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THE MATHER MEASURE AND A LARGE DEVIATION PRINCIPLE FOR THE ENTROPY PENALIZED METHOD

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We present the rate function and a large deviation principle for the entropy penalized Mather problem when the Lagrangian is generic (it is known that in this case the Mather measure μ is unique and the support of μ is the Aubry set). We assume the Lagrangian $L(x, v)$, with x in the torus $\mathbb{T}^N$ and $v \in \mathbb{R}^N$, satisfies certain natural hypotheses, such as superlinearity and convexity in v, as well as some technical estimates. Consider, for each value of ϵ and h, the entropy penalized Mather problem

$$\min\left\{\int_{\mathbb{T}^N\times\mathbb{R}^N} L(x,v)d\mu(x,v) + \epsilon S[\mu]\right\},$$

where the entropy S is given by $S[\mu] = \int_{\mathbb{T}^N\times\mathbb{R}^N} \mu(x,v)\ln\frac{\mu(x,v)}{\int_{\mathbb{R}^N}\mu(x,w)dw}dxdv$, and the minimization is performed over the space of probability densities $\mu(x, v)$ on $\mathbb{T}^N \times \mathbb{R}^N$ that satisfy the discrete holonomy constraint $\int_{\mathbb{T}^N\times\mathbb{R}^N} \varphi(x + hv) - \varphi(x)d\mu = 0$. It is known [17] that there exists a unique minimizing measure $\mu_{\epsilon,h}$ which converges to a Mather measure μ, as $\epsilon, h \to 0$. In the case in which the Mather measure μ is unique we prove a Large Deviation Principle for the limit $\lim_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A)$, where $A \subset \mathbb{T}^N\times\mathbb{R}^N$. In particular, we prove that the deviation function I can be written as $I(x, v) = L(x, v) + \nabla\phi_0(x)(v) - \bar{H}_0$, where ϕ_0 is the unique viscosity solution of the Hamilton–Jacobi equation, $H(\nabla\phi(x), x) = -\bar{H}_0$. We also prove a large deviation principle for the limit $\epsilon \to 0$ with fixed h.

Finally, in the last section, we study some dynamical properties of the discrete time Aubry–Mather problem, and present a proof of the existence of a separating subaction.

Keywords: Aubry–Mather measure; discrete Aubry–Mather problem; Large Deviation Principle; entropy penalized Mahler problem; viscosity solution; Hamilton–Jacobi equation; subaction.

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1. Introduction

Recently, several results concerning large deviations as well as asymptotic limits for Mather measures have appeared in the literature (see, for instance [1–4]). In this paper we will consider a related setting: the entropy penalized method introduced in [17], which was inspired by the seminal paper [1]. We study the rate of convergence of the entropy penalized Mather measures by establishing several large deviations results.

Let $\mathcal{M}$ denote the set of probability measures on $\mathbb{T}^N \times \mathbb{R}^N$. The Mather problem (see [20, 18, 8, 11]) consists in determining probability measures $\mu \in \mathcal{M}$, called Mather measures, which minimize the action

$$\int_{\mathbb{T}^N \times \mathbb{R}^N} L(x, v) d\mu(x, v), \tag{1}$$

among the probabilities $\mu \in \mathcal{M}$ that are invariant by the Euler–Lagrange flow for L. The Mather measures usually are not absolutely continuous with respect to Lebesgue measure and are supported in sets which are not attractors for the flow. In this way, given L, it is important to have computable methods that permit, in some way, to approximate the location of the support of these measures.

For the mechanical Lagrangian $L(x, v) = \frac{1}{2}|v|^2 - V(x)$, where V is the potential, the Mather measure is concentrated on the points of the form $(x_0, 0)$, where x_0 are the points of $\mathbb{T}^N$ which realize the maximum of the potential. This follows from the fact that, by one hand, the Dirac delta measure on each of these points is invariant by the Euler–Lagrange flow, and, by the other hand, the value $L(x, 0) = -V(x)$ can not be smaller. In this case, each point $(x_0, 0)$ is not stable for the Euler–Lagrange flow ([8]).

When considering the Lagrangian $L(x, v) = \frac{1}{2}|v|^2 + w_x(v)$, where w is a closed form, it is known that one can produce examples where the Mather measure has support in a unique periodic closed geodesic (which in general is hyperbolic), or, in other cases, in a geodesic lamination (see [19, 8]). In both cases one can get examples with positive and negative Lyapunov exponents.

In all these examples the Mather measure is singular with respect to the Lebesgue measure on $\mathbb{T}^N$.

For $h > 0$ fixed, in analogy with the continuous case, define the set of discrete holonomic measures as

$$\mathcal{M}_h := \left\{ \mu \in \mathcal{M}; \int_{\mathbb{T}^N \times \mathbb{R}^N} \varphi(x + hv) - \varphi(x) d\mu = 0, \forall\, \varphi \in C(\mathbb{T}^N) \right\}. \tag{2}$$

Any measure $\mu \in \mathcal{M}_h$ is called a discrete holonomic measure. We denote by $\mathcal{M}_h^{a.c.}$ the measures in $\mathcal{M}_h$ which admit a density.

The discrete time Aubry–Mather problem, see [13], consists in determining probability measures $\mu \in \mathcal{M}_h$ that minimize the action

$$\int_{\mathbb{T}^N \times \mathbb{R}^N} L(x, v) d\mu(x, v). \tag{3}$$

Motivated by the papers [1, 2], the entropy penalized method was introduced in [17] in order to approximate Mather measures by smooth densities. The entropy penalized Mather problem, for $\epsilon > 0$ and $h > 0$ fixed, consists in

$$\min_{\mathcal{M}_h^{a.c.}} \left\{ \int_{\mathbb{T}^N \times \mathbb{R}^N} L(x,v) d\mu(x,v) + \epsilon S[\mu] \right\},$$

where the entropy S is given by

$$S[\mu] = \int_{\mathbb{T}^N \times \mathbb{R}^N} \mu(x,v) \ln \frac{\mu(x,v)}{\int_{\mathbb{R}^N} \mu(x,w) dw} dx dv,$$

The entropy penalized method can be seen as a procedure to approximate Mather measures by absolutely continuous probability measures. These measures can be obtained as a fixed point of an operator $\mathcal{G}$, to be described later, from a discrete time process with small parameters ϵ, h. Furthermore, this fixed point can be obtained by means of iteration of the operator $\mathcal{G}$. In [16] it is shown that, for ϵ and h fixed, the velocity of convergence to the fixed point is exponentially fast.

In this paper, we assume that the Lagrangian L is such that the Mather measure is unique. Then, it follows from a result by Gomes and Valdinoci [17] that $\mu_{\epsilon,h}$ (the solution of the entropy penalized problem) converges to a discrete Mather measure μ_h, i.e. a measure that minimizes (3) over $\mathcal{M}_h$. Furthermore, by a result of Gomes (see [13, 7]), with the Lagrangian satisfying some hypotheses to be stated in the next section, the sequence of measures μ_h converges, through a subsequence, to the Mather measure μ. Hence $\mu_{\epsilon,h}$ converges, through a subsequence, to μ.

We address here the question of estimating how good is this approximation. In this way, it is natural to consider a Large Deviation Principle (L.D.P. for short) for such limit. We refer the reader to [6] for general properties of large deviation theory.

We start in the next section by describing briefly the entropy penalized Mather measure problem, as well as stating some of the results, such as the uniform semi-concavity estimates, that we will need throughout the paper. We refer the reader to [5] for general results concerning semiconcavity. In this section we also generalize a result by Gomes and Valdinoci which shows the existence, for each ϵ and h, of a density of probability $\mu_{\epsilon,h}$ on $\mathbb{T}^N \times \mathbb{R}^N$ which solves the entropy penalized Mather problem. This generalization is essential for the large deviation results later in the paper.

In the two next sections we consider Large Deviation Principles in the following three forms:

Firstly, for h fixed, as $\mu_{\epsilon,h} \rightharpoonup \mu_h$, we show the existence of a rate function I_h such that,

(a) If $A \subset \mathbb{T}^N \times \mathbb{R}^N$ is a closed (respectively open) and bounded set, then

$$\lim_{\epsilon \to 0} \epsilon \ln \mu_{\epsilon,h}(A) \leq -\inf_A I_h(x,v) \quad \text{(respectively } \geq).$$

In order to prove this result, we also need to study some dynamical properties of the discrete time Aubry–Mather problem, namely, the uniqueness of the calibrated subaction for the discrete time problem. Because of its independent interest, we present these results in a separate section in the end of the paper.

For our second large deviation result, we assume that the Mather measure is unique and the support of this measure is the Aubry set, hence there exists only one viscosity solution, say ϕ_0. Then, as $\mu_{\epsilon,h} \rightharpoonup \mu$,

(b) If $A \subset \mathbb{T}^N \times \mathbb{R}^N$ is a closed (respectively open) and bounded set such that $\pi_1(A) \cap \mathcal{A} \neq \emptyset$, where $\mathcal{A}$ is the projected Aubry set, then there exists a function $I(x,v)$ such that

$$\lim_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) \leq -\inf_A I(x,v) \quad \text{(respectively } \geq).$$

In this case we show that the deviation function I is given by

$$I(x,v) = L(x,v) + \nabla\phi_0(x)(v) - \bar{H}_0,$$

where $\bar{H}_0$ is the Mañé's critical value.

We point out that we just consider $I(x,v)$ for the points x where $\nabla\phi_0(x)$ is defined. For the others points x we declare $I(x,v) = \infty$. We remark that $\mu_{\epsilon,h}$ is absolutely continuous with respect to Lebesgue measure on the tangent bundle $\mathbb{T}^N \times \mathbb{R}^N$, and, as $\nabla\phi_0$ is Lipschitz, $\nabla\phi_0$ is differentiable almost every where in the compact manifold $\mathbb{T}^N$ where lives the x variable. In this way, all points (x,v) we consider in the support of $\mu_{\epsilon,h}$ are assumed to be such that $\nabla\phi_0(x)$ is defined.

Finally, the last case is:

(c) If $A \subset \mathbb{T}^N \times \mathbb{R}^N$ is a closed (open) and bounded set such that $\pi_1(A) \cap \mathcal{A} = \emptyset$ we will show a L.D.P. which yields an estimate for the convergence rate of

$$\lim_{\epsilon,h\to 0} \epsilon h \ln \mu_{\epsilon,h}(A).$$

In the last section we study the discrete Aubry–Mather problem under the point of view of subactions, i.e. continuous functions that satisfy

$$u(x) - u(x+hv) \leq h(L(x,v) - \bar{H}_h) \quad \forall\, (x,v) \in \mathbb{T}^N \times \mathbb{R}^N \tag{4}$$

for each $h > 0$ fixed, where $\bar{H}_h$ is the analog of the Mañé's critical value, i.e.

$$\bar{H}_h = \min_{\mathcal{M}_h} \left\{ \int_{\mathbb{T}^N \times \mathbb{R}^N} L(x,v) d\mu(x,v) \right\}.$$

There exist two important classes of subactions in which we are interested. The first class is composed of the calibrated subactions, those such that

$$u(x) = \inf_{v\in\mathbb{R}^N} \{u(x+hv) + hL(x,v) - h\bar{H}_h\}.$$

The second class of subactions consists in the separating subactions, that is, those for which the equality in (4) is attained for some (x, v) if, and only if $x \in \Omega(L)$ (this set will be defined in the last section).

Under the hypothesis that the Lagrangian is generic, we will show that there exists only one calibrated subaction, which gives the uniqueness of the deviation function I_h. Furthermore, we will establish the existence of a separating subaction, which can be considered as discrete analog of the main result of [12].

By the way, we point out that according to [12] we can add to the Lagrangian $L(x, v)$ a term $d\varphi$, where φ is differentiable C^1, in such way that the Mather measures for $\hat{L} = L + d\varphi$ are the same as for L, $\bar{H}_0$ is the same, and, moreover

$$\hat{I}(x, v) = \hat{L}(x, v) + \nabla\phi_0(x)(v) - \bar{H}_0 = 0,$$

if and only if, (x, v) is in the support of the Mather measure.

2. The Entropy Penalized Mather Problem

In [17], the Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ has the form

$$L(x, v) = K(v) - U(x), \quad \text{for } v \in \mathbb{R}^N, \ x \in \mathbb{R}^N,$$

in which K is strictly convex in v and superlinear at infinity, and the potential energy U is bounded, $\mathbb{Z}^N$-periodic and semiconvex, that is, there exists $C_U > 0$ such that

$$\inf_{x,y \in \mathbb{R}^N, y \neq 0} \frac{U(x+y) + U(x-y) - 2U(x)}{|y|^2} \geq -C_U.$$

Furthermore, K is semiconcave, i.e. that there exists C_K such that

$$\sup_{v,w \in \mathbb{R}^N, w \neq 0} \frac{K(v+w) + K(v-w) - 2K(v)}{|w|^2} \leq C_K.$$

In this work, we will need to work in slightly generalized setting. The main reason is that even if the Lagrangian has the form $L(x, v) = K(v) - U(x)$, the time-reversed Lagrangian $L(x + hv, -v)$ will not have this form in general. The time-reversed Lagrangian, however, arises naturally in our problems. Therefore we need to modify our hypothesis accordingly.

We will assume in the whole paper that the Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, is $\mathbb{Z}^N$-periodic (we can consider it as a function $L : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$), and satisfies the following estimates:

(1) Uniform superlinearity:

$$\lim_{|v| \to \infty} \frac{L(x, v)}{|v|} = +\infty, \quad \text{uniformly on } x \in \mathbb{T}^N.$$

(2) Convexity in v: the Hessian matrix $\frac{\partial^2 L}{\partial v_i \partial v_j}(x, v)$ is positive definite.

(3) There exist uniform constants $C, \Gamma > 0$ such that

$$L(x+y, v-z) + L(x-y, v+z) - 2L(x,v) \le C|y|^2 + \Gamma|z|^2.$$

We consider here, the optimal control setting, where

$$H(p,x) = \sup_v (-p \cdot v - L(x,v)).$$

Remark. In the Classical Mechanics setting, we usually define the Hamiltonian in a different way, that is

$$H(p,x) = \sup_v (p \cdot v - L(x,v)).$$

These two definitions differ by the sign of $p \cdot v$. And they are related in the following way: if, instead of $L(x,v)$, we begin with the symmetrical Lagrangian, i.e. $\check{L}(x,v) = L(x,-v)$ (see [11, § 4.5]), then

$$\check{H}(p,x) = \max_v \{p \cdot v - \check{L}(x,v)\} = \max_v \{-p \cdot v - L(x,v)\}.$$

Therefore, the results presented here also hold, of course, in the Classical Mechanics setting of Aubry–Mather theory.

Consider, for each value of ϵ and h, the following operators acting on continuous functions $\phi : \mathbb{T}^N \to \mathbb{R}$:

$$\mathcal{G}[\phi](x) := -\epsilon h \ln \left[\int_{\mathbb{R}^N} e^{-\frac{hL(x,v)+\phi(x+hv)}{\epsilon h}} dv \right],$$

and

$$\bar{\mathcal{G}}[\phi](x) := -\epsilon h \ln \left[\int_{\mathbb{R}^N} e^{-\frac{hL(x-hv,v)+\phi(x-hv)}{\epsilon h}} dv \right].$$

We point out that the ϵ in [17] correspond here to ϵh.

Remark. Let $\bar{L}$ be the Lagrangian given by $\bar{L}(x,v) = L(x+hv, -v)$, we have that $\bar{\mathcal{G}}$ is the operator $\mathcal{G}$ for the Lagrangian $\bar{L}$. Hence, it is enough to prove the properties we need for $\mathcal{G}$.

Theorem 1. *Suppose L satisfies assumptions* (1)–(3) *above. Then for ϵ and h fixed there exist $\lambda_{\epsilon,h} \in \mathbb{R}$ and $\mathbb{Z}^N$-periodic Lipschitz functions $\phi_{\epsilon,h}, \bar{\phi}_{\epsilon,h}$ so that*

$$\mathcal{G}[\phi_{\epsilon,h}] = \phi_{\epsilon,h} + \lambda_{\epsilon,h}, \tag{5}$$

and

$$\bar{\mathcal{G}}[\bar{\phi}_{\epsilon,h}] = \bar{\phi}_{\epsilon,h} + \lambda_{\epsilon,h}. \tag{6}$$

Also there exists a constant $\bar{C}$ such that the semiconcavity modulus of $\phi_{\epsilon,h}$ and $\bar{\phi}_{\epsilon,h}$ is bounded by $\bar{C}$ for all ϵ and all h sufficiently small.

Proof. We need to generalize the proof of [17, Theorem 13] to a slightly more general setting. We recall that the proof in [17] works only for $L(x,v) = K(v) - U(x)$, with suitable semiconcavity/semiconvexity on K and U.

Let u be a function with semiconcavity modulus smaller than σ. We will show that for a suitable σ, the image $\mathcal{G}(u)$ has also modulus of concavity smaller than σ.

Because $\mathcal{G}$ commutes with constants, we can look at fixed points modulus constants. The set of functions with semiconcavity modulus bounded by σ is invariant by $\mathcal{G}$. When quotiented by the constants this set is compact and therefore $\mathcal{G}$ admits a fixed point modulo constants, which is precisely the result of the theorem.

Consider

$$u_1(x) := -\epsilon h \ln \int e^{-\frac{hL(x,v)+u(x+hv)-\lambda_{\epsilon,h}}{\epsilon h}} dv,$$

$$u_1(x+hy) \;=\; -\epsilon h \ln \int e^{-\frac{hL(x+hy,v)+u(x+hy+hv)-\lambda_{\epsilon,h}}{\epsilon h}} dv,$$

and

$$u_1(x-hy) = -\epsilon h \ln \int e^{-\frac{hL(x-hy,v)+u(x-hy+hv)-\lambda_{\epsilon,h}}{\epsilon h}} dv.$$

Let $0<\theta<1$, and $t=1-\theta$. Using the change of coordinates $v\to v-\theta y$, we can write the second equation as

$$u_1(x+hy) = -\epsilon h \ln \int e^{-\frac{hL(x+hy,v-\theta y)+u(x+hty+hv)-\lambda_{\epsilon,h}}{\epsilon h}} dv,$$

whereas the third equation, through the change of coordinates $v\to v+\theta y$, can be written as

$$u_1(x-hy) = -\epsilon h \ln \int e^{-\frac{hL(x-hy,v+\theta y)+u(x-thy+hv)-\lambda_{\epsilon,h}}{\epsilon h}} dv.$$

Now using the hypothesis (3) of the Lagrangian L, we get

$$L(x+hy,v-\theta y)+L(x-hy,v+\theta y)-2L(x,v)\le Ch^2|y|^2+\Gamma\theta^2|y|^2.$$

We want to estimate the modulus of concavity of u_1 knowing that

$$u(x+hty)+u(x-hty)-2u(x)\le \sigma h^2t^2|y|^2.$$

It is also true that

$$u(x+hty+hv)+u(x-hty+hv)-2u(x+hv)\le \sigma h^2t^2|y|^2.$$

Hence using the concavity estimate of u, we can write

$$\begin{aligned}
u_1(x) &= -\epsilon h \ln \int e^{-\frac{hL(x,v)+u(x+hv)-\lambda_{\epsilon,h}}{\epsilon h}} dv \\
&\ge -\epsilon h \ln \int e^{-\frac{\substack{[hL(x+hy,v-\theta y)+u(x+hty+hv)-\lambda_{\epsilon,h}]+[hL(x-hy,v+\theta y)\\ +u(x-hty+hv)-\lambda_{\epsilon,h}]+[-Ch^3-h\Gamma\theta^2-\sigma_u h^2t^2]|y|^2}}{2\epsilon h}} dv \\
&= -\epsilon h \ln \int e^{-\frac{\substack{[\frac12(hL(x+hy,v-\theta y)+u(x+thy+hv)-\lambda_{\epsilon,h})\\ +\frac12(hL(x-hy,v+\theta y)+u(x-thy+hv)-\lambda_{\epsilon,h})]}}{\epsilon h}} dv \\
&\quad -\left[\frac{Ch}{2}+\frac{\sigma_u t^2}{2}\right]h^2|y|^2-\frac{\Gamma}{2}\theta^2|y|^2h.
\end{aligned}$$

By Cauchy–Schwartz inequality, we know that given functions a, b we have

$$\int ab \leq \left(\int a^2\right)^{\frac{1}{2}} \left(\int b^2\right)^{\frac{1}{2}},$$

hence using the expressions of $u_1(x + hy)$ and $u_1(x - hy)$ we obtain

$$u_1(x) \geq \frac{1}{2}(u_1(x + hy) + u_1(x - hy)) - \left[\frac{Ch}{2} + \frac{\sigma_u t^2}{2}\right] h^2|y|^2 - \frac{\Gamma}{2}\theta^2|y|^2 h.$$

Therefore the semiconcavity modulus of u_1 is $\sigma_{u_1} = Ch + \sigma_u t^2 + \Gamma\theta^2/h$.

We want to choose a upper bound to the semiconcavity modulus of u such that the semiconcavity modulus of u_1 is also smaller then this upper bound. We claim that $\bar{C} = C + \Gamma$ is the bound which we are looking for. Indeed, suppose $\sigma_u < \bar{C}$, by choosing $\theta = h$, and taking h small we have that

$$\sigma_{u_1} = Ch + \sigma t^2 + \Gamma h \leq (C + \Gamma)h + (C + \Gamma)(1 - h)^2 \leq C + \Gamma = \bar{C}.$$

Hence, as in [17, Theorem 26], there exist a Lipschitz function $\phi_{\epsilon,h}$ and $\lambda_{\epsilon,h} \in \mathbb{R}$ such that

$$\mathcal{G}[\phi_{\epsilon,h}] = \phi_{\epsilon,h} + \lambda_{\epsilon,h},$$

also the semiconcavity modulus of $\phi_{\epsilon,h}$ is smaller than $\bar{C}$ for all ϵ and h. □

Remark. It is easy to see that if we add a constant to each $\phi_{\epsilon,h}$ and $\bar{\phi}_{\epsilon,h}$, Eqs. (5) and (6) are also satisfied. Then, for each ϵ and h, we choose a pair of functions $\phi_{\epsilon,h}$ and $\bar{\phi}_{\epsilon,h}$ and define a new pair of uniformly bounded functions $\tilde{\phi}_{\epsilon,h} := \phi_{\epsilon,h} - \phi_{\epsilon,h}(0)$ and $\tilde{\bar{\phi}}_{\epsilon,h} := \bar{\phi}_{\epsilon,h} + c_{\epsilon,h}$ such that

$$\int_{\mathbb{T}^N} e^{-\frac{\tilde{\bar{\phi}}_{\epsilon,h}(x) + \tilde{\phi}_{\epsilon,h}(x)}{\epsilon h}} dx = 1. \tag{7}$$

As the functions $\tilde{\phi}_{\epsilon,h}, \tilde{\bar{\phi}}_{\epsilon,h}$ are uniformly Lipschitz in ϵ and h, we have that $\tilde{\phi}_{\epsilon,h}$ is uniformly bounded. Moreover, because $\tilde{\bar{\phi}}_{\epsilon,h}$ must satisfy Eq. (7), we get that $\tilde{\bar{\phi}}_{\epsilon,h}$ is also uniformly bounded in ϵ and h. From now on we will drop the symbol $\tilde{}$.

We note that most of the results in [17] do not assume the Lagrangian is of the form $L(x, v) = K(v) - U(x)$. All the results we need from [17] are true under the hypotheses (1)–(3) we mentioned above:

Theorem 2. *Let $\phi_{\epsilon,h}, \bar{\phi}_{\epsilon,h}$ and $\lambda_{\epsilon,h}$ given by Theorem 1. Also suppose that $\phi_{\epsilon,h}$ and $\bar{\phi}_{\epsilon,h}$ are uniformly bounded and satisfy (7).*

We define $\theta_{\epsilon,h} : \mathbb{T}^N \to \mathbb{R}$ as

$$\theta_{\epsilon,h}(x) = e^{-\frac{\bar{\phi}_{\epsilon,h}(x) + \phi_{\epsilon,h}(x)}{\epsilon h}}.$$

Then, the probability density

$$\mu_{\epsilon,h}(x, v) = \theta_{\epsilon,h}(x) e^{-\frac{hL(x,v) + \phi_{\epsilon,h}(x+hv) - \phi_{\epsilon,h}(x) - \lambda_{\epsilon,h}}{\epsilon h}}$$

minimizes the functional

$$\int_{\mathbb{T}^N \times \mathbb{R}^N} L(x,v)d\mu(x,v) + \epsilon S[\mu]$$

over the densities in $\mathcal{M}_h$.

Proof. Indeed, $\theta_{\epsilon,h}$ satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^N} \theta_{\epsilon,h}(x-hv)e^{-\frac{hL(x-hv,v)+\phi_{\epsilon,h}(x)-\phi_{\epsilon,h}(x-hv)-\lambda_{\epsilon,h}}{\epsilon h}}dv \\
&= \int_{\mathbb{R}^N} e^{-\frac{\bar{\phi}_{\epsilon,h}(x-hv)+\phi_{\epsilon,h}(x-hv)}{\epsilon h}} e^{-\frac{hL(x-hv,v)+\phi_{\epsilon,h}(x)-\phi_{\epsilon,h}(x-hv)-\lambda_{\epsilon,h}}{\epsilon h}}dv \\
&= e^{-\frac{\phi_{\epsilon,h}(x)}{\epsilon h}} \int_{\mathbb{R}^N} e^{-\frac{\bar{\phi}_{\epsilon,h}(x-hv)}{\epsilon h}} e^{-\frac{hL(x-hv,v)-\lambda_{\epsilon,h}}{\epsilon h}}dv \\
&= e^{-\frac{\phi_{\epsilon,h}(x)}{\epsilon h}} e^{-\frac{\bar{\phi}_{\epsilon,h}(x)}{\epsilon h}} = \theta_{\epsilon,h}(x).
\end{aligned}$$

Therefore, from [17, Theorem 32] the result follows. □

Theorem 3. *Let* $\phi_{\epsilon,h}, \bar{\phi}_{\epsilon,h}$ *and* $\lambda_{\epsilon,h}$ *given by Theorem 1. Also suppose that* $\phi_{\epsilon,h}$ *and* $\bar{\phi}_{\epsilon,h}$ *are uniformly bounded and satisfy* (7). *Then, for* h *fixed, when* $\epsilon \to 0$, *we have*

(a)

$$\bar{H}_{\epsilon,h} := \int_{\mathbb{T}^N \times \mathbb{R}^N} L(x,v)d\mu_{\epsilon,h}(x,v) + \epsilon S[\mu_{\epsilon,h}] = \frac{\lambda_{\epsilon,h}}{h}$$

and $\bar{H}_{\epsilon,h} \to \bar{H}_h$.

(b) *Through some subsequence,* $\phi_{\epsilon,h} \to \phi_h, \bar{\phi}_{\epsilon,h} \to \bar{\phi}_h$ *uniformly.* $\phi_h, \bar{\phi}_h$ *are semiconcave functions, with the semiconcavity constant bounded by* $\bar{C}$ *(as in Theorem 1), and satisfy*

$$\phi_h(x) = \inf_{v\in\mathbb{R}^N} \{\phi_h(x+hv) + hL(x,v) - h\bar{H}_h\} \tag{8}$$

and

$$\bar{\phi}_h(x) = \inf_{v\in\mathbb{R}^N} \{\bar{\phi}_h(x-hv) + hL(x-hv,v) - h\bar{H}_h\}. \tag{9}$$

(c) $\mu_{\epsilon,h} \rightharpoonup \mu_h$, *where* μ_h *is a discrete Mather measure.*

Proof. From [17, Theorems 37 and 38] and also by Theorem 2 we obtain item (a), by [17, Theorems 39 and 40] we get, respectively, (b) and (c). □

If we use the so called Hopf–Cole transformation $\phi \to e^{-\frac{\phi}{\epsilon h}} = \varphi$, the setting above can be written as the search for the eigenfunction associated to the largest eigenvalue of the Perron operator $\varphi \to \mathcal{L}(\varphi)$ acting on continuous functions φ

$$x \to \varphi(x) \Rightarrow x \to \mathcal{L}(\varphi)(x) = \int e^{-\frac{L(x,v)}{\epsilon}} \varphi(x+hv)dv.$$

The largest eigenvalue of this operator is (see [17, Corollary 27]) $e^{-\frac{\lambda_{\epsilon,h}}{\epsilon h}}$.

Definition 1. A property P is said to be generic for the Lagrangian L if there exists a generic set $\mathcal{O}$ (in the Baire sense) on the set $C^\infty(\mathbb{T}^N, \mathbb{R})$ such that if ψ is in $\mathcal{O}$ then $L + \psi$ has property P.

Theorem 4. *Given a Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(\mathbb{T}^N)$ such that*

(a) *If $\psi \in \mathcal{O}$ then there exists only one Mather measure for $L + \psi$, such measure μ is uniquely ergodic.*
(b) $\mathrm{supp}(\mu) = \hat{\mathcal{A}}(L + \psi)$, *where $\hat{\mathcal{A}}$ is the Aubry set.*

The proof of this Theorem can be found in [9].

Assumption. We will suppose that the Lagrangian $L(x, v)$ is generic, i.e. the Mather measure is unique, which we will denote by μ, and $\mathrm{supp}(\mu) = \hat{\mathcal{A}}(L)$.

Remark. As we suppose the Lagrangian is generic, we have only one static class, and the Mather measure is ergodic. Then by [8, Corollary 4-8.5] we know that the set of weak-KAM solutions (positive and negative) are unitary, modulo an additive constant. It can be shown, see [11], that $-\phi$ is a positive weak-KAM solution, if and only if, ϕ is a viscosity solution of $H(\nabla\phi(x), x) = -\bar{H}_0$ (remember we are using the definition $H(p, x) = \sup_v(-p \cdot v - L(x, v))$, and $\bar{\phi}$ is a negative weak-KAM solution, if and only if, $\bar{\phi}$ is a viscosity solution of $H(-\nabla\bar{\phi}(x), x) = -\bar{H}_0$.

Let us call ϕ_0 and $\bar{\phi}_0$, the unique viscosity solutions of $H(\nabla\phi(x), x) = -\bar{H}_0$ and $H(-\nabla\bar{\phi}(x), x) = -\bar{H}_0$, respectively.

Applying the [11, Corollary 5.3.7] and the remark above, we obtain:

Corollary 1. *Suppose that the Lagrangian L is generic, then we have that*

$$\phi_0(x) + \bar{\phi}_0(x) = h(x, x),$$

where h is the Peierls barrier.

Theorem 5. *Let $L(x, v)$ be a generic Lagrangian that satisfies the hypotheses* (1)–(3) *above. For each h, let $\phi_h, \bar{\phi}_h$ be the functions, μ_h be the measure, and $\bar{H}_h$ be the constant that are given in Theorem 3. Then, when $h \to 0$ we have*

(a) $\bar{H}_h \to \bar{H}_0 = \int_{\mathbb{T}^N \times \mathbb{R}^N} L(x, v) d\mu$,
(b) *Through some subsequence, $\phi_h \to \phi_0$ and $\bar{\phi}_h \to \bar{\phi}_0$, uniformly,*
(c) $\mu_h \rightharpoonup \mu$.

Proof. (a) See [10, 17, 13]. In order to apply [13, Theorems 7.2–7.4] we need the following remark: as the Lagrangian satisfies the hypothesis (3) we have, by item (b) of Theorem 3, that ϕ_h and $\bar{\phi}_h$ are uniformly semiconcave in h. Let

Λ be the uniform Lipschitz constant. We claim that each $v(x) = v_h(x)$ that achieves the infimum Eq. (8) is uniformly bounded in h. Indeed,

$$|L(x, v(x)) + \bar{H}_h| = \left| \frac{u(x) - u(x + hv)}{h} \right| \leq \Lambda |v(x)|,$$

then, because the Lagrangian is superlinear and we have (a), we conclude that $|v(x)| \leq K$ for some constant K that depends only on the Lagrangian L.

(b) Just note that ϕ_h and $\bar{\phi}_h$ are uniformly bounded, because they are limits of the functions $\phi_{\epsilon,h}$ that are uniformly bounded in ϵ and h, hence we can apply [13, Theorem 7.2].

(c) See [13, Theorems 7.3 and 7.4]. □

Theorem 6. *Let $L(x, v)$ be a generic Lagrangian that satisfies hypotheses* (1)–(3) *above. Suppose that $\phi_{\epsilon,h}$ and $\bar{\phi}_{\epsilon,h}$ given by Theorem 1 are uniformly bounded and satisfy* (7)*. Then, through some subsequence,*

$$\phi_{\epsilon,h} \to \phi_0 \quad \textit{and} \quad \bar{\phi}_{\epsilon,h} \to \bar{\phi}_0.$$

Proof. By item (b) of Theorem 5, we know that any collection $\overline{\{\phi_h\}}_{h\in[0,1]}$ of solutions of the $\epsilon = 0$ problem is a compact set, then if we take a sequence $\{\phi_{h_i}\}_{i\in\mathbb{N}}$ it has a subsequence that converges to ϕ_0, i.e. there exists a set $\mathcal{H}$ such that

$$\lim_{h_i \in \mathcal{H}} \phi_{h_i} = \phi_0.$$

We know by [17, Theorem 39], for each $h_i \in \mathcal{H}$ fixed (as ϕ_{ϵ,h_i} are normalized), that $\overline{\{\phi_{\epsilon,h_i}\}}_{\epsilon\in[0,1]}$ is a compact set. Then if we fix $h_1 \in \mathcal{H}$ and a sequence $\{\phi_{\epsilon_i,h_1}\}_{i\in\mathbb{N}}$, then there exists a set $\mathcal{E}_{h_1}$ such that

$$\lim_{\epsilon_i \in \mathcal{E}_{h_1}} \phi_{\epsilon_i,h_1} = \phi_{h_1}.$$

Then, if we do this for each $h_i \in \mathcal{H}$, we can find a set $\mathcal{E}_{h_i} \subset \cdots \subset \mathcal{E}_{h_2} \subset \mathcal{E}_{h_1}$. Now we define a set $\mathcal{E}$ such that the ith element of $\mathcal{E}$ is the ith element of $\mathcal{E}_i$. The set $\mathcal{E}$ has the property that

$$\lim_{\epsilon_i \in \mathcal{E}} \phi_{\epsilon_i,h_j} = \phi_{h_j} \quad \text{for each } h_j \in \mathcal{H}.$$

Finally, we have that

$$\lim_{i\to\infty} \phi_{\epsilon_i,h_i} = \phi_0.$$ □

3. A Large Deviation Principle: h Fixed and $\epsilon \to 0$

Lemma 1 (Laplace Method). *If $f_k(x, v) \to f_0(x, v)$ uniformly as $k \to 0$, then for each $A \subset \mathbb{T}^N \times \mathbb{R}^N$ closed bounded set, we have*

$$\limsup_{k\to 0} \; k \ln \int_A e^{-\frac{f_k(x,v)}{k}} dxdv \leq -\inf_A f_0(x, v),$$

and for each $A \subset \mathbb{T}^N \times \mathbb{R}^N$ *open bounded set, we have*

$$\liminf_{k \to 0} k \ln \int_A e^{-\frac{f_k(x,v)}{k}} dxdv \geq -\inf_A f_0(x,v).$$

Let us define,

$$f_{\epsilon,h}(x,v) = \frac{\bar{\phi}_{\epsilon,h}(x) + \phi_{\epsilon,h}(x)}{h} + L(x,v) + \frac{\phi_{\epsilon,h}(x+hv) - \phi_{\epsilon,h}(x)}{h} - \bar{H}_{\epsilon,h},$$

and

$$I_h(x,v) = \frac{\bar{\phi}_h(x) + \phi_h(x+hv)}{h} + L(x,v) - \bar{H}_h, \quad \text{for any } (x,v).$$

In order to have I_h defined in a unique way we need the uniqueness of ϕ_h and $\bar{\phi}_h$. In the last section we will show a sufficient condition to that.

Theorem 7. *Consider* $A \subset \mathbb{T}^N \times \mathbb{R}^N$ *a closed (respectively, open) bounded set, then*

$$\begin{aligned}\lim_{\epsilon \to 0} \epsilon \ln \mu_{\epsilon,h}(A) &= \lim_{\epsilon \to 0} \epsilon \ln \int_A e^{-\frac{f_{\epsilon,h}(x,v)}{\epsilon}} dxdv \\ &\leq - \inf_{(x,v) \in A} I_h(x,v) \quad (\textit{respectively } \geq).\end{aligned}$$

Proof. As for h fixed, the convergence of $\phi_{\epsilon,h}$ and $\bar{\phi}_{\epsilon,h}$, with $\epsilon \to 0$, to respectively, ϕ_h and $\bar{\phi}_h$, is uniform by item (b) of Theorem 3. Then, the proof follows from Lemma 1 (Laplace method). □

4. A Large Deviation Principle: $\epsilon, h \to 0$

Thanks to [12] we can assume the Lagrangian L we consider here satisfies the property that $I(x,v) = 0$, if and only if, (x,v) is in the support of the Mather measure μ.

Note that by Theorem 6 there exists a sequence $\{\epsilon_i, h_i\}_{i \in \mathbb{N}}$ such that $\epsilon_i, h_i \to 0$ and $\lim_{i \to \infty} \phi_{\epsilon_i,h_i} = \phi_0$, and $\lim_{i \to \infty} \bar{\phi}_{\epsilon_i,h_i} = \bar{\phi}_0$. For convenience we will write $\lim_{\epsilon,h \to 0}$ when we want to mean $\lim_{\epsilon_i,h_i \to 0}$.

All the results that we will obtain will be independent of the particular sequence we choose, because ϕ_0 and $\bar{\phi}_0$ are uniquely determined.

Theorem 8. *If* $x \in \operatorname{dom}(\nabla \phi_0)$, *then we have*

$$\lim_{\epsilon,h \to 0} \frac{\phi_{\epsilon,h}(x+hv) - \phi_{\epsilon,h}(x)}{h} = \nabla \phi_0(x)(v),$$

uniformly in each closed bounded subset of $\operatorname{dom}(\nabla \phi_0) \times \mathbb{R}^N$.

To prove Theorem 8 we need the following properties of semiconcave functions (see [5, Chap. 3]).

Proposition 1. *Let $u : \mathbb{T}^N \to \mathbb{R}$ be a semiconcave function. Given $x, y \in \mathbb{T}^N$ there exist $\xi \in]x, y[$ and $p \in D^+u(\xi)$ such that $u(y) - u(x) = p \cdot (y - x)$, where $D^+u(x)$ is the superdifferential of u at x.*

Proposition 2. *Let $u : \mathbb{T}^N \to \mathbb{R}$ be a semiconcave function with semiconcavity modulus C, and let $x \in \mathbb{T}^N$. Then, a vector $p \in \mathbb{R}^N$ belongs to $D^+u(x)$ if and only if*

$$u(y) - u(x) \leq p \cdot (y - x) + \frac{C}{2}|y - x|^2$$

for any point $y \in \mathbb{T}^N$.

Proof of Theorem 8. By Theorem 1, the functions $\phi_{\epsilon,h}$ are semiconcave with semiconcavity modulus uniformly bounded by some constant $\bar{C}$. Let $\{\epsilon_i, h_i\}_{i\in\mathbb{N}}$ be a sequence such that $\phi_{\epsilon_i,h_i} \to \phi_0$.

Let K be a closed bounded subset of $\mathrm{dom}(\nabla\phi_0) \times \mathbb{R}^N$. Hence, by Propositions 1 and 2, for each $(x, v) \in K$, and each ϵ_i and h_i there exist $\xi_{\epsilon_i,h_i}(x, v) \in]x, x + h_i v[$ and $p_{\epsilon_i,h_i} \in D^+\phi_{\epsilon_i,h_i}(\xi_{\epsilon_i,h_i}(x, v))$, such that

$$\frac{\phi_{\epsilon_i,h_i}(x + h_i v) - \phi_{\epsilon_i,h_i}(x)}{h_i} = p_{\epsilon_i,h_i} \cdot v.$$

Then, in order to prove the lemma it is enough show that

$$\lim_{\epsilon_i,h_i\to 0} p_{\epsilon_i,h_i} \cdot v = \nabla\phi_0(x)(v) \quad \text{uniformly for } (x, v) \in K,$$

i.e. given $\zeta > 0$ we need to find $i_0 \in \mathbb{N}$ such that for each $i > i_0$ and all $(x, v) \in K$ we have

(i) $\nabla\phi_0(x)(v) \leq p_{\epsilon_i,h_i} \cdot v + \zeta$
(ii) $\nabla\phi_0(x)(-v) \leq -p_{\epsilon_i,h_i} \cdot v + \zeta$.

Firstly, we claim that there exists i_0, such that the first inequality holds for every $i > i_0$, and every $(x, v) \in K$. We argue by contradiction to prove the claim: we suppose that there is no $i_0 > 0$, with the specified properties. Then, there exists a sequence $\{(x_n, v_n)\}$, and subsequences $\{h_n\}, \{\epsilon_n\}$ of $\{h_i\}, \{\epsilon_i\}$, such that

$$\nabla\phi_0(x_n)(v_n) > p_{\epsilon_n,h_n} \cdot v_n + \zeta, \tag{10}$$

where $p_{\epsilon_n,h_n} \in D^+\phi_{\epsilon,h}(\xi_n)$ and $\xi_n := \xi_{\epsilon_n,h_n}(x_n, v_n) \in]x_n, x_n + h_n v_n[$ are given by Proposition 1. Passing to a subsequence, if necessary, we can suppose that the sequence $\{(x_n, v_n)\}$ converges to a point (x, v) of K, then $\{\xi_n\}$ converges to x. Now, by Proposition 2, we have that, for any $\lambda > 0$

$$p_{\epsilon_n,h_n} \cdot v_n \geq \frac{\phi_{\epsilon_n,h_n}(\xi_n + \lambda v_n) - \phi_{\epsilon_n,h_n}(\xi_n)}{\lambda} - \frac{\bar{C}}{2}\lambda|v_n|^2. \tag{11}$$

Note that $\phi_{\epsilon_n,h_n} \to \phi_0$ when $n \to \infty$ uniformly, $\xi_n \to x$ and $\nabla\phi_0$ is continuous in $\mathrm{dom}(\nabla\phi_0)$. Then by Eqs. (10) and (11), we have that

$$\lim_n \nabla\phi_0(x_n)(v_n) \geq \liminf_n p_{\epsilon_n,h_n} \cdot v_n + \zeta$$
$$\geq \lim_n \frac{\phi_{\epsilon_n,h_n}(\xi_n + \lambda v_n) - \phi_{\epsilon_n,h_n}(\xi_n)}{\lambda} - \frac{\bar{C}}{2}\lambda|v_n|^2 + \zeta,$$

hence

$$\nabla\phi_0(x)(v) \geq \frac{\phi_0(x+\lambda v) - \phi_0(x)}{\lambda} - \frac{\bar{C}}{2}\lambda|v|^2 + \zeta, \quad \text{for all } \lambda > 0.$$

Then

$$\nabla\phi_0(x)(v) \geq \lim_{\lambda\downarrow 0} \frac{\phi_0(x+\lambda v) - \phi_0(x)}{\lambda} - \frac{\bar{C}}{2}\lambda|v|^2 + \zeta = \nabla\phi_0(x)(v) + \zeta,$$

and this is a contradiction. Repeating the argument with v replaced by $-v$ yields the other inequality. □

Theorem 9. *Consider ϕ_0 and $\bar{\phi}_0$ the functions given by Theorem 6 and denote by μ the Mather measure for L. Then,*

$$\pi_1(\mathrm{supp}(\mu)) = \{x : \phi_0(x) + \bar{\phi}_0(x) = 0\},$$

where π_1 is the canonical projection on the x coordinate.

Proof. This follows by Corollary 1 (as the Lagrangian is generic), because the Peierls barrier $h(x,x) = 0$, if and only if, x is in the projection of the support of the Mather measure (the projected Aubry set). □

Theorem 10. *Consider two sequences $\{\epsilon_n\}, \{h_n\}$ such that, for any $\delta > 0$, $e^{\delta/\epsilon_n} h_n^N \to \infty$, assume further that $\mu_{\epsilon_n,h_n} \rightharpoonup \mu, \bar{H}_{\epsilon_n,h_n} \to \bar{H}_0, \phi_{\epsilon_n,h_n} \to \phi_0$ and $\bar{\phi}_{\epsilon_n,h_n} \to \bar{\phi}_0$. To simplify the notation we will denote by $\mu_n = \mu_{\epsilon_n,h_n}, \bar{H}_n = \bar{H}_{\epsilon_n,h_n}, \phi_n = \phi_{\epsilon_n,h_n}$ and $\bar{\phi}_n = \bar{\phi}_{\epsilon_n,h_n}$. Then, we have that*

(a) $\liminf_{n\to\infty} \frac{\bar{\phi}_n(x)+\phi_n(x)}{h_n} \geq 0, \ \forall x \in \mathbb{T}^N$,
(b) $\lim_{n\to\infty} \frac{\bar{\phi}_n(x)+\phi_n(x)}{h_n} = \infty$, *if $x \notin \pi_1(\mathrm{supp}(\mu))$*,
(c) $\limsup_{n\to\infty} \ \inf_{x\in B_{x_0}(r)} \frac{\bar{\phi}_n(x_0)+\phi_n(x_0)}{h_n} = 0$, *if $x_0 \in \pi_1(\mathrm{supp}(\mu))$, for all $r > 0$.*

Proof. (a) Suppose by contradiction that there exists $x \in \mathbb{T}^N$ such that for a subsequence $\lim_{j\to\infty} \frac{\bar{\phi}_{n_j}(x)+\phi_{n_j}(x)}{h_{n_j}} = c < 0$, then there exists a neighborhood V of x of diameter $\bar{c}h_{n_j}$, where $\bar{c}$ is a constant, such that $\frac{\bar{\phi}_{n_j}(y)+\phi_{n_j}(y)}{h_{n_j}} \leq c/2$ for all $y \in V$. Then

$$e^{-\frac{c}{2\,\epsilon_{n_j}}} \int_V dx \leq \int_V e^{-\frac{1}{\epsilon_{n_j}} \frac{\bar{\phi}_{n_j}(x)+\phi_{n_j}(x)}{h_{n_j}}} dx < \int_{\mathbb{T}^N} e^{-\frac{1}{\epsilon_n} \frac{\bar{\phi}_n(x)+\phi_n(x)}{h_n}} dx = 1.$$

But, $e^{-\frac{c}{2\,\epsilon_{n_j}}} (\bar{c}h_{n_j})^N \to \infty$ when $n_j \to \infty$, then we get a contradiction, as $c < 0$.

(b) It follows by item (a) and Theorem 9.

(c) First, we fix a point (x_0, v_0) in the support of μ and let B be a small neighborhood of (x_0, v_0) in the phase space. As $\mu_n \rightharpoonup \mu$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then

$$1 > \int_B e^{-\frac{1}{\epsilon_n}\left(\frac{\bar{\phi}_n(x)+\phi_n(x)}{h_n} + \frac{h_n L(x,v)+\phi_n(x+h_n v)-\phi_n(x)-h_n \bar{H}_n}{h_n}\right)} dxdv > \delta_B > 0 \tag{12}$$

for some positive δ_B.

Claim. *Given $\zeta > 0$ there exists $\bar{n} \in \mathbb{N}$ and a neighborhood B of (x_0, v_0) such that, if $(x, v) \in B$ and $n > \bar{n}$ then $L(x, v) + \frac{\phi_n(x+h_n v)-\phi_n(x)}{h_n} - \bar{H}_n > -\zeta$.*

We postpone the proof of the claim. Suppose by contradiction that

$$\limsup_{n\to\infty} \inf_{x\in B_{x_0}(r)} \frac{\bar{\phi}_n(x) + \phi_n(x)}{h_n} = c > 0,$$

then there exists a subsequence such that $\lim_{j\to\infty} \inf_{x\in B_{x_0}(r)} \frac{\bar{\phi}_{n_j}(x)+\phi_{n_j}(x)}{h_{n_j}} = c$. Then, there exists j_0 such that for $j > j_0$ we have

$$\frac{\bar{\phi}_{n_j}(x) + \phi_{n_j}(x)}{h_{n_j}} > \frac{c}{2} \quad \forall\, x \in B_{x_0}(r). \tag{13}$$

Let $\tilde{B} = B_{x_0}(r)\times\mathbb{R}^N$. Now using the claim with $\zeta < c/4$, let B be the neighborhood in the claim. Take $\hat{B} = B \cap \tilde{B}$, jointing the inequalities (13) and that of the claim we have a contradiction, when $\epsilon_{n_j} \to 0$, with the inequality (12). This proves (c).

Proof of the claim. Let $\bar{C}$ be the semiconcavity bound of the functions $\phi_{\epsilon,h}$. For $\zeta > 0, \eta > 0$ there exists $\lambda > 0$ such that

$$\frac{\bar{C}}{2}(|v_0| + \eta)^2\lambda < \zeta \quad \text{and} \quad \left|\frac{\phi_0(x_0 + \lambda v_0) - \phi_0(x_0)}{\lambda} - \nabla\phi_0(x_0)(v_0)\right| < \zeta.$$

As $\phi_n \to \phi_0$ uniformly in x, there exists n_2 such that if $n > n_2$, then we have $|\phi_0(x)-\phi_n(x)| < \zeta\lambda$, for all $x \in \mathbb{T}^N$. Also there exists a neighborhood B_λ of (x_0, v_0) such that, if $(x, v) \in B_\lambda$ and $n > n_2$, then

$$\left|\frac{\phi_n(x + \lambda v) - \phi_n(x) - \phi_0(x_0 + \lambda v_0) + \phi_0(x_0)}{\lambda}\right| < 6\zeta.$$

There exists n_3 such that if $n > n_3$ and $(x, v) \in B_\lambda$ (choosing B_λ smaller if necessary) such that $|L(x_0, v_0) - L(x, v) - \bar{H}_0 + \bar{H}_n| < \zeta$ and $|v - v_0| < \eta$.

By Propositions 1 and 2 we have that

$$\frac{\phi_n(x + h_n v) - \phi_n(x)}{h_n} = p_n(x_n) \cdot v \geq \frac{\phi_n(x_n + \lambda v) - \phi_n(x_n)}{\lambda} - \frac{\bar{C}}{2}\lambda|v|^2,$$

where $x_n \in]x, x + h_n v[$, therefore there exists n_4 such that if $n > n_4$ then $(x_n, v) \in B_\lambda$.

Now, define $\bar{n} = \max\{n_2, n_3, n_4\}$, collecting all the above inequalities, for any $n > \bar{n}$ and $(x, v) \in B_\lambda$, we get

$$L(x,v) + \frac{\phi_n(x + h_n v) - \phi_n(x)}{h_n} - \bar{H}_n > L(x_0, v_0) + \nabla\phi_0(x_0)(v_0) - \bar{H}_0 - 9\zeta > -9\zeta,$$

which proves the claim. □

Let us define the deviation function I by

$$I(x,v) = L(x,v) + \nabla\phi_0(x)(v) - \bar{H}_0.$$

We recall that we just consider $I(x, v)$ for the points x where $\nabla\phi_0(x)$ is defined. For the others points x we declare $I(x, v) = \infty$.

Proposition 3. *Let ϕ_0 be a viscosity solution to $H(\nabla\phi_0(x), x) = -\bar{H}_0$. If $(x, v) \in \operatorname{supp}(\mu)$ then $\nabla\phi_0(x)(v) + L(x, v) = \bar{H}_0$.*

Proof. By [11, Theorem 4.8.3] we have that ϕ_0 is differentiable in $\pi_1(\operatorname{supp}(\mu))$. Let $(x, v) \in \operatorname{supp}(\mu)$, by [11, Corollary 4.4.13] we obtain $H(\nabla\phi_0(x), x) = -\bar{H}_0$. Therefore

$$\nabla\phi_0(x)(v) + L(x,v) \geq \bar{H}_0. \tag{14}$$

To get the other inequality, suppose, by contradiction, that there exists $(x, v) \in \operatorname{supp}(\mu)$ and $\epsilon > 0$ such that

$$\nabla\phi_0(x)(v) + L(x,v) > \bar{H}_0 + \epsilon.$$

Then there is a neighborhood V of (x, v) such that for all $(v, w) \in V$ we have

$$\nabla\phi_0(y)(w) + L(y,w) > \bar{H}_0.$$

We recall that $\int \nabla\phi_0(x, v)d\mu = 0$, then

$$\int \nabla\phi_0(x)(v) + L(x,v)d\mu > \bar{H}_0,$$

because (14) is true at any point $(x, v) \in \pi_1(\operatorname{supp}\mu) \times \mathbb{R}^N$ and at the points $(x, v) \in V$ we have the strict inequality.

This implies

$$\int L(x,v)d\mu > \bar{H}_0,$$

but this is a contradiction. □

If we fix x, we have that

$$-\inf_v I(x,v) = \sup_v(-I(x,v)) = H(\nabla\phi_0(x), x) + \bar{H}_0. \tag{15}$$

We know that if $x \in \pi_1(\operatorname{supp}(\mu))$ then $H(\nabla\phi_0(x), x) + \bar{H}_0 = 0$, and by the hypothesis that the Lagrangian is strictly convex in v, we obtain that there is just one v

which achieves the supremum in (15). Moreover, as we know that $(x,v) \in \mathrm{supp}(\mu)$, if and only if, $I(x,v) = 0$, we conclude that $I(x,v) > 0$, for all $(x,v) \notin \mathrm{supp}(\mu)$ with $x \in \pi_1(\mathrm{supp}(\mu))$.

It makes sense to look for lower and upper deviations inequalities just in the case $\inf_A I(x,v) > 0$.

Theorem 11. *Let us denote $D = \mathrm{dom}(\nabla\phi_0)$. Let $A \subset D \times \mathbb{R}^N$ be such that $D \cap \pi_1(\mathrm{supp}(\mu)) \neq \emptyset$, but $d(A, \mathrm{supp}(\mu)) \geq c > 0$. Then*

(a) *if A is a closed bounded set in $D \times \mathbb{R}^N$ we have*

$$\lim_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) \leq -\inf_{(x,v)\in A} I(x,v),$$

(b) *if A is an open bounded set in $D \times \mathbb{R}^N$*

$$\lim_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) \geq -\inf_{(x,v)\in A_1} I(x,v),$$

where $A_1 = \{(x,v) \in A \, : \, x \in \pi_1(\mathrm{supp}(\mu))\}$.

Remark on item (b). Given a set A as above, consider $B^\delta = \{(x,v) \in A$, such that $d(x, \mathrm{supp}(\mu)) \geq \delta > 0\}$, for any fixed small δ. From Theorem 10 (and Theorem 8) we have that

$$\lim_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) = \lim_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(B^\delta).$$

In this way, the lower bound $-\inf_{(x,v)\in A_1} I(x,v)$ is the precise information that makes sense. In other words, the values $I(x,v)$ outside A_1 are not relevant.

Proof. (a) Note that $A \subset D \times B_R$, where $B_R = \{v \in \mathbb{R}^N; |v| \leq R\}$, for some $R > 0$. Remember that

$$\mu_{\epsilon,h}(A) = \int_A e^{-(\frac{\bar{\phi}_{\epsilon,h}(x)+\phi_{\epsilon,h}(x)}{\epsilon h} + \frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-h\bar{H}_{\epsilon,h}}{\epsilon h})} dxdv,$$

then

$$\mu_{\epsilon,h}(A) \leq e^{-\inf_A (\frac{\bar{\phi}_{\epsilon,h}(x)+\phi_{\epsilon,h}(x)}{\epsilon h} + \frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-h\bar{H}_{\epsilon,h}}{\epsilon h})} \int_A dxdv.$$

Hence

$$\epsilon \ln \mu_{\epsilon,h}(A) \leq -\inf_A \left(\frac{\bar{\phi}_{\epsilon,h}(x) + \phi_{\epsilon,h}(x)}{h} \right) - \inf_A \left(L(x,v) + \frac{\phi_{\epsilon,h}(x+hv) - \phi_{\epsilon,h}(x)}{h} - \bar{H}_{\epsilon,h} \right) + \epsilon \ln c_1 R^N.$$

By item (c) of Theorem 10 we have

$$\limsup_{\epsilon,h\to 0} \inf_{x\in A} \frac{\{\bar{\phi}_{\epsilon,h}(x) + \phi_{\epsilon,h}(x)\}}{h} = 0.$$

This implies that

$$\limsup_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) \le -\inf_{x\in A} I(x,v).$$

(b) Let $A \subset D\times\mathbb{R}^N$ be a bounded and open set in $D\times\mathbb{R}^N$, such that $A\cap A_1 \neq \emptyset$. We fix $\delta > 0$, as $I(x,v)$ is continuous in $D\times\mathbb{R}^N$ (see [11, Theorem 4.9.2]), we can find an open set A_δ such that: $\bar{A}_\delta$ is a closed set in $D\times\mathbb{R}^N$, $\bar{A}_\delta \cap A_1 \neq \emptyset$ and

$$I(x,v) \le \inf_{A_1} I(x,v) + \delta \quad \text{for all } (x,v)\in \bar{A}_\delta.$$

Therefore

$$\mu_{\epsilon,h}(A) \ge \mu_{\epsilon,h}(\bar{A}_\delta) \ge e^{-\sup_{\bar{A}_\delta}\left(\frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-h\bar{H}_{\epsilon,h}}{\epsilon h}\right)} \times \int_{\bar{A}_\delta} e^{-\left(\frac{\bar{\phi}_{\epsilon,h}(x)+\phi_{\epsilon,h}(x)}{\epsilon h}\right)} dxdv.$$

As $\bar{A}_\delta \cap A_1 \neq \emptyset$ and A_δ is an open set there exists $c_\delta > 0$ such that

$$1 \ge \liminf_{\epsilon,h\to 0} \int_{\bar{A}_\delta} e^{-\left(\frac{\bar{\phi}_{\epsilon,h}(x)+\phi_{\epsilon,h}(x)}{\epsilon h}\right)} dxdv \ge c_\delta.$$

We have that

$$\liminf_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) \ge -\sup_{\bar{A}_\delta} I(x,v) \ge -\inf_{A_1} I(x,v) + \delta.$$

Making $\delta \to 0$ we obtain

$$\liminf_{\epsilon,h\to 0} \epsilon \ln \mu_{\epsilon,h}(A) \ge -\inf_{A_1} I(x,v).$$

□

Theorem 12. *If $A \subset D\times\mathbb{R}^N$ is such that $D\cap\pi_1(\operatorname{supp}(\mu)) = \emptyset$. If A is closed and bounded we have*

$$\lim_{\epsilon,h\to 0} \epsilon h \ln \mu_{\epsilon,h}(A) \le -\inf_{x\in D}\{\bar{\phi}_0(x) + \phi_0(x)\}.$$

And if A is open and bounded we have

$$\lim_{\epsilon,h\to 0} \epsilon h \ln \mu_{\epsilon,h}(A) \ge -\inf_{x\in D}\{\bar{\phi}_0(x) + \phi_0(x)\}.$$

Proof. We can write

$$\mu_{\epsilon,h}(A) = \int_A e^{-\frac{\tilde{f}_{\epsilon,h}(x,v)}{\epsilon h}} dxdv,$$

where

$$\tilde{f}_{\epsilon,h}(x,v) = \bar{\phi}_{\epsilon,h}(x) + \phi_{\epsilon,h}(x) + hL(x,v) + \phi_{\epsilon,h}(x+hv) - \phi_{\epsilon,h}(x) - h\bar{H}_{\epsilon,h}.$$

As $\tilde{f}_{\epsilon,h}(x,v) \to \bar{\phi}_0(x) + \phi_0(x)$ uniformly, using Lemma 1 (Laplace Method) we get the two inequalities of the theorem. □

We have some final comments about the large deviation problem. For a fixed (x, p) consider

$$Z_{\epsilon,h,p}(x) = \int e^{\frac{\langle p,v\rangle}{\epsilon}} d\mu_{\epsilon,h}(x, v)$$

and the free energy

$$c(p, x) = \lim_{\epsilon,h\to 0} \epsilon \ln Z_{\epsilon,h,p}(x),$$

where $\mu_{\epsilon,h}$ was chosen for L as above.

Theorem 13. *For each,* (x, p), *for almost everywhere* (*Lebesgue*) x

$$c(p, x) = H(\nabla\phi_0(x) - p, x) + \bar{H}_0.$$

Proof. As

$$\begin{aligned}\mu_{\epsilon,h}(x, v) &= \theta_{\epsilon,h}(x)\gamma_{\epsilon,h}(x, v)\\ &= e^{-\frac{\bar{\phi}_{\epsilon,h}(x)+\phi_{\epsilon,h}(x)}{\epsilon h}} e^{-\frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h}},\end{aligned}$$

then

$$Z_{\epsilon,h,p}(x) = \int e^{\frac{\langle p,v\rangle}{\epsilon}} \theta_{\epsilon,h}(x)\gamma_{\epsilon,h}(x, v)dxdv.$$

As H is the Legendre transform of L, the result follows from the results we obtained before (L.D.P) and the Varadhan's integral lemma ([6, Sec. 4.3]). □

Therefore, the Legendre transform of the free energy is the deviation function.

Example. An interesting example is the following:
Consider $L(x, v) = \frac{v^2}{2}$.
Then

$$\mathcal{G}[\phi](x) := -\epsilon h \ln\left[\int_{\mathbb{R}^N} e^{-\frac{h\frac{v^2}{2}+\phi(x+hv)}{\epsilon h}} dv\right],$$

satisfies

$$\mathcal{G}[0](x) = -\epsilon h \ln\left(\int_{\mathbb{R}^N} e^{-\frac{h\frac{v^2}{2}}{\epsilon h}} dv\right) = -\epsilon h \ln\sqrt{2\pi\epsilon} + 0 = \lambda_{\epsilon,h}.$$

Therefore $\theta_{\epsilon,h} = 1$ and

$$\mu_{\epsilon,h}(x, v) = e^{-\frac{\frac{v^2}{2}-\bar{H}_{\epsilon,h}}{\epsilon}}.$$

In this case,

$$S[\mu_{\epsilon,h}] = \int_{\mathbb{T}^N\times\mathbb{R}^N} \mu_{\epsilon,h}(x,v) \ln \frac{\mu_{\epsilon,h}(x,v)}{\displaystyle\int_{\mathbb{R}^N} \mu_{\epsilon,h}(x,w)dw} dxdv$$
$$= \int_{\mathbb{R}^N} e^{-\frac{h\frac{v^2}{2}-\lambda_{\epsilon,h}}{\epsilon h}} \ln \left(e^{-\frac{h\frac{v^2}{2}-\lambda_{\epsilon,h}}{\epsilon h}}\right) dv$$
$$= \int_{\mathbb{R}^N} e^{-\frac{h\frac{v^2}{2}-\lambda_{\epsilon,h}}{\epsilon h}} \left(-\frac{h\frac{v^2}{2}-\lambda_{\epsilon,h}}{\epsilon h}\right) dv$$
$$= -\ln\sqrt{2\pi\epsilon} - \frac{1}{\sqrt{2\pi\epsilon}} \int \frac{v^2}{2\epsilon} e^{-\frac{v^2}{2\epsilon}} dv = -\ln\sqrt{2\pi\epsilon} - \frac{1}{2}.$$

Therefore, the term $\epsilon S(\mu_{\epsilon,h})$ goes to 0 when $\epsilon \to 0$. We point out that $S(\mu_{\epsilon,h})$ goes to $+\infty$ when $\epsilon \to 0$ Moreover,

$$\bar{H}_0 = \lim_{h\to 0}\lim_{\epsilon\to 0} \frac{\lambda_{\epsilon,h}}{h} = 0.$$

In this case

$$I(x,v) = L(x,v) + \nabla\phi_0(x)(v) - \bar{H}_0 = \frac{v^2}{2},$$

and the equation $I(x,v) = 0$, means that, $v = 0$. The Aubry set, as it is known, in this case is the set of elements of the form $(x,0)$, for any $x \in \mathbb{T}^N$.

The Varadhan's integral lemma [6] claims the following: suppose $I(x,v) = I(v)$ is the deviation function for $\mu_{\epsilon,h}$ as above, then, if $g : \mathbb{R}^N \to \mathbb{R}$ is a continuous function $g(v)$, then

$$\lim_{\epsilon,h\to 0} \epsilon \ln \int e^{g(v)} \mu_{\epsilon,h}(v) = \sup_v \{g(v) - I(v)\} = \sup_v \left\{g(v) - \frac{v^2}{2}\right\}.$$

An interesting example is when p is fixed and we consider $g(v) = \langle p, v\rangle$. In this case, $\sup\{g(v) - \frac{v^2}{2}\} = p^2$.

5. The Discrete Time Aubry–Mather Problem

5.1. *The uniqueness of the calibrated subactions*

In this section we will study some dynamical properties of the discrete time Aubry–Mather problem (see [13]). These will be used to obtain conditions for the uniqueness of ϕ_h used in the definition of I_h.

For a $h > 0$ fixed, remember that

$$\bar{H}_h = \min_{\mathcal{M}_h} \left\{ \int_{\mathbb{T}^N\times\mathbb{R}^N} L(x,v) d\mu(x,v) \right\}$$

where

$$\mathcal{M}_h = \left\{ \mu \in \mathcal{M} : \int_{\mathbb{T}^N \times \mathbb{R}^N} \varphi(x+hv) - \varphi(x) d\mu = 0, \forall \varphi \in C(\mathbb{T}^N) \right\}.$$

A measure μ_h which attains such minimum is called a discrete Mather measure for L. Note that $\bar{H}_h$ (possibly up to a sign convention) is the analog of Mañé's critical value.

Definition 2. A continuous function $u : \mathbb{T}^N \to \mathbb{R}$ is called

(a) a forward-subaction if

$$u(x) \le u(x+hv) + hL(x,v) - h\bar{H}_h, \quad \forall (x,v) \in \mathbb{T}^N \times \mathbb{R}^N,$$

(b) a backward-subaction if

$$u(x) \le u(x-hv) + hL(x-hv,v) - h\bar{H}_h, \quad \forall (x,v) \in \mathbb{T}^N \times \mathbb{R}^N.$$

Definition 3. A continuous function $u : \mathbb{T}^N \to \mathbb{R}$ is called a calibrated forward-subaction (calibrated subaction for short) if, for any x, we have

$$u(x) = \inf_{v \in \mathbb{R}^N} \{u(x+hv) + hL(x,v) - h\bar{H}_h\}.$$

For each value x this infimum is attained by some (can be more that one) $v(x)$.

Definition 4. A continuous function $u : \mathbb{T}^N \to \mathbb{R}$ is called a calibrated backward-subaction if, for any x we have

$$u(x) = \inf_{v \in \mathbb{R}^N} \{u(x-hv) + hL(x-hv,v) - h\bar{H}_h\}.$$

By item (b) of Theorem 3, any limit of a subsequence $\lim_{\epsilon_i \to 0} \phi_{\epsilon_i,h} = \phi_h$, is a calibrated subaction for L. In general it is not known if ϕ_h is unique (up to a constant). We will establish below (Theorem 15) a condition for such uniqueness. Similar properties are true for the backward problem, that is, if $\lim_{\epsilon_i \to 0} \bar{\phi}_{\epsilon_i,h} = \bar{\phi}_h$, then $\bar{\phi}_h$ is a calibrated backward-subaction, etc.

The calibrated subaction plays the role in the discrete time Aubry–Mather problem [13] of the solution of the Lax–Oleinik problem [11]. We substitute here the continuous time path by a kind of discrete time path. In all known cases the calibrated subaction helps, via de sub-cohomological equation, to determine the support of the minimizing measure [11, 13].

Proposition 4. *Let u be a calibrated subaction to the Lagrangian L. If u is differentiable at x then*

$$\nabla u(x) = hL_x(x, v(x)) - L_v(x, v(x)).$$

This result is contained in the arguments of the proof of [13, Theorem 4.1].

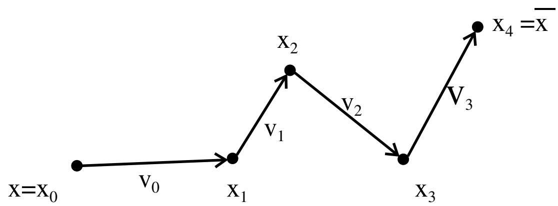

Fig. 1. Graphical representation of paths belonging to $\mathcal{P}_4(x,z)$, with $h=1$.

Assumption. We shall suppose also that the Lagrangian is such that L_x has bounded Lipschitz constant in v. Because in this case the equation $p = hL_x(x,v(x)) - L_v(x,v(x))$ has only one differentiable solution, when h is small enough. Hence by the same arguments used in [13, Theorem 5.5] we obtain that any minimizing measure μ_h is supported in a graph.

The next definitions will be considered for a fixed value of $h>0$, small enough, such that we have the graph property.

Definition 5. Given k and $x,z \in \mathbb{R}^N$, we will call a k-path beginning in x and ending at z an ordered sequence of points

$$(x_0, x_1, \ldots, x_k) \in \mathbb{R}^N \times \cdots \times \mathbb{R}^N$$

satisfying $x_0 = x, x_k = z$.

We will denote by $\mathcal{P}_k(x,z) = \mathcal{P}_k^h(x,z)$ the set of such k-paths.
For each x_j we will associate a $v_j \in \mathbb{R}^N$, such that

$$v_j = \frac{x_{j+1} - x_j}{h}, \quad \text{for } 0 \leq j < k.$$

Definition 6. For a k-path fixed $(x_0, \ldots, x_k)$ we define it action by:

$$A_{L-\bar{H}_h}(x_0, \ldots, x_k) := h \sum_{i=0}^{k-1} (L - \bar{H}_h)(x_i, v_i).$$

Remark. Let $(x_0, \ldots, x_k) \in \mathcal{P}_k(x+s,z)$ be a path, where $x,z \in \mathbb{R}^N$ and $s \in \mathbb{Z}^N$. As the Lagrangian is $\mathbb{Z}^N$-periodic we have that the path $(\tilde{x}_0, \ldots, \tilde{x}_k) \in \mathcal{P}_k(x, z-s)$ given by $\tilde{x}_i = x_i - s$ is such that $A_{L-\bar{H}_h}(x_0, \ldots, x_k) = A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_k)$.

Definition 7. A point $x \in \mathbb{T}^N$ is called non-wandering with respect to L if, given $\epsilon > 0$ there exist $k \geq 1$, $s_k \in \mathbb{Z}^N$ and a k-path $(x_0, \ldots, x_k) \in \mathcal{P}_k(x+s_k, x)$ such that

$$|A_{L-\bar{H}_h}(x_0, \ldots, x_k)| < \epsilon.$$

We will denote by $\Omega_h(L)$ the set of non-wandering points with respect to L.

Remark. $\Omega_h(L)$ is a closed set. Indeed, let $x_k \in \Omega_h(L)$ be such that $x_k \to x$. For each x_k and $\epsilon = \frac{1}{n}$ there exists j_n, s_{j_n} and $(x_0, \ldots, x_{j_n}) \in \mathcal{P}_{j_n}(x_k + s_{j_n}, x_k)$ such that $|A_{L-\bar{H}_h}(x_0, x_1, \ldots, x_{j_n})| \le \frac{1}{n}$. Hence the path $(x + s_{j_n}, x_1, \ldots, x_{j_n - 1}, x)$ has also small action, when $n \to \infty$ we get $x \in \Omega_h(L)$.

The proof for the results we describe bellow are similar to the ones presented in [14] where the discrete time symbolic dynamics version of Aubry–Mather theory is considered.

Proposition 5. *Let μ_h be a discrete-time Mather measure, then*

$$\pi_1(\mathrm{supp}(\mu_h)) \subset \Omega(L).$$

Proof. By [13] we know that μ_h is supported on a Lipschitz graph, then we can define $\psi : \mathbb{R}^N \to \mathbb{R}^N$, such that $\psi(x) = x + hv(x)$, we define $\bar{\psi} : \mathbb{T}^N \to \mathbb{T}^N$ by $\bar{\psi}(x) = \psi(x) \mod \mathbb{Z}^N$. We claim that $\mu \circ \pi_1^{-1}$ is $\bar{\psi}$-invariant.

Indeed, as μ_h is holonomic and by the definition of ψ we have that for all $\phi : \mathbb{T}^N \to \mathbb{R}$

$$\int_{\mathbb{T}^N} \phi \circ \bar{\psi}(x) d(\mu_h \circ \pi_1^{-1}) = \int_{\mathbb{T}^N \times \mathbb{R}^N} \phi(x + hv(x) \mod \mathbb{Z}^N) d\mu_h$$
$$= \int_{\mathbb{T}^N \times \mathbb{R}^N} \phi(x) d\mu_h = \int_{\mathbb{T}^N} \phi(x) d(\mu_h \circ \pi_1^{-1}).$$

Let $(x, v) \in \mathrm{supp}(\mu_h)$ and let B be an open ball centered at the point x, then $\mu \circ \pi_1^{-1}(B) > 0$, hence there exists $x_0 \in B$ such that $\bar{\psi}^j(x_0)$ returns infinitely many times to B, i.e. there exists $s_j \in \mathbb{Z}^N$ such that $\psi^j(x_0) - s_j \in B$. Because ϕ_h is a calibrated subaction for L we can write $\phi_h(x_0) - \phi_h(x_j) = h \sum_{i=0}^{j-1} (L - \bar{H}_h)(x_i, v_i)$, where $x_i := \psi^i(x_0)$. Given $\delta > 0$ and $x_j - s_j \in B$ we can construct the following path $(\tilde{x}_0, \ldots, \tilde{x}_j) = (x, x_1, \ldots, x_{j-1}, x + s_j)$ such that

$$A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_j) \le \delta.$$

Indeed,

$$A_{L-\bar{H}_h}(x_0, \ldots, x_j) = \phi_h(x_0) - \phi_h(x_j) + h\Bigg[L\bigg(x, \frac{x_1 - x}{h}\bigg) - L(x_0, v_0)$$
$$+ L\bigg(x_{j-1}, \frac{x + s_j - x_{j-1}}{h}\bigg) - L(x_{j-1}, v_{j-1})\Bigg]$$
$$\le \delta,$$

if B is small enough. Hence $x \in \Omega_h(L)$. □

Definition 8. For a fixed value $h > 0$, define

$$S_h^k(x, z) = \inf_{s \in \mathbb{Z}^N} \inf_{(x_0, \ldots, x_k) \in \mathcal{P}_k(x+s, z)} A_{L-\bar{H}_h}(x_0, \ldots, x_k).$$

Let S_h be the Mañé potential the function $S_h : \mathbb{T}^N \times \mathbb{T}^N \to \mathbb{R}$ defined by

$$S_h(x, z) = \inf_k S_h^k(x, z).$$

The Peierls barrier $\mathbf{h}_h : \mathbb{T}^N \times \mathbb{T}^N \to \mathbb{R} \cup \{\pm\infty\}$ is the function defined by

$$\mathbf{h}_h(x, z) = \liminf_{k\to\infty} S_h^k(x, z).$$

Note that

$$\Omega_h(L) = \{x \in \mathbb{T}^N : \mathbf{h}_h(x, x) = S_h(x, x) = 0\}.$$

We point out here a main difference from the continuous time Mather problem where the Mañé potential S (defined in a similar way as for instance in [11] or [8]) is zero for any pair (x, x) where x is in the configuration space. The point is that in the continuous time case we can consider trajectories with time as small as we want, whereas this is not possible in discrete time.

The functions S_h and $\mathbf{h}_h$ have the following properties:

(i) $S_h(x, z) \le S_h(x, y) + S_h(y, z)\ \forall\, x, y, z \in \mathbb{T}^N$.
(ii) $\mathbf{h}_h(x, z) \le \mathbf{h}_h(x, y) + \mathbf{h}_h(y, z)\ \forall\, x, y, z \in \mathbb{T}^N$.
(iii) $\mathbf{h}_h(x, z) \le S_h(x, y) + \mathbf{h}_h(y, z)\ \forall\, x, y, z \in \mathbb{T}^N$.

Proposition 6. *Let us fix $z \in \mathbb{T}^N$, the functions $S_h(\cdot, z)$ and $\mathbf{h}_h(\cdot, z)$ are forward subactions.*

Proof. It follows by (i) and (iii), respectively, and by the observation that

$$S_h(x, y) \le h\left[L\left(x, \frac{y - x}{h}\right) - \bar{H}_h\right]. \qquad \square$$

In order to prove that $\mathbf{h}_h(\cdot, z)$ is a calibrated subaction, we need the following lemma. Also, note that if $z \in \Omega_h(L)$ then by (iii) we have that $\mathbf{h}_h(\cdot, z)$ is finite.

Lemma 2. *Let $(x_0, \ldots, x_k) \in \mathcal{P}_k(x+s, z)$ be a path such that $A_{L-\bar{H}_h}(x_0, \ldots, x_k) = S_h^k(x, z)$. Then there exists a constant K such that $|v_i| < K$ for all $0 \le i < k$. Also, K is independent of $x, z \in \mathbb{T}^N$.*

Proof. Let $R = 2\max_{x,y\in\mathbb{T}^N} d(x, y)$, we define $A(R) = \max\{L(x, v) : |v| \le R\}$. As L is superlinear there exists K such that if $|v| \ge K$ then $L(x, v) > A(R)$.

We will show the lemma by induction. First let us prove that $|v_0| < K$: suppose by contradiction that $|x_1 - x_0| > K$. We choose $s_0 \in \mathbb{Z}^N$ such that $|x_0 + s_0 - x_1| < R$, then the path $(\tilde{x}_0, \ldots, \tilde{x}_k) = (x_0 + s_0, x_1, \ldots, x_k)$ is such that $A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_k) < A_{L-\bar{H}_h}(x_0, \ldots, x_k) = S_h^k(x, z)$, which is a contradiction. Suppose we have proved that $|v_i| < K$ for all $0 \le i < j$ and suppose by absurd that $|v_j| > K$, we choose $s_{j-1} \in \mathbb{Z}^N$ such that $|x_{j-1} + s_{j-1} - x_j| < R$, then the path $(\tilde{x}_0, \ldots, \tilde{x}_k) = (x_0 + s_{j-1}, \ldots, x_{j-1} + s_{j-1}, x_j, \ldots, x_k)$ is such that

$A_{L-\bar{H}_h}(\tilde{x}_0,\ldots,\tilde{x}_k) < A_{L-\bar{H}_h}(x_0,\ldots,x_k)$, which is a contradiction, hence $|v_i| < K$, for all $0 \le i \le j$. ☐

Proposition 7. *For any $z \in \Omega_h(L)$ the function $u(\cdot) = \mathbf{h}_h(\cdot, z)$ is a calibrated subaction.*

Proof. For a point $x \in \mathbb{T}^N$, we want to find $v \in \mathbb{R}^N$ such that

$$\mathbf{h}_h(x,z) - \mathbf{h}_h(x+hv,z) = hL(x,v) - h\bar{H}_h.$$

By the definition of Peierls barrier there exist a sequence $j_n \to \infty$ and a sequence of paths $(x_0^n,\ldots,x_{j_n}^n) \in \mathcal{P}_{j_n}(x+s_n,z)$, $s_n \in \mathbb{Z}^N$, such that

$$A_{L-\bar{H}_h}(x_0^n,\ldots,x_{j_n}^n) = S_h^{j_n}(x,z) \to \mathbf{h}_h(x,z).$$

As $|v_0^n| = |\frac{x+s_n-x_1^n}{h}| \le K$, the sequence $\{x_1^n - s_n\}$ has an accumulation point, say x_1, taking a subsequence if necessary, we can suppose that, $x_1 = \lim_n(x_1^n - s_n)$ and we define $v = \lim_n(\frac{x_1^n - x - s_n}{h})$. Then, because $z \in \Omega_h(L)$,

$$\mathbf{h}_h(x_1,z) \le S_h^{j_n-1}(x_1,z) \le A_{L-\bar{H}_h}(x_1+s_n, x_2^n,\ldots,x_{j_n}^n).$$

Hence

$$\begin{aligned} &hL(x,v) - h\bar{H}_h + \mathbf{h}_h(x_1,z) \\ &\quad \le \lim_{n\to\infty}\left[hL\left(x+s_n, \frac{x_1^n - x - s_n}{h}\right) - h\bar{H}_h + A_{L-\bar{H}_h}(x_1^n,\ldots,x_{j_n}^n)\right] \\ &\quad = \lim_{n\to\infty} A_{L-\bar{H}_h}(x_0^n,\ldots,x_{j_n}^n) = \mathbf{h}_h(x,z). \end{aligned}$$

Then

$$hL(x,v) - h\bar{H}_h \le \mathbf{h}_h(x,z) - \mathbf{h}_h(x+hv,z).$$

As $\mathbf{h}_h(\cdot,z)$ is a subaction we have the other inequality. Hence

$$hL(x,v) - h\bar{H}_h = \mathbf{h}_h(x,z) - \mathbf{h}_h(x+hv,z).$$ ☐

Remark. When $z \in \Omega_h(L)$ we have that $S_h(\cdot,z) = \mathbf{h}_h(\cdot,z)$.

Theorem 14. *For a fixed value of h, if u is a calibrated subaction, then for any x we have*

$$u(x) = \inf_{p\in\Omega_h(L)}\{u(p) + S_h(x,p)\}.$$

Proof. By the definition of calibrated subaction we have that

$$u(x) \le \inf_{p\in\Omega_h(L)}\{u(p) + S_h(x,p)\}.$$

Let us now show that $u(x) \ge \inf_{p\in\Omega_h(L)}\{u(p) + S_h(x,p)\}$: Fix $x \in \mathbb{T}^N$, we will denote $x_0 = x$. As u is a calibrated subaction there exists v_0 such that

$$u(x_0) = u(x_0 + hv_0) + hL(x_0,v_0) - h\bar{H}_h.$$

Let $x_1 := x_0 + hv_0$, we can construct a sequence $(x_0, x_1, \ldots, x_j, \ldots,)$ such that for each $j > 0$, $x_{j+1} = x_j + hv_j$, and $u(x_j) = u(x_{j+1}) + hL(x_j, v_j) - h\bar{H}_h$. We project this points in the torus, i.e. we choose $s_j \in \mathbb{Z}^N$ such that $\bar{x}_j = x_j + s_j \in \mathbb{T}^N$.

Let $p \in \mathbb{T}^n$ be a limit point of the sequence $\{\bar{x}_j\}$, we claim that $p \in \Omega_h(L)$. Indeed, suppose $\bar{x}_{j_m} \to p$. We can construct, for $n > m$, the following path: $(\tilde{x}_0, \ldots, \tilde{x}_{j_n - j_m}) =: (p - s_{j_m}, x_{j_m+1}, \ldots, x_{j_n-1}, p - s_{j_n})$, hence

$$\begin{aligned}
&A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_{j_n-j_m}) \\
&\quad = A_{L-\bar{H}_h}(x_{j_m}, \ldots, x_{j_n}) + h\Bigg(L\bigg(p - s_{j_m}, \frac{x_{j_m+1} - p + s_{j_m}}{h}\bigg) - L(x_{j_m}, v_{j_m}) \\
&\qquad + L\bigg(x_{j_n-1}, \frac{p - s_{j_n} - x_{j_n-1}}{h}\bigg) - L(x_{j_n-1}, v_{j_n-1})\Bigg).
\end{aligned}$$

As $A_{L-\bar{H}_h}(x_{j_m}, \ldots, x_{j_n}) = u(x_{j_m}) - u(x_{j_n})$, given $\epsilon > 0$, if m is large enough, then

$$|A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_{j_n-j_m})| < \epsilon,$$

i.e. $p \in \Omega_h(L)$. For this p let us show that

$$S_h(x, p) \le u(x) - u(p).$$

Indeed, we consider the following path: $(\tilde{x}_0, \ldots, \tilde{x}_{j_m}) = (x_0, \ldots, x_{j_m-1}, p - s_{j_m})$, then

$$\begin{aligned}
&A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_{j_m}) - u(x) + u(p) \\
&\quad = u(p) - u(x_{j_m}) - hL(x_{j_m-1}, v_{j_m-1}) + hL\bigg(x_{j_m-1}, \frac{p - s_{j_m} - x_{j_m-1}}{h}\bigg).
\end{aligned}$$

Hence, given $k > 0$ there exists $m_k \in \mathbb{N}$ such that if $m > m_k$ then

$$A_{L-\bar{H}_h}(\tilde{x}_0, \ldots, \tilde{x}_{j_m}) < u(x) - u(p) + \frac{1}{k}.$$

Finally, when $k \to \infty$ we obtain

$$S_h(x, p) \le u(x) - u(p),$$

and

$$u(x) \ge \inf_{p \in \Omega_h(L)} \{u(p) + S_h(x, p)\}. \qquad \square$$

Proposition 8. *$\mathcal{O}^h := \{\psi \in C^\infty(\mathbb{T}^N, \mathbb{R}) : \mathcal{M}_0^h(L+\psi) = \{\mu_h\}$ and $\pi_1(\mathrm{supp}(\mu_h)) = \Omega_h(L+\psi)\}$ is a generic set, where $\mathcal{M}_0^h$ denote the set of holonomic minimizing measures, i.e. probability measures in $\mathbb{T}^n \times \mathbb{R}^n$ such that $\int L d\mu_h = \bar{H}_h$ and $\int \varphi(x + hv) - \varphi(x) d\mu_h = 0, \forall\, \varphi \in C(\mathbb{T}^N)$.*

Proof. The proof that $\mathcal{O}_1^h := \{\psi \in C^\infty(\mathbb{T}^N, \mathbb{R}) : \mathcal{M}_0^h(L+\psi) = \{\mu_h\}\}$ is generic is similar to the one in the continuous case, see [8].

Let $\psi_0 \in \mathcal{O}_1^h$, and $\psi_1 \in C^\infty(\mathbb{T}^N, \mathbb{R})$ such that $\psi_1 \geq 0$ and $\{x : \psi_1(x) = 0\} = \pi_1(\mathrm{supp}(\mu_h))$. Then $\pi_1(\mathrm{supp}(\mu_h)) \subset \Omega_h(L + \psi_0 + \psi_1)$.

Claim. *If $x_0 \notin \pi_1(\mathrm{supp}(\mu_h))$ then $x_0 \notin \Omega_h(L + \psi_0 + \psi_1)$. Indeed, $\psi_1(x_0) > 0$, and*

$$\begin{aligned}\mathbf{h}_h^{(L+\psi_0+\psi_1)}(x_0, x_0) &= \liminf_{k\to\infty}\left(\inf_{s\in\mathbb{Z}^N}\inf_{\mathcal{P}_k(x_0+s,x_0)}\sum_{i=0}^{k-1}(L + \psi_0 + \psi_1 - \bar{H}_h)(x_i, v_i)\right)\\ &\geq \liminf_{k\to\infty}\left(\inf_{s\in\mathbb{Z}^N}\inf_{\mathcal{P}_k(x_0+s,x_0)}\sum_{i=0}^{k-1}(L + \psi_0 - \bar{H}_h)(x_i, v_i) + \psi_1(x_0)\right)\\ &= \mathbf{h}_h^{(L+\psi_0)}(x_0, x_0) + \psi_1(x_0).\end{aligned}$$

Hence $\pi_1(\mathrm{supp}(\mu_h)) = \Omega_h(L + \psi_0 + \psi_1)$. ☐

Proposition 9. *There exists a bijective correspondence between the set of calibrated subactions and the set of functions $f \in C(\Omega_h(L))$ satisfying $f(x) - f(z) \leq S_h(x, z)$, for all points x, z in $\Omega(L)$.*

We claim that the proof of this Proposition is similar to the proof of [14, Theorem 13]. Indeed, the paths connecting points for defining S here are obtained in a similar way as in [14]. In the same way, when $z \in \Omega_h(L)$ we have that $S_h(\cdot, z)$ is calibrated. These are the basic elements in the proof of the mentioned result in [14].

Proposition 10. *Let $\mu_h \circ \pi_1^{-1}$ be an ergodic measure (with respect to the flow induced by $\bar{\psi}$), and u, u' two calibrated subactions for L, then $u - u'$ is constant in $\pi_1(\mathrm{supp}(\mu_h))$.*

Proof. It was shown in [13] that the points of the support of the measure μ_h are the form $(x, v) = (x_0 + hv_0, v)$ with (x_0, v_0) in the support of μ_h. Take $x \in \pi_1\,\mathrm{supp}(\mu_h)$, then $x = x_0 + hv_0$, hence

$$u(x_0) - u(x_0 + hv_0) = hL(x_0, v_0) - h\bar{H}_h = u'(x_0) - u'(x_0 + hv_0).$$

Then $u - u' = (u - u') \circ \bar{\psi}$ in $\pi_1(\mathrm{supp}(\mu_h))$ and as $\mu_h \circ \pi_1^{-1}$ is ergodic it follows that $u - u'$ is constant in $\pi_1(\mathrm{supp}(\mu_h))$. ☐

Lemma 3. *Suppose that L is generic and let μ_h be the unique minimizing measure, then the measure $\mu_h \circ \pi_1^{-1}$ is ergodic for the map $\bar{\psi}$ (defined in the proof of the Proposition 5).*

Proof. In Proposition 5 it was proved that $\bar{\psi}$ is $\mu_h \circ \pi_1^{-1}$-invariant. Let us show that it is uniquely ergodic. Let η be a measure in the Borel sets of $\Omega_h(L) = \pi_1(\mathrm{supp}(\mu_h))$, invariant by $\bar{\psi} : \Omega_h(L) \to \Omega_h(L)$. We define, for each Borel set B of $\mathbb{T}^N \times \mathbb{R}^N$, $\mu(B) = \eta(\pi_1(B \cap \mathrm{supp}(\mu_h)))$, then μ is a probability in $\mathbb{T}^N \times \mathbb{R}^N$, such that

(i) $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu_h)$,

(ii) $\mu \circ \pi_1^{-1} = \eta$,
(iii) $\mu \in \mathcal{M}_h$.

(i) and (ii) are easily verified. (iii): Let $\varphi \in C(\mathbb{T}^N)$ be a function, we have that

$$\int_{\mathbb{T}^N \times \mathbb{R}^N} \varphi(x+hv)d\mu(x,v) = \int_{\mathbb{T}^N \times \mathbb{R}^N} \varphi(x+hv(x))d\mu(x,v) = \int_{\mathbb{T}^N} \varphi \circ \bar{\psi}(x)d\eta(x)$$
$$= \int_{\mathbb{T}^N} \varphi(x)d\eta(x) = \int_{\mathbb{T}^N \times \mathbb{R}^N} \varphi(x)d\mu(x,v).$$

Let u be a calibrated subaction, by [13, Theorem 5.4] for each point $(x,v) \in \operatorname{supp}(\mu_h)$ we have

$$hL(x,v) = u(x) - u(x+hv) + h\bar{H}_h.$$

By (i) and (iii) we have that

$$\int hL(x,v)d\mu(x,v) = \int \left(u(x) - u(x+hv) + h\bar{H}_h\right) d\mu(x,v) = h\bar{H}_h.$$

Hence μ is a minimizing measure, but as we are supposing that minimizing measure is unique, we obtain $\mu = \mu_h$. Therefore $\eta = \mu_h \circ \pi_1^{-1}$, then μ_h is uniquely ergodic. □

Note that if u is a calibrated subaction, then $U + c$. where c is a constant, is also a calibrated subaction.

Theorem 15. *If L is generic in the Mané's sense, then the set of calibrated subactions has a unique element (up to an additive constant).*

Proof. By hypothesis we have that $\pi_1(\operatorname{supp}(\mu_h)) = \Omega_h(L)$.

Let $f, f' : \Omega_h(L) \to \mathbb{R}$ be continuous functions satisfying $f(x) - f(\bar{x}) \leq S_h(x,\bar{x})$, $f'(x) - f'(\bar{x}) \leq S_h(x,\bar{x})$. As in Proposition 7, we construct two calibrated subactions $u_f, u_{f'}$ such that $f - f' = u_f - u_{f'}$ in $\Omega_h(L)$, now by the Proposition 10 $u_f - u_{f'}$ is constant in $\pi_1(\operatorname{supp}(\mu_h)) = \Omega_h(L)$.

Hence the set $\{f \in C(\Omega_h(L) : f(x) - f(\bar{x}) \leq S_h(x,\bar{x})\}$ is unitary, by Proposition 9 we conclude that the set of calibrated subactions is unitary.

Remember that from the results above each calibrated subaction is associated to a certain f. □

Remark. Note that the definition of the Lagrangian be generic depends on the property P we consider. We fix a sequence $h_n \to 0$, for each h_n we consider the property P_n given by: $\mathcal{M}_0^{h_n}(L+\psi) = \{\mu_{h_n}\}$ and $\pi_1(\operatorname{supp}(\mu_{h_n})) = \Omega_{h_n}(L+\psi)$. Then, for each n, there exists a generic set $\mathcal{O}^n \in C^\infty(\mathbb{T}^N, \mathbb{R})$ where P_n is verified.

We define

$$\mathcal{O} = \bigcap_{n \geq 0} \mathcal{O}^n.$$

Hence, if $\psi \in \mathcal{O}$ then $L + \psi$ has the property P_n for each n.

Corollary 2. *Suppose that the Lagrangian L satisfy the hypotheses* (1)–(3), *and is generic in the sense of the previous remark. Let $u_h(\cdot) = \mathbf{h}_h(\cdot, x)$, where $x \in \Omega_h(L)$, define $\tilde{u}_h = u_h - u_h(0)$. Then $\tilde{u}_h$ converges to the unique viscosity solution ϕ_0 of the H-J equation, which can be show to be $h(\cdot, z)$, where $z \in \mathcal{A}$, and h is the Peierls barrier.*

Proof. The hypothesis (3) implies that u_h is semiconcave (uniformly in h) and hence locally Lipschitz. Thus, by periodicity $\tilde{u}_h$ is an uniformly bounded and equicontinuous family. It follows, by Theorem 15, Proposition 7 and item (b) of Theorem 5. □

Here we finish the part strictly necessary for the results required by the first part of the paper.

5.2. *Existence of a separating subaction*

In this last part we are interested in showing a discrete analog of the [12], that is the existence of a separating subaction, as in [15]. We add Theorem 16 in order to have a more complete understanding of the Discrete Time Aubry–Mather problem.

For this goal we need to consider the Hamiltonian defined in the following way.

Definition 9. Let $L(x, v) : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ be the Lagrangian, we define

$$\tilde{H}(p, x) = \max_v \{p \cdot v - L(x, v)\}.$$

The equation

$$\max_v \left\{ \frac{u(x + hv) - u(x)}{h} - L(x, v) \right\} \leq -\bar{H}_h$$

can be seen as a discrete analogous of the Hamilton–Jacobi equation

$$\max_v \{\nabla u(x) \cdot v - L(x, v)\} = \tilde{H}(\nabla u(x), x) \leq -\bar{H}_0.$$

Definition 10. For a fixed value $h > 0$, a continuous function $u : \mathbb{T}^N \to \mathbb{R}$ is called a subaction if for all $x \in \mathbb{T}^N$ we have

$$\max_v \{u(x + hv) - u(x) - hL(x, v)\} \leq -h\bar{H}_h.$$

Definition 11. We say that a subaction u is separating if

$$\max_v \{u(x + hv) - u(x) - hL(x, v)\} = -h\bar{H}_h \Leftrightarrow x \in \Omega_h(L).$$

Our main result of this last part is the following:

Theorem 16. *There exists a separating subaction.*

Before proceeding with the proof, we need some preliminary results.

Lemma 4. *For any subaction u and all $x \in \Omega_h(L)$ we have*

$$\max_v \{u(x+hv) - u(x) - hL(x,v)\} = -h\bar{H}_h.$$

We will postpone the proof of the lemma.

From now on we will suppose $h = 1$, and $\bar{H} := \bar{H}_1$ (here we do not need the graph property).

Note that the definition of subaction

$$\max_v \{u(x+v) - u(x) - L(x,v)\} \leq -\bar{H}$$

is equivalent to

$$u(x_k) - u(x_0) \leq A_{L-\bar{H}}(x_0, \dots, x_k) \quad \text{for any path } (x_0, \dots, x_k). \tag{16}$$

By this characterization of the subactions, it is easy to see that $\mathbf{h}_x = \mathbf{h}(x, \cdot)$ and $S_x = S(x, \cdot)$ are subactions.

Proposition 11. *If $x \in \Omega(L)$ there exists a sequence $(x_0, x_1, \dots, x_k, \dots)$ such that $x_0 = x$ and for all k*

$$\mathbf{h}(x_k, x_0) \leq -A_{L-\bar{H}}(x_0, \dots, x_k).$$

Proof. Since $x \in \Omega(L)$ there exists a sequence of minimal paths $\{(x_0^n, \dots, x_{j_n}^n)\}_{n\in\mathbb{N}}$ such that $x_0^n = x$, $x_{j_n}^n = x + s_{j_n}$ and $j_n \to \infty$ satisfying

$$A_{L-\bar{H}}(x_0^n, \dots, x_{j_n}^n) \to 0. \tag{17}$$

As $|v_j^n| < K$ there exists a sequence $(x_0, \dots, x_k, \dots)$ which is the limit of the paths above, the convergence being uniform in each compact part.

Fixed $k \in \mathbb{N}$, for $j_n > k$ we have that

$$S^{j_n-k}(x_k, x_0) \leq L(x_k, x_{k+1}^n - x_k) - \bar{H} + A_{L-\bar{H}}(x_{k+1}^n, \dots, x_{j_n}^n),$$

and so

$$\begin{aligned} S^{j_n-k}(x_k, x_0) &- A_{L-\bar{H}}(x_0^n, \dots, x_{j_n}^n) \\ &\leq L(x_k, x_{k+1}^n - x_k) - \bar{H} - A_{L-\bar{H}}(x_0^n, \dots, x_{k+1}^n). \end{aligned}$$

Hence taking the $\lim_{n\to\infty}$ and using (17) we obtain

$$\mathbf{h}(x_k, x_0) \leq -A_{L-\bar{H}}(x_0, \dots, x_k).$$ □

Proof of Lemma 4. It follows from (16) that if u is a subaction, then $u(\bar{y}) - u(y) \leq \mathbf{h}(y, \bar{y})$.

Let $x \in \Omega(L)$ and $(x_0, \ldots, x_k, \ldots)$ be the sequence given by Proposition 11. If u is a subaction, by Proposition 11 we have

$$u(x_0) - u(x_k) \leq \mathbf{h}(x_k, x_0) \leq -A_{L-\bar{H}}(x_0, \ldots, x_k).$$

The other inequality follows from (16), hence

$$u(x_k) - u(x_0) = A_{L-\bar{H}}(x_0, \ldots, x_k),$$

in particular, for $k = 1$, this implies

$$\max_v [u(x+v) - u(x) - L(x,v)] = -\bar{H}.$$

□

Lemma 5. *The function $S_x(\cdot) = S(x, \cdot)$ is uniformly Lipschitz in x.*

Proof. We fix $x \in \mathbb{T}^N, \epsilon > 0$ and $y, z \in \mathbb{T}^N$. By the definition of S there exists a path $(x_0, \ldots, x_k) \in \mathcal{P}_k(x+s, y), s \in \mathbb{Z}^N$ such that

$$|A_{L-\bar{H}}(x_0, \ldots, x_k)| \leq S(x,y) + \epsilon,$$

we can construct the following path $(\tilde{x}_0, \ldots, \tilde{x}_k) = (x_0, \ldots, x_{k-1}, z) \in \mathcal{P}_k(x+s, z)$, the action of such path is given by

$$A_{L-\bar{H}}(\tilde{x}_0, \ldots, \tilde{x}_k) = A_{L-\bar{H}}(x_0, \ldots, x_k) + L(x_{k-1}, z - x_{k-1}) - L(x_{k-1}, y - x_{k-1}).$$

Note that $|y - x_{k-1}| \leq K$ and as $y, z \in \mathbb{T}^N$, for any $\theta \in (0,1)$, we have that $|z - x_{k-1} + \theta(y - z)| < K_1$, for any x_{k-1}, hence

$$|L(x_{k-1}, z - x_{k-1}) - L(x_{k-1}, y - x_{k-1})| \leq \max_{(x,v) \in \mathbb{T}^N \times K_1} |L_v(x,v)|\, |z - y|$$

$$= C|z - y|.$$

Then for all $\epsilon > 0$ we have that

$$S(x,z) \leq A_{L-\bar{H}}(\tilde{x}_0, \ldots, \tilde{x}_k) \leq S(x,y) + \epsilon + C|z-y|,$$

which implies $S(x,z) - S(x,y) \leq C|z-y|$. Changing the roles of y and z we get $S(x,y) - S(x,z) \leq C|z-y|$.

Therefore $|S_x(y) - S_x(z)| \leq C|z-y|$, note that the Lipschitz constant is independent of x. □

Proof of Theorem 16. Remember that the function $S_x(\cdot) = S(x, \cdot)$ is a subaction.

By the definition of S we have that

$$S(x, x+v) \leq L(x,v) - \bar{H} \quad \forall\, v.$$

Fix $x \notin \Omega(L)$, then $S(x,x) > 0$. Hence

$$S_x(x+v) - S_x(x) < L(x,v) - \bar{H} \quad \forall\, v.$$

As $\Omega(L)$ is closed, for each $x \notin \Omega(L)$ we can find a neighborhood V_x of x such that for all $y \in V_x$

$$S_x(y+v) - S_x(y) < L(y,v) - \bar{H} \quad \forall\, v.$$

We can extract from this family of neighborhoods $\{V_x\}_{x\notin\Omega(L)}$, a countable subcover $\{V_{x_j}\}_{j=1}^{\infty}$. And we define

$$\tilde{S}_{x_j}(z) = S_{x_j}(z) - S_{x_j}(0),$$

as S_{x_j} is uniformly Lipschitz we obtain that $|\tilde{S}_{x_j}(z)| \leq C|z|$, hence the series given by

$$u(z) = \sum_{j=1}^{\infty} \frac{\tilde{S}_{x_j}(z)}{2^j}$$

is well defined and uniformly convergent, as $\mathbb{T}^N$ is a compact set. Finally we show that u is a subaction:

$$\begin{aligned} u(x+v) - u(x) &= \sum_{j=1}^{\infty} \frac{\tilde{S}_{x_j}(x+v) - \tilde{S}_{x_j}(x)}{2^j} \\ &= \sum_{j=1}^{\infty} \frac{S_{x_j}(x+v) - S_{x_j}(x)}{2^j} \\ &\leq \sum_{j=1}^{\infty} \frac{L(x,v) - \bar{H}}{2^j} = L(x,v) - \bar{H}. \end{aligned}$$

Hence by Theorem 4

$$\max_v \{u(x+v) - u(x) - L(x,v)\} = -\bar{H} \quad \text{if } x \in \Omega(L)$$

and for $x \notin \Omega(L)$, there exists $k \geq 1$ such that $x \in V_{x_k}$, hence

$$S_{x_k}(x+v) - S_{x_k}(x) < L(x,v) - \bar{H} \quad \forall\, v.$$

Therefore,

$$\begin{aligned} u(x+v) - u(x) &= \left(\sum_{j\neq k} \frac{S_{x_j}(x+v) - S_{x_j}(x)}{2^j} + \frac{S_{x_k}(x+v) - S_{x_k}(x)}{2^k} \right) \\ &< \left(\sum_{j\neq k} \frac{(L(x,v) - \bar{H})}{2^j} + \frac{L(x,v) - \bar{H}}{2^k} \right) = L(x,v) - \bar{H} \quad \forall\, v, \end{aligned}$$

i.e.

$$\max_v \{u(x+v) - u(x) - L(x,v)\} < -\bar{H} \quad \text{if } x \notin \Omega(L). \qquad \square$$

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