Exponential decay of correlation for the Stochastic Process associated to the Entropy Penalized Method

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Abstract

In this paper we present an upper bound for the decay of correlation for the stationary stochastic process associated with the Entropy Penalized Method. Let $L(x, v) : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ Lagrangian of the form

$$L(x, v) = \frac{1}{2} |v|^2 - U(x) + \langle P, v \rangle.$$ 

We point out that we do not assume more differentiability of $L$ according the the dimension of the torus $\mathbb{T}^n$.

For each value of $\epsilon$ and $h$, consider the operator

$$G[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^n} e^{-\frac{hL(x, v) + \phi(x+hv)}{\epsilon h}} dv \right],$$

as well as the reversed operator

$$\bar{G}[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^n} e^{-\frac{hL(x, hv, -v) + \phi(x+hv)}{\epsilon h}} dv \right],$$

both acting on continuous functions $\phi : \mathbb{T}^n \to \mathbb{R}$. Denote by $\phi_{\epsilon,h}$ the solution of $G[\phi_{\epsilon,h}] = \phi_{\epsilon,h} + \lambda_{\epsilon,h}$, and by $\bar{\phi}_{\epsilon,h}$ the solution of $\bar{G}[\phi_{\epsilon,h}] = \bar{\phi}_{\epsilon,h} + \lambda_{\epsilon,h}$. Let $\theta_{\epsilon,h}(x) = e^{-\frac{hL(x)}{\epsilon h}}$. From [GV], it is known that

$$\mu_{\epsilon,h}(x, v) = \theta_{\epsilon,h}(x) \gamma_{\epsilon,h}(x, v) = \theta_{\epsilon,h}(x) e^{-\frac{hL(x, v) + \phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h}},$$

is a solution to the entropy penalized problem: $\min \{ \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, v) d\mu(x, v) + \epsilon S[\mu] \}$, where the entropy $S$ is given by

$$S[\mu] = \int_{\mathbb{T}^n \times \mathbb{R}^n} \mu(x, v) \ln \frac{\mu(x, v)}{\int_{\mathbb{R}^n} \mu(x, w) dw} dx dv,$$

and the minimization is made over all holonomic probability densities on $\mathbb{T}^n \times \mathbb{R}^n$, that is probabilities that satisfy $\int \varphi(x+v) - \varphi(x) \mu(x, v) dv = 0$, for all $\varphi \in C^1(\mathbb{T}^n)$. The density $\gamma_{\epsilon,h}(x, v)$ defines a Markovian transition kernel on $(\mathbb{T}^n)^N$. The invariant initial density in $\mathbb{T}^n$ is $\theta_{\epsilon,h}(x)$. In order to analyze the decay of correlation for this process we show that the operator $L(\varphi)(x) = \int e^{-\frac{hL(x,v)}{\epsilon h}} \varphi(x+hv) dv$, has a maximal eigenvalue isolated from the rest of the spectrum.
1 Definitions and the set up of the problem

Let \( T^n \) be the \( n \)-dimensional torus. In this paper we assume that the Lagrangian, \( L(x,v) : T^n \times \mathbb{R}^N \rightarrow \mathbb{R} \) has the form

\[
L(x,v) = \frac{1}{2} |v|^2 - U(x) + \langle P, v \rangle,
\]

where \( U \in C^1(T^n) \), and \( P \in \mathbb{R}^n \) is constant.

We consider here the discrete time Aubry-Mather problem \([\text{Gom}]\) and the Entropy Penalized Mather method which provides a way to obtain approximations by continuous densities of the Aubry-Mather measure. We refer the reader to \([\text{Gom}]\) and the last section of \([\text{GLM}]\) for some of the main properties of Aubry-Mather measures, subactions, Peierl’s barrier, etc...

The Entropy Penalized Mather problem (see \([\text{GV}]\) for general properties of this problem) can be used to approximate Mather measures \([\text{CI}]\) by means of absolutely continuous densities \( \mu_{\epsilon,h}(x) \), when \( \epsilon, h \rightarrow 0 \), both in the continuous case or in the discrete case. In \([\text{GLM}]\) it is presented a Large Deviation principle associated to this procedure. We briefly mention some definitions and results.

Consider, for each value of \( \epsilon \) and \( h \), the operators acting on continuous functions \( \phi \):

\[
\mathcal{G}[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^N} e^{-h L(x,v)+\phi(x+h)v} e^{h \phi(x) - \lambda_{\epsilon,h}} dv \right],
\]

and

\[
\bar{\mathcal{G}}[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^N} e^{-h L(x-hv,v)+\phi(x-hv)} e^{h \phi(x) - \lambda_{\epsilon,h}} dv \right].
\]

Denote by \( \phi_{\epsilon,h} \) the solution of \( \mathcal{G}[\phi_{\epsilon,h}] = \phi_{\epsilon,h} + \lambda_{\epsilon,h} \), and by \( \bar{\phi}_{\epsilon,h} \) the solution of \( \bar{\mathcal{G}}[\phi_{\epsilon,h}] = \bar{\phi}_{\epsilon,h} + \lambda_{\epsilon,h} \). Let

\[
\theta_{\epsilon,h}(x) = e^{\frac{\phi_{\epsilon,h}(x) + \bar{\phi}_{\epsilon,h}(x)}{\epsilon h}}
\]

By adding a suitable constant to \( \phi_{\epsilon,h} \) or \( \bar{\phi}_{\epsilon,h} \), we can assume that \( \theta_{\epsilon,h}(x) \) is a probability density on \( T^N \). From D. Gomes and E. Valdinoci, it is known that the probability measure on \( T^N \times \mathbb{R}^N \)

\[
\mu_{\epsilon,h}(x,v) = \theta_{\epsilon,h}(x) e^{-h L(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x) - \lambda_{\epsilon,h}}
\]

is a solution to the entropy penalized Mather problem:

\[
\min_{\mathcal{M}_h} \left\{ \int_{T^N \times \mathbb{R}^N} L(x,v) d\mu(x,v) + \epsilon S[\mu] \right\},
\]

where the entropy \( S \) is given by

\[
S[\mu] = \int_{T^N \times \mathbb{R}^N} \mu(x,v) \ln \frac{\mu(x,v)}{\int_{\mathbb{R}^N} \mu(x,w) dw} dx dv,
\]

and

\[
\mathcal{M}_h := \left\{ \mu \in \mathcal{M} : \int_{T^N \times \mathbb{R}^N} \varphi(x+hv) - \varphi(x) d\mu = 0, \forall \varphi \in C(T^N) \right\}.
\]
Here $\mathcal{M}$ denotes the set of probability densities on $T^N \times \mathbb{R}^N$ and we will call $\mu \in \mathcal{M}_h$ a holonomic probability measure.

We will be interested in measures that minimize the functional below (under the holonomic constrain)
\[ \int_{T^N \times \mathbb{R}^N} L(x, v) d\mu(x, v) + \epsilon S[\mu]. \] (2)

Note that, for a probability $\mu(x, v)$ the value
\[ -S[\mu] = -\int_{T^N \times \mathbb{R}^N} \mu(x, v) \ln \mu(x, v) \frac{dx}{\int_{\mathbb{R}^N} \mu(x, w) dw} \] is not necessarily positive.

This is the entropy penalized version of the discrete time Aubry-Mather problem, see [Gom], where we look for probability measures $\mu \in \mathcal{M}_h$ that minimize the action
\[ \int_{T^N \times \mathbb{R}^N} L(x, v) d\mu(x, v) \] (3)

Definition 1: The forward (non-normalized) Perron operator $L$ is defined
\[ x \to \varphi(x) \Rightarrow x \to L(\varphi)(x) = \int e^{-L(x,v)/\epsilon} \varphi(x + hv) dv, \]

In [GV] it is shown that $L$ has a unique eigenfunction $e^{-\phi_{\epsilon,h}/\epsilon}$ with eigenvalue $e^{-\lambda_{\epsilon,h}/\epsilon}$.

Definition 2: The backward operator $N$ is given by
\[ x \to \varphi(x) \Rightarrow x \to N(\varphi)(x) = \int e^{-L(x-hv,v)/\epsilon} \varphi(x - hv) dv, \]

In [GV] it is shown that $N$ has a unique eigenfunction $e^{-\bar{\phi}_{\epsilon,h}/\epsilon}$ with eigenvalue $e^{-\lambda_{\epsilon,h}/\epsilon}$.

Definition 3: The operator
\[ g(x) \to F(g)(x) = \int e^{\frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h} g(x + hv) dv}, \]
is the normalized forward Perron operator.

From [GV] we have that given a continuous function $g : T^n \to \mathbb{R}$, then $F^n(g)$ converges to the unique eigenfunction $k$ as $m \to \infty$. We show in this paper that for $\epsilon$ and $h$ fixed, the convergence is exponentially fast.

Our notation:
\[ \theta = \theta_{\epsilon,h}(x) = e^{\frac{\phi_{\epsilon,h}(x)}{\epsilon h}}, \]
\[ \gamma(x, v) = \gamma_{\epsilon,h}(x, v) = e^{\frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h}}, \]
in such way that $\mu_{\epsilon,h} = \theta_{\epsilon,h}(x) \gamma_{\epsilon,h}(x, v)$. 

3
2 Reversed Markov Process and Adjoint Operator

In this section we define the reversed Markov process and compute the adjoint of \( F \) in \( L^2(\theta) \). We assume \( h = 1 \) from now on.

We can consider the stationary forward Markovian process \( X_n \) according to the initial probability \( \theta(x) \) and transition \( \gamma(x,v) \). For example
\[
P(X_0 \in A_0) = \int_{x \in T^n \cap A_0} \theta(x) dx,
\]
\[
P(X_0 \in A_0, X_1 \in A_1) = \int_{x \in T^n \cap A_0, (x+v) \in A_1} \theta(x) \gamma(x,v) dx dv,
\]
and so on. Define the backward transfer operator \( F^* \) acting on continuous functions \( f(x) \) by
\[
F^*(f)(x) = \int \frac{\theta(x-v) \gamma(x-v,v)}{\theta(x)} f(x-v) dv.
\]
The backward transition kernel is given by
\[
Q(x,v) = \frac{\theta(x-v) \gamma(x-v,v)}{\theta(x)}.
\]
The fact that for any \( x \) we have \( \int Q(x,v) dv = 1 \) follows from Theorem 32 in [GV]. We will show in Corollary 1 that \( \theta \) is an invariant measure for the process with transition kernel \( Q \), more precisely, that
\[
\int g d\theta = \int F^*(g) d\theta,
\]
for any \( g \in L^2(d\theta) \).

**Theorem 1.** \( F^* \) is the adjoint of \( F \) in \( L^2(\theta) \), that is for all \( f, g \in L^2(\theta) \) then
\[
\int f(x) F g(x) \theta(x) dx = \int g(x) F^* f(x) \theta(x) dx.
\]

**Proof.** Consider \( f, g \in L^2(\theta) \), then
\[
\int g(x) [F^*(f)(x)] \theta(x) dx = \int g(x) \left[ \int \frac{\theta(x-v) \gamma(x-v,v)}{\theta(x)} f(x-v) dv \right] \theta(x) dx
\]
\[
= \int g(x) \left[ \int \theta(x-v) \gamma(x-v,v) f(x-v) dv \right] dx
\]
\[
= \int \left[ \int g(x) \theta(x-v) \gamma(x-v,v) f(x-v) dx \right] dv
\]
\[
= \int \left[ \int g(x) \theta(x) \gamma(x,v) f(x) dx \right] dv
\]
\[
= \int f(x) \left[ \int \gamma(x,v) g(x+v) dv \right] \theta(x) dx
\]
\[
= \int f(x) \left[ \int e^{-\frac{L(x,v)+\phi_{1}(x+v)-\phi_{1}(x)+\lambda_{1}}{\epsilon}} g(x+v) dv \right] \theta(x) dx
\]
\[
= \int f(x) [F(g)(x)] \theta(x) dx,
\]
where we use above the change of coordinates \( x \to x - v \) and the fact that \( \mu \) is holonomic.

**Corollary 1.** Consider the inner product \( \langle \cdot, \cdot \rangle \) in \( L^2(\theta) \). Then \( \mathcal{F} \) leaves invariant the orthogonal space to the constant functions: \( \{ g \mid \langle g, 1 \rangle = \int g \, d\theta = 0 \} \). Furthermore

\[
\int g \, d\theta = \int \mathcal{F}^*(g) \, d\theta.
\]

**Proof.** Note that \( \mathcal{F}(1) = 1 \), therefore

\[
\int g \, d\theta = \int g(1) \, d\theta = \int \mathcal{F}^*(g) \, d\theta.
\]

Thus if \( \int g \, d\theta = 0 \) it follows \( \int \mathcal{F}^*(g) \, d\theta = 0 \). 

3 Spectral gap, exponential convergence and decay of correlations

From \([GV]\) it is known that \( \mathcal{L} \) has a unique (normalized) eigenfunction \( e^{-\frac{\phi_{\epsilon,h}}{h}} \) corresponding to the largest eigenvalue \( e^{-\frac{\lambda_{\epsilon,h}}{h}} \), in the next theorem we prove the this eigenvalue is separated from the rest of the spectrum.

**Theorem 2.** The largest eigenvalue of \( \mathcal{L} \) is at a positive distance from the rest of the spectrum.

**Proof.** We will prove the result for the normalized operator

\[
g(x) \to \mathcal{F}(g)(x) = \int e^{-\frac{hL(x,v)+\phi_{\epsilon,h}(x+h,v)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{h}} g(x+h,v) \, dv.
\]

Recall from \([GV]\) that the functions \( \phi_{\epsilon,h}(x) \) and \( \bar{\phi}_{\epsilon,h}(x) \) are differentiable. In this way we consider a new Lagrangian (adding \( \phi_{\epsilon,h}(x+h,v) - \phi_{\epsilon,h}(x) - \lambda_{\epsilon,h} \)) in such way \( \mathcal{L} = \mathcal{F} \). We also assume \( \epsilon = 1 \) and \( h = 1 \) from now on.

Therefore,

\[
g(x) \to \mathcal{F}(g)(x) = \int e^{-L(x,v)} g(x+v) \, dv,
\]

the eigenvalue is 1, and, by the results in \([GV]\), the corresponding eigenspace is one-dimensional and is generated by the constant functions.

Suppose there exist a sequence of \( f_p \in L^2(\theta) \), \( p \in \mathbb{N} \) such that

\[
\mathcal{F}(f_p) = \lambda_p(f_p),
\]

\( \langle f_p, 1 \rangle = 0 \), \( \lambda_p \to 1 \) and \( \|f_p\| = 1 \). If the operator is compact, then the theorem follows from the classical argument: through a subsequence \( f_p \to f \), and since \( \lambda_p \to 1 \) we have \( \mathcal{F}(f) = f \). Furthermore, since \( \langle f_p, 1 \rangle = 0 \), it follows \( \langle f, 1 \rangle = 0 \), which is a contradiction. Therefore we proceed to establish the compactness of the operator \( \mathcal{F} \).
To establish compactness, consider \( g \in L^2(\theta) \). We claim that \( f = \mathcal{F}(g) \) is in the Sobolev space \( \mathcal{H}^1 \) (see [E] for definition and properties). Indeed, for a fixed \( x \), we will compute the derivative of \( f \). Integrating by parts we have

\[
\frac{d}{dx} f(x) = \frac{d}{dx} \left( \mathcal{F}(g)(x) \right)
\]

\[
= \int \left( \frac{d}{dx} g(x + v) \right) e^{-L(x,v)} - L(x,v) \left[ \frac{d}{dx} e^{-L(x,v)} \right] g(x + v) \, dv
\]

\[
= \int \left( \frac{d}{dv} g(x + v) \right) e^{-L(x,v)} - L(x,v) \left[ \frac{d}{dx} e^{-L(x,v)} \right] g(x + v) \, dv
\]

\[
= \int \left( \frac{d}{dv} e^{-L(x,v)} \right) g(x + v) - L(x,v) \left[ \frac{d}{dx} e^{-L(x,v)} \right] g(x + v) \, dv
\]

\[
= \int \left( \frac{d}{dv} e^{-L(x,v)} \right) - L(x,v) \left[ \frac{d}{dx} e^{-L(x,v)} \right] g(x + v) \, dv.
\]

From the hypothesis about \( L \), if \( g \in L^2(\theta) \), then indeed \( \frac{d}{dx} f \) is also in \( L^2(\theta) \) (with the above derivative).

Note that, for \( v \) uniformly in a bounded set

\[
\left\| \frac{d}{dx} f \right\|_2 \leq \left\| \frac{d}{dx} g \right\|_\infty \leq \left\| \frac{d}{dv} e^{-L(x,v)} \right\|_2 \| g \|_2.
\]

Therefore, \( f \) is in the Sobolev space \( \mathcal{H}^1 \).

By iterating the procedure described above, we have that

\[
g_j = \mathcal{F}^j(g) \in \mathcal{H}^j.
\]

It is known that if \( j > \frac{n}{2} \), where \( n \) is the dimension of the torus \( \mathbb{T}^n \), then \( g_j \) is continuous H"{o}lder continuous [E]. Thus the operator \( \mathcal{F} \) is compact and \( g_j \) is differentiable for a much more larger \( j \). From the reasoning described before, \( f_p \to f \), and \( \mathcal{F}(f) = f \), \( \langle f, 1 \rangle = 0 \) and \( f \) is differentiable. It is easy to see that the modulus of concavity of \( f \) is bounded (the iteration by \( \mathcal{F} \) does not decrease it). We can add a constant to \( f \) and by linearity of \( \mathcal{F} \) we also get a new fixed point for \( \mathcal{F} \) (note that \( \mathcal{F}(1) = 1 \)). Therefore, we can assume \( f = e^{-g} \) for some \( g \).

In this way, we obtain a contradiction with the uniqueness in Theorem 26 in [GV].

Suppose \( \int g(x) \theta(x)dx = 0 \). For \( \epsilon, h \) fixed, then it follows from above that \( \mathcal{F}^n(g) \to 0 \) with exponential velocity (according to the spectral gap).

Consider the backward stationary Markov process \( Y_n \) according to the transition \( Q(x,v) \) and initial probability \( \theta \) as above.

**Theorem 3.** Given \( f(x), g(x) \) with \( \int f(x) \theta(x)dx = \int g(x) \theta(x)dx = 0 \), it follows

\[
\int g(Y_0) f(Y_n) \, dP \to 0,
\]

with exponential velocity.
Proof. Note that

\[
\int g(Y_0) f(Y_1) dP = \int g(x) \left( \int Q(x, v) f(x - v) dv \right) \theta(x) dx = \\
\int g(x) \left( F^*(f)(x) \right) \theta(x) dx = \int f(x) \left( F(g)(x) \right) \theta(x) dx.
\]

In the same way, for any \( n \)

\[
\int g(Y_0) f(Y_n) dP = \int f(x) \left( F^n(g)(x) \right) \theta(x) dx.
\]

The exponential decay of correlation follows from this.

\[\square\]

**Theorem 4.** Let \( f(x), g(x) \in L^2(\theta) \) be such that \( \int f(x) \theta(x) dx = \int g(x) \theta(x) dx = 0 \). Then

\[
\int g(X_0) f(X_n) dP \to 0,
\]

with exponential velocity.

**Proof.** Now, for analyzing the decay of the forward system, \( X_n \), with transition \( \gamma(x, v) \), we have to consider the backwark operator \( F^* \), use the fact that its exponential convergent, that is \( (F^*)^n(g) \to 0 \), if \( \int g(x) \theta(x) dx = 0 \), and the result follows in the same way.

\[\square\]

**References**


