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The Theta Group and the Continued Fraction Expansion with Even Partial Quotients

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Abstract F. Schweiger introduced the continued fraction with even partial quotients. We will show a relation between closed geodesics for the theta group (the subgroup of the modular group generated by $z+2$ and $-1/z$) and the continued fraction with even partial quotients. Using thermodynamic formalism, Tauberian results and the above mentioned relation, we obtain the asymptotic growth number of closed trajectories for the theta group. Several results for the continued fraction expansion with even partial quotients are obtained; some of these are analogous to those already known for the usual continued fraction expansion related to the modular group, but our proofs are by necessity in general technically more difficult.

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§0. Introduction

The purpose of this paper is to study the length spectra of Fuchsian groups of the first kind with parabolic elements. We shall concern ourselves with the problem of computing the asymptotic growth number of closed geodesics on the Riemann surface generated by a certain Fuchsian group (we are considering a metric of constant negative curvature on the Riemann surface). This problem was analyzed from the point of view of a dynamical system generated by a one-dimensional map on the unit circle in the case of compact Riemann surfaces by M. Pollicott ([15],[17],[18] and [19]), S.P. Lalley[8] and others. In this case the one-dimensional system built by glueing together several pieces of the graph of the generators of the group acting on the unit circle is expanding. This follows from the work of several people, among them R. Bowen and C. Series (see [21] and [22] for references). In fact, the results of Bowen and Series are quite general, in the sense that they can be applied to non-compact surfaces, but the extension of Pollicott's results for the general non-compact Riemann surface depends on a better understanding of thermodynamic formalism of non-expanding maps.

In Lopes [9], some results for non-expanding one-dimensional maps with a parabolic point (a fixed point with derivative of modulus one) are obtained. We are interested in applying some of the ideas of [9] to the one-dimensional system (continued fraction expansion with even partial quotients) related to the Riemann surface with only parabolic points.

For the modular surface, the one-dimensional map obtained from the action of the generators on the unit circle can be basically understood in terms of the Gauss (or regular) continued fraction expansion in the interval $[0, 1]$. There is a close relation between closed geodesic trajectories on the modular surface and real numbers in $[0, 1]$ with periodic continued fraction expansion; see Hardy-Wright ([4], p.142), for the concept of equivalence of numbers modulo the modular group. It is also known that the length of closed geodesics is related to the derivative at a fixed point of a fractional linear transformation in the modular group. These considerations are due to several authors and it is difficult for us to mention them all; we just mention C.Series ([21],[22]), D.H.Mayer ([10],[11]), M.Pollicott ([18],[19]), T. Pignataro [14] and S.P.Lalley [8], who made important contributions in this direction and in whose work we learned the above results.

For the case of Riemann surfaces with infinite volume, the reader can find an extensive analysis in T. Pignataro's thesis [14]. In this work the map on the boundary is also expanding and the limit set has a fractal structure. When there exist parabolic elements in the group, the analysis requires to understand properties of non-expanding maps or maps with infinitely many branches (see Bowen-Series[2]). The Gauss map is expanding, but has also an infinite cardinal of inverse branches. The continued fraction expansion is not exactly the induced map on the boundary, but is obtained by a procedure of first return of the iterates of the map (see [21]).

The map induced in the boundary we will consider here is not expanding.

The modular surface has elliptic points, and this makes the analysis more cumbersome. For instance, the pressure $P(s)$ associated with potentials of the kind $-s\log|f^{'}(x)|$ is a quite natural object to analyze in order to understand the main problem that interests us; here f denotes the one-dimensional map on the boundary of the disk (the unit circle). Using the work of D.Ruelle [20] and, more specifically, the work of W.Parry and M. Pollicott ([12],[13]) on the use of Tauberian Theorems for the Ruelle Zeta Function, one can compute the asymptotic with t of the number of periodic trajectories of period n with mean value of $\log|f^{'}(x)|$ smaller than t ([12],[13] and [18]).

The pressure $P(s)$ of the Gauss continued fraction expansion map has a logarithmic singularity at some value s_0 (see D.Mayer [11]); this is a very different situation from the case of expanding maps with a finite number of branches , where the pressure is real analytic on s $\in \mathbf{R}$.

Tha classical theory of Thermodynamic Formalism works fine when the map consider is expanding, as in the case of the induced map in the boundary of the Modular Group. In the present situation we have to consider a non-expanding one-dimensional map acting on the boundary of the disk. The map has parabolic fixed points, and the analysis can be carried out using similar techniques (not exactly equal), to the ones used in Lopes [9].

Instead of using the regular continued fraction expansion, as it is possible in the modular case, we will show that the so called continued fraction with even partial quotients (see[23]) can be very helpful in the understanding of the Theta group. We will have to prove several results for the continued fraction expansion with even partial quotients analogous to the ones already known for the regular continued fraction expansion. The proofs of these results are technically more difficult than those for the regular continued fraction expansion case.

We believe that the methods used in Section 2 are quite general, but there we will analyze a particular group : the Theta group. This is the subgroup Θ of the modular group generated by

$$-\tfrac{1}{z} \quad \text{and} \quad z+2,$$

with z in the upper half plane. In terms of matrices, Θ is the subgroup of the modular group $SL(2,Z)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a = d \pmod 2$ and $b = c \pmod 2$. Note that in this case, a is odd if and only if b is even ([1], p.92).

We will prove the following theorem:

Theorem 0: *If we let $\pi(r)$ denote the cardinal of the set of closed geodesics of the Theta Group such that $e^l < r$ (where l is the length of the trajectory) then*

$$\lim_{r\to\infty} \frac{\pi(r)}{\frac{r}{\log r}} = 1.$$

The above theorem is already known but the novelty here is that the result is obtained using Thermodynamic Formalism and also the interesting relation with the coding of closed geodesics by means of the continued fraction expansion with even partial quotients.

The Riemann surface obtained taking the quotient of the upper semiplane by the Theta group is topologically a sphere with two cuspidal ends (see D.Sullivan [25] for general references). The surface is a noncompact finite volume hyperbolic manifold. The spectrum of the natural Laplacian coming from the metric of negative curvature has, in this case, a discrete and a continuous part.

We would like to thank I. Efrat, T. Pignataro and L. F. da Rocha for helpful conversations regarding actions of discrete groups. T. Pignataro pointed out to the authors that some of our results here are implicitly presented as a limit case in his thesis [14].

In a forthcoming paper (see [2]) a more geometric proof of some of the results presented here will appear. We will show the existence of polynomial decay of correlation for the billiard associated to the Theta group.

In [3] we will analyse trace class properties of the Ruelle-Perron-Frobenius operator associated with indiferent fixed points. This result applied to the Theta group is similar to the one obtained in [11] for the modular group.

§1. Continued Fraction Expansion with Even Partial Quotients

Continued fractions with even (and also odd) partial quotients were analyzed by F. Schweiger ([23], [24]); here we will mention some of the properties of continued fractions with even partial quotients, which are the only ones we are interested in.

We partition the unit interval $[0,1]$ into $B(-1,k) =]\frac{1}{2k}, \frac{1}{2k-1}], k \geq 1$, and $B(+1,k) =]\frac{1}{2k+1}, \frac{1}{2k}], k \geq 1$. Consider the map T defined by

$$T(x) \quad := e(\frac{1}{x} - 2k), \quad x \in B(e,k), \quad e \in \{\pm 1\}.$$

One of Schweiger's results states that this map T on the unit interval $[0,1]$ is ergodic and has a σ-finite invariant measure μ with infinite mass. This measure μ has a density $d(x)$ given by

$$d(x) \quad = \quad \frac{1}{x+1} - \frac{1}{x-1}.$$

The natural extension of $([0,1], \mu, T)$ is defined on $\Omega = [0,1] \times [-1,1]$ by the map

$$\tau(x,y) \quad := \quad (Tx, \frac{e}{a(x)+y}), \qquad (x,y) \in B(e,k) \times [-1,1] \ ,$$

where $a(x) := 2k$, for $x \in B(e,k)$. The density $D(x,y)$ of the invariant measure ρ is given by

$$D(x,y) = (1+xy)^{-2};$$

see also [24]. The map T generates (in a way analogous to the regular continued fraction expansion) an expansion of any real number x in [0,1] :

$$x = \cfrac{1}{a_1 + \frac{e_2}{a_2 + \frac{e_3}{a_3 + ..}}}.$$

The numbers e_i are equal to 1 or -1 , and the numbers a_i are now even; this is the reason for the name *continued fraction with even partial quotients* (shortly E.C.F.).

For a real number x not necessarily in $[0,1]$, we have the same kind of expansion:

$$x = a_0 + \cfrac{e_1}{a_1 + \frac{e_2}{a_2 + \frac{e_3}{a_3 + \dots}}} \ . \tag{1.1}$$

Here $a_0 \in Z$ is even and $e_1 \in \{\pm 1\}$ is such that $e_1(x - a_0) =: \xi \in [0,1]$ has as E.C.F. expansion

$$\xi = \cfrac{1}{a_1 + \frac{e_2}{a_2 + \frac{e_3}{a_3 + \dots}}} \ .$$

Denote the n^{th} convergent of x by p_n/q_n, that is,

$$\frac{p_n}{q_n} = a_0 + \cfrac{e_1}{a_1 + \frac{e_2}{a_2 + \dots + \frac{e_{n-1}}{a_{n-1} + \frac{e_n}{a_n}}}} \ ,$$

where $(p_n, q_n) = 1$, $(n \geq -1)$, $q_n > 0\,(n \geq 0)$, and where the sequences $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ satisfy the recurrence relations

$$p_{-1} := 1; \quad p_0 := a_0; \quad p_n = a_n p_{n-1} + e_n p_{n-2} \quad n \geq 1,$$

and

$$q_{-1} := 0; \quad q_0 := 1; \quad q_n = a_n q_{n-1} + e_n q_{n-2} \quad n \geq 1; \tag{1.2}$$

see also ([7], (1.8)).

In general we will denote a continued fraction expansion of the form (1.1) by

$$x \;=\; [\,a_0;\, e_1/a_1,\, e_2/a_2,\, e_3/a_3, \ldots\,]$$

and we put, following ([7], Section 1) :

$$t_n \;=\; t_n(x) \;:=\; [\,0;\, e_{n+1}/a_{n+1},\, e_{n+2}/a_{n+2}, \ldots\,], \tag{1.3}$$

with $n \geq 0$ and the obvious restriction on n in case the expansion (1.1) of x is finite.

Remark 1- It follows at once from the definition of the operator T that the E.C.F. expansion of a real irrational number x is infinite. Moreover, due to the fact that $a_n \in Z$ is even (for $n \geq 0$), and

$$|t_n| = T^n(e_1(x - a_0)) \in [0, 1] \setminus Q, \quad n \geq 0,$$

we see that the E.C.F. expansion of x is unique.

The E.C.F. expansion of a rational number P/Q, where $(P, Q) = 1$, is finite (and unique) if and only if $P \not\equiv Q(mod 2)$; see also Proposition 2 below.

Remark 2- For our purposes it is convenient to see how the E.C.F. expansion of a real number x can be obtained from the *regular continued fraction* (R.C.F.) expansion of x. The idea of a *singularization,* as described in ([7], Section 2), is essential; let us recall, briefly, its main idea. Let

$$x = [\,a_0;\, e_1/a_1,\, e_2/a_2, \ldots\,],\; a_n \in N,\; n > 0;\; e_i \in \{\pm 1\},\; i \geq 1$$

be a continued fraction expansion of x; finite truncation yields the sequence of convergents $(r_k/s_k)_{k\geq -1}$. Suppose that for some $n \geq 0$ one has

$$a_{n+1} = 1;\; e_{n+2} = 1\,,$$

i.e.,

$$[\,a_0;\, e_1/a_1, \ldots\,] \;=\; [\,a_0; e_1/a_1, \ldots\,,\, e_n/a_n,\, e_{n+1}/1,\, 1/a_{n+2}, \ldots\,].$$

The transformation σ_n which changes this continued fraction into the continued fraction

$$[\,a_0;\, e_1/a_1, \ldots, e_n/(a_n + e_{n+1}), -e_{n+1}/(a_{n+2}+1), \ldots], \tag{1.4}$$

is called a *singularization* ; (1.4) is again a continued fraction expansion of x, with convergents denoted by, say, $(p_k/q_k)_{k\geq -1}$. One easily shows that the sequence of vectors $\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1}$ is obtained from $\begin{pmatrix} r_k \\ s_k \end{pmatrix}_{k\geq -1}$ by removing the term $\begin{pmatrix} r_n \\ s_n \end{pmatrix}$ from the latter; see ([7], (2.6)).

We will also need the inverse of a singularization in a special case; suppose that for some $n \geq 0$ one has

$$a_{n+1} > 1;\; e_{n+1} = 1.$$

The transformation τ_n which changes the continued fraction

$$[\,a_0;\, e_1/a_1, \ldots,\, e_n/a_n,\, 1/a_{n+1}, \ldots] \tag{1.5}$$

into

$$[\,a_0;\, e_1/a_1, \ldots, e_n/(a_n+1),\, -1/1,\, 1/(a_{n+1}-1), \ldots],$$

is called an *insertion* ; its image is again a continued fraction expansion of x and we will denote its convergents by, say, $(p_k/q_k)_{k\geq -1}$. If $(r_k/s_k)_{k\geq -1}$ is the sequence of convergents connected with (1.5) then, using matrix identities one easily shows that the sequence of vectors $\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1}$ is obtained from $\begin{pmatrix} r_k \\ s_k \end{pmatrix}_{k\geq -1}$ by *inserting* the term $\begin{pmatrix} r_k + r_{k-1} \\ s_k + s_{k-1} \end{pmatrix}$ before the n^{th} term of the latter sequence, i.e.,

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1} \equiv \begin{pmatrix} r_{-1} \\ s_{-1} \end{pmatrix}, \begin{pmatrix} r_0 \\ s_0 \end{pmatrix}, \ldots, \begin{pmatrix} r_{n-1} \\ s_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} r_n + r_{n-1} \\ s_n + s_{n-1} \end{pmatrix}, \begin{pmatrix} r_n \\ s_n \end{pmatrix}, \begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix}, \ldots .$$

Now let x be an irrational number, with R.C.F. expansion

$$x = [\,B_0;\, B_1,\, \ldots\,,\, B_n,\, \ldots],\;\; B_0 \in Z;\; B_n \in N,\; n \geq 1,$$

and let $(P_k/Q_k)_{k\geq -1}$ be the sequence of regular convergents of x. (In this case we write $[\,B_0;\, B_1, \ldots]$ instead of $[\,B_0;\, 1/B_1, \ldots]$, since $e_n = 1$, for all $n \geq 1$.) Next we describe an algorithm which yields the E.C.F. expansion of a number x from its R.C.F. expansion.

(I)Suppose $n \geq 0$ is the first index for which B_n is odd (if such an index does not exist we're done).We can discern the following two cases :

(i) In case $B_{n+1} = 1$: singularize B_{n+1}, that is, replace the R.C.F. expansion of x by

$$\sigma_{n+1}([\,B_0;\, B_1,\, \ldots\,,\, B_n,\, 1,\, B_{n+2},\, \ldots]) =$$

$$[\,B_0;\,1/B_1,\,\dots\,,\,1/(B_n+1),\,-1/(B_{n+2}+1),\,1/B_{n+3},\,\dots].$$

(ii) In case $B_{n+1}>1$: replace $[\,B_0;\,B_1,\,\dots\,,\,B_n,\,B_{n+1},\,\dots]$ by

$$\tau_{n+B_{n+1}-1}(\dots\,(\tau_{n+1}(\tau_n([\,B_0;\,B_1;\,\dots\,,\,B_n,\,B_{n+1},\,\dots]))\,\dots)=$$

$$[\,B_0;\,1/B_1,\,\dots\,,1/(B_n+1),\,\underbrace{-1/2,\,\dots\,,\,-1/2}_{(B_{n+1}-2)-times},\,-1/1,\,1/1,\,1/B_{n+2},\,\dots].\quad (1.6)$$

Now singularize in (1.6) the partial quotient with index $n+B_{n+1}$.

Doing so, we find in both cases a continued fraction expansion $[\,a_0;\,e_1/a_1,\,\dots]$ of x of the form :

$$[B_0;1/B_1,\dots,1/(B_n+1),\underbrace{-1/2,\dots,-1/2}_{(B_{n+1}-1)-times},-1/(B_{n+2}+1),1/B_{n+3},\dots].\quad (1.7)$$

Let $(r_k/s_k)_{k\geq -1}$ be the sequence of convergents of (1.7). Then, if $B_{n+1}=1$:

$$\begin{pmatrix} r_k \\ s_k \end{pmatrix}_{k\geq -1} \equiv \begin{pmatrix} P_{-1} \\ Q_{-1} \end{pmatrix},\begin{pmatrix} P_0 \\ Q_0 \end{pmatrix},\dots\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix},\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix}=$$

$$\begin{pmatrix} P_n+P_{n-1} \\ Q_n+Q_{n-1} \end{pmatrix},\begin{pmatrix} P_{n+2} \\ Q_{n+2} \end{pmatrix},\dots$$

and if $B_{n+1}>1$:

$$\begin{pmatrix} r_k \\ s_k \end{pmatrix}_{k\geq -1} \equiv \begin{pmatrix} P_{-1} \\ Q_{-1} \end{pmatrix},\begin{pmatrix} P_0 \\ Q_0 \end{pmatrix},\dots,\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix},\begin{pmatrix} P_n+P_{n-1} \\ Q_n+Q_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} 2P_n+P_{n-1} \\ 2Q_n+Q_{n-1} \end{pmatrix},\dots,\begin{pmatrix} (B_{n+1}-1)P_n+P_{n-1} \\ (B_{n+1}-1)Q_n+Q_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix}=\begin{pmatrix} B_{n+1}P_n+P_{n-1} \\ B_{n+1}Q_n+Q_{n-1} \end{pmatrix},\begin{pmatrix} P_{n+2} \\ Q_{n+2} \end{pmatrix},\dots.$$

(Notice that for all $k\geq n+B_{n+1}+1$ we have : $e_k=+1$.)

(II) Suppose $m\geq n+B_{n+1}$ is the first index in $[\,a_o;\,e_1/a_1,\,\dots]$ for which a_m is odd (if such an index does not exist, that is, if a_m is even for all m, then we're done and $[\,a_0;\,e_1/a_1,\,\dots]$ is the E.C.F. expansion of x). Now repeat the procedure in (I), with $[\,B_0;\,B_1,\,\dots]$ replaced by $[\,a_0;\,e_1/a_1,\,\dots]$.

Notice that the above steps (I) and (II) form an algorithm which yields the E.C.F. expansion of x from its R.C.F. expansion. We moreover have, if $(p_k/q_k)_{k\geq -1}$ denotes the sequence of E.C.F. convergents of x, that

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq 1} \subset \cup_{i=0}^{\infty}\left\{\begin{pmatrix} jP_i+P_{i-1} \\ jQ_i+Q_{i-1} \end{pmatrix}\;;\;1\leq j\leq B_{i+1}\right\},$$

i.e., each E.C.F. convergent is either a principal or a median convergent of x. Due to this algorithm (or due to the second recurrence relation in (1.2), and the fact that a_k is even for $k \geq 0$) we also have :

$$q_k > q_{k-1} > 0\,, \;\; k \geq 1. \tag{1.8}$$

Furthermore we have for each $k \geq 0$:

$$\frac{P_k}{Q_k} \;\; is \;\; not \;\; an \;\; E.C.F.convergent \Rightarrow \frac{P_{k+1}}{Q_{k+1}} \;\; is \;\; an \;\; E.C.F.convergent. \tag{1.9}$$

Remark 3- Now let $x = P/Q$, where $P, Q \in Z$, $Q > 0$ and $x = [\,B_0;\, B_1,\, \dots\,,\, B_n\,]$ is the R.C.F. expansion of x. (Notice that x has actually two regular expansions, for if $B_n > 1$ one has $[\,B_0;\, B_1,\, \dots\,,\, B_n\,] = [\,B_0;\, B_1,\, \dots\,,\, B_n - 1,\, 1\,]$.) Applying the algorithm in Remark 2 to the R.C.F. expansion of x, after finitely many steps we obtain a finite continued fraction

$$[\,a_0;\, e_1/a_1,\, \dots\,,\, e_m/a_m\,] \tag{1.10}$$

of x, with $a_0 \in Z\, a_i \in N$, $e_i \in \{\pm 1\}$, $1 \leq i \leq m$ and a_i even, $0 \leq i \leq m-2$. We can discern the following three cases :
(i) a_{m-1} and a_m are even. Then (1.10) is the (unique) E.C.F. expansion of x.
(ii) a_{m-1} is odd, $a_m = 1$ and $e_m = 1$. In this case we can rewrite (1.10) as

$$[\,a_0;\, e_1/a_1,\, \dots\,,\, e_{m-2}/a_{m-2},\, e_{m-1}/(a_{m-1}+1)\,]\,,$$

which is the (unique) E.C.F. expansion of x.
(iii) a_{m-1} is even and a_m is odd. In this case we may assume that $a_m \geq 3$, for if $a_m = 1$ we must have that $e_m = +1$. But then we can rewrite (1.10) as

$$[\,a_0;\, e_1/a_1,\, \dots\,, e_{m-1}/(a_{m-1}+1)\,].$$

Now

$$1 = [\,0;\, 1/2,\, -1/2,\, \dots\,,\, -1/2,\, \dots] = [\,0;\, 1/2,\, \overline{-1/2}\,]$$

and

$$1 = [\,2;\, -1/2,\, \dots\,,\, -1/2,\, \dots] = [\,2;\, \overline{-1/2}\,],$$

the bar indicating the period. Thus x has in this case two E.C.F. expansions, viz.,

$$x = [\,a_0;\, e_1/a_1,\, \dots\,,\, e_m/(a_m-1),\, 1/2,\, \overline{-1/2}\,]$$

and

$$x = [\,a_0;\, e_1/a_1,\, \dots\,,\, e_m/(a_m+1),\, \overline{-1/2}\,].$$

(See also Proposition 2 below.)
(iv) a_{m-1} is odd and a_m is even. Using the first part of the algorithm of Remark 2, it can

be easily seen that this case actually reduces to the case (ii).

We will now describe the relation between the E.C.F. expansion and the theta group.

Proposition 1- *Let x be a real number, with E.C.F. expansion*

$$x = [a_0; e_1/a_1, \ldots, e_n/a_n, \ldots]$$

and let p_n/q_n denote its n^{th} convergent. Then any matrix of the form

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}, \ n \geq 1,$$

is an element of the theta group Θ.

Proof- It is shown in ([7], Corollary (1.10)), that

$$det \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \delta_n, \ n \geq 0,$$

where $\delta_0 := 1$ and $\delta_n := (-1)^n e_1 \ldots e_n$, $n \geq 1$, therefore it is sufficient to show that the only options are either

$$p_{n-1} \ \textit{and} \ q_n \ \textit{are even}, \ \textit{and} \ p_n \ \textit{and} \ q_{n-1} \ \textit{are odd}$$

or

$$p_{n-1} \ \textit{and} \ q_n \ \textit{are odd}, \ \textit{and} \ p_n \ \textit{and} \ q_{n-1} \ \textit{are even.}$$

But this follows at once from the recurrence relations (1.2) and the fact that for the E.C.F. expansion of x the partial quotients a_n (for $n \geq 0$) are always even; as a matter of fact we have, for $n \geq -1$:

$$p_n \ \textit{is odd} \Leftrightarrow n \ \textit{is odd}$$

and

$$q_n \ \textit{is odd} \Leftrightarrow n \ \textit{is even.}$$

This proves Proposition 1.

Definition- We say that two real numbers x and y are theta equivalent if there exists an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Theta,$$

such that

$$y = \frac{ax+b}{cx+d} \ . \tag{1.11}$$

This definition is analogous to the usual one for the modular group [4].

We now state the main result of this section.

Theorem 1- *Two irrational numbers x and y are theta equivalent if and only if there exist two non-negative integers m and n, such that $t_m(x) = t_n(y)$, that is, the E.C.F. expansions of x and y are, respectively,*

$$[\, a_0;\, \alpha_1/a_1,\, \dots\, ,\, \alpha_m/a_m,\, \gamma_1/c_1,\, \dots]$$

and

$$[\, b_0;\, \beta_1/b_1,\, \dots\, ,\, \beta_n/b_n,\, \gamma_1/c_1,\, \dots]\ .$$

The proof of this theorem follows to some extent the proof of Hardy-Wright([4], Theorem 175), but the point with eigenvalue 1 of our map f will require a different kind of argument in part of our proof. We need some further results before we can prove Theorem 1.

Proposition 2- *Let $P, Q \in Z$, $Q > 0$ and suppose moreover that P or Q is odd. Then $P \not\equiv Q(mod2)$ if and only if the E.C.F. expansion of P/Q is finite.*

Proof- Let $P \not\equiv Q(mod2)$ and suppose that the E.C.F. expansion of P/Q is infinite. Then P/Q has a finite continued fraction of the form

$$[\, a_0;\, e_1/a_1,\, \dots\, ,\, e_m/a_m\,],$$

where $a_0 \in Z$, $a_i \in N$, $e_i \in \{\pm\}$, $1 \leq i \leq m$; a_i is even for $0 \leq i \leq m-1$ and a_m is odd (see also Remark 3). Put $p_i/q_i := [\, a_0;\, e_1/a_1,\, \dots\, ,\, e_i/a_i\,]$, $0 \leq i \leq m$; then Proposition 1 implies that

$$\begin{pmatrix} p_{m-2} & p_{m-1} \\ q_{m-2} & q_{m-1} \end{pmatrix} \in \Theta\, .$$

From this property, the fact that a_m is odd and the recurrence-relations (1.2) it at once follows that p_m and q_m are both odd. Since

$$\frac{P}{Q} = \frac{p_m}{q_m},$$

$(p_m, q_m) = 1$ and P or Q is odd, we have that

$$(P, Q)\ \ is\ \ odd.$$

But then $P \equiv Q(mod2)$, which is a contradiction.

Conversely, if P/Q has a finite E.C.F. expansion, say

$$\frac{P}{Q} = [\, a_0;\, e_1/a_1,\, \dots\, ,\, e_m/a_m\,],$$

with convergents $(p_i/q_i)_{-1 \leq i \leq m}$, then

$$\frac{P}{Q} = \frac{p_m}{q_m}\, .$$

Due to Proposition 1 we have that either p_m is odd and q_m is even or p_m is even and q_m is odd and again we find that

$$(P,Q) \ \ is \ \ odd,$$

hence

$$P \not\equiv Q(mod2)\,,$$

which proves Proposition 2.

Theorem 2- *Let the real number x be defined by*

$$x \,:=\, \frac{Ry+P}{Sy+Q},$$

where

$$\begin{pmatrix} R & P \\ S & Q \end{pmatrix} \in \Theta,$$

$y \in [-1,1]$ and where we furthermore assume that $Q > S > 0$. Then R/S and P/Q are two consecutive convergents of the E.C.F. expansion of x.

Moreover, if R/S is the $(m-1)^{th}$ and P/Q is the m^{th} E.C.F. convergent of x, then $y = t_m$, with $t_m = t_m(x)$ is defined as in (1.3) .

Proof- Notice that $\begin{pmatrix} R & P \\ S & Q \end{pmatrix} \in \Theta$ implies that $P \not\equiv Q$ (mod 2). Due to Remark 3 and Proposition 2 we can therefore develop P/Q in a unique and finite E.C.F. expansion $\frac{P}{Q} \,=\, [\,a_0;\, e_1/a_1,\, \dots\,,\, e_m/a_m\,]$. Let $(p_i/q_i)_{-1\le i\le m}$ be the sequence of convergents of this expansion. Since $(P,Q)=1$ and $Q>0$ we have

$$P = p_m,\; Q = q_m\,.$$

and therefore

$$p_mS - q_mR \,=\, PS - QR \,=\, \pm(p_mq_{m-1} - p_{m-1}q_m)\,.$$

(I) Suppose first that

$$p_mS - q_mR \,=\, p_mq_{m-1} - p_{m-1}q_m\,.$$

In this case, $p_m(S-q_{m-1}) \,=\, q_m(R-p_{m-1})$ and, since $(p_m,q_m)=1$, we have

$$q_m \,|\, (S-q_{m-1})\,.$$

Now $q_m = Q > S > 0$ and $q_m > q_{m-1} > 0$ (see (1.8)) imply that $|S-q_{m-1}| < q_m$, which, combined with $q_m \,|\, (S-q_{m-1})$, at once yields that $S = q_{m-1}$. Therefore $R = p_{m-1}$,

$$x \,=\, \frac{p_{m-1}y + p_m}{q_{m-1}y + q_m}\ ,$$

and one easily shows (see also [7], Section 1), that

$$x \,=\, [\,a_0;\, e_1/a_1,\, \dots\,,\, e_m/(a_m+y)\,]\,.$$

Let

$$y = [0;\, e_{m+1}/a_{m+1},\, e_{m+2}/a_{m+2},\, \ldots]$$

be an E.C.F. expansion of y. It follows that

$$x = [a_0;\, e_1/a_1,\, \ldots\, ,\, e_m/a_m,\, e_{m+1}/a_{m+1},\, \ldots]$$

is an E.C.F. expansion of x and since

$$x = \frac{p_{m-1}t_m + p_m}{q_{m-1}t_m + q_m}\,,$$

we at once find that $y = t_m$.

(II) Now suppose that

$$p_m S - q_m R = p_{m-1}q_m - p_m q_{m-1};$$

then

$$p_m(S + q_{m-1}) = q_m(p_{m-1} + R)$$

and $(p_m, q_m) = 1$ implies that $q_m \mid (S + q_{m-1})$. Now $q_m = Q > S > 0$ and $q_m > q_{m-1} > 0$ imply that $S + q_{m-1} < 2q_m$ and therefore $S + q_{m-1} = q_m$. Hence

$$p_m q_m = (p_{m-1} + R)q_m,$$

which yields $p_{m-1}+R = p_m$. From $p_{m-1}+R = p_m,\ q_{m-1}+S = q_m$ and $\begin{pmatrix} p_{m-1} & p_m \\ q_{m-1} & q_m \end{pmatrix} \in \Theta$ it follows that both P and Q are odd, which is impossible since $\begin{pmatrix} R & P \\ S & Q \end{pmatrix} \in \Theta$.

This proves Theorem 2.

Proof of Theorem 1- First assume that x and y have the following E.C.F. expansions :

$$x = [a_0;\, \alpha_1/a_1,\, \ldots\, ,\, \alpha_m/a_m,\, \gamma_1/c_1,\, \gamma_2/c_2,\, \ldots]$$

and

$$y = [b_0;\, \beta_1/b_1,\, \ldots\, ,\, \beta_n/b_n,\, \gamma_1/c_1,\, \gamma_2/c_2,\, \ldots].$$

Write

$$z = [0;\, \gamma_1/c_1,\, \gamma_2/c_2,\, \ldots].$$

Then we have, if $(p_k/q_k)_{k\geq -1}$ is the sequence of E.C.F. convergents of x and $(r_k/s_k)_{k\geq -1}$ is the sequence of E.C.F. convergents of y, that

$$x = \frac{p_{m-1}z + p_m}{q_{m-1}z + q_m}\ ,\ \ y = \frac{r_{n-1}z + r_n}{s_{n-1}z + s_n}$$

and therefore

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} r_{n-1} & r_n \\ s_{n-1} & s_n \end{pmatrix} \begin{pmatrix} p_{m-1} & p_m \\ q_{m-1} & q_m \end{pmatrix}^{-1}$$

belongs to Θ and a simple calculation shows that (1.11) is satisfied.

Conversely, if x and y are two theta equivalent numbers, there exists an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Theta$$

such that

$$y = \frac{ax+b}{cx+d} \,.$$

We may assume that $cx+d>0$, since we can always replace the coefficients in $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by their negatives. When we develop x as an E.C.F. expansion, we obtain

$$x = [a_0; e_1/a_1, \ldots, e_{k-1}/a_{k-1}, e_k/a_k, \ldots] = [a_0; e_1/a_1, \ldots, e_k/(a_k+t_k)], \,,$$

where t_k, $k \geq 0$ is defined as in (1.3). Notice that $x = a_0 + t_0$.

Let $(p_k/q_k)_{k\geq -1}$ be the sequence of E.C.F. convergents of x; with the above notation we therefore have

$$x = \frac{p_{k-1}t_k + p_k}{q_{k-1} + q_k} \,, \; k \geq 0 \,.$$

Hence

$$y = \frac{Rt_k + P}{St_k + Q} \,, \; k \geq 0 \,, \tag{1.12}$$

where

$$R = ap_{k-1} + bq_{k-1} \,; \; P = ap_k + bq_k$$

and

$$S = cp_{k-1} + dq_{k-1} \,; \; Q = cp_k + dq_k \,.$$

Since Θ is a group, Proposition 1 implies that

$$\begin{pmatrix} R & P \\ S & Q \end{pmatrix} \in \Theta \,;$$

notice that this element of Θ depends on k.

Now, we want to prove the following property: for sufficiently large k we have

$$Q > S > 0 \,.$$

In order to obtain such result, we need first the following Lemma.

Lemma 1- *Let x be a real irrational number, and let $(p_k/q_k)_{k\geq -1}$ be its sequence of E.C.F. convergents. Define the constants $\rho_n = \rho_n(x)$, $n \geq 1$, by*

$$\rho_n := p_n - q_n x \,, \; n \geq 1 \,.$$

Then

$$\lim_{n\to\infty} \rho_n = 0 .$$

Proof- Let $x = [\,a_0;\, e_1/a_1,\, \ldots]$ be the E.C.F. expansion of x. We have (see also [7], (1.20)) :

$$\rho_k = \frac{-\delta_k t_k}{q_k + t_k q_{k-1}} \ , \ k \geq 1,$$

where $\delta_k = (-1)^k e_1 \, \ldots \, e_k$, and, as before, $t_k = e_{k+1}T^k(e_1(x{-}a_0))$, $k \geq 1$. Hence $|-\delta_k t_k| \leq 1$ and we have moreover that $q_k + t_k q_{k-1} \geq q_k - q_{k-1}$. In order to prove the Lemma it is therefore sufficient to show that

$$\lim_{k\to\infty} \frac{1}{q_k - q_{k-1}} = 0. \tag{1.13}$$

Let $x = [\,B_0;\, B_1,\, \ldots\, ,\, B_n,\, \ldots]$ be the R.C.F. expansion of x, and let $(P_n/Q_n)_{n\geq -1}$ be its sequence of regular convergents. It is well-known (see e.g. [7], Section 3), that the sequence $(Q_n)_{n\geq 1}$ is a monotonically increasing sequence of positive integers. Hence, for each $\epsilon > 0$ there exists a positive integer $n_0 = n_0(\epsilon)$, such that $1/Q_n < \epsilon$, for all $n \geq n_0$. In order to prove (1.13) we will show that, for each $\epsilon > 0$, there exists an integer $k_0 = k_0(n_0(\epsilon))$ such that, for all $k \geq k_0$,

$$q_k - q_{k-1} \geq Q_{n_0} \ .$$

To prove this, we use Remark 2 and (1.9) and discern the following four cases.

1. If both p_{k-1}/q_{k-1} and p_k/q_k are R.C.F. convergents, then either there exists an integer n such that

$$\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix}, \begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} P_n \\ Q_n \end{pmatrix},$$

in which case

$$q_k - q_{k-1} = Q_n - Q_{n-1} = (B_n Q_{n-1} + Q_{n-2}) - Q_{n-1} \geq Q_{n-2}\,,$$

or there exists an integer n such that

$$\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} = \begin{pmatrix} P_{n-2} \\ Q_{n-2} \end{pmatrix}, \begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} P_n \\ Q_n \end{pmatrix},$$

and therefore

$$q_k - q_{k-1} = Q_n - Q_{n-2} = B_n Q_{n-1} \geq Q_{n-1}\,.$$

2. If p_{k-1}/q_{k-1} is an R.C.F. convergent, and p_k/q_k is not, then there exists an integer n such that

$$\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix}, \begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} P_n + P_{n-1} \\ Q_n + Q_{n-1} \end{pmatrix}$$

and we find

$$q_k - q_{k-1} \;=\; Q_n\,.$$

3. If p_{k-1}/q_{k-1} is not an R.C.F. convergent, and p_k/q_k is an R.C.F. convergent, then there exists an integer n, such that

$$\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} \;=\; \begin{pmatrix} (B_{n+1}-1)P_n + P_{n-1} \\ (B_{n+1}-1)Q_n + Q_{n-1} \end{pmatrix}\,,\quad \begin{pmatrix} p_k \\ q_k \end{pmatrix} \;=\; \begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix}\,,$$

hence

$$q_k - q_{k-1} \;=\; Q_{n+1} - (Q_{n+1} - Q_n) \;=\; Q_n\,.$$

4. If neither p_{k-1}/q_{k-1} nor p_k/q_k is a R.C.F. convergent, then it follows from Remark 2 that both convergents must belong to the same "block of inserted median convergents", i.e., there exists an integer n such that

$$\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} \;=\; \begin{pmatrix} (i-1)P_n + P_{n-1} \\ (i-1)Q_n + Q_{n-1} \end{pmatrix}\,,\quad \begin{pmatrix} p_k \\ q_k \end{pmatrix} \;=\; \begin{pmatrix} iP_n + P_{n-1} \\ iQ_n + Q_{n-1} \end{pmatrix}\,,$$

for some i satisfying $2 \le i \le B_{n+1} - 1$ and we obtain

$$q_k - q_{k-1} \;=\; Q_n\,.$$

Now define $k_0 = k_0(n_0(\epsilon))$ by

$$\begin{pmatrix} p_{k_0} \\ q_{k_0} \end{pmatrix} \;=\; \begin{cases} \begin{pmatrix} P_{n_0+2} \\ Q_{n_0+2} \end{pmatrix}, & \text{if } \dfrac{P_{n_0+2}}{Q_{n_0+2}} \text{ is an E.C.F. convergent,} \\ \begin{pmatrix} P_{n_0+3} \\ Q_{n_0+3} \end{pmatrix}, & \text{if } \dfrac{P_{n_0+2}}{Q_{n_0+2}} \text{ is not an E.C.F. convergent.} \end{cases}$$

Notice that k_0 is well-defined, due to (1.9). It now at once follows from (1.1–1.4), (1.9) and from the fact that both $(Q_n)_{n\ge1}$ and $(q_n)_{n\ge1}$ are monotonically increasing sequences of positive integers, that for each $k \ge k_0$ there is a positive integer n, $n \ge n_0 + 2$, such that

$$q_k - q_{k-1} \;\ge\; Q_{n-2} \;\ge\; Q_{n_0}\,.$$

This proves Lemma 1.

Remark 4- Let x and ρ_k be as in Lemma 1. Then

$$\lim_{k\to\infty} \frac{p_k}{q_k} = x, \tag{1.14}$$

since (1.8) gives

$$|x - \frac{p_k}{q_k}| \;=\; \frac{1}{q_k}\,|\rho_k|\,,\;\; k \ge 0.$$

If $x = P/Q$ is a rational number with a finite E.C.F. expansion, the situation is trivial. However, if $(P, Q) = 1$ and $P \equiv Q(mod2)$, we saw in Remark 3 that x has two (infinite) E.C.F. expansions. It is not difficult to show that the conclusion of Lemma 1 (and therefore also (1.14)) is valid for each of these two expansions. The sequences of convergents of these two E.C.F. expansions of x are closely related; for instance, the sequences of E.C.F. convergents of 1 are

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$$

and

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{n+1}{n}, \dots .$$

Details are left to the reader.

Now we will show that $Q > S > 0$ for sufficiently large k.

In order to prove this property, notice that we have

$$Q = (cx + d)q_{k+1} + c\rho_{k+1}$$

and

$$S = (cx + d)q_k + c\rho_k ,$$

for $k \geq 0$ and that we may assume that $c \neq 0$, by (1.8). Now let $k(x) \in \mathbf{N}$ be such that

$$|\rho_k| < \frac{cx + d}{2|c|} \quad for \ \ all \ \ k \geq k(x) .$$

But $q_{k+1} - q_k \geq 1$ (see (1.8)) and $cx + d > 0$; therefore, for $m \geq k(x)$,

$$Q - S = (cx + d)(q_{m+1} - q_m) + c(\rho_{m+1} - \rho_m)$$

$$\geq cx + d - |c|(|\rho_{m+1}| + |\rho_m|) > 0 ,$$

proving $Q > S$ for k large enough (the last inequality comes from Lemma 1).

Now we will proveTheorem 1.

From Theorem 2 we conclude that R/S and P/Q are two consecutive elements of the sequence $(r_k/s_k)_{k\geq -1}$ of E.C.F. convergents of y, that is, there exists a positive integer $n = n(m)$ such that

$$\begin{pmatrix} R & P \\ S & Q \end{pmatrix} = \begin{pmatrix} r_{n-1} & r_n \\ s_{n-1} & s_n \end{pmatrix} .$$

Hence (1.12) and

$$y = \frac{r_{k-1}t_k(y) + r_k}{s_{k-1}t_k(y) + s_k} , \ k \geq 0$$

now yield, for all $m \geq k(x)$:

$$t_m(x) = t_{n(m)}(y)\,.$$

This proves Theorem 1.

We will consider now two propositions (propositions 3 and 4) of independent interest, that will be also very important for the proof of our main result Theorem 0. For the regular continued fraction we have the classical result of Lagrange, that x is a quadratic surd if and only if the R.C.F. expansion of x is eventually periodic (see[21]). For the E.C.F. expansion we have a similar result.

Proposition 3.-*Let x be a real number. Then x is a quadratic surd if and only if the E.C.F. expansion of x is eventually periodic and x is not theta equivalent to 1.*

The proof of proposition 3 will be presented later in this paper.

The proof of this Proposition is an adaptation to the present situation of the proof of [4, Theorem 177]. Essential in Hardy's and Wright's proof is the fact that, if x is an irrational number with R.C.F. convergents P_n/Q_n, $n \geq 1$, then, for $n \geq 1$,

$$0 < Q_n|Q_n x - P_n| < 1\,.$$

This property does not hold for the continued fraction with even partial quotients. Let x be an irrational number with E.C.F. convergents p_k/q_k, $k \geq 1$, and define the *approximation coefficients* $\theta_k = \theta_k(x)$, $k \geq 1$, by

$$\theta_k := q_k|q_k x - p_k|\,.$$

The following theorem is due to F. Schweiger [24].

Theorem 3 (Schweiger).– *Given* $z \geq 0$, *for almost all* x

$$\lim_{N\to\infty} \frac{1}{N}\,\#\{1 \leq j \leq N;\, \theta_j(x) \leq z\,\} = 0\,.$$

That the idea behind the proof of Hardy and Wright can still be applied, after some minor modifications, is due to the following Lemma.

Lemma 2.- *Let x be a quadratic surd. Then there exists a positive constant $M = M(x)$, such that*

$$0 < \theta_k < M\,,\ for\ \ all\ \ k \geq 1\,.$$

Proof- Let $(p_k/q_k)_{k\geq -1}$ be the sequence of E.C.F. convergents of x; moreover, let $x = [\,B_0;\, B_1,\, \ldots\,,\, B_n,\, \ldots]$ be the R.C.F. expansion of x. By the above mentioned Theorem 177 from [4], there exists a constant $A = A(x) \in N$, such that

$$B_n \leq A,\ for\ \ each\ \ n \geq 1\,.$$

Put $M = M(x) := A^2 + A + 1$, and let p_k/q_k be some E.C.F. convergent of x. Either p_k/q_k is a principal convergent of x, i.e., there exists an integer n, such that

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} P_n \\ Q_n \end{pmatrix},$$

in which case $\theta_k = Q_n|Q_n x - P_n| < 1 \leq M$, or else p_k/q_k is a mediant convergent of x, i.e., there exist integers n and λ, $\lambda \in \{0, \dots, B_{n+1} - 1\}$, such that

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} \lambda P_n + P_{n-1} \\ \lambda Q_n + Q_{n-1} \end{pmatrix}.$$

In this case we have

$$\theta_k \leq (\lambda Q_n + Q_{n-1})(\lambda|Q_n x - P_n| + |Q_{n-1}x - P_{n-1}|,$$

and then the Lemma at once follows from

$$Q_k|Q_{k-1}x - P_{k-1}| + Q_{k-1}|Q_k x - P_k| = 1; \frac{Q_{k-1}}{Q_k} < 1, fork \geq 1.$$

Apart from the theorem of Lagrange there is also the classical theorem of Galois on purely periodic quadratic surds. Below we will obtain a similar result for the E.C.F. expansion, but for the sake of reference we will first state Galois' theorem.

Theorem 4 (Galois). *A real number $\alpha > 1$ has a purely periodic continued fraction expansion if and only if α is a reduced quadratic surd (that is, if the conjugate root $\bar{\alpha}$ satisfies $-1 < \bar{\alpha} < 0$). If*

$$\alpha = [\overline{n_1; n_2, \dots, n_{2r}}],$$

then

$$\frac{-1}{\bar{\alpha}} = [\overline{n_{2r}; n_{2r-1}, \dots, n_1}].$$

(For a proof of this theorem, see e.g. [21], Theorem 3.3.4).

Proposition 4.-

[A] *A real number α has a purely periodic continued fraction expansion with even partial quotients if and only if α is a reduced quadratic surd.*

[B] *If $ECF(\alpha) = [\overline{a_0; e_1/a_1, \dots, e_{\ell-1}/a_{\ell-1}}]$, then*

$$ECF(\frac{-1}{\bar{\alpha}}) = [\overline{a_{\ell-1}; e_{\ell-1}/a_{\ell-2}, \dots, e_2/a_1, e_1/a_0}].$$

Proof-

[A1] Let α be a reduced quadratic surd. Due to the above mentioned theorem of Galois we have that the R.C.F. expansion of α is purely periodic, i.e.,

$$RCF(\alpha) = [\overline{B_0; B_1, \ldots, B_{L-1}}].$$

Now suppose there exists an integer $i \in \{0, \ldots, L-1\}$ such that B_i is odd (if such an i does not exist we're done). The idea of the proof is to chop up the R.C.F. expansion of α into pieces to which the algorithm from Remark 2 can be applied individually. To this end we define

$$i(1) := min\{i; B_i \ is \ odd\};$$

$$j(1) := min\{i; i > i(1), i - i(1) \ \ is \ \ even, B_i \ \ is \ \ odd\}$$

and recursively

$$i(n+1) := min\{i; i \geq j(n)+1, B_i \ \ is \ \ odd\}, \ n \geq 1;$$

$$j(n+1) := min\{i; i > i(n+1), i - i(n+1) \, is \ \ even, \ \ B_i \, is \ \ odd\}, n \geq 1.$$

Furthermore, define the ordered sets of symbols $(\mathcal{A})_{n\geq 0}$ and $(\mathcal{B})_{n\geq 1}$ (which we will call *blocks*) by

$$\mathcal{A}_0 := \{B_0, \ldots, B_{i(1)-1}\}; \mathcal{A}_n := \{B_{j(n)+1}, \ldots, B_{i(n+1)-1}\}, \ n \geq 1$$

and

$$\mathcal{B}_n := \{B_{i(n)}, \ldots, B_{j(n)}\}, \ n \geq 1.$$

Since RCF(α) is purely periodic, we have that each block $\mathcal{B}_n$ has length at most 2L; by definition, $\mathcal{B}_n$ is never empty. The blocks $\mathcal{A}_n$ have length smaller than L, and might be empty. Let

$$k := \#\{i; 0 \leq i \leq L-1, B_i \ is \ \ odd\};$$

then there exists an integer $n_0 \in \{1, \ldots, k\}$, such that

$$i(n_0) \equiv i(k+1)(mod \, L).$$

But then the fact that the R.C.F. expansion of α is purely periodic implies that

$$\mathcal{B}_{n_0+m} = \mathcal{B}_{k+m+1}, \ m \geq 0,$$

that is, the sequence of blocks $(\mathcal{B}_n)_{n\geq 1}$ is (ultimately) periodic.

Claim- *The sequence of blocks $(\mathcal{B}_n)_{n\geq 1}$ is purely perodic; i.e., there exists an integer ℓ, $\ell \geq 1$, such that $\mathcal{B}_n = \mathcal{B}_{n+\ell}$, $n \geq 1$.*

Proof- Let $\ell := i(k+1) - i(n_0)$, where $i(n_0)$ and $i(k+1)$ are as before and put

$$n_0^* := min\{n; \mathcal{B}_n = \mathcal{B}_{n+\ell}\}.$$

Now suppose that the sequence of blocks $(\mathcal{B}_n)_{n\geq 1}$ is not purely periodic, i.e., suppose that $n_0^* > 1$. Notice that $n_0^* > 1$ implies that $i(n_0^*) \geq 4$. Since $\mathcal{B}_{n_0^*} = \mathcal{B}_{n_0^*+\ell}$, we have

$$B_{i(n_0^*)} = B_{i(n_0^*+\ell)} ,$$

but then the fact that the R.C.F. expansion of α is purely periodic implies that

$$B_{i(n_0^*)-1} = B_{i(n_0^*+\ell)-1}; \ldots ; B_0 = B_{i(n_0^*+\ell)-i(n_0^*)}$$

and we find, due to the definition of $\mathcal{B}_n$ and the assumption that $n_0^* > 1$, that

$$\mathcal{B}_{n_0^*-1} = \mathcal{B}_{n_0^*+\ell-1} ,$$

which is in conflict with the definition of n_0^*. This proves the Claim.

Concatenating the blocks $\mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \mathcal{B}_2, \mathcal{A}_2, \ldots$ now yields that

$$RCF(\alpha) = [\mathcal{A}_0, B_1, \mathcal{A}_1, \ldots , \mathcal{B}_\ell, \mathcal{A}_\ell, \mathcal{B}_{\ell+1} = \mathcal{B}_1, \mathcal{A}_{\ell+1} = \mathcal{A}_1, \ldots]$$

and setting

$$\mathcal{A}_\ell^\flat := \{ B_{j(\ell)+1}, \ldots , B_{j(\ell)+r} \} ,$$

where $r := L - 1 - (j(\ell)(mod\, L))$, we have

$$\mathcal{A}_\ell = \{ B_{j(\ell)+1}, \ldots , B_{j(\ell)+r} = B_{L-1}, B_0, \ldots , B_{i(1)-1} \}$$

and therefore

$$RCF(\alpha) = [\overline{\mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \ldots , \mathcal{B}_\ell, \mathcal{A}_\ell^\flat}] .$$

Due to the definition of the sequences of blocks $(\mathcal{A}_n)_{n\geq 0}$, $(\mathcal{B}_n)_{n\geq 1}$ it is clear that the algorithm from Remark 2 yields the following: for each $n \geq 0$ replace in RCF(α) the block $\mathcal{A}_n$ by

$$\mathcal{C}_0 := \{ B_0; 1/B_1, \ldots , 1/B_{i(1)-1} \};\ \mathcal{C}_n := \{1/B_{j(n)+1}, \ldots , 1/B_{i(n+1)-1} \}, n \geq 1$$

and replace each block $\mathcal{B}_n$ by

$$\mathcal{D}_n := \{ 1/(B_{i(n)} + 1), \underbrace{-1/2, \ldots , -1/2}_{(B_{i(n)+1}-1)-times}, -1/(B_{i(n)+2} + 2), \ldots\ldots$$

$$\ldots\ldots , \underbrace{-1/2, \ldots , -1/2}_{(B_{j(n)-1}-1)-times}, -1/(B_{j(n)} + 1) \} .$$

Thus we find

$$ECF(\alpha) = [\mathcal{C}_0, \mathcal{D}_1, \mathcal{C}_1, \ldots , \mathcal{D}_\ell, \mathcal{C}_\ell, \mathcal{D}_{\ell+1} = \mathcal{D}_1, \mathcal{C}_{\ell+1} = \mathcal{C}_1, \ldots]$$

and, setting

$$\mathcal{C}_\ell^\flat \ := \ \{\,1/B_{j(\ell)+1},\, \ldots\, ,\, 1/B_{j(\ell)+r}\,\}\,,$$

where r is defined as before, we obtain

$$ECF(\alpha) \ = \ [\,\overline{\mathcal{C}_0,\, \mathcal{D}_1,\, \mathcal{C}_1,\, \ldots\, ,\, \mathcal{D}_\ell,\, \mathcal{C}_\ell^\flat}\,]\ .$$

[A2] Conversely, let $\alpha > 1$ and suppose that the E.C.F. expansion of α is purely periodic, i.e.,

$$ECF(\alpha) \ = \ [\,\overline{a_0;\, e_1/a_1,\, \ldots\, ,\, e_{\ell-1}/a_{\ell-1}}\,]$$

(here and in the sequel we assume that $e_{n\ell} = +1$, for all $n \geq 0$). The idea now is to invert the algorithm from Remark 2.

Notice that $\alpha > 1$, a_0 is even and ECF(α) is purely periodic imply that α is not theta equivalent to 1. But then we at once have that α is a quadratic surd. In view of Galois' theorem it is therefore sufficient to show that the R.C.F. expansion of α is purely periodic.

Put $e_0 := +1$. We may assume that there exists an integer i, such that

$$e_i \ = \ +1;\ e_{i+1} \ = \ -1\,;$$

(If such an integer i does not exists then $e_{\ell+n+1} \ = \ +1$, for each $n \geq 0$, implies that $e_m \ = \ +1$, for all m, and we're done.)

In order to invert the algorithm from Remark 2 we actually invert the approach from part [A1] of this proof. To this end we define again two sequences of blocks $(\mathcal{C}_n)_{n\geq 0}$ and $(\mathcal{D}_n)_{n\geq 1}$ as follows: first define

$$i(1) \ := \ min\{\,i;\, e_i = 1 = -e_{i+1}\,\};\ j(1) \ := \ \{\,i;\, e_{i+1} = 1 = -e_i\,\}$$

and then, recursively,

$$i(n+1) \ := \ min\{\,i;\, i \geq j(n)+1,\, e_i = 1 = -e_{i+1}\,\},\, n \geq 1\,;$$

$$j(n+1) \ := \ min\{\,i;\, i > i(n+1),\, e_{i+1} = 1 = -e_i\,\},\, n \geq 1\,.$$

Next define

$$\mathcal{C}_0 \ := \{a_0; 1/a_1,\, \ldots\, , 1/a_{i(1)-1}\};\, \mathcal{C}_n \ := \{1/a_{j(n)+1},\, \ldots\, , 1/a_{i(n+1)-1}\}, n \geq 1$$

and

$$\mathcal{D}_n \ := \ \{\,e_{i(n)}/a_{i(n)},\, e_{i(n)+1}/a_{i(n)+1},\, \ldots\, ,\, e_{j(n)}/a_{j(n)}\,\}$$

$$= \ \{\,1/a_{i(n)},\, -1/a_{i(n)+1}/a_{i(n)+1},\, \ldots\, ,\, -1/a_{j(n)}\,\},\ \ n \geq 1\,. \qquad (*)$$

(Notice that some, or even all, of the blocks $\mathcal{C}_n$ might be empty.) It follows from $e_\ell = 1$, that the sequence of blocks $(\mathcal{D}_n)_{n\geq 1}$ is purely periodic with period length, say, k. Therefore

$$ECF(\alpha) = [\mathcal{C}_0, D_1, \mathcal{C}_1, \ldots, \mathcal{D}_{k+1} = \mathcal{D}_1, \mathcal{C}_{k+1} = \mathcal{C}_1, \ldots].$$

Putting

$$\mathcal{C}_k^\flat := \{ 1/a_{j(k)+1}, \ldots, 1/a_{\ell-1} \},$$

we have

$$ECF(\alpha) = [\overline{\mathcal{C}_0, \mathcal{D}_1, \mathcal{C}_1, \ldots, \mathcal{D}_k, \mathcal{C}_k^\flat}] .$$

Now replace the blocks $\mathcal{C}_n$ and $\mathcal{D}_n$ by $\mathcal{A}_n$ resp. $\mathcal{B}_n$, where the blocks $\mathcal{A}_n$ are defined by (if we put $j(0) := -1$)

$$A_n := \{ a_{j(n)+1}, \ldots, a_{i(n+1)-1} \}, \; n \geq 0 .$$

(Notice again that some, or all, of the blocks $\mathcal{A}_n$ might be empty).

In order to define the blocks $\mathcal{B}_n$ we do the following; let $\mathcal{D}_n$ be as in (*), that is,

$$\mathcal{D}_n = \{ 1/a_{i(n)}, -1/a_{i(n)+1}, \ldots, -1/a_{j(n)} \}.$$

Then

1. replace $1/a_{i(n)}$ by $1/(a_{i(n)}+1)$ and replace $-1/a_{j(n)}$ by $-1/(a_{j(n)}+1)$;

2. for each $i \in \{ i(n), \ldots, j(n)-1 \}$ for which $a_i, a_{i+1} \geq 3$: replace the subblock $\pm 1/a_i, -1/a_{i+1}$ by

$$\pm 1/a_i, -1/1, -1/a_{i+1};$$

Notice that for $i = i(n)$, a_i actually has the value $a_{i(n)}+1$; a similar remark holds for the case $i = j(n)-1$.

3. for each $i \in \{ i(n)+1, \ldots, j(n)-1 \}$ for which $a_{i-3} \geq 3$, $a_i = 2$ we put

$$j_i := min\{ j; j \geq i, a_j = 2, a_{j+1} \geq 3 \}.$$

Now replace the subblock

$$-1/a_i, \ldots, -1/a_{j_i} \equiv \underbrace{-1/2, \ldots, -1/2}_{(j_i-i)-times}$$

by

$$-1/(j_i - i + 1) .$$

Applying this algorithm yields a block $\mathcal{E}_n$, given by

$$\mathcal{E}_n = \{ 1/c_{n,1}, -1/c_{n,2}, \ldots, -1/c_{n,m} =\}$$

$$\{ 1/(a_{i(n)}+1),\, -1/c_{n,2},\, -1/(a_{i(n)+2}+2),\, \ldots\,,\, -1/c_{n,m-1},\, -1/(a_{j(n)}+1) \}\ .$$

One easily shows (as the above expression for $\mathcal{E}_n$ already suggests) that m is odd. Moreover, for each $i \in \{2, \ldots, m-1\}$, i odd, one has that $c_{n,i}$ is even.

Now replace $\mathcal{E}_n$ by $\mathcal{B}_n$, where

$$\mathcal{B}_n \ := \ \{ c_{n,1}-2,\, c_{n,2},\, c_{n,3}-2,\, \ldots\,,\, c_{n,m-1},\, c_{n,m}-2 \},\ n \geq 1\,.$$

We have

$$RCF(\alpha) \ = \ [\mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \ldots\,, \mathcal{B}_k, \mathcal{A}_k, \mathcal{B}_{k+1} = \mathcal{B}_1, \mathcal{A}_{k+1} = \mathcal{A}_1, \ldots \}\ .$$

Putting $\mathcal{A}_k^\flat \ = \ \{a_{j(k)+1}, \ldots\,, a_{\ell-1}\,,\}$ one easily shows that

$$RCF(\alpha) \ = \ [\,\overline{\mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \ldots\,, \mathcal{A}_{k-1}, \mathcal{B}_k, \mathcal{A}_k^\flat}\,]\ ,$$

that is, $\alpha > 1$ is a purely periodic quadratic surd, and is therefore reduced.

[B] Let $ECF(\alpha) \ = \ [\,\overline{a_0;\, e_1/a_1, \ldots\,, e_{\ell-1}/a_{\ell-1}}\,]$. We now want to show that

$$ECF(\frac{-1}{\bar{\alpha}}) \ = \ [\,\overline{a_{\ell-1};\, e_{\ell-1}/a_{\ell-2}, \ldots\,, e_1/a_0}\,]\,.$$

Let the sequences $(\mathcal{A}_n)_{n\geq 0}$, $(\mathcal{B}_n)_{n\geq 1}$ $(\mathcal{C}_n)_{n\geq 0}$ and $(\mathcal{D}_n)_{n\geq 1}$, and the blocks $\mathcal{C}_k^\flat$, $\mathcal{D}_k^\flat$ be defined as in part [A2] of this proof; i.e.,

$$RCF(\alpha) \ = \ [\,\overline{\mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \ldots\,, \mathcal{B}_k, \mathcal{A}_k^\flat}\,]$$

and

$$ECF(\alpha) \ = \ [\,\overline{\mathcal{C}_0, \mathcal{D}_1, \mathcal{A}_1, \ldots\,, \mathcal{D}_k, \mathcal{C}_k^\flat}\,]\,.$$

Let $\mathcal{G}$ and $\mathcal{H}$ be ordered blocks, given by

$$\mathcal{G} \ = \ \{ g_1, \ldots\,, g_m\};\ \mathcal{H} \ = \ \{ 1/h_0;\, e_1/h_1, \ldots\,, e_n/h_n \}\,.$$

Then we define the ordered blocks $\overline{\mathcal{G}}$, $\overline{\mathcal{H}}$ by, respectively,

$$\overline{\mathcal{G}} \ := \{ g_m, \ldots\,, g_1 \}\ ; \overline{\mathcal{H}} \ := \{ 1/h_n;\, e_n/h_{n-1}, \ldots\,, e_1/h_0 \}\ .$$

Since $\frac{-1}{\bar{\alpha}} \ = \ [\,\overline{\overline{\mathcal{A}_k^\flat},\, \overline{\mathcal{B}_k}, \ldots\,, \overline{\mathcal{A}_1},\, \overline{\mathcal{B}_1},\, \overline{\mathcal{A}_0}}\,]$, it now follows at once from [A1] that

$$ECF(\frac{-1}{\bar{\alpha}}) \ = \ [\,\overline{\overline{\mathcal{C}_k^\flat},\, \overline{\mathcal{D}_k}, \ldots\,, \overline{\mathcal{C}_1},\, \overline{\mathcal{D}_1},\, \overline{\mathcal{C}_0}}\,] = \ [\,\overline{a_{\ell-1};\, e_{\ell-1}/a_{ell-2}, \ldots\,, e_1/a_0}\,]\,.$$

This proves Proposition 4.

Some examples:

(I) Let

$$RCF(\alpha) = [\overline{1;\, 2,\, 3,\, 4,\, 5}\,] =$$

$$= [\underbrace{1,2,3}_{\mathcal{B}_1},\underbrace{\ 4\ }_{\mathcal{A}_1},\underbrace{5,1,2,3,4,5,1}_{\mathcal{B}_2},\underbrace{\ 2\ }_{\mathcal{A}_2},\underbrace{3,4,5}_{\mathcal{B}_3},\underbrace{1,,2,3}_{\mathcal{B}_4=B_1},\ \ldots],$$

then the algorithm from part [A1] of the above proof yields

$$ECF(\alpha) = [\overline{\mathcal{D}_1,\, \mathcal{C}_1,\, \mathcal{D}_2,\, \mathcal{C}_2,\, \mathcal{D}_3}\,]$$

$$= [\,\overline{2;-1/2,-1/4,1/4,1/6,-1/4,-1/2,-1/2,-1/6,\underbrace{-1/2,\ \ldots\ ,\ -1/2,}_{4-times}}$$

$$\overline{-1/2,1/2,1/4,-1/2,-1/2,-1/2,-1/6}\,].$$

(II) Let

$$ECF(\alpha) = [\,\overline{6;1/4,1/4,\underbrace{-1/2,\ \ldots\ ,-1/2}_{9-times},1/2}\,] =$$

$$= [\,\underbrace{6;1/4}_{\mathcal{C}_0},\underbrace{1/4,-1/2,\ \ldots\ ,-1/2}_{\mathcal{D}_1},\underbrace{1/2,1/6,1/4}_{\mathcal{C}_1},\underbrace{1/4,\ \ldots\ ,-1/2}_{\mathcal{D}_2=\mathcal{D}_1},\ \ldots].$$

then the algorithm from [A2] yields that

$$RCF(\alpha) = [\overline{6;\, 4,\, 3,\, 9,\, 1,\, 2}\,].$$

(III) Let

$$RCF(\alpha) = [\,4;\, 2,\, 1,\, \overline{2,\, 6,\, 8}\,] =$$

$$= [\,\underbrace{4;\, 2}_{\mathcal{A}_0},\ \underbrace{1,\ 2,\ 6,\ 8,\ 2,\ 6,\ \ldots}_{\mathcal{B}_1}\,],$$

then we have

$$ECF(\alpha) = [\,4;1/2,1/2,-1/2,-1/8\underbrace{-1/2,\ \ldots\ ,-1/2}_{7-times},-1/4,\underbrace{-1/2,\ \ldots\ ,-1/2}_{5-times},$$

$$-1/10,-1/8,\underbrace{-1/2,\ \ldots\ ,-1/2}_{7-times},\ \ldots\ldots] =$$

$$== [\,4;1/2,1/2,-1/2,\overline{-1/8,(-1/2)^7,-1/4,(-1/2)^5,-1/10}\,].$$

Here $(-1/2)^k$ is short for $\underbrace{-1/2,\ \ldots\ ,-1/2}_{k-times}$

A proof of Proposition 3 can be given similar to the proof of Proposition 4[A]; some care with the definition of the blocks $\mathcal{B}_n$ is however necessary, as example (I) above shows.

Another proof of Proposition 3, following the lines of the proof of [4, Theorem 177], will be presented now.

Proof of Proposition 3- Let x be a quadratic surd. Then there exist integers A, B, C such that

$$Ax^2 + Bx + C = 0. \tag{1.15}$$

Since x is irrational we have $A, C \neq 0$ and $B^2 - 4AC > 0$. If

$$x = [a_0; e_1/a_1, e_2/a_2, \dots, e_n/a_n, \dots]$$

is the E.C.F. expansion of x, with E.C.F. convergents p_n/q_n, $n \geq -1$, then

$$x = \frac{p_n + t_n p_{n-1}}{q_n + t_n q_{n-1}}, \tag{1.16}$$

where $t_n = t_n(x)$ is defined as in (1.3). Substitution of (1.16) into (1.15) yields

$$A_n t_n^2 + B_n t_n + C_n = 0,$$

where

$$A_n = Ap_{n-1}^2 + Bp_{n-1}q_{n-1} + Cq_{n-1}^2\,;$$

$$B_n = 2Ap_{n-1}p_n + B(p_n q_{n-1} + p_{n-1}q_n) + 2Cq_{n-1}q_n$$

and

$$C_n = Ap_n^2 + Bp_n q_n + Cq_n^2\,.$$

Notice that since x is irrational, we must have that $A_n \neq 0$. It is easy to show that

$$B_n^2 - 4A_nC_n = B^2 - 4AC\,. \tag{1.17}$$

Since x is a quadratic irrational number, we have, due to Lagrange's theorem, that the R.C.F. expansion of x is eventually periodic, i.e., there exist integers $n_0 \geq 0$ and $\ell \geq 1$ such that

$$x = [B_0; B_1, \dots, B_{n_0}, \overline{B_{n_0+1}, \dots, B_{n_0+\ell}}]$$

(the bar indicates the period). But then there exists a positive integer K, such that

$$B_n \leq K,\ for\ \ all\ \}n \geq 1.$$

Put

$$M = M(x) := K^2 - K + 1,$$

and define the sequence of real numbers $\psi_n = \psi_n(x)$, $n \geq 1$, by

$$\psi_n := \rho_n q_n\,,\ n \geq 1,$$

where ρ_n is defined as in Lemma 1. Notice that $|\psi_n| = \theta_n$, where θ_n is defined in Lemma 2. By Lemma 2 we then have

$$|\psi_n| < M.$$

It follows that

$$C_n = A(q_n x + \frac{\psi_n}{q_n})^2 + Bq_n(q_n x + \frac{\psi_n}{q_n}) + Cq_n^2$$

$$= 2A\psi_n x + A\frac{\psi_n^2}{q_n^2} + B\psi_n ,$$

and therefore, for n sufficiently large, we have

$$|C_n| < (2|Ax| + |A| + |B|)M.$$

Since $A_n = C_{n-1}$ we also have, for sufficiently large n, that

$$|A_n| < (2|Ax| + |A| + |B|)M.$$

Due to (1.17) we at once obtain

$$B_n^2 \leq 4|A_n C_n| + |B^2 - 4AC|$$

$$< 4(2|Ax| + |A| + |B|)^2 M^2 + |B^2 - 4AC|.$$

Hence the absolute values (for n sufficiently large) of A_n, B_n and C_n are less than constants independent of n. But then there are only a finite number of triplets (A_n, B_n, C_n), and we can find a triplet (A^*, B^*, C^*), which occurs at least three times, say as $(A_{n_1}, B_{n_1}, C_{n_1})$, $(A_{n_2}, B_{n_2}, C_{n_2})$ and $(A_{n_3}, B_{n_3}, C_{n_3})$.

But then t_{n_1}, t_{n_2} and t_{n_3} are roots of

$$A^* y^2 + B^* y + C^* = 0;$$

and therefore at least two of them must be equal; say we have $t_{n_1} = t_{n_2}$. By induction we obtain

$$t_{n_1+\kappa} = t_{n_2+\kappa} , \ \kappa \geq 0.$$

Since t_n completely determines the values of e_n and a_n, we thus see that the sequence $(e_k, a_k)_{k\geq 1}$ is eventually periodic, i.e. there exist integers n_0 and ℓ, $\ell \geq 1$, such that

$$e_{n_0+i} = e_{n_0+i+k\ell}, \ a_{n_0+i} = a_{n_0+i+k\ell}, \ \ k \geq 0, \ 0 \leq i \leq \ell - 1.$$

Notice that x cannot be theta equivalent to 1, since x is irrational.

Now assume that x has an E.C.F. expansion which is eventually periodic, i.e.,

$$x = [\, a_0; \, e_1/a_1, \, \ldots \, , \, e_{n_0}/a_{n_0}, \, \overline{e_{n_0+1}/a_{n_0+1}, \, \ldots \, , \, e_{n_0+\ell}/a_{n_0+\ell}} \,] \, .$$

(Again the bar indicates the period; we may assume that n_0 and ℓ are minimal.) Since x is not theta equivalent to 1, we have

$$t_{n_0} = t_{n_0+\ell} = [\,0;\, \overline{e_{n_0+1}/a_{n_0+1},\, \ldots\, ,\, e_{n_0+\ell}/a_{n_0+\ell}}\,]$$
$$\neq [\,0;\, \overline{-1/2}\,],$$

and due to

$$x \;=\; \frac{p_{n_0} + t_{n_0} p_{n_0-1}}{q_{n_0} + t_{n_0} q_{n_0-1}} \;=\; \frac{p_{n_0+\ell} + t_{n_0} p_{n_0+\ell-1}}{q_{n_0+\ell} + t_{n_0} q_{n_0+\ell-1}}\ ,$$

we see that t_{n_0} satisfies a quadratic equation with integer coefficients. Clearly t_{n_0} is not rational and therefore we have at once that x is a quadratic surd. This proves Proposition 3.

§3. Lenght Spectra

In this section we will analyse the coding of closed geodesics of the Theta Group by means of the continued fraction expansion with even quotients and prove the Theorem 0 stated in section 0.

We will explain now the following claim: there exists a relation among closed geodesics and periodic trajectories of the map T (defined in section 1), furthermore this relation also presents a nice way to translate the lengthts of closed geodesics to an expression related to the Continued Fraction Expansion with even partial quotients.

The claim is obtained after we prove results analogous to the ones used for the case of the Modular Group and the regular Continued Fraction Expansion[17],[21]. We will indicate now the outline of the proof of the claim.

We point out first that we can assume without lost of generality that any closed oriented geodesic we will consider for the Theta Group, is such that its ends $\gamma_{-\infty}$ and γ_{∞} (in the boundary of the Hyperbolic Semiplane) are chosen in such manner that the first has modulus larger than one and the second has modulus smaller than one.

Consider the bi-infinite sequence

$$[..., e_{-2}/a_{-2}, e_{-1}/a_{-1}, a_0; e_1/a_1, e_2/a_2, e_3/a_3, ...]. \tag{2.1}$$

where a_0 is an even number in Z, the others a_i are all even and positive and e_i are 1 or -1.

This bi-infinite sequence will determine the two end points of the geodesic as we will see later.

Let's consider first an specific example to simplify our reasoning.

Suppose this sequence is periodic with period, say, 2. We will also assume for simplicity that the e_i are all equal to 1. Consider

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_1 + \ldots}}} = \overline{[0; e_1/a_1, e_2/a_2]}.$$

Note that x has period 2 for T because

$$T(x) = \frac{1}{a_2 + \frac{1}{a_1 + \ldots}} \quad \textit{and} \quad T^2(x) = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = x$$

We will associate to x the element of the Theta Group

$$\theta_2 u \theta_1 u = \beta \tag{2.2}$$

where $\theta_1 = (z + a_1), \theta_2 = (z - a_2)$ and $u(z) = -1/z$.

It is easy to see that $\beta(\overline{[0; e_1/a_1, e_2/a_2]}) = \overline{[0; e_1/a_1, e_2/a_2]}$, because

$$\theta_1 u(x) = -\frac{1}{a_2 + \frac{1}{a_1 + \ldots}} \tag{2.3}$$

and

$$\beta(x) = \theta_2 u \theta_1 u(x) = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = x. \tag{2.4}$$

Note also that T^2 and β are different maps. Here we have a situation quite different from the modular case because the map T do not have an expression as an element of the Theta Group in any sense.

The element β in the Theta Group is hyperbolic and leaves invariant a semicircle S (a geodesic in the Hyperbolic Semiplane) with ends $\gamma_{-\infty}$ and γ_{∞} . From the above γ_{∞} is equal to $\overline{[0; e_1/a_1, e_2/a_2]} = x$ and x is a quadratic surd (this follows from the equation $\beta(x) = x$). Now from Proposition 3 and 4 we know that the conjugate solution of x is $\gamma_{-\infty}$ and satifies $-1/\gamma_{-\infty} = \overline{[0; e_2/a_2, e_1/a_1]}$.

Therefore S is determined by $\gamma_{\infty} = \overline{[0, e_1/a_1, e_2/a_2]}$
and $-1/\gamma_{-\infty} = \overline{[0, e_2/a_2, e_1/a_1]}$.

Two closed geodesics in the Hyperbolic space lying over the same one in the Riemann Surface generated by the Theta Group are mapped to symbols that are cyclic permutations of each other. In other words, there exist a one-to-one correspondence between conjugacy classes of elements of the group and cycles of the coding. The proof of this property is a straigthforward adaptation of the usual arguments for the Modular Group [17,21,22]. A detailed proof of this claim for the Theta Group, using a more geometric point of view, is presented in [2].

In this way we associate to the string (2.1) the closed geodesic s (in the quotient of the Hyperbolic Semiplane by the Theta Group) obtained as quotient of S by β. The length of the closed geodesic s can be given in terms of (2.2) and (2.3) (applying the map $-1/z^2$ and taking products) and in this case can be obtained as (see[17])

$$log \frac{1}{\overline{[0; 1/a_1, 1/a_2]}^2} \frac{1}{(-\overline{[0; 1/a_2, 1/a_1]})^2} = log \frac{1}{\overline{[0; 1/a_1, 1/a_2]}^2} \frac{1}{\overline{[0; 1/a_2, 1/a_1]}^2}. \tag{2.5}$$

This value can be also obtained using the map T, in the following way: first note that

$$T^{2'}(x) = T^{'}(T(x)) T^{'}(x)$$

and that

$$|T'(T(x))| = \frac{1}{T(x)^2} = \frac{1}{\overline{[0, 1/a_2, 1/a_1]}^2} \quad and \quad T^{'}(x) = \frac{1}{\overline{[0; 1/a_1, 1/a_2]}^2}.$$

Therefore in the present example

$$log|T^{2'}(x)| \quad = \quad length \quad of \quad S.$$

The essential point is that the appearance of a minus sign in (2.3) do not create any difference between (2.5) to the telescopic products $log \quad |T^{'}(T(x))||T^{'}(x)|$ because we will apply each time the map $(-1/z^2)$.

Now let's consider the more general situation.

The general case is basically the same, that means given x, $0 < x < 1$, with the periodic even expansion

$$x = \overline{[0; e_1/a_1, e_2/a_2, ..., e_n/a_n]}, \tag{2.6}$$

we associate the element

$$\beta = \theta_n u....\theta_1 u$$

where $\theta_{i+1} = (z - a_{i+1})$ or $\theta_{i+1} = (z + a_{i+1})$ according to the fact that after applying $\theta_i u...\theta_1 u$ we obtain

$$-\frac{1}{a_{i+1} + \frac{e_{i+2}}{a_{i+2}+...}} \tag{2.7}$$

or

$$\frac{1}{a_{i+1} + \frac{e_{i+2}}{a_{i+2}+...}}. \tag{2.8}$$

In the same way as before $T^n(x) = x$, and the length of the geodesic S corresponding to β will be also given in terms of

$$log \quad |T^{'}(T^{n-1}(x))...T^{'}(T(x))T^{'}(x)|. \tag{2.9}$$

The appearance of a minus sign as in (2.7) will not create any difference between (2.9) and (2.10) bellow, by the same reason as before (that is, we will apply each time $-1/z^2$).

The problem that can happen is that when we apply β to x we can sometimes obtain (- x).

For example consider $x = \overline{[0; 1/a_1, -1/a_2]}$. In this case we should associate to x the element $\theta_2 u \theta_1 u$ where $\theta_1 = (z + a_1)$ and $\theta_2 = (z + a_2)$. In this way

$$\theta_1 u \theta_0 u(x) = -\overline{[0; 1/a_1, -1/a_2]} = -x.$$

This problem can be solved applying β^* to -x, where $\beta^* = \theta_n^* u...\theta_1^* u$ and $\theta_i^* = (z - a_i)$ if $\theta_i = (z + a_i)$ and $\theta_i^* = (z + a_i)$ if $\theta_i = (z - a_i)$.

In this way we can recover x, but now we can not say anymore that $\beta(x) = x$ .In fact , now we have $\beta^*\beta(x) = x$. Half of the periodic trajectories of T of period n (2.6) will require to use β and half will require to use $\beta^*\beta$.

In the first case the length of the closed geodesic s corresponding to β is (see[17] Prop.1)

$$-2 \quad log \quad \Pi_{i=1}^n \overline{[0; e_i/a_i, e_{i+1}/a_{i+1}, ..., e_n/a_n, e_1/a_1, ..., e_{i-1}/a_{i-1}]} \tag{2.10}$$

and in the second case the length of the closed geodesic s corresponding to $\beta^*\beta$ is

$$-4 \ \ log \ \ \Pi_{i=1}^{n}\overline{[0; e_i/a_i, e_{i+1}/a_{i+1}, ..., e_n/a_n, e_1/a_1, ..., e_{i-1}/a_{i-1}]}.$$

In the first case we relate the length of s to $log|T^{n'}(x)|$ (where x is considered to have period n) and in the second case to $log|T^{2n'}(x)|$ (where now x is considered to have period 2n).

In any case, in the same way as before (using proposition 3 and 4) the geodesic s associated to (2.1) of period n is determined by $\gamma_\infty = x$ and
$-1/\gamma_{-\infty} = \overline{[0; e_n/a_n, ..., e_2/a_2, e_1/a_1]}$

If x is such that $-1 < x < 0$,then we can use symetry arguments to show that we have similar conclusions.

Therefore we conclude that we can tranfer results about the number of periodic trajectories such that $log|T^{n'}(x)| < \log r$ ($l < \log r$ or in an alternative expression $e^l < r$) , where x is a periodic orbit of T with period n, to results about the number of closed trajectories with length smaller than r. In the proof of Theorem 0 we will explore this fact.

We want to show that the assymptotic of $\Pi(r)$ the number of closed geodesics γ such that $l(\gamma)$ smaller than $\log r$, is of order $\frac{r}{\log r}$.

(a) First note that any closed geodesic of the Theta Group is a closed geodesic of the Modular Group with the same lenght, therefore the assymptotic for the Theta Group is not larger than $\frac{r}{\log r}$ (this value is already known for the Modular Group) .

(b) We will show bellow that the assymptotic with r of the number of periodic orbits for T such that $T^n(x) = x$ and $log|T^{n'}(x)| < \log r$ is of order $\frac{r}{\log r}$.

Now from the reasoning above we know that the number of closed geodesics for the Theta Group with length smaller than r is larger that the number of periodic trajectories such that $T^n(x) = x$ and $log|T^{n'}(x)| < \log r$.

Now from (a) and the proof of (b), in the reasoning described bellow, we will be able to conclude that $\Pi(r)$ is of order $\frac{r}{\log r}$.

After giving the general outline of the proof of our main result we will proceed to the formal proof of Theorem 0.

Theorem 0: *If we let $\pi(r)$ denote the cardinal of the set of closed geodesics of the Theta Group such that $e^l < r$ (where l is the length of the trajectory) then*

$$\lim_{r\to\infty} \frac{\pi(r)}{\frac{r}{\log r}} = 1.$$

Proof : We want to analyze the assymptotic growth number of closed geodesic of the Surface M obtained as the quotient of the upper half-plane by the Θ Group using the reasoning presented above.

Recall that we denote by μ the unique invariant measure equivalent to the Lebesgue measure on [0,1]. This measure has infinite mass due to the singularity of the kind $\frac{1}{1-x}$ around 1. The point 1 is a fixed point with eigenvalue 1, and therefore T is not expanding. This is a quite different situation from the Modular group case.

In Lopes[9], the dynamics of a one dimensional map f with a fixed point with eigenvalue 1 is considered. Results about the assymptotic growth number of n-periodic trajectories x such that $log|f^{n'}(x)| < r$ in terms of r are derived.

In the present situation if we want to analyze the map T, we have to change some of the arguments of [9], because now the map is infinite to one, and not two to one as in [9].

The purpose of the reasoning described below, is to extend the results of [L] to the present situation.

We will be able to obtain results about the lenght spectra of M, thanks to the relation of $l(\gamma)$ and eigenvalues of periodic trajectories mentioned above.

First, in order to have a setting more close to the case considered in [9] , we will perform the change of coordinates 1 - x to T. We will obtain a new map that we will denote by f, defined from [0,1] to itself and given in analytic form in the following way:

consider for k =1,2,3,... the sets A(-1,k)= [$\frac{2k-2}{2k-1}, \frac{2k-1}{2k}$) and A (+1,k) = [$\frac{2k-1}{2k}, \frac{2k}{2k+1}$) .

On A(+1, k) , k = 1,2,3..., the map f is given by :

$$f(x) = \frac{2k - (2k+1)x}{1-x},$$

and on A(-1,k) , k= 1,2,3,... by :

$$f(x) = \frac{2 - 2k + (2k-1)x}{1-x}$$

The map f is continuous and has the graph given in fig. 1.

The sets A(-1,k) and A(+1,k) correspond by means of the change of coordinates respectively to B(-1,k) and B(+1,k), k=1,2,3... .

fig.1

Note that now 0 is a fixed point with eigenvalue 1. The analytic expression of f around 0 is given in the interval A(-1,1) = [0, 0.5) by:

$$f(x) = \frac{x}{1-x} = x + x^2 + x^3 + x^4 + ...$$

In terms of the notation used for the Manneville-Pomeau map in [9], this correspond to the case s = 1 (or $\gamma = 2$ in Theorem 1 in [9]).

We will show that we have a functional equation for the pressure as in [9], and from this equation, we can derive results about asympotic number of periodic trajectories. Before we can show this result we need to define a certain potential function g and consider the pressure associated to g.

First we need to define a partition of [0,1], and define g in each element of the partition.

Consider q a fixed real value and M(n) = [$\frac{1}{n+2}, \frac{1}{n+1}$) for n = 1,2,3,...., and also define M(0) = [0.5,1] =$A(+1,1) \cup A(-1,2) \cup A(+1,2) \cup A(-1,3) \cup A(+1,3)$.... .

Remark 5- Note that f(M(n)) = M(n-1), for n= 2,3,4,... and also, f(M(1)) = [0.5,1] = M(0).

Define g on M(n) , n =2,3,... by the constant function

$$g(x) = -q log \frac{n+2}{n}$$

On M(1) = ($\frac{1}{3}, \frac{1}{2}$) we define g as - q log(2 3 c). The constant value c will be specified later.

Now let's define g on

$$M(0) = [\frac{1}{2}, \frac{2}{3}) \cup [\frac{2}{3}, \frac{3}{4}) \cup [\frac{3}{4}, \frac{4}{5}), ...$$

Denote C(k) = [$\frac{k}{k+1}, \frac{k+1}{k+2}$) k=1,2,3,... , and define g as

$$g(x) = -qlog(|f^{'}(x)|D)$$

The value D will be specified later.

Remark 6- It follows from remark 5 that $f^n(M(n)) = [0.5, 1]$ and f^n is injective on M(n). In the present situation due to the existence of the fixed point with eigenvalue 1, we can say (see [26] prop. 2.4) that for all x in M(n), and independent of n, we have that $|f^{n^{'}}(x)|^{-1}$ has the same order as the size of M(n), that is there exist c_1, c_2 such that for all n=1,2,...

$$c_1 \frac{1}{(n+1)(n+2)} < |f^{n^{'}}(x)|^{-1} < c_2 \frac{1}{(n+1)(n+2)}$$

for $x \in M(n)$.

The reason for all these remarks is the fact that will be explained soon that the potential g, when $q = 1$, is a "constant by part version" in the interval [0, 0.5] of the potential - log $|f^{'}(x)|$.

As $\cup_{n=0}^{\infty} M(n) = [0, 1]$, g is now already defined on the whole unit interval.

Denote by C the space of real continuous functions on [0, 1] with the supremum norm ||.||. For a fixed potential g, we define the Ruelle-Perron-Frobenius operator L_g (RPF operator for short) associated to g, the operator on C, such that for s $\in$ C, $L_g(s)(x) = \sum_{y \in f^{-1}(x)} e^{g(y)} s(y)$.

It is easy to see that

$$L_g^n(s)(x) = \sum_{y \in f^{-n}(x)} e^{\sum_{i=0}^{m-1} s(f^i(y))} s(y).$$

We refer the reader to [16] for general properties of the RPF operator.

The pressure associated with g is by definition the limit

$$P(g) = \lim_{n \to \infty} \frac{1}{n} log \;\; L_g^n(1)(x),$$

where we choose x in (0, 1) , and were we apply the RPF operator to the constant function equal 1. In our situation the above limit exist, and this follows from a straightforward extension of the results of Hofbauer[5] and Lopes([9], proposition in section 1).

Recall that the potentials g also depend on a value q. That is, for each q, we have a different g. We are interested on the dependence of the pressure on q. Therefore we will use the notation $P(q), q \in R$, for the P(g), $g \in$ C, associated to the value q .

In the same way as in Hofbauer[5] or Lopes [9], using the Schauder-Tychonov theorem there exist a probability measure ν, that is an eigenvalue for the dual of the operator L_g. As in [5], first we will compute the values $\nu(M(n)), n \in N$.

Denote λ the eigenvalue associated to ν. Now using the fact that λ is the eigenvalue associated to ν, for the dual L_g^* of L_g, we have

$$\lambda\nu(M(0)) = \lambda \int I_{M_0}(x)d\nu(x) = \int L_g(I_{M_0})(x)d\nu(x) = \int \sum_{y\in f^{-1}(x)} e^{g(y)} I_{M_0}(y)d\nu(x)$$

Denote by B_1 the last integral above. This value is of order

$$B_1 \approx D^{-q} \sum_{k=1}^{\infty} ((k+1)(k+2))^{-q}$$

We will denote by B_2 the value $B_1 D^q$.

Now we proceed by induction as in Hofbauer [5], but in order to make the argument more clear let's compute ν (M(1)).

Using the same reasoning as before, we have

$$\lambda\nu(M(1)) = \lambda \int I_{M_1}(x)d\nu(x) = \int L_g(I_{M_1})(x)d\nu(x) = \int \sum_{y\in f^{-1}(x)} e^{g(y)} I_{M_1}(y)d\nu(x) =$$

$$\nu(M(0))(23c)^{-q} = B_1\lambda^{-1}(23)^{-q}c^{-q}.$$

Now by induction, we have that:

$$\lambda\nu(M(n)) = B_1\lambda^{-n}((n+1)(n+2))^{-q}c^{-q}.$$

In conclusion $\nu(M(n)) = B_1\lambda^{-n-1}((n+1)(n+2))^{-q}c^{-q}$.

As ν is a probability, we have that

$$1 = \sum_{n\in N} \nu(M(n)) = B_1(\lambda^{-1} + \sum_{n=1}^{\infty} \lambda^{-(n+1)}((n+1)(n+2))^{-q})c^{-q}.$$

Now we use the fact [5] that $\lambda = e^{P(g)}$, and then we have the functional equation :

$$c^q B_1^{-1} - e^{-P(q)} = c^q D^q B_2^{-1} - e^{-P(q)} = \sum_{n=1}^{\infty} \frac{e^{-nP(q)}}{((n+1)(n+2))^q}. \tag{2.11}$$

Let D be chosen such that :

$$cDB_2^{-1} - 1 = \sum_{n=1}^{\infty} ((n+1)(n+2))^{-1}$$

In this way P(1) = 0 .

The above expression (2.11) is very close to the expression

$$\zeta(2)^t = \sum_{n=1}^{\infty} \frac{e^{-nP(t)}}{n^{2t}}$$

obtained in [9]. Using same techniques as in ([9],Theorem1), one can show that for $q > 0$ and q close to zero $P(q) \approx K(1-q)/log(1/(1-q))$, where K is a constant

Now using the same proof as in ([9],paragraph 2), one can show results about the growth number of periodic orbits under some restrictions related to the mean value of g (with q=1) in the periodic orbit (we will not repeat the proof here and we refer the reader to [9] for the proof).

In order to relate the above results to the lenghts of closed geodesics , we have to relate g and $-log|f^{'}(x)|$. We will explain now more carefully the reasons for the need of analyzing such relationship between g and - log $|f^{'}(x)|$.

We will denote by p(t) (it is different from P(q)) the pressure associated to the value $t \in \mathbf{R}$, the expression :

$$p(t) = \lim_{n\to\infty} \frac{1}{n} log \sum_{y \in f^{-n}(x)} |f^{n^{'}}(y)|^{-t} =$$

$$\lim_{n\to\infty} \frac{1}{n} log \sum_{y \in f^{-n}(x)} e^{-t \sum_{i=1}^{n-1} log|f^{'}(f^i(y))|} \quad tag2.12,$$

where $x \in$ (0,1).

Note the resemblance of the last expression with $P(q)$.

Claim- It is also possible to write p(t) as

$$p(t) = \lim_{n\to\infty} \frac{1}{n} log \sum_{y \quad such \quad that \quad f^n(y)=y} |f^{n^{'}}(y)|^{-t} \tag{2.13}$$

The above mentioned claim, in the situation we are considering here, follows from straightforward changes in the argument for the expanding case.

Using Tauberian type of results, one can derive information about the growth number of n-periodic trajectories y such that log $|f^{n^{'}}(x)| < r$, if one knows the local behavior of $p(t)$ (as defined in (2.13)) around t = 0. It follows from the reasoning in [9] that this number growths like $\frac{r}{logr}$

In this way, finally, we are able to derive information about the growth of the number of closed geodesics γ with length $l(\gamma)$ smaller than r, in terms of r.

Now we only have to show the relation of $P(q)$ and $p(t)$. We will consider a certain value x fixed in all our considerations. For each n $\in N$ and y $\in f^{-n}(x)$ the orbit $y, f(y), f^2(y), ..., f^{n-1}(y)$ can be decomposed in strings that are totally in M(0)= [0.5,1] and others strings that are totally in A(-1,1)= [0,0.5].

We will use the same notation as in proposition in paragraph 1 in [9]. One can write a real number y in some kind of binary expansion $\{a_0, a_1, a_2, ...\}$ according to the rule $a_i = 0 \ \ if \ \ f^i(y) \ \in \ [0.5,1], \ \ and \ \ a_i = 1 \ \ if \ \ f^i(y) \ \in \ [0,0.5), \ \ i = 0,1,2,..$ .This is ussualy called the Markov Partition expansion related to f. Therefore $y \in f^{-n}(x)$ has the first n elements of its expansion as

$$\{a_0, a_1, ..., a_{n-1}\} = \{\underbrace{0,0,...,0,0}_{n_0}, \underbrace{1,1,...,1,1}_{m_0}, ..., \underbrace{1,1,..,1,1}_{n_r}, \underbrace{0,0,...,0,0}_{m_r}\} \quad (2.14).$$

We will denote the cardinal of the first string os ones by n_0, the first strings of zeroes by m_0, the second strings of ones by n_1, the second string of zeroes by m_1, and so on until the last string os ones will be n_r and the last string of zeroes will be m_r.

We could also have that the above expression of y in (2.14) is in such way that begins with zeroes. We can choose x in [0.5,1] in such way we can be sure that the last string is certainly constituted only of of zeroes. In order not to complicate the notation, let's suppose we have that y is under the above case (2.4).

Remark 7- Each time a certain string of ones appear with size n_i , i=0,1,2,..r , then we know that $x_i \doteq f^{n_0+m_0+n_1+m_1+m_2+...+n_{i-1}+m_{i-1}}(y) \in M(n_i)$ (see Rermark 5). The derivative $|f^{n_i'}(x_i)|$ on this point $x_i \in M(n_i)$ will be of order $(n_i+2)(n_i+1)$ (see Remark 6). That is, there exist c_1, c_2 such that, independent of n_i we have

$$c_1 f^{n_i'}(x_i) < e^{-\sum_{j=0}^{n_i-1} g(f^j(x_i))} = ((n_i+2)(n_i+1)) < c_2 f^{n_i'}(x_i). \tag{2.15}$$

Here we are considering g with $q = 1$.

Remark 8- In the strings of zeroes the two potentials g and $-qlog|f'(x)|$ are the same .

One can analyze the pressure p(t) just analyzing the potentials of the form g(x) as we defined before. One have to consider the constant c in the definition of g, the two possible values c_1^{-1} and c_2^{-1} (see Remark 7 and 8). Then use the well known fact that if $\Phi < \Psi$, then $P(\Phi) < P(\Psi)$ (see[27]) and (2.15). Therefore the results obtained concerning the kind of singularity of P(q) around $q = 0$, will be of the same kind as of p(t) for t=0 .

Therefore we have a good control of the singularity of p(t), and in this way we can obtain the growth number of T-periodic trajectories x of period n (such that for x in this periodic trajectory we have $log|T^{n'}(x)| < r$) in terms of r (see[9]). Finnaly, using the relation about size of geodesics and periodic trajectories of T we obtain our final result about the growth number of closed geodesics subject to have lenght smaller than r in terms of r. This number growths like $\frac{r}{\log r}$ as we said before.

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