = 0 \,\forall\, \mu \in \mathcal{P}(X)$. We call $\mathcal{S} = (\mathcal{P}(X), \phi, q)$ a max-plus IFS (mpIFS) (see [MO24]).

Definition 37. *To each mpIFS $\mathcal{S} = (\mathcal{P}(X), \phi, q)$ we assign the following operators:*
1. $\mathcal{L}_{\phi,q} : C(\mathcal{P}(X), \mathbb{R}) \to C(\mathcal{P}(X), \mathbb{R})$, defined by

$$\mathcal{L}_{\phi,q}(f)(\mu) := \bigoplus_{\nu\in\mathcal{P}(X)} [q_\nu(\mu) \odot f(\phi_\nu(\mu))]. \tag{36}$$

2. $\mathcal{M}_{\phi,q} : I(\mathcal{P}(X)) \to I(\mathcal{P}(X))$*, defined by*

$$\mathcal{M}_{\phi,q}(\ell) := \bigoplus_{\nu\in\mathcal{P}(X)} [\ell(q_\nu \odot (f \circ \phi_\nu))]. \tag{37}$$

3. $L_{\phi,q} : U(\mathcal{P}(X)) \to U(\mathcal{P}(X))$*, defined by*

$$L_{\phi,q}(\lambda)(\mu) := \bigoplus_{(\nu,\eta)\in\phi^{-1}(\mu)} [q_\nu(\eta) \odot \lambda(\eta)]. \tag{38}$$

Next theorem establishes the relation between the three operators in Definition 37.

Theorem 38. *[MO24] Given a density function* $\lambda \in U(\mathcal{P}(X))$ *satisfying* $\oplus_\mu \lambda(\mu) = 0$ *and the associated idempotent pressure* $\ell = \bigoplus_{\mu\in\mathcal{P}(X)} \lambda(\mu) \odot \delta_\mu \in I(\mathcal{P}(X))$ *we have that* $\mathcal{M}_{\phi,q}(\ell) = \bigoplus_{\mu\in\mathcal{P}(X)} L_{\phi,q}(\lambda)(\mu) \odot \delta_\mu$*, that is,* $\mathcal{M}_{\phi,q}(\mu)$ *has density* $L_{\phi,q}(\lambda)$ *where* λ *is the density of* ℓ*. Furthermore*

$$\mathcal{M}_{\phi,q}(\ell)(f) = \ell(\mathcal{L}_{\phi,q}(f)),$$

for any $f \in C(X,\mathbb{R})$*, that is,* $\mathcal{M}_{\phi,q}$ *is the max-plus dual of* $\mathcal{L}_{\phi,q}$*.*

Definition 39. *An idempotent pressure* $\ell \in I(\mathcal{P}(X))$ *with density* $\lambda \in U(\mathcal{P}(X))$ *is called invariant (with respect to the mpIFS) if it satisfies any of the following equivalent conditions:*
1. $\mathcal{M}_{\phi,q}(\ell) = \ell$*;*
2. $L_{\phi,q}(\lambda) = \lambda$*;*
3. $\ell(\mathcal{L}_{\phi,q}(f)) = \ell(f)$*, for any* $f \in C(\mathcal{P}(X),\mathbb{R})$*.*

Question: Consider an idempotent pressure function ℓ satisfying $\ell(0) = 0$ and with density entropy $h := h_\ell$. Is there a family of weights $q_\nu(\mu) \leq 0$, such that $\sup_{\nu\in X} q_\nu(\mu) = 0$, and $\ell = \sup_{\mu\in\mathcal{P}(X)}[h(\mu) + \delta_\mu]$ is invariant for the max-plus IFS $(\mathcal{P}(X), \phi_\nu, q_\nu)$?
The density entropy, as an invariant density, must verify $L_{\phi,q}(h) = h$, that is,

$$h(\mu) = \sup_{(\nu,\eta)\in\phi^{-1}(\mu)} [q_\nu(\eta) \odot h(\eta)].$$

Since $\phi_\nu(\mu) = \nu$, $\forall \nu \in \mathcal{P}(X), \mu \in \mathcal{P}(X)$, we obtain $(\nu,\eta) \in \phi^{-1}(\mu)$ if, and only if $\nu = \mu$ and η is arbitrary, thus,

$$h(\mu) = \sup_{\eta\in\mathcal{P}(X)} [q_\mu(\eta) + h(\eta)]. \tag{39}$$

Note that $q_\nu(\zeta) = h(\nu) - h(\zeta)$ is not necessarily a candidate, because $h(\nu) - h(\zeta)$ is not necessarily bounded from above, neither continuous. Actually, for each ν, we have that $q_\nu(\zeta)$ is a l.s.c. function.
Let us suppose that the entropy h is continuous, non-positive and that there exists $\mu_0 \in \mathcal{P}(X)$ such that $h(\mu_0) = 0$. In this case, a solution is

$$q_\mu(\eta) := h(\mu),\ \forall \eta, \mu \in \mathcal{P}(X).$$

We can check that this function solves Equation 39.
Indeed,

$$h(\mu) = \sup_{\eta \in \mathcal{P}(X)} [q_\mu(\eta) + h(\eta)] = \sup_{\eta \in \mathcal{P}(X)} [h(\mu) + h(\eta)]$$

is always true because $h \leq 0$ and for $\eta = \mu_0$ we have $h(\eta) = 0$.

7 Max-plus dynamics

Suppose K is a compact metric space, and $T : K \to K$ is continuous. Consider an ergodic T-invariant probability μ on K and a continuous function $f : K \to \mathbb{R}$. It is natural to consider the asymptotic behavior, as $n \to \infty$, of the sums

$$f(x) \oplus f(T(x)) \oplus ... \oplus f(T^{n-1}(x)),$$

for μ-almost every $x \in K$. Among other things, we will introduce the max-plus partition function and we will describe large deviation properties.

The study of the properties of large deviations is a topic of great importance in classical ergodic theory. For Level-2 large deviation properties see Chapter VIII in [Ell06] for the i.i.d case, and [Kif90], [Lop90] or Section 8 in [LTF25] for more general cases. In the classical case (classical algebra), once an ergodic probability μ and certain potential $A : X \to \mathbb{R}$ are fixed, and taking into account the corresponding properties related to the ergodic theorem (using $+$ and not $\oplus$), in most of the cases the resulting deviation function that is obtained has support in an open interval on the real line, which leads to different finite rates of exponential decay to zero (unless μ is uniquely ergodic, producing a rate that is better than any exponential decay). In the Max-Plus case, due to the expression

$$\lim_{n\to\infty} f(x) \oplus (T(x)) \oplus ... \oplus f(T^{n-1}(x)) = \sup_{x \in X} f(x), \tag{40}$$

(see equation (41) below) we have a completely different situation for two reasons:

I) the rates are better than any exponential decay, an atypical behavior in the classical theory (see expression (44) in Proposition 45). Example 46 aims to clarify the main differences.

II) The corresponding Max-Plus sum in an orbit (see (40)) has a different feature when compared to the classical case: we consider $\oplus$ and not $+$, and the left-hand side of (40) is not divided by n.

The results of this section can be applied to the case where T is the push-forward map and $K = \mathcal{P}(X)$, where X is a compact metric space. A version of next result for uniquely ergodic dynamical systems can be found in Corollary 4 in [KM89], which is the max-plus analogue to Furstenberg's theorem [Fur61].

Lemma 40. *Suppose K is a compact metric space and $T : K \to K$ is continuous. Consider an ergodic T-invariant probability μ which is positive on open* *non-empty sets and a continuous function $f : K \to \mathbb{R}$. Then, for μ-almost every $x \in K$ we have*

$$\lim_{n\to\infty} f(x) \oplus (T(x)) \oplus ... \oplus f(T^{n-1}(x)) = \sup_{x\in K} f(x). \tag{41}$$

Proof. Let $x_0 \in K$ be such that $f(x_0) = \sup_{x\in K} f(x)$. Given $k \in \mathbb{N}$, let $\delta > 0$ be such that $|f(x) - f(x_0)| < 1/k$ for any $x \in B(x_0, \delta)$.

As $\mu(B(x_0, \delta)) > 0$, by Poincaré's recurrence theorem, for μ-almost every point $x \in K$, the sequence $x, T(x), T^2(x), ...$ will visit the set $B(x_0, \delta)$ infinitely many times. Therefore, there exists a set $U_k \subseteq K$ such that $\mu(U_k) = 1$ and for any $x \in U_k$,

$$\liminf_{n\to\infty} f(x) \oplus f(T(x)) \oplus ... \oplus f(T^{n-1}(x)) \geq f(x_0) - 1/k.$$

Taking $U = \cap_{k\in\mathbb{N}} U_k$ we get $\mu(U) = 1$ and

$$\liminf_{n\to\infty} f(x) \oplus f(T(x)) \oplus ... \oplus f(T^{n-1}(x)) \geq f(x_0) \ \ \forall x \in U.$$

On the other hand, as $f(x_0) = \sup_{x\in K} f(x)$, for any $n \in \mathbb{N}$ and $x \in U$ we have

$$f(x) \oplus f(T(x)) \oplus ... \oplus f(T^{n-1}(x)) \leq f(x_0),$$

so

$$\limsup_{n\to\infty} f(x) \oplus f(T(x)) \oplus ... \oplus f(T^{n-1}(x)) \leq f(x_0).$$

□

Let K be a compact set, $T : K \to K$ be continuous, $f : K \to \mathbb{R}$ be a continuous function and μ be an ergodic T-invariant probability which is positive on open sets. For each value $t \in \mathbb{R}$, we define $c(t) := \limsup_{n\to\infty} c_n(t)$, where

$$c_n(t) := \frac{1}{n} \log \int e^{n\,t\,(f(x)\oplus f(T(x))\oplus f(T^2(x))\oplus...\oplus f(T^{n-1}(x)))} d\mu(x). \tag{42}$$

Proposition 41. *For $t \geq 0$ we have $c(t) = t \cdot \sup_{x\in K} f(x)$ and $c(-t) \geq (-t) \cdot \sup_{x\in K} f(x)$.*

Proof. Clearly, $c(0) = 0$. Suppose $t > 0$ and let $x_0 \in K$ be such that $f(x_0) = \sup_{x\in K} f(x)$. For each $\epsilon > 0$ let $\delta > 0$ be such that $f(x) > f(x_0) - \epsilon$ for any $x \in B(x_0, \delta)$. Then we have

$$\begin{aligned} c(t) &\geq \limsup_{n\to+\infty} \frac{1}{n} \log \int_{B(x_0,\delta)} e^{ntf(x)} d\mu(x) \\ &\geq \limsup_{n\to+\infty} \frac{1}{n} \log \left(e^{nt(f(x_0)-\epsilon)} \cdot \mu(B(x_0, \delta)) \right) \\ &= t(f(x_0) - \epsilon) = t(\sup_{x\in K} f(x)) - t\epsilon. \end{aligned}$$

As ϵ is arbitrary, we get $c(t) \geq t \sup_{x\in K} f(x)$. The opposite inequality is trivial. Furthermore,

$$
\begin{aligned}
c(-t) &= \limsup_{n\to+\infty} \frac{1}{n} \log \int e^{-n\,t\,(f(x)\oplus\ldots\oplus f(T^{n-1}(x))} d\mu(x) \\
&\geq \limsup_{n\to+\infty} \frac{1}{n} \log \int e^{-n\,t\,f(x_0)} d\mu = -tf(x_0) = -t(\sup_{x\in K} f(x)).
\end{aligned}
$$

□

Remark 42. *In Example 46 we exhibit a case where* $c(-t) \neq (-t)\cdot\sup_{x\in K} f(x)$.

Remark 43. *If* $f \geq 0$ *then* c *is non-decreasing and consequently* $c(s \oplus t) = c(s) \oplus c(t)$. *If* $f \geq 1$ *and* $\alpha < 0$ *then* $c(\alpha \odot t) \leq \alpha \odot c(t)$. *Indeed,*

$$
c(\alpha + t) \leq \limsup_{n\to\infty} \frac{1}{n} \log \int e^{(n\,\alpha\cdot 1)+n\,t\,(f(x)\oplus\ldots\oplus f(T^{n-1}(x))} d\mu(x) = \alpha + c(t).
$$

Consequently, if $f \geq 1$ *then* c *is max-plus convex, that is, if* $\alpha \oplus \beta = 0$ *then*

$$
c((\alpha \odot t) \oplus (\beta \odot s)) \leq (\alpha \odot c(t)) \oplus (\beta \odot c(s)).
$$

In what follows we would like to estimate the growth with n of the expression

$$
\int e^{n\,t\,(f(z)\oplus f(\sigma(z))\oplus f(\sigma^2(z))\oplus\ldots\oplus f(\sigma^{n-1}(z))} d\mu(z), \tag{43}
$$

which is the max-plus version of the so-called partition function (see expression (5.7) in [Sal01] or (4.5) in [Ell06]). This concept helps to define the probability distribution of the canonical ensemble (see (5.29) in [Sal01]), which provides the probability maximizing the analogous concept of pressure (see expression (C.8) in Section C.5 in [Ell06]) for a non-dynamical framework. (5.29) in [Sal01] is sometimes called Gibbs distribution.

Lemma 44 (Chebyshev's inequality)**.** *Let* $g : K \to \mathbb{R}$ *be a measurable function and* $h : \mathbb{R} \to \mathbb{R}$ *be a non-negative, non-decreasing function such that* $\int h(g(x))d\mu(x) < +\infty$. *Then, for any value* d *such that* $h(d) > 0$,

$$
\mu(\{x \mid g(x) \geq d\}) \leq \frac{\int h(g(x))d\mu(x)}{h(d)}.
$$

In the classical case, the upper bound large deviation follows from Chebyshev's inequality (see [Ell06]). In the max-plus case we will also apply Chebyshev's inequality, but we have to consider a small variation of it, taking into account our definition of $c_n(t)$ (for the dynamical max-plus sum, in the exponential term in (43), there is an extra term n multiplying t, which does not appear in the case of the classical dynamical sum, as stated for $c(t)$ in page 535 in [Lop90].

Proposition 45. *[Upper large deviation bounds] Let K be a compact metric space, $T : K \to K$ be continuous, $f : K \to \mathbb{R}$ be a continuous function and μ be an ergodic T-invariant probability which is positive on open sets. Then, for any value $b < \sup_{x \in K} f(x)$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log(\mu\{x|(f(x) \oplus ... \oplus f(T^{n-1}(x)) \leq b\}) \leq \inf_{t\geq 0}[tb + c(-t)]. \qquad (44)$$

Proof. We can suppose $f \geq 0$. Indeed, if we add a constant on both f and b then both sides of (44) remains equal. For each $n \in \mathbb{N}$ and $t > 0$, let us consider

$$h(x) = e^{n\,t\,x},\ \ g(x) = -(f(x) \oplus ... \oplus f(T^{n-1}(x)) \ \text{ and } \ d = -b.$$

From Chebychev inequality, for $t \geq 0$

$$\mu\{x| - (f(x) \oplus ... \oplus f(T^{n-1}(x)) \geq -b\} \leq \frac{\int e^{-nt(f(x)\oplus ... \oplus f(T^{n-1}(x)))}\, d\mu(x)}{e^{-ntb}}.$$

Then

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu\{x| - (f(x) \oplus ... \oplus f(T^{n-1}(x)) \geq -b\} \leq c(-t) + tb.$$

Consequently

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu\{x|(f(x) \oplus ... \oplus f(T^{n-1}(x)) \leq b\} \leq \inf_{t\geq 0}[c(-t) + tb].$$

□

Example 46. *Let us suppose that $K = \{0,1\}^{\mathbb{N}}$ and $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is given by*

$$f(x_1, x_2, x_3, ...) = \begin{cases} 0 & \text{if } x_1 = 0 \\ 1 & \text{if } x_1 = 1 \end{cases}.$$

Let $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map σ and μ be the Bernoulli probability $(p, 1-p)$, where $0 < p < 1$. This means that, for any cylinder set $[x_1, x_2, ..., x_n]$, we have $\mu[x_1, x_2, ..., x_n] = P_{x_1} \cdot P_{x_2} \cdots P_{x_n}$ where $P_0 = p$ and $P_1 = 1 - p$.

We remark that, for any $x \in \{0,1\}^{\mathbb{N}}$ we have that $(f(x) \oplus ... \oplus f(\sigma^{n-1}(x)))$ is equal to zero or one. Furthermore, it is zero iff $x \in [\underbrace{0, 0, .., 0}_{n}]$. Therefore, for any $0 < b < 1$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\mu\{x|(f(x) \oplus ... \oplus f(\sigma^{n-1}(x))) \leq b\})$$
$$= \lim_{n\to\infty} \frac{1}{n} \log(\mu[\underbrace{0, 0, .., 0}_{n}]) = \lim_{n\to\infty} \frac{1}{n} \log(p^n) = \log(p).$$

From now on we want to estimate the upper bound large deviation described in Proposition 45 for $0 < b < 1$. As we will see, the right hand side of inequality (44) does not match with $\log(p)$. Consequently, inequality (44) can not be *replaced by an equality. In this way, initially we need to estimate* $c(-t)$ *for each* $t \geq 0$*. For fixed* $n \geq 0$ *and* $t \geq 0$ *we have*

$$\begin{aligned}
&\int e^{n\,(-t)\,(f(x)\oplus f(\sigma(x))\oplus \ldots \oplus f(\sigma^{n-1}(x))}d\mu(x) \\
&= \int_{[1]} e^{-n\,t\,(f(x)\oplus \ldots \oplus f(\sigma^{n-1}(x))}d\mu(x) \\
&\quad + \sum_{j=1}^{n-1} \int_{[\underbrace{0,0,..,0}_{j},1]} e^{-n\,t\,(f(x)\oplus \ldots \oplus f(\sigma^{n-1}(x))}d\mu(x) \\
&\quad + \int_{[\underbrace{0,0,..,0}_{n}]} e^{-n\,t\,(f(x)\oplus \ldots \oplus f(\sigma^{n-1}(x))}d\mu(x) \\
&= e^{-nt}(1-p) + \sum_{j=1}^{n-1} e^{-nt}p^j(1-p) + e^0 p^n \\
&= e^{-nt}(1-p)(\frac{1-p^n}{1-p}) + p^n \\
&= e^{-nt}(1-p^n) + p^n.
\end{aligned}$$

It follows that

$$\begin{aligned}
c(-t) &= \limsup_{n\to+\infty} \frac{1}{n}\log(e^{-nt}(1-p^n)+p^n) \\
&= \limsup_{n\to+\infty} \frac{1}{n}\log(e^{-nt}(1-p^n)+e^{\log(p)n}) = \sup\{-t,\log(p)\}.
\end{aligned}$$

Consequently, from inequality (44) *we get, for* $0 < b < 1$,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log(\mu\{x|(f(x)\oplus \ldots \oplus f(\sigma^{n-1}(x))\leq b\}) \\
&\qquad \leq \inf_{t\geq 0}[tb + \sup\{-t,\log(p)\}] = \inf_{t\geq 0}[\sup\{-t(1-b), tb+\log(p)\}].
\end{aligned}$$

The function $t \mapsto -t(1-b)$ *is decreasing and the function* $t \mapsto tb + \log(p)$ *is increasing. Then, the above infimum is attained by* t *satisfying* $tb + \log(p) = -t(1-b)$*, that is,* $t = -\log(p)$*. Therefore, inequality* (44) *can be rewritten as*

$$\limsup_{n\to\infty}\frac{1}{n}\log(\mu\{x|(f(x)\oplus \ldots \oplus f(\sigma^{n-1}(x))\leq b\}) \leq (1-b)\log(p).$$

On the other hand, as we saw above,

$$\lim_{n\to\infty}\frac{1}{n}\log(\mu\{x|(f(x)\oplus \ldots \oplus f(\sigma^{n-1}(x))\leq b\}) = \log(p),$$

being $\log(p) < (1-b)\log(p)$ *because* $\log(p) < 0$.

8 Appendix - Motivation and related results

A topic of great importance in applications is the idempotent version of the Perron Theorem, as mentioned in [Chou87], which concerns the additive eigenvalue problem of physics. The concept of calibrated subaction of ground states in ergodic optimization is also linked to this problem (see [BLL13] and [LM26]).

Applications of tropical geometry (idempotent analysis) to Classical Mechanics appear in [Litvi10] and [Roublev05], to the Schrödinger equation in [McE19] and [Kol97], and in Statistical Mechanics in [Mas16] and [Kirov09].

One can show (see [Ken20]) that Feynmans Path Integral designed for quantum mechanics has an analogue in classical mechanics, the so-called (min, +) Path Integral. The former is built on (min, +)-algebra and (min, +)-analysis which permits to handle in a linear way non-linear problems occurring in mathematical physics.

The introduction of the study of Tropical Semi-Ring Structures is natural in the analysis of some problems related to Mixed Systems of Bosons and Fermions in quantum mechanics (see Section 4 in [Che23]).

Flows (continuous-time dynamics) in the space of probabilities are well-known and important topics of investigation; for example, the geodesic flow in the 2-Wassertein space of probabilities over a certain fixed metric space X (the Otto-Wasserstein geometry), as described in[Vil03]. Via optimal transport, an initial measure μ_0 is moved to a final measure μ_1, along the shortest path. Many systems in applications can be interpreted as gradient flows in the 2-Wasserstein space, where the flow eventually converges to an equilibrium state μ (a probability on X) that is a minimizer of a specific energy functional (e.g., the Kullback-Leibler divergence to a reference measure π). In this case μ is invariant for the flow. The natural corresponding version of discrete-time dynamics acting on the space of probabilities is the push-forward map. In the Tropical Geometry context, in terms of dynamical evolution, it is natural to consider idempotent measures that are invariant for the action of the push-forward map; the Max-Plus level-2 setting.

Somewhat in line with what we considered above in our text, in [KL13] the author considers an expanding dynamics described by a function $\Phi(x) = d\,x \pmod 1), d \in \mathbb{N}$, acting on the unit circle $\mathbb{S}_1$, and moreover the associated dynamics of the pushforward map $\Phi_\sharp$ acting on probabilities in the circle. He is able to calculate (see Theorem 5.1) the derivative in the sense of Gâteaux of the action of $\Phi_\sharp$ at level 2 acting on the set of probabilities on the unit circle. Estimates of the metric mean dimension associated with the action of $\Phi_\sharp$ in the p-Wasserstein metric are the object of Theorem 1.1 in [KL13].

Now, we present an example of a topic which can be viewed as part of our general formalism. Consider $X = \{1, 2, 3\}$, in this case the probabilities on X are indexed by $p = (p_1, p_2, p_3) \in \mathcal{P}(X)$, where $p_1, p_2, p_3 \geq 0$, and $p_1+p_2+p_3 = 1$. Given a continuous function $g : \mathcal{P}(X) \to \mathbb{R}$ we set $\ell(g) = \sup_{p\in\mathcal{P}(X)} g(p)$. In this case entropy h in

$$\ell(g) = \sup_{p\in\mathcal{P}(X)} [g(p) + h(p)]$$

is constant equal to zero.

Set $\hat{p} = (1/3, 1/3, 1/3)$, and for fixed $n > 0$, consider a random sample $y = (y_1, y_2, ..., y_n)$ of size n, obtained from the independent probability associated to $\hat{p}$. We denote $C_n(y)$ the cylinder set $[y_1, y_2, ..., y_n] \in \{1, 2, 3\}^{\mathbb{N}}$. The function g_n will be defined as the maximum likelihood estimator

$$g_n(p) = \frac{1}{n} \log\left(\frac{p(C_n(y))}{\hat{p}(C_n(y))}\right).$$

This function is commonly used in Bayesian Statistics (see [AR13]). For fixed n, one is interested in the probability p^n maximizing g_n. The loss function is $-\,g_n$

Assume the string y has n_j occurrences of each symbol $j = 1, 2, 3$. In this case $n_1 + n_2 + n_3 = n$. Using Lagrange multipliers one can show that the probability p^n maximizing the above expression is such that

$$p^n_j = \frac{n_j}{n}, j = 1, 2, 3.$$

Finally

$$\ell(g_n) = \frac{1}{n}(\sum_{j=1}^{3} n_j \log n_j) - \log n.$$

One can show that $p^n \to \hat{p}$, when $n \to \infty$.

A simple example to illustrate the dissimilarity of our context to the setting of [BCMV22, BCM+23] in the level-2 case is the following: consider $X = \{1, 2\}$, in this case the probabilities on X are indexed by $p = (p_1, p_2) \in \mathcal{P}(X)$, where $p_1, p_2 \geq 0$, and $p_1 + p_2 = 1$. We can parametrize the probabilities $p \in \mathcal{P}(X)$ by p_1, and for each continuous function $g(p) = g(p_1)$, we set the idempotent pressure function $\ell(g) = \sup_{p_1\in[0,1]} g(p_1)$. In this case entropy h in

$$\ell(g) = \sup_{\mu\in\mathcal{P}(X)} [g(\mu) + h(\mu)]$$

is constant equal to zero. Moreover, given a continuous function $g : \mathcal{P}(X) \to \mathbb{R}$, in our setting an equilibrium state is any probability $p \in \mathcal{P}(X)$ such that $g(p_1)$ attains the maximal value of the function $g \in C_0(\mathcal{P}(X), \mathbb{R})$. In our case, when more than one equilibrium state exists, let's say $\tilde{p}_1, \tilde{p}_2 \in \mathcal{P}(X)$, the convex combination $\lambda\,\tilde{p}_1 + (1-\lambda)\,\tilde{p}_2$, $\lambda \in (0, 1)$, is not necessarily an equilibrium state. In the setting of Theorem 1 in [BCMV22, BCM+23], we get that the entropy $\mathfrak{h}$ is zero, and $\delta_{\tilde{p}_1}, \delta_{\tilde{p}_2}$ are equilibrium states for g. Diversely, the convex combination of equilibrium states $\lambda\,\delta_{\tilde{p}_1} + (1-\lambda)\,\delta_{\tilde{p}_2}$, $\lambda \in (0, 1)$, is also an equilibrium state in the sense of [BCMV22, BCM+23].

Regarding questions related to the comparison of the definitions of the Ruelle operator in both settings, note that in [BCMV22] it is assumed that

$$\sup_{x\in X}\{\#f^{-1}(x)\} < \infty, \tag{45}$$

which is not the case when considering the pushforward map in Section 5. The entropy in Corollary 7.15 in [BCMV22] if is finite due to (45), a condition that we do not require. Given a probability $\mu \in \mathcal{P}(X)$, the cardinality of the set of pre-images of μ, under the pushforward map $T^\sharp$, is ∞ (see [LO24]). In Proposition 23 we get a concept of entropy h_ℓ which is finite in the setting of the pushforward map $T^\sharp$.

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