ON THE SPECTRAL DENSITY OF A CLASS OF CHAOTIC TIME SERIES

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Abstract: The purpose of this paper is to show explicitly the spectral density function of the stationary stochastic process determined by a certain class of two-dimensional maps F_α defined below (α is a parameter in $(0,1)$), the random variable $\phi(x,y) = x$ and the invariant probability ν described below.

We first define the transformation $T_\alpha : [0,1] \to [0,1]$ given by

$$T_\alpha(x) = \begin{cases} \frac{x}{\alpha}, & \text{if} \quad 0 \le x < \alpha \\ \\ \frac{\alpha(x-\alpha)}{1-\alpha}, & \text{if} \quad \alpha \le x \le 1, \end{cases}$$

where $\alpha \in (0,1)$ is a constant. The map T_α describes a model for a particle (or the probability of a certain kind of element in a given population) that moves around, in discrete time, in the interval $[0,1]$.

The results presented here can be stated either for T_α or for F_α but we will prefer the later. The results for T_α can be obtained from the more general setting described by F_α.

The map F_α is defined from $K = ([0,1] \times (0,\alpha)) \cup ([0,\alpha] \times [\alpha,1]) \subset \mathbf{R^2}$ to itself and it is given by $F_\alpha(x,y) = (T_\alpha(x), G_\alpha(x,y))$, for $(x,y) \in K$ where

$$G_\alpha(x,y) = \begin{cases} \alpha\, y, & \text{if} \quad 0 \le x < \alpha \\ \alpha + \left(\frac{1-\alpha}{\alpha}\right) y, & \text{if} \quad \alpha \le x \le 1. \end{cases}$$

The spectral density function of the stationary process with probability ν (invariant for F_α and absolutely continuous with respect to the Lebesgue measure)

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0, Y_0)) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z},$$

where $(X_0, Y_0) \in \mathbf{R}^2$ and $\{\xi_t\}_{t \in \mathbf{Z}}$ is a white noise process, will be given explicitly (see Theorem 1 in Section 3) by

$$f_Z(\lambda) = f_X(\lambda) + \frac{\sigma_\xi^2}{2\pi} = \frac{1}{2\pi\, Var(X_t)}[\gamma(e^{i\lambda}) + \gamma(e^{-i\lambda}) - C(0)] + \frac{\sigma_\xi^2}{2\pi},$$

for all $\lambda \in [0, 2\pi)$, where $Var(X_t) = (\alpha^2 - \alpha + 1)(\alpha^2 - 5\alpha + 5)[12(2-\alpha)^2]^{-1}$, γ is given by the last equality in (2.10) at Proposition 5 and $C(0) = (1 + \alpha^2 - \alpha^3)[3(2-\alpha)]^{-1}$.

If one defines the spectral density as

$$\frac{1}{2\pi}\rho_X(0) + \frac{1}{\pi}\sum_{k=0}^{\infty} \cos(k\lambda)\rho_X(k),$$

where $\rho_X(k)$ is the correlation of order $k > 0$, then one can consider that our results apply to the map T (not invertible) instead of F (invertible).

The explicit expression of the spectral density gives a very important information about ressonances in the stationary process.

We point out that the spectral density function has poles in the same values of the poles of the zeta function. Therefore, an explicit expression of this form it is also important for this reason.

We shall also estimate the parameter α based on a time series.

Keywords: Spectral density; chaotic time series; dynamical systems.

1. INTRODUCTION

We shall present the spectral analysis of the stationary stochastic process

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0, Y_0)) + \xi_t = \phi(F_\alpha(X_{t-1}, Y_{t-1}) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z}, \tag{1.1}$$

where ϕ is the continuous function that describes the position in the x variable, that is $\phi(x, y) = x$, ξ_t is a white noise process, and F_α is the transformation defined below.

The map F_α is defined from $K = ([0,1] \times (0,\alpha)) \cup ([0,\alpha] \times [\alpha,1]) \subset \mathbf{R^2}$ to itself and it is given by $F_\alpha(x,y) = (T_\alpha(x), G_\alpha(x,y))$ where the transformation $T_\alpha : [0,1] \to [0,1]$ is defined by

$$T_\alpha(x) = \begin{cases} \frac{x}{\alpha}, & \text{if} \quad 0 \leq x < \alpha \\ \\ \frac{\alpha(x-\alpha)}{1-\alpha}, & \text{if} \quad \alpha \leq x \leq 1, \end{cases} \tag{1.2}$$

with $\alpha \in (0,1)$ as a constant, and

$$G_\alpha(x,y) = \begin{cases} \alpha\, y, & \text{if} \;\; 0 \leq x < \alpha \\ \alpha + \left(\frac{1-\alpha}{\alpha}\right) y, & \text{if} \;\; \alpha \leq x \leq 1. \end{cases} \tag{1.3}$$

The stationary process (1.1) will be considered with respect to a certain stationary (or invariant) measure ν that will be defined on Section 3.

The graph of the map T_α is shown in Figure 1. The action of the piecewise diffeomorphism F_α is presented in Figure 2. The transformation F_α is a modification of the well known Baker transformation.

The map T_α describes a model for a particle that moves around in the interval [0,1]. If the particle is at position x, then after a unit of time it jumps to $T_\alpha(x)$ and so on. According to the model considered here suppose the spatial position of the particle is $T_\alpha^t(x) = X_t$, $t \in \mathbf{N}$, in the interval [0,1]. If the particle X_t is in the interval $[0,\alpha)$, it has a uniformly spread possibility to jump to any point X_{t+1} in [0,1]. However, if it is in the interval $[\alpha, 1)$ it has a uniformly spread possibility to jump to any point X_{t+1} in the interval $[0,\alpha)$.

We are primarily interested in the map T_α, but for defining the spectral density in the classical form (see Brockwell and Davis (1987))

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-ik\lambda)\rho_X(k) \geq 0, \quad 0 \leq \lambda < 2\pi,$$

we need to consider $\rho_X(k)$ for negative k and hence we need a bijective map. Therefore, we have to consider F_α, *the natural extension of* T_α (Bogomolny and Carioli (1995)).

The piecewise diffeomorphism F_α leaves invariant (see definition in Lopes and Lopes (1995)) an ergodic probability ν on $K \subset \mathbf{R}^2$, absolutely continuous with respect to the Lebesgue measure, that will be described in Section 3.

Choosing a point $(x_0, y_0) \in R^2$ at random, according to the Lebesgue measure (or according to ν), the spectral properties of the process Z_t will be analyzed.

More precisely, we shall present explicitly the analytic expression of the spectral density function of such stochastic process (see Section 3).

We refer the reader to Lasota and Mackey (1994) and to Lopes and Lopes (1995) for general definitions and more detailed explanations for the context of the class of problems considered here.

In Grossmann and Thomae (1977) a recursive relation for the autocorrelation coefficients is presented for a different class of maps: tent maps. These results allows one to obtain in that case an explicit closed form for the autocorrelation coefficients and an explicit expression for the spectral density (see also Lopes and Lopes (1995)). The results obtained here can not be obtained from Grossmann and Thomae (1977) due to the different structure of branches and the fact that the invariant measure is not a Lebesgue measure anymore. In Lopes, Lopes and Souza (1996) a generalization of the result of Grossmann and Thomae (1977) is presented.

In Section 2 we present the basic results for a map T_α that are used in Section 3 for obtaining results for the map F_α.

The main results of this paper is the expression (3.1) for the spectral density function of the stochastic process $X_t = \phi(F_\alpha^t)$ (in terms of radial limits in the unit complex disk) presented in Theorem 1 in Section 3. For more general results (see Lopes, Lopes and Souza (1996)).

Note that the noise does not play a very cruscial role in our analysis.

The explicit expression of the spectral density function (as obtained here) of a stochastic process allows one to analyze the efficiency of a given numerical method for estimating the spectrum, based on the closeness of the estimation obtained from the method compared to the true spectral density function.

We also estimate the parameter α at the end of Section 2.

The explicit expression of the spectral density gives a very important information about ressonances in the stationary process.

We point out that the spectral density function has poles in the same values of the poles of the zeta function (see Pollicott (1985)(1990)). Therefore, an explicit expression of this form it is also important for this reason.

We refer the reader to Lopes, Lopes and Souza (1996) for results about the periodogram of times series obtained from expanding maps.

2. THE AUTOCORRELATION FUNCTION

Before considering the transformation F_α we will need to consider the transformation T_α.

Let the transformation $T_\alpha : [0,1] \to [0,1]$ be given as in (1.2), where $\alpha \in (0,1)$ is a constant. The derivative of $T_\alpha(x)$ at x is $a = 1/\alpha$ if $0 \le x < \alpha$ and $b = \alpha/(1-\alpha)$ if $\alpha \le x \le 1$.

A piecewise monotonic differentiable transformation T is *expanding*, if there exists $\beta > 1$ such that $T^{'}(x) > \beta$ for all x where T is differentiable.

One observes that $\underline{a}$ is always greater than 1, however $b \le 1 \Leftrightarrow \alpha \le 1/2$ and $b > 1 \Leftrightarrow \alpha > 1/2$. The transformation T_α is an expansive map (see Robinson (1995)) when $\alpha > 1/2$. It is easy to show that when $\alpha < 1/2$, T_α^2 is an expanding map.

We will be interested here in finding the invariant measure μ absolutely continuous with respect to the Lebesgue measure (see Lasota and Yorke (1973); Parry and Pollicott (1990); Robinson (1995)) and also in analyzing the autocorrelation function associated with the stationary stochastic process (T_α^t, μ).

The general existence of absolutely continuous invariant measures for expanding maps is known by the Lasota-Yorke Theorem (see Lasota and Yorke (1973)). We need explicit expressions and we obtained this density in a closed form in (2.1) below.

The invariant measure for the transformation T_α is of the form (see Lopes, Lopes and Souza (1995)) $d\mu = g(x)dx$ where the function g is defined by

$$g(x) = \begin{cases} \frac{1}{\alpha(2-\alpha)}, & \text{if} \quad 0 \le x < \alpha \\ \\ \frac{1}{2-\alpha}, & \text{if} \quad \alpha \le x \le 1. \end{cases} \tag{2.1}$$

$g(x)$ is the unique density such that defines an absolutely continuous invariant probability for T.

Consider in the sequel the following notation

$$c = \frac{1}{\alpha(2-\alpha)} \quad \text{and} \quad d = \frac{1}{2-\alpha}. \tag{2.2}$$

When T (or T^2) is an expanding map the measure μ is ergodic (see Parry and Pollicott (1990)). Hence, the measure μ given by the expression (2.1) is an ergodic measure (applying the last statement to T_α or T_α^2).

In an analogous way as in Lopes and Lopes (1995), consider $F_\alpha : K \to K$ (K will be defined later), *the natural extension of* T_α (see Bogomolny and Carioli (1995)). The one-dimensional measure μ can be extended to a two-dimensional measure ν (defined later in Section 3) invariant for F_α. We shall give the explicit expression for the spectral density function of the stationary stochastic process $\phi(F_\alpha^t) = X_t$, the random variable $\phi(x, y) = x$ and the measure ν.

Consider now the stationary stochastic process given by (1.1), where $\{\xi_t\}_{t\in\mathbf{Z}}$ is a noise process. For simplicity of the exposition we consider $\xi_t \sim N(0, \sigma_\xi^2)$, for any $t \in \mathbf{Z}$, that is, a Gaussian white noise process. The noise ξ_t has no dynamic characteristic in the model considered here and could be, in fact, eliminated from the model in (1.1). The model with noise is nevertheless more general and we will leave the noise in the model.

We assume that $\{(X_t, Y_t)\}_{t\in\mathbf{Z}}$ and $\{\xi_t\}_{t\in\mathbf{Z}}$ are uncorrelated processes.

Define the *autocorrelation function of order* k of the process $\{X_t\}_{t\in\mathbf{Z}}$ by

$$\rho_X(k) = \frac{Cov(X_t, X_{t+k})}{\sqrt{Var(X_t)Var(X_{t+k})}}$$

$$= \frac{E[X_t\phi(F_\alpha^k(X_t, Y_t))] - E(X_t)E[\phi(F_\alpha^k(X_t, Y_t))]}{\sqrt{Var(X_t)Var[\phi(F_\alpha^k(X_t, Y_t))]}}$$

$$= \frac{E(xT_\alpha^k(x)) - [E(x)]^2}{Var(x)}. \tag{2.3}$$

Our goal is to derive the spectral density function of the process $\{X_t\}_{t\in\mathbf{Z}}$

$$f_X(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \exp(-ik\lambda)\rho_X(k), \quad \text{for any} \quad \lambda \in [0, 2\pi).$$

Hence, one needs to derive the autocorrelation coefficients $\rho_X(k)$, $k \in \mathbf{Z}$, defined in (2.3).

By abuse of the notation, we shall denote $\phi(x) = x$ and $\phi(x, y) = x$ by the same letter ϕ.

In an analogous way as in Lopes and Lopes (1995; Section 2), we will show that for positive k the autocorrelation function of lag k of the dynamical systems $(F_\alpha(x, y), \phi(x, y), \nu)$ and $(T_\alpha(x), \phi(x), \mu)$ are the same. For negative values of k the autocorrelation function of order k of F_α is equal to the corresponding autocorrelation function $-k$ for (F_α, ϕ, ν) (or for (T_α, ϕ, μ)). These properties will be described in Section 3.

There is no meaning for the autocorrelation function of T_α at negative lag k because T_α is not an invertible map.

If one defines the spectral density as

$$\frac{1}{2\pi}\rho_X(0) + \frac{1}{\pi}\sum_{k=0}^{\infty}\cos(k\lambda)\rho_X(k),$$

where $\rho_X(k)$ is the correlation of order $k > 0$, then one can consider that our results apply to the map T (not invertible) instead of F (invertible).

First one needs three technical propositions involving the transformation T_α.

The proofs of Propositions 1 to 5 will not be presented here (see Lopes, Lopes and Souza (1995)).

The next proposition gives a characterization of the k-th iterated of the transformation $T_\alpha(x)$ by a recursive formula.

PROPOSITION 1: *The k-th iteration of the transformation $T_\alpha(x)$ defined by the expression (1.2) is given by*

$$T_\alpha^k(x) = \begin{cases} T_\alpha^{k-1}\left(\frac{x}{\alpha}\right), & \text{if} \quad 0 \le x < \alpha \\ \\ T_\alpha^{k-2}\left(\frac{x-\alpha}{1-\alpha}\right), & \text{if} \quad \alpha \le x \le 1, \end{cases} \tag{2.4}$$

for any integer $k \ge 2$.

We need Proposition 1 to prove Proposition 2 that will be used in Proposition 3.

PROPOSITION 2: *The integral $A(k) = \int_0^1 T_\alpha^k(x)dx$ satisfies the recursive equation*

$$A(k) = \alpha\, A(k-1) + (1-\alpha)\, A(k-2), \tag{2.5}$$

for any integer $k \ge 2$, with initial values $A(0) = 1/2$ and $A(1) = (2-\alpha)\alpha/2$.

The next proposition shows a three term relation that will be used in Proposition 4.

PROPOSITION 3: *The integral* $B(k) = \int_0^1 xT_\alpha^k(x)dx$ *satisfies the recursive equation*

$$B(k) = \alpha^2 B(k-1) + (1-\alpha)^2 B(k-2) + \alpha\,(1-\alpha)\,A(k-2), \tag{2.6}$$

for any integer $k \geq 2$, *with initial values,* $B(0) = 1/3$ *and* $B(1) = (1+\alpha)(2-\alpha)\alpha/6$.

From Propositions 2 and 3 we shall derive recursively the autocorrelation function $\rho_X(k)$, $k \geq 0$, of the process $\{X_t\}_{t\in\mathbf{N}}$.

PROPOSITION 4: *The autocorrelation function of order* k *of the process* $\{X_t\}_{t\in\mathbf{N}}$ *defined in expression (1.1) is given by*

$$\rho_X(k) = \frac{E[X_t T_\alpha^k(X_t)] - \left(\frac{1+\alpha-\alpha^2}{2(2-\alpha)}\right)^2}{\frac{(\alpha^2-\alpha+1)(\alpha^2-5\alpha+5)}{12(2-\alpha)^2}}, \tag{2.7}$$

where $E[X_t T_\alpha^k(X_t)]$, *denoted by* $C(k)$, *is given by the three term relation*

$$C(k) = \alpha^2\, c\, B(k-1) + (1-\alpha)^2 d\, B(k-2) + \alpha(1-\alpha) d\, A(k-2), \tag{2.8}$$

for any integer $k \geq 2$, *with* $A(k)$ *given by (2.5),* $B(k)$ *given by (2.6) and the constants* c *and* d *are defined in the expression (2.2). Moreover, the initial values* $C(0)$ *and* $C(1)$ *are given by*

$$C(0) = \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)} \quad \textit{and} \quad C(1) = \frac{\alpha(4-\alpha-\alpha^2)}{6(2-\alpha)}.$$

From (2.7) we obtain the quantity $C(k)$ recursively but not in a closed form. This is not enough for obtaining explicitly the spectral density function of the process $\{X_t\}_{t\in\mathbf{Z}}$. One can, equivalently, describe the quantities $A(k)$, $B(k)$ and $C(k)$ by the following power series

$$\varphi(z) = \sum_{k\geq 0} A(k)z^k, \quad \Psi(z) = \sum_{k\geq 0} B(k)z^k \quad \text{and} \quad \gamma(z) = \sum_{k\geq 0} C(k)z^k. \tag{2.9}$$

The previous proposition (the three term relation (2.8)) implies a relation between the functions $\varphi(z), \Psi(z)$ and $\gamma(z)$ defined above (see Proposition 5) and this property will permit to obtain $\gamma(z)$. From Proposition 5 one can derive the spectral density function in a closed form (see Section 3).

PROPOSITION 5: *The power series for $A(k)$, $B(k)$ and $C(k)$ as in expression (2.9) are given, respectively, by*

$$\begin{aligned}
\varphi(z) &= \frac{1+\alpha z(1-\alpha)}{2[(1-\alpha)z+1](1-z)}, \\
\psi(z) &= \frac{2-\alpha z(\alpha^2+\alpha-2)+6\alpha(1-\alpha)z^2\varphi(z)}{6[1-\alpha^2 z-(1-\alpha)^2 z^2]} \qquad \textit{and} \\
\gamma(z) &= \frac{2\alpha^2(1-\alpha)+2+\alpha z(2-\alpha-\alpha^2)}{6(2-\alpha)} \\
&+ \left[\frac{\alpha z+(1-\alpha)^2 z^2}{2-\alpha}\right]\psi(z)+\frac{\alpha(1-\alpha)z^2}{2-\alpha}\varphi(z). \qquad (2.10)
\end{aligned}$$

A few words about estimation of the parameter α from a times series with N observations. Note that the estimation of α for the stationary stochastic process (1.1) can be obtained from Proposition 4. This follows from the fact that

$$\frac{\alpha(4-\alpha-\alpha^2)}{6(2-\alpha)} = C(1) = \int_0^1 x T_\alpha^0(x) d\mu(x).$$

By considering the times series $\{Z_t\}_{t=1}^N$ and by using the Birkhoff's Ergodic Theorem one can estimate α by solving the equation

$$\hat{C(1)} = \frac{1}{N}\sum_{t=1}^{N-1} Z_t Z_{t+1} = \frac{\hat{\alpha}(4-\hat{\alpha}-\hat{\alpha}^2)}{6(2-\hat{\alpha})},$$

in the variable $\hat{\alpha}$.

We are using the method of moments above.

3. THE SPECTRAL DENSITY FUNCTION

In this section we shall use the results for T_α (obtained in the previous section) for the map F_α.

One observes that F_α is a homeomorphism of K and F_α^n is of the form

$$F_\alpha^n(x,y) = (T_\alpha^n(x), G_{\alpha,n}(x,y)),$$

that is, the action of F_α in the first variable is just the action of T_α.

The explicit expression of $G_{\alpha,n}(x,y)$ will not be necessary for our reasoning.

Now we shall define the F_α-invariant measure ν on K, absolutely continuous with respect to the Lebesgue measure $dxdy$.

For sets of the form $A_1 \times A_2$, where $A_1 \subset (0,\alpha)$ and $A_2 \subset (\alpha,1)$ or $A_1 \subset (\alpha,1)$ and $A_2 \subset (0,\alpha)$, we define $\nu(A_1 \times A_2) = (2-\alpha)\,\mu(A_1)\,\mu(A_2)$.

For sets of the form $A_1 \times A_2$, where $A_1 \subset (0,\alpha)$ and $A_2 \subset (0,\alpha)$, we define $\nu(A_1 \times A_2) = (2-\alpha)\,\alpha\,\mu(A_1)\,\mu(A_2)$.

It is not difficult to see that ν is invariant for F_α and it is absolutely continuous with respect to the Lebesgue measure. The measure ν satisfies $\nu(A \times (0,1)) = \mu(A)$, when $A \subset (0,\alpha)$ and $\nu(A \times (0,\alpha)) = \mu(A)$, when $A \subset (\alpha,1)$.

THEOREM 1: *The spectral density function of the process*

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0,Y_0)) + \xi_t, \quad \textit{for } \ t \in \mathbf{Z},$$

is given by

$$f_Z(\lambda) = \frac{1}{2\pi\, Var(X_t)}[\gamma(e^{i\lambda}) + \gamma(e^{-i\lambda}) - C(0)] + \frac{\sigma_\xi^2}{2\pi}, \quad \textit{for any} \quad \lambda \in [0,2\pi), \tag{3.1}$$

where $Var(X_t) = \frac{(\alpha^2-\alpha+1)(\alpha^2-5\alpha+5)}{12(2-\alpha)^2}$, $C(0) = \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)}$ *and* $\gamma(z)$ *is given by the third equality in expression (2.10) of Proposition 5. The point* (X_0,Y_0) *is chosen randomly according to the measure* ν *(or according to the Lebesgue measure).*

PROOF: It is easy to see that the integral of any integrable function $H(x,y)$ with respect to ν, such that $H(x,y) = H(x)$ depends only on the x

variable satisfies

$$\int H(x) d\nu(x,y) = \int H(x) d\mu(x). \tag{3.2}$$

One observes that $\phi(x,y) = x$ is a function variable and $F_\alpha : K \to K$ defines a stationary stochastic process (see for instance Durret) $X_t = \phi(F_\alpha^t(X_0, Y_0))$ with respect to the invariant probability ν defined above.

From the expression (3.2) and for any positive $t \in \mathbf{N}$,

$$\int \phi(F_\alpha^t(x,y))\phi(x,y) d\nu(x,y) = \int \phi(T_\alpha^t(x))\phi(x) d\mu(x).$$

For any positive $t \in \mathbf{N}$ (that is, when $-t$ is negative)

$$\int \phi(F_\alpha^{-t}(x,y))\phi(x,y) d\nu(x,y) = \int \phi(x,y)\phi(F_\alpha^t(x,y)) d\nu(x,y)$$

because ν is invariant for F_α. Therefore, from (3.2) and for any positive $t \in \mathbf{N}$

$$\int \phi(F_\alpha^{-t}(x,y))\phi(x,y) d\nu(x,y) = \int \phi(T_\alpha^t(x))\phi(x) d\mu(x). \tag{3.3}$$

The conclusion is that the autocorrelation coefficients $C(t) = C(-t)$, $t \in \mathbf{N}$ of the stochastic process given by the function $\phi(x,y) = x$, the transformation F_α and the probability ν can be obtained from the autocorrelation coefficients obtained previously for the stochastic process given by the function $\phi(x) = x$, the transformation T_α and the probability μ.

The spectral density function of the process $\{X_t\}_{t \in \mathbf{Z}}$ is given by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-ik\lambda)\rho_X(k) \geq 0,$$

for all $\lambda \in [0, 2\pi)$ (see Brockwell and Davis (1987)). Therefore, the spectral density function of the process (1.1) is given by (3.1).

From Parry and Pollicott (1990) it is known that $\rho_X(k)$ decays exponentially to zero, hence $f_X(\lambda)$ is an analytic function, for any $\lambda \in [0, 2\pi)$.

REMARK: The power series $\gamma(z)$ is an analytic function on the disc $\{z \in \mathbf{C} | \parallel z \parallel < 1\}$ and the expression (3.1) has the meaning of the radial limit

$$\lim_{r \to 1} r e^{i\lambda} = e^{i\lambda} = z.$$

In this sense, the series

$$\sum_{n \in \mathbf{Z}} e^{in\lambda} = 2\,\mathcal{R}\mathrm{e}\left(\frac{1}{1-e^{i\lambda}}\right) - 1 = 0, \quad \text{for} \quad \lambda \neq 0,$$

eventhough the series $\sum_{n \in \mathbf{Z}} e^{in\lambda}$ does not converge. We are using this fact in the expression (3.1).

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