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LARGE DEVIATIONS AND AUBRY-MATHER MEASURES SUPPORTED IN NONHYPERBOLIC CLOSED GEODESICS

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Abstract. We obtain a large deviation function for the stationary measures of twisted Brownian motions associated to the Lagrangians $L_\lambda(p,v) = \frac{1}{2} g_p(v,v) - \lambda\omega_p(v)$, where g is a C^∞ Riemannian metric in a compact surface (M,g) with nonpositive curvature, ω is a closed 1-form such that the Aubry-Mather measure of the Lagrangian $L(p,v) = \frac{1}{2} g_p(v,v) - \omega_p(v)$ has support in a unique closed geodesic γ; and the curvature is negative at every point of M but at the points of γ where it is zero. We also assume that the Aubry set is equal to the Mather set. The large deviation function is of polynomial type, the power of the polynomial function depends on the way the curvature goes to zero in a neighborhood of γ. This results has interesting counterparts in one-dimensional dynamics with indifferent fixed points and convex billiards with flat points in the boundary of the billiard. A previous estimate by N. Anantharaman of the large deviation function in terms of the Peierl's barrier of the Aubry-Mather measure is crucial for our result.

1. **Introduction.** Large deviations of families of measures are important in many physical applications where one would like to estimate observables at exceptional conditions using "physical" measures in the phase space. A typical setting in dynamical systems would be to estimate an invariant measure supported in a singular set with respect to the Lebesgue measure (for instance, a measure minimizing some variational principle) in terms of a family of absolutely continuous measures containing a sequence converging to this singular measure. There is a vast literature on the subject in mathematical physics, assuming in most of the cases a hyperbolicity condition for the dynamical system and/or dimension one for the configuration space. The subject of the present article is to study large deviations in the nonhyperbolic setting and higher dimensions with the help of weak KAM theory (see [1] [2] [3] [4] for related results). We will consider a family of surfaces initially described in [11] which have negative curvature everywhere up to the points along a closed geodesic where the curvature vanishes. Since the hyperbolicity of orbits

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arises from non-parallel Jacobi fields, and the curvature vanishes along γ, the orbit corresponding to γ is not hyperbolic.

As a motivation for studying such problem we point out that there classes of problems (which do not present full hyperbolicity) where a special orbit plays an important role. For transformations with a fixed indifferent point (like the Maneville-Pomeau map, see [35], [33]) this point is associated with the phenomena of phase transition and polynomial decay of correlation. For special billiards, where a cusp point can make a trajectory stay for arbitrary long time close to this point, this is also associated with polynomial decay of correlation [27]. The careful analysis of the evolution on time of these special trajectories determines singular characteristics of these special cases of dynamics. Here we analyze the phenomena of large deviation associated to a special closed geodesic. The family of probabilities indexed by λ which is considered here is associated to critical solutions of the Evans action (see [17] [24]), and, the limit, when $\lambda \to \infty$, is usually known as the semiclassical limit (see [3] [30]).

One of the important results we get is the link between Peierls barrier and Busemann functions, which allow us to make accurate analytic estimates of the former from the geometry of the surface closed to the vanishing curvature geodesic γ. We point out that our work considers a pathological case, and some kind of analytical control over the lack of hyperbolicity is essential to get meaningful large deviation estimates. This is similar to the case of the investigation of the ergodic properties of dynamical systems with fixed indifferent points [35] [33] or billiards with cusps [27].

We denote by d the distance on the manifold induced by the Riemannian metric.

From these estimations we get the main result of the paper which is the following:

Theorem 1. Let (M, g) be a compact surface with $K \leq 0$ such that:

1. There is a closed geodesic γ where $K \equiv 0$ whose orbit supports the (unique) Aubry-Mather measure of $L(p, v) = \frac{1}{2} g_p(v, v) - \omega_p(v)$. The Aubry set and the Mather set of the Aubry-Mather measure coincide.
2. $K < 0$ in the complement of γ.
3. There exists $m > 0$ such that for every geodesic $\beta : (-\epsilon, -\epsilon) \longrightarrow M$ perpendicular to γ at $\beta(0) = \beta \cap \gamma$ we have that m is the least integer where $\frac{\partial^m}{dt^m} K(\beta(t))|_{t=0} \neq 0$.

$\mu_{\lambda\omega}$ be the stationary measure for the $\lambda\omega$-twisted Brownian motion, $\lambda > 0$. Then $\exists$ $D > 0$ such that $\forall$ open ball $A \subset M$ (which does not intersect the closed geodesic γ),

$$-(1/D) \inf_{x \in A} d(x, \gamma)^{2+\frac{m}{2}} \leq \lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln(\mu_{\lambda\omega}(A))$$

and

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} \ln(\mu_{\lambda\omega}(\bar{A})) \leq -D \inf_{x \in \bar{A}} d(x, \gamma)^{2+\frac{m}{2}}.$$

To a given form ω, corresponds by duality in Mather Theory, a homology class $[h]$ (see for instance [12] [26] [29]). In our case this $[h]$ is the homology of the curve γ. A brief account of some basic results of Brownian motion and Aubry-Mather theory is made in the first three sections of the paper.

Several precise results about large deviations for hyperbolic dynamical systems are known [2] [4]. In the case of a probability supported on a proper invariant hyperbolic set, the large deviation function is typically a linear function in the distance to the support of the measure; its slope depending on the hyperbolicity of the system (this in the context of Lagrangian dynamics on the torus see also [1]). The thermodynamical formalism is the main tool used to get large deviation formulae in the presence of hyperbolicity.

Let us give an outline of the proof of Theorem 1. N. Anantharaman in [1] [2] and [3] considers the family of measures mentioned in Theorem 1: stationary measures for the twisted Brownian motions arising from twisting a Riemannian Laplacian by multiples of a closed one form ω. By the general theory of harmonic analysis, such measures are absolutely continuous probabilities on the configuration space (the manifold M). These probabilities approximate the Aubry-Mather measure associated to the Lagrangian given by the kinetic energy of a Riemannian metric plus the closed form ω. Under certain assumptions (uniqueness of the Aubry-Mather measure), N. Anantharaman also considers a large deviation principle for this family, and exhibits a deviation function which is given by the Peierl's barrier. This nice geometric result tells roughly that the deviation function at a point p depends on how far from being minimizers of the Lagrangian action are closed loops based at p. We consider here an special example of this setting where one can have a sharp control of the Peierls barrier in a neighborhood of a certain closed, non-hyperbolic geodesic (the curve γ in Theorem 1). In [4], among other things, it is analyzed a similar problem for a hyperbolic periodic trajectory of the geodesic flow. General references for the Aubry-Mather theory are [29] [26] [12] [18].

Our estimate of the Peierls barrier comes from sharp bounds for the Busemann function (see [7]) of the Riemmanian metric (M, g) associated to lifts of the geodesic γ in the universal covering. These bounds are obtained by comparing the metric (M, g) in a neighborhood of γ with an annulus of revolution (under the assumptions of Theorem 1). Notice that the Peierls barrier is defined in terms of the weak KAM solutions of the considered Lagrangian (see Sections 4 and subsequent). Therefore, one of the main issues of the proof of Theorem 1 is to relate Busemann functions and Peierls barrier. There is a natural generalization of Busemann functions to convex, superlinear Lagrangians (see section 4.9 of [12] for instance) in the context of weak KAM theory. Such generalized Busemann functions are used to exhibit fixed points of the Lax-Oleinik operator (backward and forward) with infinite critical value (in [12] [10] these functions are called Busemann weak KAM solutions). We would like to point out that the Busemann functions we use are just the Riemannian ones, we do not need to apply this general notion in our argument.

Let us make some comments about the assumption in Theorem 1 concerning the equivalence between the Aubry-Mather set and the Mather set. This condition is of topological nature, as observed in [28] for surfaces. Particularly important for us are the results of section 4 in [28] where it is considered the case where the Mather set is a single periodic orbit: if γ separates M (see case 1.2) then the two sets are equal. In fact, the coincidence of the Aubry-Mather set and the Mather set is generic in homology (see Theorem 3 in [28]).

Let us finish the Introduction with some further remarks and problems. First of all, notice that Theorem 1 describes how a family of absolutely continuous measures in the **configuration** space approaches a Dirac measure concentrated in the so-called projected Mather set (see Section 3 for the definition). The terminology Mather measure may have two meanings: the projected Mather measure (with support on the manifold) and the one in the tangent bundle. So Theorem 1 is a large deviation for the projected Mather measure. A more recent stream of ideas give us some hints of how we can obtain a L. D. P. for the Mather measure on the tangent bundle. The so-called entropy penalized method presented in [22] shows other ways to obtain approximations of the Mather measure on the tangent bundle by absolutely continuous probabilities. Under some conditions ($M = T^n$, convex superlinear Lagrangians, uniqueness of the Aubry-Mather measure), a large deviation principle for such procedure is described in [23]. Another way to get a L. D. P. for the Mather measure on the tangent bundle is via semi-classical limit of Wigner functions [24]. We believe that these procedures can also be extended to our setting.

Finally, let us observe that the geodesic flows considered here are particular cases of expansive geodesic flows in manifolds with non-positive curvature [15] [5] [31] [32] [11] [25]. The topological dynamics of expansive geodesic flows in manifolds without conjugate points is well understood, it is about the same Anosov topological dynamics. Moreover, the universal covering is a Gromov hyperbolic space. However, the ergodic theory of expansive geodesic flows with non-positive curvature is almost the same ergodic theory of rank one manifolds. The ergodicity of rank one manifolds is a very hard open problem, as well as for expansive geodesic flows with non-positive curvature. The family of examples considered in Theorem 1 are perhaps the simplest non-Anosov geodesic flows of rank one, they are ergodic an even Bernoulli by Pesin theory. So we could ask if it is possible to give a sharper description of the ergodic properties of invariant measures in this case (Liouville measure, Gibbs measures). In [21] the Holder class of the horocycle flow for the metrics considered in Theorem 1 is presented. We believe that from these estimates, and some of the ideas described in the present paper, one can detect the so called concentration of measure phenomena, or even calculate the decay of correlation of the Liouville measure. This will be the purpose of a future work.

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We refer the reader to [14] for general properties of large deviations.

2. Preliminaries about diffusion and Brownian motion.

2.1. Diffusion and Brownian motion in a Riemannian manifold.

Let (M, g) be a compact C^∞ Riemannian manifold, let $(\tilde{M}, \tilde{g})$ be its universal covering endowed with the pullback of g by the covering map. Let Δ be the Laplace operator of (M, g), $\tilde{\Delta}$ be the lift of the Laplace operator to $(\tilde{M}, \tilde{g})$.

The operator Δ gives rise to a stochastic process in (M, g), the Brownian motion. It is linked to Δ in the following way: given $x \in M$, let $C(\mathbb{R}, M)$ be the space of continuous paths, $\mathbb{P}_x$ be the Wiener measure, let $X_t : \gamma \longrightarrow \gamma(t)$ be a realization of the Brownian motion starting at x. Then

$$P^t(f(x)) = e^{\frac{1}{2}t\Delta}(f(x)) = \mathbb{E}_x(f(X_t)).$$

for every C^∞ function $f : M \longrightarrow \mathbb{R}$. The operator P^t is the **Heat semigroup** of (M, g), i.e., the solution of the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$ in (M, g).

We refer the reader to [16] or [34] for general results on Brownian motion and diffusions.

2.2. **Twisting the Laplacian by closed 1-forms.** We refer the reader to [1] for general results on twisted Laplacians.

Let ω be a C^∞ closed 1-form in M. The Laplacian **twisted** by the 1-form ω is

$$\begin{aligned}\Delta_\omega(f(x)) &= \tilde{\Delta}_{\tilde{x}}(\tilde{f}(\tilde{x})) \\ &= e^{-\int_p^{\tilde{x}}\tilde{\omega}}\tilde{\Delta}(e^{\int_p^{\tilde{x}}\tilde{\omega}})\tilde{f}(x).\end{aligned}$$

where $p \in \tilde{M}$ is a base point, $\tilde{\omega}$ is any lift of ω to $\tilde{M}$, f is a C^∞ function in M and $\tilde{f}$ is any lift of f. Twisted Laplace operators are used to study asymptotic properties of the number of closed orbits in a fixed homology class of geodesic flows of negative curvature (via Selberg's trace formula) taking $Re(\omega) = 0$. The semigroup

$$P_\omega^t = e^{\frac{1}{2}\Delta_\omega t}$$

gives the solution of the **twisted** heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_\omega u$ for the **twisted Lagrangian** $L_\omega(p,v) = \frac{1}{2}g_p(v,v) - \omega(v)$.

The twisted Laplacian appears in a natural way when we want to consider the Schrodinger operator for a Mechanical Lagrangian to which we add a closed form (the magnetic term).

2.3. **Stationary probability for the twisted Brownian motion.** The operator P_ω^t acts on the space of measures by $(P_\omega^t)^*$:

$$\int_M f d(P_\omega^t)^*\mu) = \int_M P_\omega^t f d\mu.$$

The action preserves positive measures and there exist $\Lambda(\omega)$ and a measure μ_ω such that

$$(P_\omega^t)^*\mu_\omega = e^{\Lambda(\omega)t}\mu_\omega.$$

Let $h_\omega(x) = \int_M K_{-\omega}^1(y,x)d\mu_{-\omega}$, where K_ω^t is the Kernel of P_ω^t.

Theorem 2.1. *There exists a (unique up to normalization) measure $\nu_\omega = f_\omega dx$ that is a fixed point of the twisted Brownian motion*

$$Q^t f(x) = e^{-t\Lambda(\omega)}h_\omega(x)^{-1}P_\omega^t(h_{-\omega}f)(x).$$

The measure ν_ω will be called the stationary measure of the twisted Brownian motion.

3. Preliminaries about Aubry-Mather measures.

3.1. **Aubry-Mather measures.** Consider M a compact C^∞ Riemannian manifold. Let $L : TM \to \mathbb{R}$ be a C^∞ convex, superlinear Lagrangian. The action of L in an absolutely continuous curve $c : I \to M$ is $A_L(c) = \int_I L(c(t), c'(t))dt$.

Let $\mathscr{M}(L)$ be the set of invariant probability measures of the E-L flow of L. The **action** of L in $\mathscr{M}(L)$ is defined by

$$A_L(\mu) = \int L d\mu.$$

The **homology class** (Mather, Mañé) $\rho(\mu)$ of the measure μ is given by

$$< \rho(\mu), \omega >= \int \omega d\mu,$$

where ω is a closed 1-form. (Recall that the homology group $H_1(M,\mathbb{R})$ is the dual of the cohomology group $H^1(M,\mathbb{R})$).

A measure $\mu \in \mathscr{M}(L)$ is called **minimizing in its homology class** if

$$A_L(\mu) = \inf\{A_L(\nu), \rho(\nu) = \rho(\mu)\}.$$

An **Aubry-Mather measure** μ is defined by

$$A_L(\mu) = \inf\{A_L(\nu), \nu \in \mathscr{M}(L)\}.$$

The union of the supports of all Aubry-Mather measures is called the Mather set for L.

Theorem 3.1. *(Mather, Mañé): The support of a minimizing measure is a Lipschitz graph over an invariant set of global minimizers of the action.*

3.2. **Critical energy values and minimizing measures.** Both globally minimizing measures in homology and Aubry-Mather measures arise as minimum (and hence critical) points of the Lagrangian action on holonomic invariant measures. Moreover, the support of an ergodic invariant measure has constant energy, and the support of globally minimizing measures are minimizing orbits of the Euler-Lagrange flow by Theorem 3.1. So it is natural to expect that the energy levels of the supports of Aubry-Mather measures have critical properties somehow.

Definition 3.1. *The critical value $c(L)$ of L (see section 2.1 [12]) is defined by*

$$c(L) = \sup_{k\in\mathbb{R}}\{A_{L+k}(\beta) < 0 \text{ for some closed curve } \beta\}.$$

An holonomic invariant measure μ is called **globally minimizing** if $A_L(\mu) = -c(L)$.

Definition 3.2. *The strict critical value $c_0(L)$ of L (see page 798 [13]) is given by*

$$c_0(L) = \min_{\alpha\in H^1(M,\mathbb{R})} c(L-\alpha) = -\beta(0),$$

where $\beta : H_1(M,\mathbb{R}) \longrightarrow \mathbb{R}$ is

$$\beta(h) = \min_{\rho(\mu)=h} A_L(\mu).$$

Notice that $c_0(L) \geq c(L)$. The strict critical level is the relevant one regarding Aubry-Mather measures.

Theorem 3.2. [8]*: The support of an Aubry-Mather measure is contained in the energy level $E = c_0(L)$.*

There are many equivalent geometric characterizations of the strict critical level, it is, for instance, the infimum of the energy levels containing a globally minimizing orbit of the Lagrangian action with nontrivial (real) homology class. This is why the strict critical value is the critical value of the lift of the Lagrangian action to the abelian cover of the manifold (see [18], for instance).

Let us give some examples. The critical value of geodesic flows is clearly 0 while the strict critical value is nonzero, if and only if, the first homology group of the manifold is nontrivial. The critical value of a mechanical Lagrangian is the opposite value of the maximum of the potential, and the Euler-Lagrange flow in energy levels above this value can be reparametrized to give the geodesic flow of a Riemannian metric (Maupertuis' principle). The strict critical value of $L(p,v) = \frac{1}{2}g_p(v,v) - \omega_p(v)$, where ω is a closed 1-form, is $\frac{1}{2} \parallel \omega \parallel_s^2$, where $\parallel \omega \parallel_s$ is the stable norm. In particular, under our hypothesis, the Aubry-Mather measure of $L(p,v) = \frac{1}{2}g_p(v,v) - \omega_p(v)$ is supported in the closed orbit $(\gamma(t),\gamma'(t))$, $t \in [0, Per(\gamma)]$, and the form ω is dual to the homology class of γ. So the stable norm of ω is just the period of γ.

4. Large deviations and weak KAM theory.

4.1. Large deviations of stationary measures and Peierl's barrier.

In this section we state the main tool we use to obtain the deviation function for the stationary measures of Brownian motions twisted by multiples of a close 1-form. Let us start with some basic analytic definitions in the context of weak KAM theory. Our main references are [18] [12]. Through the section, M will be a compact C^∞ manifold, and $L : TM \longrightarrow \mathbb{R}$ will be a C^∞ convex, superlinear Lagrangian.

Definition 4.1. The Lax-Oleinik operators T_t^-, T_t^+. *Given a continuous function $f : M \to (-\infty, \infty)$, and $t > 0$, define the function $T_t^-(f) : M \to \mathbb{R}$ by*

$$T_t^-(f)(x) = \inf_\gamma [f(\gamma(0)) + \int_0^t L(\gamma(t), \gamma'(t))dt],$$

where $\gamma : [0, t] \to M$ is an absolutely continuous curve with $\gamma(t) = x$. Let

$$T_t^+(f)(x) = \sup_\gamma [f(\gamma(t)) - \int_0^t L(\gamma(t), \gamma'(t))dt],$$

where $\gamma : [0, t] \to M$ is an absolutely continuous curve with $\gamma(0) = x$.

The Lax-Oleinik operators T_t^- form a continuous time semigroup family of operators, as well as the operators T_t^+. They enjoy very nice properties (see [19]), in particular, the family of $T_t^- - c_0(L)t$ has a fixed point u^- which is a viscosity solution of the Hamilton-Jacobi equation (see Definition 7.2.3 and Proposition 7.2.7 in [18] and also [12]).

It is also true that u^- is a Lipschitz function (see Theorem 4.4.6 and Corollary 4.4.13 [18]), and Lebesgue almost everywhere we have that

$$H(x, d_x u^-) = c_0(L).$$

The function u^- is is differentiable along the projection $\mathscr{M}_0$ of the Mather set. Analogously, the family of operators $T_t^+ + c_0(L)t$ has a fixed point u^+ which is Lipschitz and a weak (in the above sense) solution of a Hamilton-Jacobi equation (the so called conjugated Hamilton-Jacobi equation) [18]. Let S^- be the set of fixed points of $T_t^- - c_0(L)t$, and let S^+ be the set of fixed points of $T_t^+ - c_0(L)t$.

Definition 4.2. *A pair of functions $u^- \in S^-$, $u^+ \in S^+$ is called a conjugate pair if $u^-(x) = u^+(x)$ for every $x \in \mathscr{M}_0$.*

According to Fathi [19] [18], for each weak solution u^- of the Hamilton-Jacobi equation there exists a unique solution u^+ such that u^-, $u+$ form a conjugate pair. We have that the differences $u^- - u^+$ of conjugate pairs are always nonnegative, and they vanish at the projected Mather set. One of the most interesting questions in weak KAM theory is wether the set of zeroes of the differences of conjugate pairs coincides with the projected Mather set. In this case, we can characterize the projected Mather set (and hence the set of global minimizers of the action) as the set of *true* critical points of the difference of two C^1-*smooth* sub-solutions of the Hamilton-Jacobi equation [20]. This property is very useful for applications, and motivates the key idea of the proof of our main theorem.

So it looks very tempting to try to characterize analytically the projected Mañé set in terms of the differences of conjugate pairs. However, we have to be careful in this point.

Definition 4.3. *The second Peierl's barrier is*

$$P(x,x) = \inf\{u^+(x) - u^-(x)\}$$

where the infimum is taken over all conjugate pairs u^+, u^-.

Definition 4.4. *As in* [18]*, we denote the projected Aubry set by* $\pi(\hat{\Sigma}(L))$*, the intersection of the set of zeroes of all conjugate pairs of the Hamilton-Jacobi equation.*

The projected Aubry set is the canonical projection of certain orbits of the Euler-Lagrange flow (see [18] for instance), whose union is called the Aubry set $\hat{\Sigma}(L)$. Moreover, the canonical projection $\pi : \hat{\Sigma}(L) \longrightarrow \pi(\hat{\Sigma}(L))$ is a Lipschitz homeomorphism (see [18], this is a version of the well known Mather's graph Theorem).

Lemma 4.1. *The projected Aubry set contains* M_0.

However, the inclusion of M_0 in $\pi(\hat{\Sigma}(L))$ might be strict: this is often the case when there exist globally minimizing connections between different non-wandering components of the Mather set. The projected Aubry set has the analytic characterization we would like to have for the Mather set (see [18], [19], [2] [3] and section 3.7 in [12]).

Proposition 4.2. *The second Peierl's barrier* $P(x,x)$ *is zero, if and only if,* x *is in the projected Aubry set.*

Finally, we are able now to state the main result of the section, which is one of the main tools used in the proof of Theorem 1.

Theorem 4.3. [2] [3] *Let* (M,g) *be a compact surface with* $K \leq 0$*, let* γ, ω *be as in the assumptions of the main theorem. Then, the measures* $\mu_{\lambda\omega}$ *satisfy the following large deviation type formulae*

$$\lim_{\lambda\to+\infty} \frac{1}{\lambda} ln(\mu_{\lambda\omega}(A)) \leq -\inf_{x\in A} P(x,x),$$

for every closed set $A \in M$*, and*

$$\lim_{\lambda\to+\infty} \frac{1}{\lambda} ln(\mu_{\lambda\omega}(B)) \geq -\inf_{x\in B} P(x,x),$$

for every open set B.

In other words, we know that the measures $\mu_{\lambda\omega}(A)$ tend to zero as $\lambda \to \infty$; so, the Peierl's barrier gives an estimate of the logarithmic rate of convergence. The logarithmic rates became worst as the sets approach the Aubry set, and, if the closure of an open set meets the Aubry set, then the above rates are just 0. This result can be interpreted in the present situation as a concentration of the stationary measures around a Dirac measure in the space of continuous paths which assigns measure one to the closed geodesic in the Mather set, and zero to any set not containing this geodesic We stated Theorem 4.3 suited to our purposes, as it is in [3]: we are assuming that the Aubry set and the Mather set coincide, and that there is a unique Mather measure. A more general result in [2] grants that only the first one of the inequalities in Theorem 4.3 holds.

5. **Busemann functions.**

Manifolds with nonpositive curvature are special examples of manifolds without conjugate points (the exponential map at every point is nonsingular). So every geodesic in $\tilde{M}$ is globally minimizing, and the convexity of the metric yields the existence of two lagrangian, invariant foliations whose leaves are locally graphs of the canonical projection. A well known way to define such foliations in terms of closed 1-forms is through the so-called *Busemann functions*: given $\theta = (p, v) \in T_1\tilde{M}$ the *Busemann function* $b^\theta : \tilde{M} \to \mathbb{R}$ associated to θ is defined by

$$b^\theta(x) = \lim_{t\to+\infty}(\tilde{d}(x, \gamma_\theta(t)) - t),$$

where $\tilde{d}$ is the metric on $\tilde{M}$. From now on we will also denote such distance by d.

The level sets of b^θ are the *horospheres* $H_\theta(t)$ where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$ (notice that $\gamma_\theta(t)$ intersects each level set of b^θ perpendicularly at only one point). The next lemma summarizes some basic properties of horospheres (which can be found in [31] [32] [15], for instance).

Lemma 5.1. *Let (M, g) be a compact C^∞ manifold without conjugate points.*

1. *b^θ is a C^1 function for every θ. If (M, g) has nonpositive curvature b^θ is a C^2 function for every θ.*
2. *The gradient ∇b^θ has norm equal to one at every point.*
3. *Every horosphere is a C^{1+K}, embedded submanifold of dimension $n-1$ (C^{1+K} means K-Lipschitz normal vector field), where K is a constant depending on curvature bounds. If the curvature of (M, g) is nonpositive each horosphere is a C^2 submanifold.*
4. *The orbits of the integral flow of $-\nabla b^\theta$, $\psi^\theta_t : \tilde{M} \longrightarrow \tilde{M}$, are geodesics which are everywhere perpendicular to the horospheres H_θ. In particular, the geodesic γ_θ is an orbit of this flow and we have that*
$$\psi^\theta_t(H_\theta(s)) \;=\; H_\theta(s + t)$$
for every $t, s \in R$.

A geodesic β is *asymptotic* to a geodesic γ in $\tilde{M}$ if there exists a constant $C > 0$ such that $d(\beta(t), \gamma(t)) \leq C$ for every $t \geq 0$. Nonpositive curvature implies that every two integral orbits of $-\nabla b^\theta$ are asymptotic. Item (2) tells us that Busemann functions are special, exact solutions of the Hamilton-Jacobi equation

$$\tilde{H}(p, d_p b^\theta) = 1,$$

where $\tilde{H} : T^*\tilde{M} \to \mathbb{R}$ is just the pullback $\tilde{g}$ of the metric g by the covering map, $\tilde{H}(p, v) = \frac{1}{2}\tilde{g}_p(v, v)$. Moreover, if (M, g) is a compact surface without conjugate points and $\gamma_\theta \subset \tilde{M}$ is a lift of a closed geodesic, the set of points $x \in \tilde{M}$ where $b^\theta(x)+b^{-\theta}(x) = 0$ is just the set of lifts of closed geodesics homotopic to $\pi(\gamma_\theta)$ which are axes of T_{γ_θ}. This elementary observation is crucial for the section: when $\pi(\gamma_\theta)$ is unique in its homotopy class the functions b^θ, $b^{-\theta}$ behave like a pair of conjugate solutions of the Lax-Oleinik operator. Namely, $b^\theta + b^{-\theta}$ takes its minimum value zero just at the points of γ_θ, and it is positive everywhere else.

6. **Peierl's barrier and Busemann functions.**

Definition 6.1. *The Peierl's barrier is the function $h : M \times M \longrightarrow \mathbb{R}$ given by*

$$h(x, y) = \lim_{T\to\infty} inf_{\alpha\in C_T(x,y)}\{A_{L+c_0(L)}(\alpha)\},$$

where $C_T(x,y)$ is the set of C^1 curves $\alpha : [0,T] \longrightarrow M$ such that $\alpha(0) = x$, $\alpha(T) = y$.

The above definition given by R. Mané [26] is based in an analogous definition due to J. Mather [29]. We introduce the Peierl's barrier by two reasons. First of all, it is naturally connected to the second Peierl's barrier defined in the previous section; and secondly, the Peierl's barrier is defined in a more geometric way than the second Peierl's barrier; it's actually really close to Busemann functions. The purpose of the section is to describe in detail the major issues. Let us begin with some basic properties of the Peierl's barrier (see [18]).

Lemma 6.1. *Let M be a C^∞ compact manifold, and L be a C^2 Lagrangian that is strictly convex and superlinear in each tangent space T_pM. Let $h(x,y)$ be the Peierl's barrier of L. Then*

1. *The function $h(x,y)$ is Lipschitz continuous.*
2. *A point x is in the projected Aubry set if and only if $h(x,x) = 0$.*
3. *Given $x \in M$, there exists a sequence $\gamma_n : [0,t_n] \longrightarrow M$ of minimizers of the action of L such that*
   - *$\gamma_n(0) = \gamma_n(t_n) = x$ for every $n > 0$.*
   - *$\lim_{n\to\infty} t_n = \infty$.*
   - *$h(x,x) = \lim_{n\to\infty}(A_{L+c_0(L)}(\gamma_n))$.*

So $h(x,x)$ vanishes at the projected Aubry set, like the differences of conjugate pairs of the Hamilton-Jacobi equation. The geometry of the manifold shows up in item (2) of the above lemma: if $L(p,v) = \frac{1}{2}g_p(v,v) - \omega$, the minimizers are just geodesics of (M,g), so item (2) tells us that the value of $h(x,x)$ is the limit of the values of the action of $L + c_0(L)$ evaluated on a sequence of minimizing loops based on x whose lengths go to infinity. Looking closer at the relationship between first and second Peierl's barriers we have (see for instance [19]):

Lemma 6.2. *The following assertions hold:*

1. *For every pair of conjugate functions u^-, u^+ we have*
$$u^-(x) - u^+(x) \leq h(x,y)$$
for every $x, y \in M$.
2. *$h(x,y) = \sup_{u^-,u^+}\{u^-(y) - u^+(x)\}$ where the supremum runs over all pair of conjugate pairs.*

6.1. **Static classes and uniqueness of the weak KAM solutions.** The dynamics of the set of global minimizers determines the uniqueness of the solutions of the Hamilton-Jacobi equation. In order to be more precise about this assertion we need some definitions. Let us recall that a semistatic curve $\beta : [a,b] \longrightarrow M$ is an absolutely continuous curve such that

$$A_{L+c[0]}(\beta) = h(\beta(a), \beta(b)).$$

A static curve $\alpha : [a,b] \longrightarrow M$ is an absolutely continuous curve such that

$$A_{L+c[0]}(\beta) = -h(\beta(b), \beta(a)).$$

A static curve is always semistatic. The curve $\beta : I \longrightarrow M$ is static, if and only if, it is static restricted to any interval contained in I. Since $h(x,y) + h(y,x)$ is the infimum of the action of the Lagrangian at the critical level $c[0]$, such curves are minimizers of the action. In the case of mechanical Lagrangians, static curves are the maximum points of the potential, while semistatic curves are projections of orbits of the Euler-Lagrange flow which tend to the singularities. Observe that [12] the Aubry set is the set of static curves. So in our case, the only static curve is the closed geodesic γ supporting the Mather measure.

Two points θ_1, θ_2 in TM are in the same static class if

$$h(\pi(\theta_1, \theta_2)) + h(\pi(\theta_2, \theta_1)) = 0.$$

According to our assumptions, there is only one static class, whose elements are the points of γ. We need the following result from [12]:

Lemma 6.3. *Suppose that the set of static classes is unique. Then there exist (up to additive constant) a unique pair of conjugate solutions of the Hamilton-Jacobi equation.*

This yields

Corollary 6.4. *Suppose that there exists just one static class. Then $h(x,x) = P(x,x)$.*

6.2. **Peierl's barrier in terms of the differences between Busemann functions.** Now, we are in shape to prove the main result of the section. We are going to link the Busemann functions of the metric g in $\tilde{M}$ to the second Peierl's barrier $P(x,x)$. For this purpose we shall prove that Busemann functions are naturally related with the Peierl's barrier $h(x,x)$, and then apply the above results. Throughout the subsection, $\gamma = \gamma_\theta$ will be the geodesic in the statement of Theorem 1, namely, the support of the Mather measure, $\theta \in T_1M$ is the initial condition of γ_θ. We choose a lift $\gamma_{\tilde{\theta}}$ of γ_θ in $\tilde{M}$, and a tubular neighborhood $\tilde{N}$ of $\gamma_{\tilde{\theta}}$ such that the covering map $\Pi : \tilde{M} \longrightarrow M$ restricted to $\tilde{N}$ is a diffeomorphism into a tubular neighborhood $N(\gamma)$ of γ.

Proposition 6.5. *Let (M,g) satisfy the assumptions of the main theorem. Then,*

1. *$h(x,x) = P(x,x) = b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x})$ for every $x \in N$ and $\tilde{x} \in \tilde{N}$ such that $\Pi(\tilde{x}) = x$.*
2. *There exist positive constants A, B such that*

$$A \inf_{p \notin \tilde{N}} (b^{-\theta}(p) + b^{\theta}(p)) \leq h(\Pi(p), \Pi(p)) \leq B \inf_{p \notin \tilde{N}} (b^{-\theta}(p) + b^{\theta}(p)).$$

We prove the proposition in many steps.

Lemma 6.6. *Let $x \in N(\gamma)$, let $T > 0$ be the minimum period of γ, and let $\beta : [0, T_n] \longrightarrow N_\gamma$ be a closed geodesic loop parametrized by arc length such that*

1. *$\beta(0) = x = \beta(T_n)$,*
2. *β is homotopic to $n[\gamma]|$.*

Then there exists a function $\delta(n)$, with $\lim_{n\to+\infty} \delta(n) = 0$, such that

$$|A_{L+c[0](L)}(\beta) - (b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x}))| \leq \delta(n),$$

where $\tilde{x} \in \tilde{N}$ is any lift of x in $\tilde{N}$.

Proof. Let $\tilde{\beta} \subset \tilde{N}$ be a lift of β, and let $\tilde{x} = \tilde{\beta}(0)$. Let $T_{\gamma_{\tilde{\theta}}}$ be the covering translation preserving $\gamma_{\tilde{\theta}}$. Then, the assumption implies that

$$\tilde{\beta}(T_n) = T^n_{\gamma_{\tilde{\theta}}}(\tilde{x}).$$

Since the curvature of (M,g) is negative but at the points of γ we have that

1. The (unique) geodesic $[\tilde{p}, T^n_{\gamma_{\tilde{\theta}}}(\tilde{p})]$ joining $\tilde{p}$ to $T^n_{\gamma_{\tilde{\theta}}}(\tilde{p})$ is contained in $\tilde{N}$ (because of the convexity of the metric),
2. The minimum distance from $\tilde{\beta}[0, T_n]$ to $\gamma_{\tilde{\theta}}$ must converge to 0 as $n \to +\infty$. Namely, given $\epsilon > 0$, there exist $n > 0$ such that for every $\tilde{p} \in \tilde{N}$, we have
$$\inf_{q \in [\tilde{p}, T^n_{\gamma_{\tilde{\theta}}}(\tilde{p})]} d(q, \gamma_{\tilde{\theta}}) \leq \epsilon.$$

Let $\beta_{x,+}[0, +\infty) \subset \tilde{N}$ be the geodesic asymptotic to $\gamma_{\tilde{\theta}}$ with $\beta_{x,+}(0) = \tilde{x}$. Let $\beta_{x,-}(-\infty, 0]$ be the geodesic asymptotic to $\gamma_{-\tilde{\theta}}$ with $\beta_{x,-}(0) = T^n_{\gamma_{\tilde{\theta}}}(\tilde{x})$. Let us denote by $[p, q]$ the geodesic joining the points p, q in $\tilde{M}$. We can assume without loss of generality that $\gamma_{\tilde{\theta}}(0)$ is the closest point in $\gamma_{\tilde{\theta}}$ to $\tilde{x}$. Since $\beta_{x,+}$, $\beta_{x,-}$ are asymptotic to $\gamma_{\tilde{\theta}}$ there exist $\mu(n) \to 0$ if $n \to +\infty$, and a number $s_n > 0$ satisfying
$$d(\beta_{x,+}(s_n), \beta_{x,-}(-s_n)) \leq \mu(n).$$
Let us consider the broken geodesic $\tilde{\alpha}_n$ given by
$$\tilde{\alpha}_n = \beta_{x,+}[0, s_n] \cup [\beta_{x,+}(s_n), \beta_{x,-}(-s_n)] \cup \beta_{x,-}[-s_n, 0].$$

By the convexity of the metric, $\tilde{\alpha}_n$ is contained in the region bounded by $\tilde{\beta}[0, T_n]$, $\gamma_{\tilde{\theta}}$, and the geodesics $[\tilde{x}, \gamma_{\tilde{\theta}}(0)]$, $[T^n_{\gamma_{\tilde{\theta}}}(x), \gamma_{\tilde{\theta}}(nT)]$.

Moreover, the length $l(\tilde{\alpha})$ of $\tilde{\alpha}$ is $2s_n + d(\beta_{x,+}(s_n), \beta_{x,-}(-s_n))$, and $s_n \to +\infty$ if $n \to +\infty$, so by the definition of ϵ we have
$$|T_n - 2s_n| \leq \epsilon.$$

Let $a_x > 0$ be defined by
$$\beta_{x,+}(a_{x,+}) = H_{\tilde{\theta}}(0) \cap \beta_{x,+},$$
and let $a_{x,-} > 0$ be defined by
$$\beta_{x,-}(-a_{x,-}) = H_{-\tilde{\theta}}(nT) \cap \beta_{x,-}.$$

Let us still define $s_{n,+} > 0$ by
$$\beta_{x,+}(s_n) \in H_{\tilde{\theta}}(s_{n,+}),$$
and $s_{n(b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x})),-} > 0$ by
$$\beta_{x,-}(-s_n) \in H_{-\tilde{\theta}}(-nT + s_{n,-}).$$

Notice that $|(s_{n,+} + s_{n,-}) - nT| \leq \epsilon$. The above definitions and the convexity of the metric yield
$$s_n = a_{x,+} + s_{n,+} = a_{x,-} + s_{n,-}.$$
And observe that
$$a_{x,+} = -b^{\tilde{\theta}}(\beta_{x,+}(a_{x,+})) + b^{\tilde{\theta}}(\tilde{x}),$$
$$a_{x,-} = -b^{-\tilde{\theta}}(\beta_{x,-}(-a_{x,-})) + b^{-\tilde{\theta}}(T^n_{\gamma_{\tilde{\theta}}}(\tilde{x})).$$
Since by definition,

1. $b^{\tilde{\theta}}(\beta_{x,+}(a_{x,+})) = 0$,
2. $b^{-\tilde{\theta}}(\beta_{x,-}(-a_{x,-})) = +nT$,
3. $b^{-\tilde{\theta}}(T^n_{\gamma_{\tilde{\theta}}}(\tilde{x})) = b^{-\tilde{\theta}}(\tilde{x}) - nT$,

we have

$$a_{x,+} + a_{x,-} = (b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x})).$$

Therefore, the length $l(\tilde{\alpha}) = 2s_n + d(\beta_{x,+}(s_n), \beta_{x,-}(-s_n))$ satisfies

$$\begin{aligned} 2s_n &= ((a_{x,+} + a_{x,-}) + s_{n,+} + s_{n,-} \\ &= (s_{n,+} + s_{n-}) + (b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x})) \end{aligned}$$

which yields

$$\begin{aligned} &|(2s_n - nT) - (b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x}))| \\ &\qquad\qquad = \quad |(s_{n,+} + s_{n,-}) - nT| \leq \epsilon. \end{aligned}$$

Hence, the length T_n of $\tilde{\beta}$ satisfies

$$|(T_n - nT) - (b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x}))| \leq 2\epsilon.$$

The same estimate holds for the curve β that is homotopic to $n[\gamma]$. To calculate the action of $L(p,v) + c[0](L) = \frac{1}{2}(v,v) - \omega_p(v) + c[0](L)$ in β, take without loss of generality $\parallel \gamma \parallel_{st} = 1 = T$, and observe that

$$\int_\beta \omega = \int_\gamma \omega = n.$$

So we have

$$A_{L+c[0](L)}(\beta) = \frac{1}{2}l(\beta) - n + \frac{1}{2}l(\beta) = l(\beta) - n.$$

Thus,

$$|A_{L+c[0](L)}(\beta) - ((b^{-\tilde{\theta}}(\tilde{x}) + b^{\tilde{\theta}}(\tilde{x}))| \leq 2\epsilon.$$

As $n \to +\infty$, we can take ϵ arbitrarily small, and this implies the Lemma. □

Lemma 6.7. *There exists a tubular neighborhood $V \subset N(\gamma)$ of γ with the following property: Let $x \in V$, and let γ_n be the sequence defined in Lemma 6.1 of closed geodesic loops based on x such that $h(x,x) = \lim_{n\to\infty}(A_{L+c_0(L)}(\gamma_n))$. Then there exists $m(x) > 0$ such that $\gamma_n \subset N(\gamma)$ for every $n \geq m(x)$.*

Proof. This is will follow from the fact that the Aubry set and the Mather set are equal to the closed geodesic γ. Indeed, suppose that the statement is not true. Then we can choose a sequence x_i of points converging to γ with the following property:

Let γ_n^i be a sequence of closed geodesic loops based on x_i with $h(x_i,x_i) = \lim_{n\to\infty}(A_{L+c_0(L)}(\gamma_n^i))$ (Lemma 6.1. Then there exist a subsequence $\gamma_{n_i}^i$, $n_i \to \infty$ if $i \to \infty$, such that

1. $\gamma_{n_i}^i$ is not contained in $N(\gamma)$ for every i,
2. $\lim_{i\to\infty} |h(x_i,x_i) - (A_{L+c_0(L)}(\gamma_{n_i}^i))| = 0$.

Since the loops $\gamma_{n_i}^i$ are minimizers based at x_i where $h(x_i,x_i)$ tends to zero with $i \to \infty$, and their domains tend to $\mathbb{R}$, there exists a subsequence of them converging to a global minimizer β. The geodesic β would have a point outside $N(\gamma)$ and, by the continuity of $h(x,x)$, β would also contain a point p with $h(q,q) = 0$. By Lemma 6.1 the point q is in the Aubry set which coincides with the Mather set: the closed geodesic γ. This is clearly a contradiction. □

Lemma 6.8. *There exists a constant $C > 0$ such that if $x \notin N(\gamma)$, then $h(x,x) \geq C \inf_{p\notin\tilde{N}}\{b^{-\tilde{\theta}}(p) + b^{\tilde{\theta}}(p)\}$.*

Proof. Since the function $h(x,x)$ is zero just at the points of the Aubry set (the closed geodesic γ) and is continuous, this implies that outside the tubular neighborhood $N(\gamma)$ it must be strictly positive. In the same way, the function $b^{-\tilde{\theta}}(p)+b^{\tilde{\theta}}(p)$ is strictly positive outside any tubular neighborhood of $\gamma_{\tilde{\theta}}$. The periodicity of the Busemann function of $\gamma_{\tilde{\theta}}$, and the convexity of the metric, imply that the function $d(p) = b^{-\tilde{\theta}}(p) + b^{\tilde{\theta}}(p)$ is convex in $\tilde{M}$ and attains its minimum value outside $\tilde{N}$ at the boundary of $\tilde{N}$. This minimum value is positive, so $h(x,x)$ and $d(\tilde{x})$ have analogous behaviour and the comparison stated in the lemma follows. $\square$

Proof of Proposition 6.5. By Lemma 6.7, if we want to estimate $h(x,x)$ at points $x \in V \subset N(\gamma)$ it is enough to consider the loops based at x contained in $N(\gamma)$, where V is the tubular neighborhood of lemma 6.7. Then, lemma 6.6 combined with this observation yields item (1). Item (2) follows from Lemma 6.8.

Remark. Proposition 6.5 has some interesting consequences regarding the regularity of the Peierl's barrier that is closely related to the regularity of the solutions of the Hamilton-Jacobi equation. The proposition tells us that the function $h(x,x)$ is, in a tubular neighborhood of γ, as regular as the Busemann functions of any lift of the geodesic γ in $\tilde{M}$. In particular, if the manifold (M,g) has non-positive curvature, the regularity of the Busemann functions and Peierl's barrier is C^3 [15]. The local regularity of Peierl's barrier and subsolutions of the Hamilton-Jacobi equation close to the support of the Mather measure is quite exceptional, it holds for instance when the support is a finite collection of closed hyperbolic orbits [6].

7. Proof of the main Theorem.

7.1. **Surfaces of revolution as models.** The main idea to show Theorem 1 is to apply our estimates of the Peierl's barrier in terms of Busemann functions in order to obtain explicit bounds for the deviation function for surfaces of revolution with nonpositive curvature. Then, to get bounds for the deviation function for the surfaces in the statement of Theorem 1 we use comparison theorems to get information about the asymptotic behavior of geodesics.

We choose polynomial functions as test functions to generate annulus of revolution: If (x,y,z) are the cartesian coordinates in $\mathbb{R}^3$ let us consider the curves

$$r(z) = (a + z^{2+k}, 0, z),$$

where $k \in \mathbb{N}$. (The exponent $2+k$ grants the existence of Gaussian curvature). We would like to point out that

- If the annulus is C^∞ then k is even.
- We can also consider $k \in \mathbb{R}^+$, but in this case the annulus might not be C^∞.

7.2. **Estimates for Busemann functions in surfaces of revolution.** To estimate the Busemann functions we use the **Clairaut equation**

$$r(\gamma(t))cos(\theta(\gamma(t))) = c$$

where $\gamma(t)$ is a geodesic parametrized by arc length, $r(p)$ is the distance from p in the annulus to the revolution axis $\overrightarrow{z}$, and $\theta(\gamma(t))$ is the angle formed by the geodesic γ and the parallel $z = z(\gamma(t))$ (in our case, $c = a$).

It follows from a result by P. Eberlein that the Busemann functions in nonpositive curvature are C^2, so we can write them as functions of z in the following way (in fact, by the symmetries of the annulus, it is enough to consider the generating curve):

$$\begin{aligned} u^+(z) + u^-(z) &= u^+(0) + u^-(0) + \int_0^z < \nabla u^+(t) + \nabla u^-(t), \frac{\delta}{\delta t} > dt \\ &= 2\int_0^z sin(\theta(t))dt \\ &= 2\int_0^z \sqrt{1 - (\frac{a}{r(t)})^2}dt \\ &= 2\int_0^z \frac{t^{1+\frac{k}{2}}}{a + t^{2+k}}\sqrt{2a + t2 + k}dt \end{aligned}$$

from which follows the estimate $h(z,z) = Cz^{2+\frac{k}{2}} + o(z^{3+\frac{k}{2}})$.

7.3. **Comparison theory and estimates for Busemann functions close to non-hyperbolic geodesics.** The purpose of the subsection is to apply the estimate of the previous subsection to surfaces of nonpositive curvature (M, g) with $K \leq 0$ like in Theorem 1. Namely,

1. There is a closed geodesic γ of period T where $K \equiv 0$ whose orbit supports the Mather measure of $L(p, v) = \frac{1}{2}g_p(v, v) - \omega_p(v)$.
2. $K < 0$ in the complement of γ.
3. There exists $m > 0$ such that for every geodesic $\beta : (-\epsilon, -\epsilon) \to M$ perpendicular to γ at $\beta(0) = \beta \cap \gamma$ we have that m is the least integer where $\frac{\partial^m}{dt^m}K(\beta(t))|_{t=0} \neq 0$.

Such surfaces might not have an annulus of revolution containing the geodesic γ as a waist, so the estimates in the previous subsection might not apply immediately. We shall show that in fact, the sum of Busemann functions $u^+(z) + u^-(z)$ is a function of the **angle** formed by the two geodesics through $z \in \tilde{M}$ which are respectively, forward and backward asymptotic to a lift of γ. Then, using CAT comparison theorems we shall be able to compare this function with its counterpart in the model surface considered in the previous subsection. This will yield a comparison of the sums of Busemann functions in both surfaces.

To begin with, let us consider a Fermi coordinate system $\Phi : S^1_T \times (-\epsilon, \epsilon) \to M$, where S^1_T is a circle of length T parametrized by arc length, such that $\Phi(t, 0) = \gamma(t)$ for every $t \in [0, T]$. The curves $\Phi_t(s) = \Phi(t, s)$, $s \in (-\epsilon, \epsilon)$, are geodesics which are perpendicular to γ.

Let A_b be the annulus of revolution generated by rotating the curve

$$r_b(z) = (a + bz^{2+k}, 0, z),$$

around the vertical axis in $\mathbb{R}^3$. The meridians of the annulus of revolution are geodesics, and we can use the same map Φ to parametrize a subset of the annulus containing the waist $\gamma_0(t) = R_t(r_b(0))$, where R_t is the rotation around the z-axis of angle t (We can take $a > 0$ such that the length of γ_0 is T). Item (3) above implies that

Lemma 7.1. *Given $\sigma > 0$ there exist a tubular neighborhood V_γ and constants $b_1 < b_2$ with $|b_2 - b_1| < \sigma$, such that*

$$K_{b_1}(\Phi(p)) \leq K(\Phi(p)) \leq K_{b_2}(\Phi(p)),$$

where K_b is the Gaussian curvature of A_b.

Proof. From the assumptions on the surface (M, g), the curvature K when restricted to a geodesic $\beta : (-\epsilon, -\epsilon) \longrightarrow M$ perpendicular to γ at $\beta(0) = \beta \cap \gamma$ satisfies

$$\frac{\partial^m}{ds^m} K(\beta(s))|_{s=0} \neq 0.$$

where m is the least integer $k > 0$ where the k-th derivative of $K(\beta(s))$ is different from 0. Also from the assumptions, the number m is the same for every geodesic β as before. So we can find two functions of the form $f_1(s) = a + b_1 s^{2+m}$, $f_2(s) = a + b_2 s^{2+m}$, such that

1. The annulus of revolution A_{b_1}, A_{b_2} generated by the graphs of f_1, f_2 respectively, have waists γ_1, γ_2 of period $2\pi a = T$, where T is the minimum positive period of γ.
2. $-\frac{\partial^{m+2}}{ds^{m+2}} f_2(0) < K(\beta(0)) < -\frac{\partial^{m+2}}{ds^{m+2}} f_1(0) < 0$, for every geodesic $\beta : (-\epsilon, \epsilon) \longrightarrow M$ perpendicular to γ.

By the theory of surfaces of revolution, the curvatures of A_{b_i} at their waists γ_i are just the opposites of the second derivatives of the f_i at $t = 0$. By continuity, there exists a tubular neighborhood of γ in the parametrization Φ satisfying the assertion of the lemma. $\square$

The estimates for Busemann functions of A_b are completely analogous (up to multiplication by constants) to the estimates for A_1 showed in the previous subsection. Now, we state a version of the comparison theorem for angles of geodesic triangles, suited for our purposes. It is based in the well known comparison theorems for geodesic triangles due to A. D. Alexandrov and V. A. Toponogov, which compare the geometry of geodesic triangles in a Riemannian manifold (M, g) with the geometry of geodesic triangles in spaces of constant curvature (the book of Cheeger and Ebin [9] is a great reference for the subject). We shall use the surfaces of revolution A_b described in the previous subsection instead of manifolds with constant curvature as comparison spaces. Since this is not the usual version of CAT theorems, we give a proof of the result at the end of the section.

Theorem 7.2. *(Comparison theorem for angles) Let $S = \mathbb{R} \times (-\epsilon, \epsilon)$ be a strip, and let $S_1 = (S, g_1)$, $S_2 = (S, g_2)$ be two C^∞ Riemannian metrics in S such that*

1. *The curvatures K_1, K_2 of S_1, S_2 respectively, satisfy $K_2(t, s) \leq K_1(t, s) \leq 0$ for every $(t, s) \in S$.*
2. *The curves $c(t) = (t, 0)$, $t \in \mathbb{R}$, and $\sigma_t(s) = (t, s)$, $s \in (-\epsilon, \epsilon)$, are geodesics in S_1 and S_2 for every $t \in \mathbb{R}$.*
3. *$c(t)$ has unit speed in S_1 and S_2, σ_t has unit speed in S_1 for every $t \in \mathbb{R}$, and σ_0 is perpendicular to $c(t)$ in S_1 and S_2.*

Let $\Delta_i(t, s)$ be the S_i-geodesic triangles whose common vertices are $c(0), \sigma_0(s), c(t)$. Let $[\sigma_0(s), c(t)]_i$ be the S_i-geodesic joining $\sigma_0(s)$ and $c(t)$. Let $\alpha_i(t, s)$ be the S_i-angle formed by $\sigma_0'(s)$ and $[\sigma_0(s), c(t)]_i$ at the point $\sigma_0(s)$. Then

1. *$\Delta_2(t, s) \subset \Delta_1(t, s)$ for every t, s.*
2. *We have that $\alpha_2(t, s) \leq \alpha_1(t, s)$. Moreover, they coincide if and only if $\Delta_1(t, s) = \Delta_2(t, s)$ and $K_1(p) = K_2(p)$ for every p inside $\Delta_i(t, s)$.*

Now, we can apply Theorem 7.2 for angles to bound from above and from below the distance from γ to its asymptotes. Namely, let us take β like before, with $\beta(0) = \gamma(t_0)$, and we consider $\beta(a) = \Phi(p)$, $a \in [0, \epsilon)$, $t \neq t_0$. Let $\Delta(p, t)$ be the geodesic triangle in (M, g) with vertices $\Phi(p)$, $\gamma(t_0) = \Phi(p_0)$, $\gamma(t) = \Phi(p_t)$; and let

$\Delta_b(p,t)$ be the geodesic triangle in A_b with the same vertices (let us remind that we are parametrizing a tubular neighborhood of γ and a subannulus of A_b by the map Φ). Then by Theorem 7.2, if $K_b(\Phi(x)) \leq K(\Phi(x))$ for every x in $(-\epsilon,-\epsilon)\times[0,T]$, we have that $\Delta_b(p,t) \subset \Delta(p,t)$ for every $t\in\mathbb{R}$. So if we let $t\to\infty$, the same property remains true for the ideal triangles in both annuli with vertices $\Phi(t_0)$, $\Phi(p)$.

Therefore, the angles formed by β and the geodesic γ_p through $\Phi(p)$ that is asymptotic to γ is at least the angle formed by β and the geodesic $\gamma_{b,p}$ in A_b asymptotic to γ_b, whenever $K_b(\Phi(x)) \leq K(\Phi(x))$ for every x.

This application of comparison theory leads us to a comparison between Busemann functions in the annuli A_b, V_γ.

Indeed, let us recall that the formula for the sum of the Busemann functions in the previous subsection was

$$u^+(s)+u^-(s) = u^+(0)+u^-(0)+\int_0^s < \nabla u^+(\rho)+\nabla u^-(\rho), \frac{\delta}{\delta\rho} > d\rho =$$
$$\int_0^s < \nabla u^+(\rho)+\nabla u^-(\rho), \frac{\delta}{\delta\rho} > d\rho.$$

The same formula essentially holds for the tubular neighborhood V_γ in (M,g), taking $\frac{\delta}{\delta s}$ as the Fermi coordinate vector field tangent to the geodesics $\Phi_t(s)$ (we assume that the pair $\gamma'(t)$, $\frac{\delta}{\delta s}$ has the canonical orientation). Namely, given $t\in[0,T]$ we have

$$u^+(\Phi(t,s))+u^-(\Phi(t,s)) = \int_0^s < \nabla u^+(\Phi(t,\rho))+\nabla u^-(\Phi(t,\rho)), \frac{\delta}{\delta\rho} > d\rho.$$

So we get

$$u^+(\Phi(t,s))+u^-(\Phi(t,s)) = \int_0^s (\cos(\theta^+(t,\rho)))+\cos(\theta^-(t,\rho))))dt,$$

where $\theta^+(t,s)$ is the angle formed by the forward asymptote of γ containing $\Phi(t,s)$ with the vector $\frac{\delta}{\delta s}(\Phi(t,s))$ (respectively, $\theta^-(t,s)$ is the angle formed by the backward asymptote of γ and $\frac{\delta}{\delta s}(\Phi(t,s))$).

Let u_b^+, u_b^- be the Busemann functions in A_b, and let $\theta_b^+(t,s)$, $\theta_b^-(t,s)$ be the angles formed by the forward (resp. backward) asymptotes of γ_0 and the vector field $\frac{\delta}{\delta s}$ at the point $\Phi(t,s)$. By Theorem 7.2 we have that if $K_b(\Phi(t,s)) < K(\Phi(t,s))$ then

$$\theta_b^+(t,s) < \theta^+(t,s),$$

and

$$\theta_b^-(t,s) < \theta^+(t,s).$$

Moreover, if $K(\Phi(t,s)) < K_b(\Phi(t,s))$ then

$$\theta^+(t,s) < \theta_b^+(t,s),$$
$$\theta^+(t,s) < \theta_b^+(t,s).$$

Replacing these inequalities in the above integrals we get lower and upper bounds for the sum of the Busemann functions in V_γ in terms of the formulae obtained in the previous subsection. This finishes the proof of Theorem 1.

Proof of Theorem 7.2. Roughly speaking, the less curved is the space the more convex is the distance between geodesics and narrower are the angles of geodesic triangles. Such features of comparison geometry are well known since the famous

CAT comparison theorems. We shall give an elementary proof using Gauss-Bonnet theorem for geodesic polygons for the sake of completeness.

First some general remarks about angles and lengths. Angles are conformal invariants, and since two metrics in an open, simply connected set of the plane are conformally equivalent we have that the angles with respect to S_1, S_2 formed by two vectors v, w tangent to a point in S are the same. Moreover, the S_2-length of a curve $c \subset S$ is at least its S_1-length, because the decrease of the curvature increases the norm of vectors and length.

The Gauss-Bonnet theorem for a geodesic triangle Δ with inner angles a, b, c tells us that

$$(a+b+c) = \int_{\Delta} K + \pi,$$

and the Gauss-Bonnet theorem for geodesic quadrilaterals $\Box$ with inner angles a, b, c, d is

$$(a+b+c+d) = \int_{\Box} K + 2\pi.$$

Now, let us consider $t = -T$, for $T > 0$, let us denote $[\sigma_0(s), c(-T)]_i = \gamma_i^s$, and notice that γ_i^s can be parametrized in terms of the parameter t since these geodesics are everywhere transversal to the vertical geodesics σ_t. So let $\gamma_i^s(t) = \gamma_i^s \cap \sigma_t$. The triangles $\Delta_i(-T, s)$ have two sides in common, the geodesics $c(t), t \in [-T, 0]$, and $\sigma_0(r)$, $r \in [0, s]$. The geodesics γ_i^s are the other sides of $\Delta_i(t, s)$, and we claim

Claim 1: γ_2^s is contained in $\Delta_1(-T, s)$.

For suppose that γ_2^s is not contained in $\Delta_1(-T, s)$. Then there exist $t_0 \in (-T, 0]$, $s_0 \in (0, s)$ such that

1. $\sigma_{t_0}(s_0) \in \gamma_1^s \cap \gamma_2^s$,
2. the angle formed by $\frac{d}{dt}\gamma_2^s$ and $\sigma'_{t_0}(s_0)$ is at least the angle formed by $\frac{d}{dt}\gamma_1^s$ and $\sigma'_{t_0}(s_0)$.

Since $\gamma_i^s(-T) = c(-T)$ there exists a minimum value $r_0 < t_0$ such that

1. $\gamma_2^s(t)$ is not in $\Delta_1(-T, s)$ for every $t \in (r_0, t_0)$,
2. $\gamma_2^s(r_0) = \gamma_1^s(r_0)$.

Then, there exists $z > s$ such that the geodesic γ_1^z is tangent to γ_2^s at some point $\gamma_1^z(t_1)$, where $t_1 \in (r_0, t_0)$. Let $s_1 < s$ be such that $\sigma_{t_1}(s_1) = \gamma_1^s(t_1)$. Let $z_1 > s_1$ be such that $\gamma_2^s(t_1) = \sigma_{t_1}(z_1)$. Let us consider the geodesic quadrilaterals $\Box_1$, $\Box_2$ with the following geodesic sides and angles:

1. The sides of $\Box_1$ are the S_1-geodesics $c(t), t \in [t_1, t_0]$, $\sigma_{t_1}([0, s_1])$, $\sigma_{t_0}([0, s_0])$ and $\gamma_1^s([t_1, t_0])$. The inner angles are a_1, the angle formed by $c(t)$ and σ_{t_1}; a_2, the angle formed by $c(t)$ and σ_{t_0}; a_3, the angle formed by σ_{t_1} and γ_1^s; and a_4 the angle formed by σ_{t_0} and γ_1^s.
2. The sides of $\Box_2$ are the S_2-geodesics $c(t), t \in [t_1, t_0]$, $\sigma_{t_1}([0, z_1])$, $\sigma_{t_0}([0, s_0])$ and $\gamma_2^s([t_1, t_0])$. The inner angles are b_j, $j = 1, 2, 3, 4$, where b_1 is the angle formed by $c(t)$ and σ_{t_1}; b_2, the angle formed by $c(t)$ and σ_{t_0}; b_3, the angle formed by σ_{t_1} and γ_2^s; and b_4 the angle formed by σ_{t_0} and γ_2^s.

Observe that $\Box_1 \subset \Box_2$, and that $a_1 = a_2 = b_1 = b_2 = \frac{\pi}{2}$. So the sum of the inner angles of $\Box_1$ is

$$\pi + (a_3 + a_4).$$

The sum of the inner angles of $\Box_2$ is

$$\pi + (b_3 + b_4).$$

Moreover, since γ_1^z is tangent to γ_2^s at $\gamma_1^z(t_1) = \sigma_{t_1}(z_1)$, and $z > s$, we have that

$$b_3 \geq a_3.$$

By Gauss-Bonnet we get that

$$\sum_{j=1}^{4} b_j - \sum_{j=1}^{4} a_j = (b_3 - a_3) + (b_4 - a_4) = \int_{\square_2} K_2 - \int_{\square_1} K_1.$$

This implies that

$$\begin{aligned}(b_4 - a_4) &= \int_{\square_2} K_2 - \int_{\square_1} K_1 - (b_3 - a_3)\\ &\leq \int_{\square_2} K_2 - \int_{\square_1} K_1.\end{aligned}$$

Since, by assumption, the tangent vector of γ_2^s at $\gamma_2^s(t_0)$ points into $\Delta_1(t,s)$, we know that $b_4 \geq a_4$, or $b_4 - a_4 \geq 0$. So we would have that the difference of the integrals in the right is positive. However,

Claim 2: The number $\int_{\square_2} K_2 - \int_{\square_1} K_1$ is negative.

In fact, we have

$$\int_{\square_2} K_2 - \int_{\square_1} K_1 = \int_{\square_1} K_2 - \int_{\square_1} K_1 - \int_{\square_2 - \square_1} K_2.$$

So

$$\int_{\square_2} K_2 - \int_{\square_1} K_1 \leq \int_{\square_1} K_2 - \int_{\square_1} K_1.$$

The integral of K_i is calculated in terms of the area form of S_i. If we put both integrals in terms of $dtds$, the coordinates of the strip S, we get

$$\int_{\square_1} K_1 = \int_{\square_1} J_1(t,s) K_1(t,s) dtds,$$

$$\int_{\square_1} K_2 = \int_{\square_1} J_2(t,s) K_2(t,s) dtds,$$

where $J_i(t,s)$ is the Jacobian of the coordinate change of the S_i-area form. Since $K_2(t,s) \leq K_1(t,s)$ the Jacobians satisfy $J_2(t,s) \geq J_1(t,s)$. Thus, $\int_{\square_1} K_2 - \int_{\square_1} K_1 \leq 0$ and the claim is proved.

The Claim contradicts $b_4 - a_4 \geq 0$, which was a consequence of the assuming that γ_2^s is not contained in the triangle $\Delta_1(-T,s)$. This shows Claim 1.

Theorem 7.2 follows from Claim 1 (that is item (1) in the statement), since $\Delta_2(-T,s) \subset \Delta_1(-T,s)$ obviously implies that the inner angles of $\Delta_2(-T,s)$ are bounded above by the inner angles of $\Delta_1(-T,s)$. Gauss-Bonnet implies metric rigidity if the two triangles coincide. The argument for $T < 0$ is completely analogous.

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