# Sub-actions for Anosov diffeomorphisms

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**Abstract**

We show a positive Livciz type theorem for $\mathcal{C}^2$ Anosov diffeomorphisms $f$ on a compact boundaryless manifold $M$ and Hölder observables $A$. Given $A : M \to \mathbb{R}$, $\alpha$-Hölder, we show there exist $V : M \to \mathbb{R}$, $\beta$-Hölder, $\beta < \alpha$ and a probability measure $\mu$, $f$-invariant such that

$$A \leq V \circ f - V + \int A \, d\mu.$$

*Dedicated to Jacob Palis*

## 1 Introduction

We consider a compact riemannian manifold $M$ of dimension $d \geq 2$ without boundary and a $\mathcal{C}^2$ transitive Anosov diffeomorphism $f : M \to M$. The tangent bundle $TM$ admits a continuous $Tf$-invariant splitting $TM = E^u \bigoplus E^s$ of expanding and contracting tangent vectors. We assume $M$ is equiped with a riemannian metric and there exist constants $C(M)$, depending only on $M$ and the metric and constants depending on $f$

$$0 < \Lambda_s < \lambda_s < 1 < \lambda_u < \Lambda_u$$

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such that for all $n \in \mathbb{Z}$

$$\begin{cases} C(M)^{-1}\lambda_u^n \le \|T_x f^n \cdot v\| \le C(M)\Lambda_u^n & \text{for all } v \text{ in } E_x^u, \\ C(M)^{-1}\Lambda_s^n \le \|T_x f^n \cdot v\| \le C(M)\lambda_s^n & \text{for all } v \text{ in } E_x^s. \end{cases}$$

Livciz theorem [4] asserts that, if $A : M \to M$ is a given Hölder function and satisfies $\int A\, d\mu = 0$ for all $f$-invariant probability measure $\mu$, then $A$ is equal to a coboundary $V$ (which is Hölder too), that is:

$$A = V \circ f - V.$$

What hapens if we only assume $\int A\, d\mu \ge 0$ for all $f$-invariant probability measure $\mu$? We denote by $\mathcal{M}(f)$, the set of $f$-invariant probability measures. We prove the following:

**Theorem 1** *Let $f : M \to M$ be a $\mathcal{C}^2$ transitive Anosov diffeomorphism on a compact manifold $M$ without boundary. For any given $\alpha$-Hölder function $A : M \to \mathbb{R}$, there exists a $\beta$-Hölder function $V : M \to \mathbb{R}$, that we call sub-action, such that:*

$$A \le V \circ f - V + m(A, f),$$

*where $m(A, f) = \sup\{ \int f\, d\mu \mid \mu \in \mathcal{M}(f) \}$, $\mathcal{M}(f)$ is the set of $f$-invariant probability measures and*

$$\beta = \alpha \frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}, \quad \text{Höld}_\beta(V) = \frac{C(M)}{\min(1 - \lambda_u^{-\alpha}, 1 - \lambda_s^{\alpha})^2} \text{Höld}_\alpha(A)$$

*where $C(M)$ is some constant depending only on $M$ and the metric.*

By analogy with Hamiltonian mechanics and the way we define $V$ from $A$, we may interpret $A$ as a lagrangian and $V$ as a sub-action. This result extends a similar one we obtained in [3] for expanding maps of the circle. Although the proof we give is specific for smooth systems, the same result holds for doubly infinite subshifts of finite type.

**Corollary 2** *The hypothesis are the same as in theorem 1. The following statements are equivalent:*

*(i) $A \ge V \circ f - V$ for some bounded measurable function $V$,*

*(ii)* $\int A\,d\mu \geq 0$ *for all* $f$*-invariant probability measure* $\mu$,

*(iii)* $\sum_{k=0}^{p-1} A \circ f^k(x) \geq 0$ *for all* $p \geq 1$ *and point* $x$ *periodic of period* $p$,

*(iv)* $A \geq V \circ f - V$ *for some Hölder function* $V$.

The proof of that corollary is straitforward and uses (for (iii) $\Rightarrow$ (ii)) the fact that the convex hull of periodic measures is dense in the set of all $f$-invariant probability measures for topological dynamical systems satisfying the shadowing lemma (see Lemma 5). F. Labourie suggested to us the following corollary:

**Corollary 3** *The hypothesis are the same as in theorem 1. If* $A$ *satisfies* $\int A\,d\mu \geq 0$ *for all* $\mu \in \mathcal{M}(f)$ *and* $\sum_{k=0}^{p-1} A \circ f^k(x) > 0$ *for at least one periodic orbit* $x$ *of period* $p$ *then* $\int A\,d\lambda > 0$ *for all probability measure* $\lambda$ *giving positive mass to any open set.*

Again the proof is straitforward: $R = A - V \circ f + V \geq 0$ for some continuous $V$ and $\int R\,d\lambda = 0$ for such a measure $\lambda$ implies $R = 0$ everywhere and in particular $\sum_{k=0}^{p-1} A \circ f(x) = 0$ for all periodic orbit $x$.

Any measure $\mu$ satisfying $\int A\,d\mu = m(A, f)$ is called a maximizing measure and since $A$ is continuous, such a measure always exists. It is then natural to ask the following two questions : For which $A$, the set of maximizing measures is reduced to a single measure ? In the case there exists a unique maximizing measure, to what kind of compact set, the support of this measure looks like ?

The following theorem gives a partial answer for "generic" functions $A$.

**Theorem 4** *Let* $f : M \to M$ *be a* $\mathcal{C}^2$ *transitive Anosov diffeomorphism and* $\beta < \ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$*. Then there exists an open set* $\mathcal{G}_\beta$ *of* $\beta$*-Hölder functions ( open in the* $\mathcal{C}^\beta$*-topology ) such that:*

*(i) any* $A$ *in* $\mathcal{G}_\beta$ *admits a unique maximizing measure* $\mu_A$*;*

*(ii) the support of* $\mu_A$ *is equal to a periodic orbit and is locally constant with respect to* $A \in \mathcal{G}_\beta$*;*

*(iii) any* $\alpha$*-Hölder function with* $\alpha > \beta \ln(\Lambda_u/\lambda_s)/\ln(1/\lambda_s)$ *is contained in the closure of* $\mathcal{G}_\beta$ *(the closure is taken with respect to the* $\mathcal{C}^\beta$*-topology).*

The proof of Theorem 4 is a simplification of what we gave in [3] in the one-dimensional setting. The existence of sub-actions is in both cases the main ingredient of the proof.

The plan of the proof of Theorem 1 is the following: Given a finite covering of $M$ by open sets $\{U_1, \dots, U_l\}$ with sufficiently small diameter, we construct a Markov covering (and not a Markov partition) $\{R_1, \dots, R_l\}$ of rectangles: each $R_i$ contains $U_i$ and satisfies

$$x \in U_i \cap f^{-1}(U_j) \Rightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j),$$

where $W^s(x, R_i)$ denotes the local stable leaf through $x$ restricted to $R_i$. We then associate to each $R_i$ a local sub-action $V_i$, defined on $R_i$ by:

$$V_i(x) = \sup\{S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geq 0, \quad y \in W^s(x, R_i)\}$$

where $\Delta^s(y, x)$ is a kind of cocycle along the stable leaf $W^s(x)$:

$$\Delta^s(y, x) = \sum_{n \geq 0} (A \circ f^n(y) - A \circ f^n(x)).$$

This family $\{V_1, \dots, V_l\}$ of local sub-actions satisfies the inequality:

$$x \in U_i \cap f^{-1}(U_j) \Rightarrow V_i(x) + A(x) - m \leq V_j \circ f(x)$$

and enable us to construct a global sub-action $V$:

$$V(x) = \sum_{i=1}^{l} \theta_i(x) V_i(x)$$

where $\{\theta_1, \dots, \theta_l\}$ is a smooth partition of unity associated to the covering $\{U_1, \dots, U_l\}$. The main difficulty is to prove that each $V_i$ is Hölder on $R_i$.

# 2 Existence of sub-actions

We continue our description of the dynamics of transitive Anosov diffeomorphisms (for details information, see Bowen's monography [2]). All the results we are going to use depend on a small constant of expansiveness $\epsilon^* > 0$ depending on $f$ and $M$ in the following way:

$$\epsilon^* = C(M)^{-1} \min\left(\frac{\lambda_u - 1}{\|D^2 f\|_\infty}, \frac{1 - \lambda_s}{\|D^2 f\|_\infty}\right)$$

where $C(M) \geq 1$ is a constant depending only on $M$ and the riemannian metric. At each point $x$, one can define its local stable manifold $W^s_\epsilon(x)$ for every $\epsilon < \epsilon^*$:

$$W^s_\epsilon(x) = \{\, y \in M \mid d(f^n(x), f^n(y)) \leq \epsilon \quad \forall n \geq 0 \,\}$$

which are $\mathcal{C}^2$ embeded closed disks of dimension $d^s = \dim E^s_x$ and tangent to $E^s_x$. In the same manner, $W^u_\epsilon(x)$ is defined replacing $f$ by $f^{-1}$. If two points $x, y$ are close enough, $d(x, y) < \delta$, then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ have a unique point in common, called $[x, y]$:

$$[x, y] = W^s_\epsilon(x) \cap W^u_\epsilon(y) = W^s_{\epsilon^*}(x) \cap W^u_{\epsilon^*}(y),$$

where $\epsilon = K^*\delta$ and $K^*$ is again a large constant depending on $M$ and $f$:

$$K^* = \frac{C(M)}{\min(1 - \lambda_u^{-1}, 1 - \lambda_s)}.$$

This estimate is in fact a particular case of Bowen's shadowing lemma:

**Lemma 5 (Bowen)** *If $\delta$ is small enough, $\delta < \epsilon^*/K^*$, if $(x_n)_{n\in\mathbb{Z}}$ is a bi-infinite $\delta$-pseudo-orbit, that is, $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$, then there exists a unique true orbit $\{f^n(x)\}_{n\in\mathbb{Z}}$ which $\epsilon$-shadow $(x_n)_{n\in\mathbb{Z}}$, that is $d(f^n(x), x_n) < \epsilon$ for all $n \in \mathbb{Z}$ with $\epsilon = K^*\delta$.*

This lemma is the main ingredient for constructing (dynamical) rectangles. A rectangle $R$ is a closed set of diameter less than $\epsilon^*/K^*$ satisfying:

$$x, y \in R \Rightarrow [x, y] \in R.$$

We will not use the notion of proper rectangles but will use instead the notion of Markov covering.

**Definition 6** Let $\mathcal{U} = \{U_1, \ldots, U_l\}$ be a covering of $M$ by open sets of diameter less than $\epsilon^*/(K^*)^2$. We call Markov covering associated to $\mathcal{U}$, a finite set $\mathcal{R} = \{R_1, \ldots, R_l\}$ of rectangles of diameter less than $\epsilon^*/K^*$ satisfying:

$$\begin{aligned}
&U_i \subset R_i \\
&x \in U_i \cap f^{-1}(U_j) \Rightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j) \\
&y \in f(U_i) \cap U_j \quad\ \Rightarrow f^{-1}(W^u(y, R_j)) \subset W^u(f^{-1}(y), R_i)
\end{aligned}$$

where $W^s(x, R_i) = W^s_{\epsilon^*}(x) \cap R_i$ and $W^u(y, R_j) = W^u_{\epsilon^*}(y) \cap R_j$.

An easy consequence of the shadowing lemma shows there always exist such Markov coverings:

**Proposition 7** *For every covering $\mathcal{U}$ of $M$ by open sets such that the diameter of each $U_i$ is less than $\epsilon^*/(K^*)^2$, there exists a Markov covering $\mathcal{R}$ by rectangles of diameter less than $\epsilon^*/K^*$.*

*Proof.* Given $\mathcal{U} = \{U_1, \ldots, U_l\}$ such a covering, we define the following compact space of $\epsilon^*/(K^*)^2$ pseudo-orbits:

$$\Sigma = \{\omega = (\ldots, \omega_{-2}, \omega_{-1} \mid \omega_0, \omega_1, \ldots) \quad \text{s.t.} \quad U_{\omega_n} \cap f^{-1}(U_{\omega_{n+1}}) \neq \emptyset\}.$$

Here $\omega$ is a sequence of indices in $\{1, \ldots, l\}$ and $\Sigma$ is a subshift of finite type where $i \to j$ is a possible transition iff $U_i \cap f^{-1}(U_j)$ is not empty. Given such $\omega \in \Sigma$, we choose for all $n \in \mathbb{Z}$, $x_n \in U_{\omega_n}$ so that $f(x_n) \in U_{\omega_{n+1}}$. Then $(x_n)_{n\in\mathbb{Z}}$ is a $\epsilon^*/(K^*)^2$ pseudo-orbit which corresponds to a unique true orbit $(f^n(x))_{n\in\mathbb{Z}}$ satisfying:

$$d(f^n(x), U_{\omega_n}) < \epsilon^*/K^* \quad \forall n \in \mathbb{Z}.$$

Since $\epsilon^*$ is a constant of expansiveness, there can exists at most one point $x$ satisfying the previous inequality for all $n$. We call that point $\pi(\omega)$ and notice that the map

$$\pi : \Sigma \to M$$

is surjective (for $\mathcal{U}$ is a covering), commutes with the left shift $\sigma$, $f \circ \pi = \pi \circ \sigma$, is continuous by expansiveness (in fact Hölder if $\Sigma$ is equiped with the standard metric). Also notice that $\pi$ may not be finite-to-one. We first construct a Markov cover on $\Sigma$ as usual by the braket

$$[\omega, \omega'] = (\cdots, \omega'_{-2}, \omega'_{-1} \mid \omega_0, \omega_1, \cdots)$$

where $\omega = (\omega_n)_{n\in\mathbb{Z}}$, $\omega' = (\omega'_n)_{n\in\mathbb{Z}}$ and $\omega'_0 = \omega_0$. By uniqueness in the construction of $\pi(\omega)$, we get

$$\begin{aligned}
&\pi([\omega, \omega']) = [\pi(\omega), \pi(\omega')] \\
&\pi([i]) = R_i \quad \text{is a rectangle of } M \text{ containing } U_i \\
&\pi(W^s(\omega, [i])) = W^s(\pi(\omega), R_i) \quad \text{whenever } \omega_0 = i
\end{aligned}$$

where $[i]$, $i = 1, \cdots, l$, is the cylinder $\{\omega \in \Sigma \mid \omega_0 = i\}$ and $W^s(\omega, [i])$ is the symbolic stable set $\{\omega' \in \Sigma \mid \omega'_n = \omega_n \ \forall n \geq 0\}$. (For the proof of the last equality, we just notice : if $x = \pi(\omega)$, $y \in W^s(x, R_i)$ and $y = \pi(\omega')$ then $\pi([\omega, \omega']) = y$ and $[\omega, \omega'] \in W^s(\omega, [i])$.) To finish the proof we only show

$$x \in U_i \cap f^{-1}(U_j) \Rightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j).$$

Indeed, $x = \pi(\omega)$ for some $\omega = (\cdots, \omega_{-1} \mid i, j, \omega_2, \cdots)$ and

$$\sigma(W^s(\omega, [i]) \subset W^s(\sigma(\omega), [j]).$$

To conclude, we apply $\pi$ on both sides. ■

**Definition 8** Let $\mathcal{R} = \{R_1, \cdots, R_l\}$ be a Markov covering of $M$ associated to some open covering $\mathcal{U} = \{U_1, \cdots, U_l\}$. We define a local sub-action by

$$V_i(x) = \sup\{\, S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geq 0,\ y \in W^s(x, R_i)\,\}$$

where $S_n B = \sum_{k=0}^{n-1} B \circ f^k$, $\Delta^s(y, x) = \sum_{k \geq 0}(A \circ f^k(y) - A \circ f^k(x))$ and the supremum is taken over all $n \geq 0$ and points $y \in W^s(x, R_i)$.

Before showing $V_i$ is a (finite!) Hölder function on each $R_i$, let's conclude the proof of Theorem 1:

**Poof of Theorem 1.** Let $\mathcal{U} = \{U_1, \cdots, U_l\}$ be an open covering of $M$, $\{R_1, \cdots, R_l\}$ a Markov covering associated to $\mathcal{U}$ and $\{\theta_1, \cdots, \theta_l\}$ a partition of unity adapted to $\mathcal{U}$. Let $\{V_1, \cdots, V_l\}$ constructed as above and

$$V = \sum_i \theta_i V_i.$$

Suppose we have proved that $x \in U_i \cap f^{-1}(U_j)$ implies

$$V_i(x) + (A - m)(x) \leq V_j \circ f(x).$$

Multiplying this inequality by $\theta_i(x)\theta_j \circ f(x)$ and summing over $i$ and $j$, we get

$$V(x) + (A - m)(x) \leq V \circ f(x) \quad (\forall x \in M).$$

We now prove the local sub-cohomological equation: if $x \in U_i \cap f^{-1}(U_j)$ and $y \in W^s(x, R_i)$, then $f(y) \in W^s(f(x), R_j)$ and

$$\begin{aligned} &S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) + (A - m)(x) \\ &\qquad = S_{n+1}(A - m) \circ f^{-(n+1)} \circ f(y) + \Delta^s(f(y), f(x)) \leq V_j \circ f(x). \end{aligned}$$

Taking the supremum over all $n \geq 0$ and all $y \in W^s(x, R_i)$, we get indeed

$$V_i(x) + (A - m)(x) \leq V_j \circ f(x).$$

That finishes the proof of theorem 1. ■

We now come to our main technical lemma. We notice that, even in the case where $A$ is Lipschitz, we only obtain a Hölder sub-action.

**Lemma 9** *If $A$ is $\alpha$-Hölder on $M$, $R$ is a rectangle and $V$ is defined as in Definition 8, then $V$ is $\beta$-Hölder on $R$ with exponent*

$$\beta = \alpha \frac{|\ln \lambda_s|}{\Lambda_u + |\ln \lambda_s|} < \alpha.$$

*Proof.* We divide the proof into four steps:
**Step one.** If $d(x, x') < \epsilon^*$ and $x$, $x'$ are on the same stable leaf, then

$$\Delta^s(x, x') \leq \sum_{n \geq 0} |A \circ f^n(x) - A \circ f^n(x')| \leq C(M) \frac{\text{Höld}_\alpha(A)}{1 - \lambda_s^\alpha} d(x, x')^\alpha,$$

for some constant $C(M)$ depending only on $M$ and the metric.

Indeed, it follows from the contraction $d(f^k(x), f^k(x')) \leq C(M)\lambda_s^k d(x, x')$ for $k \geq 0$ and the fact that $A$ is $\alpha$-Hölder.

**Step two.** For every $n \geq 1$, $x, x' \in M$ such that $d(f^k(x), f^k(x')) < \epsilon^*/K^*$ for all $0 \leq k \leq n$, then

$$\sum_{k=0}^{n-1} |A \circ f^k(x) - A \circ f^k(x')| \leq K(M, f) \max(d(x, x')^\alpha, d(f^n(x), f^n(x'))^\alpha),$$

$$\text{where} \quad K(M, f) = C(M) \frac{\text{Höld}_\alpha(A)}{\min(1 - \lambda_u^{-\alpha}, 1 - \lambda_s^\alpha)^2}.$$

Indeed, one can build $w = [x, x']$; then on the one hand, $d(x, w) \leq \epsilon^*$ and $x, w$ are on the same stable leaf; on the other hand, $d(f^n(w), f^n(x')) \leq \epsilon^*$ and $f^n(w)$ and $f^n(x')$ are on the same unstable leaf. We conclude by applying step one and the estimates:

$$d(x, w) \leq K^* d(x, x'), \quad d(f^n(w), f^n(x')) \leq K^* d(f^n(x), f^n(x')).$$

**Step three.** We show that $V(x)$ is finite for every $x \in R$. It is precisely here that the choice of the normalizing constant $m(A,f)$ is important.

Indeed, since a transitive Anosov diffeomorphism is mixing, there exists an integer $\tau^* \geq 1$ such that, for every finite orbit $\{f^{-n}(y), \cdots, f^{-1}(y), y\}$, n arbitrary, $f^{\tau^*}(B(y, \epsilon^*/K^*))$ contains $f^{-n}(y)$. Thanks to the shadowing lemma, there exists a periodic orbit $z$, of period $n+\tau^*$, satisfying

$$d(f^{-k}(z), f^{-k}(y)) \leq \epsilon^* \qquad (\forall k = 0, 1, \cdots, n).$$

Using step two, $\sum_{k=1}^{n}(A \circ f^{-k}(y) - A \circ f^{-k}(z))$ is uniformly bounded in $n$ by some constant $C(M,f)$ and using $\sum_{k=1}^{n+\tau^*} A \circ f^{-k}(z) \leq (n+\tau^*)m(A,f)$, we get

$$\begin{aligned}\sum_{k=1}^{n} A \circ f^{-k}(y) &\leq C(M,f) + \sum_{k=1}^{n+\tau^*} A \circ f^{-k}(z) + \tau^* \|A\|_\infty \\ &\leq C(M,f) + 2\tau^* \|A\|_\infty.\end{aligned}$$

**Step four.** We finally prove that $V$ is Hölder on $R$. Let $n \geq 0$, $x, x' \in R$, $y \in W^s(x,R)$ and define $y' = [x', y]$ belonging to $R$ since $R$ is a rectangle and to the same local unstable manifold as $y$. Then for some $N$ we are going to choose soon: let $B = A - m(A,f)$,

$$\begin{aligned} S_n B \circ f^{-n}(y) + \Delta^s(y,x) \leq S_n B \circ f^{-n}(y') &+ \Delta^s(y', x') \\ &+ \sum_{k=-n}^{N-1} |A \circ f^k(y) - A \circ f^k(y')| \qquad (= \Sigma_1) \\ &+ \sum_{k=0}^{N-1} |A \circ f^k(x) - A \circ f^k(x')| \qquad (= \Sigma_2) \\ &+ |\Delta^s(f^N(y), f^N(x))| \qquad (= \Sigma_3) \\ &+ |\Delta^s(f^N(y'), f^N(x'))| \qquad (= \Sigma_4) \end{aligned}$$

We now bound from above each $\Sigma_i$ with respect to $d(x,x')$:

$$\Sigma_1 \le C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_u^{-\alpha}} d(f^N(y), f^N(y'))^\alpha,$$
$$\Sigma_2 \le C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha}, 1-\lambda_s^\alpha)^2} \max(d(x,x')^\alpha, d(f^N(x), f^N(x'))^\alpha),$$
$$\Sigma_3 \le C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha} d(f^N(y), f^N(x)),$$
$$\Sigma_4 \le C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha} d(f^N(y'), f^N(x'))^\alpha.$$

We now choose $N = N(x,x')$ by $\lambda_s^t \epsilon^* = \Lambda_u^t d(x,x')$, $N = [t]+1$ and then choose $\tilde\epsilon \ge \epsilon^*$ so that $\lambda_s^N \tilde\epsilon = \Lambda_u^N d(x,x')$. Then

$$d(f^N(x), f^N(x')) \le C(M)\Lambda_u^N d(x,x') \le C(M)\lambda_s^N\tilde\epsilon,$$
$$d(f^N(y), f^N(x)) \text{ or } (f^N(y'), f^N(x')) \le C(M)\lambda_s^N\epsilon^* \le C(M)\lambda_s^N\tilde\epsilon.$$

In particuliar, we get first $d(f^N(y), f^N(y')) \le 3C(M)\lambda_s^N\tilde\epsilon$ and next:

$$\Sigma_1 + \cdots + \Sigma_4 \le 6C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha}, 1-\lambda_s^\alpha)^2}(\lambda_s^N\tilde\epsilon)^\alpha = K(M,f)(\lambda_s^N\tilde\epsilon)^\alpha,$$
$$S_nB\circ f^{-n}(y) + \Delta^s(y,x) \le S_nB\circ f^{-n}(y') + \Delta^s(y',x') + K(M,f)(\lambda_s^N\tilde\epsilon)^\alpha,$$
$$V(x) \le V(x') + K(M,f)(\lambda_s^N\tilde\epsilon)^\alpha.$$

But
$$\lambda_s^N\tilde\epsilon = d(x,x')^{\ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)}.$$

■

**Remark 10** We have not used explicitely the fact that the stable foliation $W^s$ is Hölder but our proof (step four) is close to showing $W^s$ is Hölder of exponent $\gamma = \ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

*Proof.* We show that if $\epsilon < \epsilon^*/K^*$, $d(x,x') \le \epsilon$, $y \in W_\epsilon^s(x)$, $y' \in W_\epsilon^s(x')$ and $y \in W_{\epsilon^*}^u(y')$ then
$$d(y,y') \le 3C(M)^2 d(x,x')^\gamma$$
where $\gamma = \ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

Indeed we choose $t > 0$ real such that $\lambda_s^t \epsilon = \Lambda_u^t d(x, x')$, $N = [t] + 1$, and $\tilde{\epsilon}$ close to $\epsilon$ so that $\lambda_s^N \tilde{\epsilon} = \Lambda_u^N d(x, x')$ where $\tilde{\epsilon}/\epsilon$ varies between 1 and $\Lambda_u/\lambda_s$. Then

$$d(f^N(x), f^N(y)) \text{ or } d(f^N(x'), f^N(y')) \text{ or } d(f^N(x), f^N(x')) \leq C(M)\lambda_s^N \tilde{\epsilon},$$
$$d(f^N(y), f^N(y')) \leq 3C(M)\lambda_s^N \tilde{\epsilon},$$
$$d(y, y') \leq 3C(M)^2 (\lambda_s/\lambda_u)^N \tilde{\epsilon} = 3C(M)^2 d(x, x')^\gamma.$$

■

# 3 Maximizing periodic measures

The proof of Theorem 4 requires two ingredients: the first one is the notion of sub-actions we have already studied, the second is the notion of strongly non-wandering points we are going to explain.

**Definition 11** Given $A \in \mathcal{C}^\beta(M)$ and $m = m(A, f)$, a point $x \in M$ is said to be strongly non-wandering with respect to $A$, if for any $\epsilon > 0$, there exist $n \geq 1$ and $y \in M$ such that

$$y \in B(x, \epsilon), \quad f^n(y) \in B(x, \epsilon) \quad \text{and} \quad |\sum_{k=0}^{n-1} (A - m) \circ f^k(y)| < \epsilon$$

where $B(x, \epsilon)$ denotes the ball centered at $x$ and radius $\epsilon$. We call $\Omega(A, f)$ the set of strongly non-wandering points.

The first non-trivial but easy observation is that $\Omega(A, f)$ is non-empty; more precisely:

**Lemma 12** *The set $\Omega(A, f)$ is compact forward and backward $f$-invariant and contains the support of any maximizing measure.*

*Proof.* If $\mu$ is maximizing, by Atkinson's theorem [1], for almost $\mu$-point $x$, the Birkhoff's sums $\sum_{k=0}^{n-1} (A - m) \circ f^k$ are recurrent ( in the sense of random walk theory ) to $\int (A - m)\, d\mu = 0$: that is, for any Borel set $B$ of positive $\mu$-measure and for any $\epsilon > 0$, the set

$$\left\{ x \in B \mid \exists\, n \geq 1 \quad f^n(x) \in B \quad \text{and} \quad |\sum_{k=0}^{n-1} (A - m) \circ f^k(x)| < \epsilon \right\}$$

has positive $\mu$-measure. Since by definition of the support of a measure, any ball $B(x,\epsilon)$ has positive $\mu$-measure, we have proved that $\text{supp}(\mu)$ is included in $\Omega(A,f)$. ■

The second observation is that any Hölder function $A$ is cohomologuous to $m(A,f)$ on $\Omega(A,f)$, more precisely:

**Lemma 13** *Let $A$ be a $\mathcal{C}^0$-function and assume $A$ admits a $\mathcal{C}^0$ sub-action $V$, then*

$$\Omega(A,f) \subseteq \Sigma_V(A,f) = \big\{x \in M \mid A - m = V \circ f - V\big\}$$

*and any $f$-invariant measure $\mu$ whose support in contained in $\Omega(A,f)$ is maximizing.*

The set $\Sigma_V(A,f)$ will play an important role later and it is convenient to to give it a name:

**Definition 14** Let $A$ be a $\mathcal{C}^0$-function and $V$ be a sub-action of $A$.

(i) We call the set $\Sigma_V(A,f) = \big\{x \in M \mid A - m = V \circ f - V\big\}$, the $V$-action-set of $A$.

(ii) Two points $x$, $y$ of the $V$-action-set are said to be $V$-connected and we shall write $x \stackrel{V}{\to} y$, if for every $\epsilon > 0$, there exist $n \geq 1$ and $z \in M$ ( not necessarily in $\Sigma_V(A,f)$ ) such that

$$x \in B(z,\epsilon), \quad y \in B(f^n(z),\epsilon), \quad |S_N(A-m)(z) - (V(y) - V(x))| < \epsilon.$$

Notice that, if $V$ is $\beta$-Hölder for some $\beta > 0$, using the shadowing lemma, one can prove that $x \stackrel{V}{\to} y$ and $y \stackrel{V}{\to} z$ imply $x \stackrel{V}{\to} z$.

*Proof of Lemma 13.* Define $R = V \circ f - V - A + m$ and choose $x \in \Omega(A,f)$. Then $\sum_{k=0}^{n_i-1}(A-m)\circ f^k(y_i)$ converges to 0 for a sequence of points $y_i$ and a sequence of integers $n_i$ such that $y_i$ converges to $x$, $n_i$ converges to $+\infty$ and $f^{n_i}(y_i)$ converges to $x$. Since $R$ is non-negative,

$$0 \leq R(y_i) \leq \sum_{k=0}^{n_i-1} R \circ f^k(y_i) = V \circ f^{n_i}(y_i) - V(y_i) - \sum_{k=0}^{n_i-1}(A-m)\circ f^k(y_i)$$

converges to 0 and by continuity of $R:\ R(x) = 0$. ■

**Definition 15** For any $\beta > 0$, define

$$\mathcal{G}_\beta = \{A \in \mathcal{C}^\beta(M) \mid \Omega(A, f) \text{ is a periodic orbit } \}.$$

Our next goal is to show that $\mathcal{G}_\beta$ is open in $\mathcal{C}^\beta$. We could have choosen a bigger set : the set of $A$ in $\mathcal{C}^\beta(M)$ such that $\Omega(A, f)$ is minimal and is dynamically isolated (i.e. there exists $U$, open, containing $\Omega(A, f)$ as the only $f$-invariant compact set inside $U$) and the proof below would again be the same.

**Lemma 16** *For any $\beta > 0$, $\mathcal{G}_\beta$ is open in $\mathcal{C}^\beta$ and $\Omega(A, f)$ is locally constant as a function of $A$ in $\mathcal{G}_\beta$.*

*Proof.* Let $A \in \mathcal{G}_\beta$. We want to show that $\Omega(A, f) = \Omega(B, f)$ whenever $B$ is sufficiently close to $A$ in the $\mathcal{C}^\beta$ topology. By contradiction : let $U$ be an isolating open set of the periodic orbit $\Omega(A, f) = \mathrm{orb}(p)$ and $\{A_n\}$ be a sequence of $\beta$-Hölder observables converging to $A$ in the $\mathcal{C}^\beta$ topology such that $\Omega(A, f)$ is not included in $U$ for each $n$.

Each $A_n$ admits (Theorem 1) a $\gamma$-Hölder subaction $V_n$ with $\gamma$-Hölder norm uniformly bounded and $\gamma = \beta \ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$. By Ascoli, $\{V_n\}$ admits a subsequence converging in the $\mathcal{C}^0$ topology to some $\gamma$-Hölder function $V$. Since the set of non-empty compact sets is compact with respect to the Hausdorff topology, we may assume that $\{\Omega(A_n, f)\}$ has a sub-sequence converging to some compact invariant set $K$. Each $A_n$ satisfies :

$$\begin{aligned} A_n - m(A_n, f) &\le V_n \circ f - V_n \qquad (\forall\, x \in M), \\ A_n - m(A_n, f) &= V_n \circ f - V_n \qquad (\forall\, x \in \Omega(A_n, f)). \end{aligned}$$

By continuity of $m(A, f)$ with respect to $A$ (for the $\mathcal{C}^0$ topology),

$$\begin{aligned} A - m(A, f) &\le V \circ f - V \qquad (\forall\, x \in M) \\ A - m(A, f) &= V \circ f - V \qquad (\forall\, x \in K). \end{aligned}$$

We have assumed that each $\Omega(A_n, f) \setminus U$ is not empty, then $K \setminus U$ is not empty too. Let $x_0 \in K \setminus U$, the $\omega$-limit set $\omega(x_0)$ and the $\alpha$-limit set $\alpha(x_0)$ of $x_0$ are compact invariant sets included in $\Omega(A, f)$, necessarily :

$$\omega(x_0) = \alpha(x_0) = \mathrm{orb}(p) \subset \overline{\mathrm{orb}(x_0)} \subset \Sigma_V(A, f).$$

Since $p$ is $V$-connected to $x_0$ and $x_0$ is $V$-connected to $p$, $x_0$ is $V$-connected to itself which is equivalent to $x_0 \in \Omega(A, f)$. We just have obtained a contradiction. ■

*Proof of Theorem 4.* Let $\beta$ given and $A$, $\alpha$-Hölder with:

$$\beta < \tilde{\beta} = \alpha \frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}.$$

According to Theorem 1, there exists $V$, $\tilde{\beta}$-Hölder, satisfying :

$$A - m \leq V \circ f - V \qquad (\forall\, x \in M).$$

Define $R = V \circ f - V - A + m$, $\phi_n = \min(R, 1/n)$ and $B_n = A + \phi_n$. Then $\phi_n$ is $\tilde{\beta}$-Hölder with $\text{Höld}_{\tilde{\beta}}(\phi_n) \leq \text{Höld}_{\tilde{\beta}}(R)$ and

$$A - m \leq B_n - m \leq V \circ f - V \qquad (\forall\, x \in M)$$
$$B_n - m = V \circ f - V \qquad (\forall\, x \in \{R < 1/n\}).$$

In particular $m(B_n, f) = m(A, f)$ and the $V$-action set of $B_n$ contains a neighborhood $\{R < 1/n\}$ of $\Omega(A, f)$. Using the shadowing lemma, we construct a periodic orbit $\text{orb}(p)$ inside $\{R < 1/n\}$ and we just have proved a perturbation $B_n$ of $A$ satisfies

$$\text{orb}(p) \cup \Omega(A, f) \subset \Omega(B_n, f).$$

Let $\psi_n$ be any $\tilde{\beta}$-Hölder function with small $\tilde{\beta}$-Hölder norm satisfying:

$$\psi_n(x) = 0 \qquad (\forall\, x \in \text{orb}(p))$$
$$\psi_n(x) > 0 \qquad (\forall\, x \in M \setminus \text{orb}(p)).$$

Then $A_n = B_n - \psi_n = A + \phi_n - \psi_n$ satisfies $\Omega(A_n, f) = \text{orb}(p)$, has small $\mathcal{C}^0$ norm and (possibly large) uniform $\tilde{\beta}$-Hölder norm. Therefore $(A_n)$ converges to $A$ in the $\mathcal{C}^\beta$-topology and each $A_n$ has a unique maximizing measure which is supported on a periodic orbit. ■

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