The Dirac operator for the Ruelle-Koopman pair on L^p-spaces: an interplay between Connes distance and symbolic dynamics

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Abstract

Here $\sigma : \Omega = \{0,1\}^{\mathbb{N}} \hookleftarrow$ denotes the shift, $\boldsymbol{\mu}$ the maximal entropy measure, $\mathcal{L}$ the associated normalized Ruelle operator, and $\mathcal{K}$ is the Koopman operator, all acting on $L^p(\boldsymbol{\mu})$, $p \geq 1$. We study a family of dynamical Dirac operators $\mathcal{D}_p$, associated with the Ruelle-Koopman pair, and an associated spectral triple in line with a Connes distance d_p. A diagonal representation is used. The explicit estimate of the L^p-operator norm for the commutator of $\mathcal{D}_p$ with multiplication operators M_f ($f \in \mathcal{C}(\Omega)$), shows a relation with the discrete-time derivative $f \circ \sigma - f$ (and also with the backward derivative). For $p = 2$, this commutator norm coincides with an expression related to a pressure problem; for general p, we have inequalities. We also compute $\mathcal{D}_p(\mathcal{K}^n \mathcal{L}^n)$. The d_p distance is compared with the Wasserstein distance of probabilities (associated to a certain symbolic metric on Ω): $d_1 = d_\infty$ coincides with the Wasserstein distance; for arbitrary p they are equivalent. We also show a duality formula reminiscent of Kantorovich's duality for a minimization problem on tensor products.

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1 Introduction

Within noncommutative geometry, spectral triples provide a framework for translating concepts of Riemannian geometry into operator-algebraic terms.

In particular, Connes distance formula defines a pseudometric on the space of states of a C*-algebra, analogous to the Monge-Kantorovich distance between probability measures on a compact metric space. In certain cases the two distances coincide, but even for commutative finite-dimensional C*-algebras they may differ [24]. It remains an open problem to characterize precisely when they agree, and whether a noncommutative analogue of the Wasserstein distance exists for which Connes distance arises as the dual formulation.

In order to define a Connes distance, we will consider a certain Dirac operator $\mathcal{D}$, which in some settings plays the role of the momentum acting on self-adjoint operators in C^*-algebras, but alternatively has also an interpretation as the inverse of the line element. We will investigate the connection of a dynamically defined Dirac operator with the discrete time forward (and backward) derivative for the shift acting on the symbolic space $\Omega = \{0,1\}^{\mathbb{N}}$. The novelty, when compared with [10], is the introduction of a diagonal representation taking into account the Ruelle and the Koopman operators. This provides a more simple and elegant formalism and a natural relation with forward and backward discrete time derivatives.

On a parallel, but related question, the appendix contains a proof for a duality formula analogous to the Kantorovich duality for the *naive* C*-algebraic generalization of the optimal transport problem (which, as pointed out in [5], does not necessarily produce a metric).

The recent developments on the theory of L^p-spectral triples [14, 3] (see [20] for more background on L^p-operator algebras) suggest one considers variations of the Connes distance in terms of a parameter $p \geq 1$, and our goal here is to illustrate by means of an example how do changes in p affect the pseudometric. Adopting a similar approach to other works, which relate to Spectral Triples and Ergodic Theory [21, 28, 29, 9, 10], such an example will take place on a symbolic dynamical setting.

Here $\boldsymbol{\mu}$ denotes the maximal entropy measure for the shift σ, $\mathcal{L}$ the associated normalized Ruelle operator, and $\mathcal{K}$ is the Koopman operator, all acting on $L^p(\boldsymbol{\mu})$, $p \geq 1$ (see (5) for definition). We will construct a family of pseudometrics on the space of shift invariant probability measures on the symbolic space space $\Omega = \{0,1\}^{\mathbb{N}}$. This will be done by introducing a Dirac operator $\mathcal{D}_p$ built from the Ruelle and Koopman operators, similar to [9], and by letting the L^p-space, on which this operator acts, vary according to a parameter $p \geq 1$. Roughly speaking, the Koopman operator $\mathcal{K}$ shifts observables forward, while the Ruelle operator $\mathcal{L}$ averages observables over preimages. Their precise definitions are given in Section 2. Given a continuous function $f : \Omega \to \mathbb{R}$ we denote by $(f \circ \sigma) - f$ the discrete time forward derivative of f. We refer the reader to [25], [23] and [6] for general results in Thermodynamic Formalism.

The C*- and L^p-operator algebra under consideration will be $\mathcal{C}(\Omega)$, the continuous complex-valued functions on Ω acting *via* diagonal multiplication operators on the L^p-space of *pairs* of p-integrable functions on Ω (w.r.t. the maximal entropy measure $\boldsymbol{\mu}$). The diagonal representation, denoted by π, is a key ingredient here. It allows us to take advantage of the duality between the Koopman and Ruelle operators when defining $\mathcal{D}_p$.

Given $f \in \mathcal{C}(\Omega)$, we denote by M_f the operator multiplication by f. The value $\|[\mathcal{D}_p, \pi(M_f)]\|$, in the setting of general algebras, plays the role of the estimate of *the Lipschitz constant of the operator* M_f.

Considering probabilities as operators acting on continuous functions on Ω, we can show d_p is equivalent to the Wasserstein distance with respect to a certain cost. This cost, in turn, will be a pseudometric d^∞ on Ω which roughly measures how many shifts away two sequences are from having the same orbit; d^∞ can take the value ∞.

More precisely, let $\mu, \nu \in \mathcal{P}(\Omega)$ be any two regular Borel probability measures on Ω, and let d_p denote the pseudometric induced by $\mathcal{D}_p$ (details on Section 4). Then, for $p \geq 1$ and $\lambda = \max\{p, p'\}$:

$$W_{d^\infty}(\mu, \nu) \leq d_p(\mu, \nu) \leq \sqrt[\lambda]{2} W_{d^\infty}(\mu, \nu), \tag{1}$$

where W_c denotes the Wasserstein distance w.r.t. the cost c, and d^∞ is given by:

$$d^\infty(x, y) := \min_{\substack{m,n \in \mathbb{N} \\ \sigma^m(x) = \sigma^n(y)}} m + n.$$

Furthermore, let $f \in \mathcal{C}(\Omega)$. We refer to the quantity $\|[\mathcal{D}_p, \pi(M_f)]\|$ as the Lipschitz seminorm of f (in the sense of [26]). Given $f \in \mathcal{C}(\Omega)$, we show that $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$. In the process of proving (1), we will show the following estimate for the Lipschitz seminorm induced by $\mathcal{D}_p$:

$$|\mathcal{K}f - f|_\infty \geq \|[\mathcal{D}_p, \pi(M_f)]\| \geq |f - \mathcal{L}f|_\infty, \tag{2}$$

with equality when $\mathcal{K}f - f$ does not depend on the first coordinate. Notice $\mathcal{K}f - f = f \circ \sigma - f$ can be interpreted as a *discrete-time forward derivative* of f.

We will show the following identity for it as well:

$$\|[\mathcal{D}_p, \pi(M_f)]\| = \max_{\lambda \in \{p,p'\}} \left| \sqrt[\lambda]{\mathcal{L}\,|f \circ \sigma - f|^\lambda} \right|_\infty . \tag{3}$$

Notice:

$$\left[\sqrt[\lambda]{\mathcal{L}\,|f \circ \sigma - f|^\lambda} \right](x) = \sqrt[\lambda]{\begin{array}{c} \frac{|f(x)-f(0x)|^\lambda}{2} + \\ + \frac{|f(x)-f(1x)|^\lambda}{2} \end{array}} ,$$

can be interpreted as a *discrete-time backward derivative* of f.

Additionally, we will relate the Lipschitz seminorm to a variational principle by proving the estimate:

$$\|[\mathcal{D}_p, \pi(M_f)]\| \geq r(M_{f\circ\sigma - f}\mathcal{K}) =$$

$$\max_{\lambda \in \{p,p'\}} \sqrt[\lambda]{2} \sup_{\mu \in \mathcal{P}(\sigma)} \exp \left(\int \log |f \circ \sigma - f| \, \mathrm{d}\mu + \frac{h_\mu(\sigma)}{\lambda} \right) . \tag{4}$$

with equality in case $p = 2$.

2 Notation

Let $\Omega = \{0,1\}^{\mathbb{N}}$ be equipped with the product topology and the corresponding Borel σ-algebra. Typical sequences are written $x, y \in \Omega$. Consider the action of the shift map $\sigma : \Omega \to \Omega$ defined by $x = (x_n)_{n\in\mathbb{N}} \to \sigma(x) = (x_{n+1})_{n\in\mathbb{N}}$, and denote by $\boldsymbol{\mu}$ the maximal entropy measure.

Let $\mathcal{C}(\Omega)$ be the algebra (both C* and L^p) of continuous complex-valued functions $f : \Omega \to \mathbb{C}$. Its dual space is given by the regular Borel measures on Ω. We write $\mathcal{P}(\Omega)$ for the regular Borel probability measures on Ω.

Let $L^p = L^p(\boldsymbol{\mu})$ be the Banach space of p-integrable complex-valued functions $g : \Omega \to \mathbb{C}$ with respect to the maximal entropy measure $\boldsymbol{\mu}$.

Let $\mathcal{C}(\Omega)$ act on L^p by multiplication operators. For a given $f \in \mathcal{C}(\Omega)$, the multiplication operator $M_f : L^p \to L^p$, is given by $g \mapsto M_f(g) := fg$.

Given $p \geq 1$, let $p' \geq 1$ be the number implicitly defined by $\frac{1}{p} + \frac{1}{p'} = 1$, so that $L^{p'}$ is the dual space of L^p.

The Koopman and Ruelle operators, denoted $\mathcal{K}$ and $\mathcal{L}$ respectively, are characterized by:

$$\mathcal{K}f := f \circ \sigma, \text{ and: } \mathcal{L}[f](x) := \frac{1}{2}\left(f(0x) + f(1x)\right), \tag{5}$$

for all continuous functions $f \in \mathcal{C}(\Omega)$; they may be closed with respect to any p-norm, and we will use the same notation, $\mathcal{K}$ and $\mathcal{L}$ still.

General results on the Ruelle and Koopman operators can be found in [25]. They are dual to each other in the case of $L^2(\boldsymbol{\mu})$ (see [9]).

3 Connes Distance

In [14, Definition 3.2] the authors pose a generalization for the definition of a spectral triple, where an L^p-*spectral triple* is said to be an ordered triple $(A, L^p(\mu), D)$, such that:

1. $L^p(\mu)$ is an arbitrary L^p-space;
2. A is an L^p-operator algebra; π is a representation of A on $L^p(\mu)$.
3. D is an unbounded linear operator on $L^p(\mu)$, such that:
   (a) $\{a \in A \mid \pi(a) \operatorname{dom} D \subseteq \operatorname{dom} D \text{ and } \|[D, \pi(a)]\| < +\infty\}$ is a norm dense involutive subalgebra of A.
   (b) $(\mathrm{Id} + D^2)^{-1}$ is a compact operator.
   (c) For any complex λ not in the spectrum of D, $(D - \lambda \,\mathrm{Id})^{-1}$ is a compact operator.

Then, the operator D is called a Dirac operator.

By also adopting the standard definition of states on a unital Banach algebra, they define the space of states in the particular case of an L^p-operator algebra A as [14, Definition 3.3]:

$$\mathcal{S}(A) := \{\eta \in A' \mid \|\eta\| = \eta(1) = 1\}. \tag{6}$$

Then, they are able to consider the Connes distance between a pair of states $\eta, \xi \in \mathcal{S}(A)$ given by the standard formula:

$$d_D(\eta, \xi) := \sup_{\substack{a \in A \\ \|[D, \pi(a)]\| \leq 1}} |\eta(a) - \xi(a)|. \tag{7}$$

Recall that this may be considered an operator algebra version of the Wasserstein, or Monge-Kantorovich, distance (see (16)).

In the particular case where $A = \mathcal{C}(\Omega)$, we have that $\mathcal{S}(A) = \mathcal{P}(\Omega)$, and therefore we may use (7) to define a pseudometric on the set of regular Borel probabilities on Ω. Then, expression (7) may be directly compared with the Wasserstein distance W_c w.r.t. a cost c. For details on the Wasserstein distance, see [30].

In fact, this construction can be exploited even when D is bounded and does not satisfy the compact resolvent hypothesis (see [11, Proposition 3, 4] where this has been originally pointed out). Therefore, the triple $(A, L^p(\mu), D)$ needs not be an L^p-spectral triple (see [27] for another recent example). The resulting formalism still fits into the picture described in [26]. We will verify this for our example in Section 5.

Remark 1. *There is also a definition of Banach spectral triple, which can be found in [2]. It should be noted that while the remarks from the previous paragraph still apply, the Dirac operator we are going to consider is bisectorial and does admit a bounded holomorphic functional calculus as required in [2]. This is the content of Proposition 14.*

4 The Dirac Operator

Let $A := \mathcal{C}(\Omega)$. In this section we frequently identify a continuous function $f \in \mathcal{C}(\Omega)$ with the bounded linear operator $M_f \in \mathcal{B}(L^p(\boldsymbol{\mu}))$. In this way, we often think of π as a representation of $\mathcal{C}(\Omega)$, while, rigorously, it is $\pi \circ M_{()}$ that is so.

Let $\boldsymbol{\mu} \sqcup \boldsymbol{\mu}$ denote the measure over the disjoint space $\Omega \sqcup \Omega$ which restricts to either component as $\boldsymbol{\mu}$; let $\mathcal{B}(L^p(\boldsymbol{\mu}))$ act on $L^p(\boldsymbol{\mu} \sqcup \boldsymbol{\mu}) \cong L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu})$ *via* a diagonal representation $\pi : \mathcal{B}(L^p(\boldsymbol{\mu})) \to \mathcal{B}(L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu}))$, in such a way that given $f \in \mathcal{C}(\Omega)$:

$$\pi(M_f) := \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix};$$

and let $D = \mathcal{D}_p$ be the linear operator acting on $L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu})$ by:

$$\mathcal{D}_p := \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{bmatrix}. \tag{8}$$

In order to compute expression (7) it helps to know which $f \in \mathcal{C}(\Omega)$ satisfy $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$. As a first step in that direction, notice that for any $f \in \mathcal{C}(\Omega)$:

$$
\begin{aligned}
[\mathcal{D}_p, \pi(M_f)] &= \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{bmatrix} \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix} - \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix} \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mathcal{K}M_f - M_f\mathcal{K} \\ \mathcal{L}M_f - M_f\mathcal{L} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & M_{f\circ\sigma - f}\mathcal{K} \\ \mathcal{L}M_{f - f\circ\sigma} & 0 \end{bmatrix}. \qquad (9)
\end{aligned}
$$

Consequently:

$$
\begin{aligned}
\|[\mathcal{D}_p, \pi(M_f)]\| &= \max\left\{\|M_{f\circ\sigma-f}\mathcal{K}\|_p, \|\mathcal{L}M_{f-f\circ\sigma}\|_p\right\} \\
&= \max\left\{\|M_{f\circ\sigma-f}\mathcal{K}\|_p, \|M_{f\circ\sigma-f}\mathcal{K}\|_{p'}\right\} \\
&= \max_{\lambda\in\{p,p'\}} \|M_{f\circ\sigma-f}\mathcal{K}\|_\lambda. \qquad (10)
\end{aligned}
$$

Equation (9) shows that the Lipschitz seminorm of a given function $f \in \mathcal{C}(\Omega)$ with respect to $\mathcal{D}_p$ is completely characterized by a weighted transfer operator with weight given by a discrete-time forward dynamical derivative of f, namely $M_{f\circ\sigma-f}\mathcal{K}$. Then, the theory of weighted transfer operators applies (see [1] and [3]). In particular, there is a lower bound for the Lipschitz seminorm $\|[\mathcal{D}_p, \pi(M_f)]\|$ given by the variational principle for the spectral radius:

Proposition 2. *For any continuous function $f \in \mathcal{C}(\Omega)$:*

$$
\|[\mathcal{D}_p, \pi(M_f)]\| \geq r(M_{f\circ\sigma-f}\mathcal{K}) =
$$

$$
\max_{\lambda\in\{p,p'\}} \sqrt[\lambda]{2} \sup_{\mu\in\mathcal{P}(\sigma)} \exp\left(\int \log|f\circ\sigma - f|\,\mathrm{d}\mu + \frac{h_\mu(\sigma)}{\lambda}\right). \qquad (11)
$$

Proof. Apply [3] or [4]. □

Remark 3. *When $p = 2$, the equality holds, since the norm of an anti-selfadjoint operator is equal to its spectral radius. Note also that*

$$
\frac{1}{p} \sup_{\mu\in\mathcal{P}(\sigma)} \int p \log|f\circ\sigma - f|\, d\mu + h(\mu)
$$

is not exactly the classical Pressure problem (as in [25]) due to the fact that $\log|f\circ\sigma - f|$ *can take the value* $-\infty$. *One can show the existence of ergodic* *probabilities maximizing* (11) *due to the fact that* $x \to \log|(f\circ\sigma)(x) - f(x)|$ *is upper semicontinuous (See Theorem A.3.12 in [16]); indeed, as* $\mu \to \int \log|f\circ\sigma - f|\, d\mu$ *is upper semi-continuous and affine, we get that the maximal value is attained at extremal points of* $\mathcal{P}(\sigma)$*.*

Remark 4. *Combining Proposition 2 and Birkhoff's ergodic theorem, it follows that for* $\boldsymbol{\mu}$*-almost every* $x \in \Omega$*:*

$$\begin{aligned}\|[\mathcal{D}_p, \pi(M_f)]\| &\geq \exp \int \log|f\circ\sigma - f|\,\mathrm{d}\boldsymbol{\mu} \\ &= \exp \sum_{n=0}^{+\infty} \log\left|f\circ\sigma^{n+1}(x) - f\circ\sigma^n(x)\right| \\ &= \prod_{n=0}^{+\infty} \left|f\circ\sigma^{n+1}(x) - f\circ\sigma^n(x)\right|.\end{aligned}$$

Some of the present results can also be deduced from the abstract point of view of [3], [4] or [1]. For example, the reader should compare (12) and [1, Equation (98)]

Lemma 5. *For any* $f \in \mathcal{C}(\Omega)$*:*

$$[\mathcal{D}_p, \pi(M_f)] = 0 \iff f\circ\sigma - f = 0.$$

The latter implies that f *is constant.*

Proof. The proof is analogous to the one in [9]. If $f\circ\sigma - f = 0$, then:

$$\|[\mathcal{D}_p, \pi(M_f)]\| = \max_{\lambda\in\{p,p'\}} \|M_{f\circ\sigma - f}\mathcal{K}\|_\lambda = \max_{\lambda\in\{p,p'\}} \|M_0\mathcal{K}\|_\lambda = 0.$$

In the other direction, if $[\mathcal{D}_p, \pi(M_f)] = 0$, then:

$$\max_{\lambda\in\{p,p'\}} \|M_{f\circ\sigma - f}\mathcal{K}\|_\lambda = 0,$$

and in particular:

$$\begin{aligned}\max_{\lambda\in\{p,p'\}} |M_{f\circ\sigma - f}\mathcal{K}(1)|_\lambda &= \max_{\lambda\in\{p,p'\}} |f\circ\sigma - f|_\lambda \\ &= 0,\end{aligned}$$

which means $f\circ\sigma - f = 0$. ∎

Lemma 6. *For any $f \in \mathcal{C}(\Omega)$:*

$$|f|_\infty \geq \sup_{|g|_p=1} |f\mathcal{K}g|_p \geq |\mathcal{L}f|_\infty .$$

Furthermore, if $f \in \mathcal{C}(\Omega)$ does not depend on the first coordinate (that is, if f is $\sigma^{-1}(\Sigma)$-measurable), then all above inequalities are equalities.

Proof. The proof is similar to the one in [9], except for the convex function to which we apply Jensen's inequality for conditional expectations is now $|\cdot|^p$ instead of $|\cdot|^2$. □

Remark 7. *Lemma 6 holds for p and p' with the same bounds.*

Theorem 8. *Replacing f by $f \circ \sigma - f$ in Lemma 6, in view of (10) and Remark 7, we get for any $f \in \mathcal{C}(\Omega)$:*

$$|\mathcal{K}f - f|_\infty \geq \|[\mathcal{D}_p, \pi(M_f)]\| \geq |f - \mathcal{L}f|_\infty .$$

Moreover, if $f \circ \sigma - f$ does not depend on the first coordinate we get the equalities:

$$|\mathcal{K}f - f|_\infty = \|[\mathcal{D}_p, \pi(M_f)]\| = |f - \mathcal{L}f|_\infty .$$

Proposition 9. *For any $f \in \mathcal{C}(\Omega)$:*

$$\|[\mathcal{D}_p, \pi(M_f)]\| = \max_{\lambda \in \{p,p'\}} \left| \sqrt[\lambda]{\mathcal{L}\,|f \circ \sigma - f|^\lambda} \right|_\infty . \tag{12}$$

Expression (12) can be written as:

$$\max_{\lambda \in \{p,p'\}} \left| \sqrt[\lambda]{\mathcal{L}\,|f \circ \sigma - f|^\lambda} \right|_\infty = \max_{\lambda \in \{p,p'\}} \sup_{x \in \Omega} \sqrt[\lambda]{\begin{array}{c} \frac{|f(x) - f(0x)|^\lambda}{2} + \\ + \frac{|f(x) - f(1x)|^\lambda}{2} \end{array}} . \tag{13}$$

The right-hand side of (13) is a form of the supremum of discrete-time backward derivative.

Proof. Analogous to [9]. We have:

$$\sup_{|g|_\lambda=1} |f\mathcal{K}g|_\lambda = \sup_{|g|_\lambda=1} \left(\int |f\mathcal{K}g|^\lambda \,\mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}}$$

$$
\begin{aligned}
&= \sup_{|g|_\lambda=1} \left(\int |f|^\lambda \, |\mathcal{K}g|^\lambda \, \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}} \\
&= \sup_{|g|_\lambda=1} \left(\int |f|^\lambda \left(\mathcal{K} \, |g|^\lambda \right) \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}} \\
&= \sup_{|g|_\lambda=1} \left(\int \left(\mathcal{L} \, |f|^\lambda \right) |g|^\lambda \, \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}} \\
&= \sup_{|g|_\lambda=1} \left| \left(\sqrt[\lambda]{\mathcal{L} \, |f|^\lambda} \right) g \right|_\lambda \\
&= \left| \sqrt[\lambda]{\mathcal{L} \, |f|^\lambda} \right|_\infty ,
\end{aligned}
$$

then we substitute f for $f \circ \sigma - f$. $\square$

Corollary 10. *Notice that:*

$$
\max_{\lambda \in \{p,p'\}} \sqrt[\lambda]{\begin{array}{r} \frac{|f(x)-f(0x)|^\lambda}{2} + \\ + \frac{|f(x)-f(1x)|^\lambda}{2} \end{array}} = \sqrt[\max\{p,p'\}]{\begin{array}{r} \frac{|f(x)-f(0x)|^{\max\{p,p'\}}}{2} + \\ + \frac{|f(x)-f(1x)|^{\max\{p,p'\}}}{2} \end{array}} ,
$$

and that $\max\{p,p'\} \geq 2$, *so:*

$$
\begin{aligned}
\min \left\{ \begin{array}{r} |f(x) - f(0x)| \, , \\ |f(x) - f(1x)| \end{array} \right\} &\leq \frac{2}{\frac{1}{|f(x)-f(0x)|} + \frac{1}{|f(x)-f(1x)|}} \\
&\leq \sqrt{\begin{array}{r} |f(x) - f(0x)| \times \\ \times |f(x) - f(1x)| \end{array}} \\
&\leq \frac{1}{2} \left(\begin{array}{r} |f(x) - f(0x)| + \\ + |f(x) - f(1x)| \end{array} \right) \\
&\leq \sqrt{\begin{array}{r} \frac{|f(x)-f(0x)|^2}{2} + \\ + \frac{|f(x)-f(1x)|^2}{2} \end{array}} \\
&\leq \sup_{x \in \Omega} \sqrt{\begin{array}{r} \frac{|f(x)-f(0x)|^2}{2} + \\ + \frac{|f(x)-f(1x)|^2}{2} \end{array}} \\
&\leq \| [\mathcal{D}, \pi(M_f)] \|
\end{aligned}
$$

$$\leq \sup_{x \in \Omega} \max \left\{ \begin{array}{c} |f(x) - f(0x)|, \\ |f(x) - f(1x)| \end{array} \right\}$$
$$= |f \circ \sigma - f|_\infty .$$

This reasoning also provides an alternative proof for the first inequality in Theorem 8.

Remark 11. *Corollary 10 shows that:*

$$\begin{aligned} \|[\mathcal{D}_p, \pi(M_f)]\| &= \max_{\lambda \in \{p,p'\}} \|M_{f\circ\sigma - f}\mathcal{K}\|_\lambda \\ &= \|M_{f\circ\sigma - f}\mathcal{K}\|_{\max\{p,p'\}}. \end{aligned}$$

Henceforth, we set $\lambda := \max\{p, p'\}$, as this will cause no confusion.

To conclude the characterization of the functions $f \in \mathcal{C}(\Omega)$ that have $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$, first we will exhibit a sufficient condition:

Proposition 12. *For any $f \in \mathcal{C}(\Omega)$:*

$$|f \circ \sigma - f|_\infty \leq 1 \implies \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$$

Proof. Apply Theorem 8. □

And, lastly, we present a necessary condition:

Proposition 13. *For any $f \in \mathcal{C}(\Omega)$:*

$$\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1 \implies |f \circ \sigma - f|_\infty \leq \sqrt[\lambda]{2}.$$

Proof. Notice that for any $x \in \Omega$:

$$\begin{aligned} & \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1 \\ \iff \quad & \sqrt[\lambda]{\begin{array}{c} \frac{|f(x)-f(0x)|^\lambda}{2} \\ +\frac{|f(x)-f(1x)|^\lambda}{2} \end{array}} \leq 1 \\ \iff \quad & \frac{|f(x) - f(0x)|^\lambda}{2} \leq 1 - \frac{|f(x) - f(1x)|^\lambda}{2} \\ \iff \quad & |f(x) - f(0x)|^\lambda \leq 2\left(1 - \frac{|f(x) - f(1x)|^\lambda}{2}\right) \end{aligned}$$

$$\iff \qquad |f(x) - f(0x)|^\lambda \leq 2 - |f(x) - f(1x)|^\lambda$$
$$\implies \qquad |f(x) - f(0x)| \leq \sqrt[\lambda]{2}.$$

Then, exchanging 0 and 1 in the previous argument we prove our main claim. □

The specific form of $\mathcal{D}_p$ we are considering also makes it convenient to compute the Lipschitz seminorm for operators of the form $\mathcal{K}^n\mathcal{L}^n$.

We will show that

$$\|[\mathcal{D}_p, \pi(\mathcal{K}\,\mathcal{L})]\| = 1, \tag{14}$$

which is a particular case of

$$\|[\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)]\| = 1. \tag{15}$$

In order to get that all the elements in the above expressions are well defined, we consider the identification of f with M_f.

In order to show (15), first notice the same computations at the end of [9, Section 2] hold for $\mathcal{D}_p$. To recall:

$$\begin{aligned}
[\mathcal{D}_p, \pi(\mathcal{K}^n\mathcal{L}^n)] &= \begin{pmatrix} 0 & \mathcal{K}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{K} \\ \mathcal{L}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{L} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{K}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^{n-1} \\ \mathcal{K}^{n-1}\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{L} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{K}\left(\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^{n-1}\mathcal{L}^{n-1}\right) \\ \left(\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n\right)\mathcal{L} & 0 \end{pmatrix}.
\end{aligned}$$

Furthermore:

$$\begin{aligned}
\left(\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n\right)^2 = &\ \mathcal{K}^{n-1}\mathcal{L}^{n-1}\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^{n-1}\mathcal{L}^{n-1}\mathcal{K}^n\mathcal{L}^n \\
&-\mathcal{K}^n\mathcal{L}^n\mathcal{K}^{n-1}\mathcal{L}^{n-1} + \mathcal{K}^n\mathcal{L}^n\mathcal{K}^n\mathcal{L}^n \\
= &\ \mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n \\
&-\mathcal{K}^n\mathcal{L}^n + \mathcal{K}^n\mathcal{L}^n \\
= &\ \mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n.
\end{aligned}$$

Also, notice that, because $\boldsymbol{\mu}$ is invariant, $\mathcal{K} : L^p \to L^p$ is an isometry for any $1 \leq p \leq +\infty$. Therefore, $\|\mathcal{K}T\| = \|T\|$ for any bounded linear transformation $T : L^p \to L^p$. In particular, when T is a projection, such as $\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n$, $\|\mathcal{K}T\| = 1$. This shows $\|[\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)]\| = 1$.

We consider this fact important because it could be a starting point to eventually computing the Connes distance induced by the $\mathcal{D}_p$ on the space of states of a C*-algebra such as the Exel-Lopes C*-algebra (see [17, 18]), since it is generated by elements of the form:

$$\sum_{i=1}^{N} M_{f_{n_i}} \mathcal{K}^{n_i} \mathcal{L}^{n_i} M_{g_{n_i}},$$

where $N, n_i \in \mathbb{N}$, and $f_{n_i}, g_{n_i} \in \mathcal{C}(\Omega)$, for every $1 \leq i \leq N$.

We also consider it important to prove the following:

Proposition 14. *The operator $\mathcal{D}_p$ is bisectorial and admits a bounded holomorphic functional calculus [2].*

Proof. Notice that $\mathcal{D}_p^2$ is a (nontrivial) projection. As such, its spectrum is the set $\{0, 1\} \subseteq \mathbb{C}$. Employing the Spectral Mapping Theorem [12, Chapter VII 4.10] we conclude that the spectrum of $\mathcal{D}_p$ is contained in the set $\{-1, 0, 1\}$. The reader may check that they are identical, but we will not use this fact here. Therefore, the operator $\mathcal{D}_p$ is θ-bisectorial for any $0 < \theta < \frac{\pi}{2}$.

Decomposing the space $L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu})$ as $\operatorname{Im} \mathcal{D}_p^2 \times \operatorname{Ker} \mathcal{D}_p^2$ instead, we see that, in such decomposition, our operator $\mathcal{D}_p$ takes the form:

$$\mathcal{D}_p = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

where $T_1^2 = \mathrm{Id}$ and $T_2^2 = 0$.

This decomposition allows us to see that $\mathcal{D}_p$ admits a holomorphic functional calculus which is the direct sum of the holomorphic functional calculi of T_1 and T_2. The first is bounded for holomorphic functions on any (open) bisector, since these contain the finite set $\{-1, 1\}$, on which the spectrum of T_1 is contained. The second is in fact trivial since $\operatorname{Ker} \mathcal{D}_p^2 = \operatorname{Ker} \mathcal{D}_p$, that is, $T_2 = 0$. Therefore, the holomorphic functional calculus for $\mathcal{D}_p$ is bounded. $\square$

5 Pure States - $d_p(\delta_x, \delta_y)$

In the particular case of the L^p-operator algebra $\mathcal{C}(\Omega)$ the set of states (defined in (6)) is exactly the same as for the C*-algebra $\mathcal{C}(\Omega)$. That is because the L^p-operator norm of a multiplication operator M_f is $|f|_\infty$ regardless of which p one chooses. In this case $A = \mathcal{C}(\Omega)$; and the states are the Borel probability measures defined on Ω. In this section, we consider the particular case of pure states: the Dirac deltas δ_x on points $x \in \Omega$. In the next section, we will consider a more general case.

A natural question is to know when $d_p(\delta_x, \delta_y) < \infty$, for $x, y \in \Omega$ (see Theorem 28); which is somehow related to homoclinic equivalence relations.

There is a way to fit our results into the setting of [26]: in their notation, our normed space is $A = \mathcal{C}(\Omega)$ equipped with the supremum norm $|\cdot|_\infty$. All of our elements are Lipschitz, so $\mathcal{L} = A = \mathcal{C}(\Omega)$. Our Lipschitz seminorm is given by $L(a) = \|[\mathcal{D}_p, \pi(a)]\|$. Its zero locus is the space of constant functions $\mathcal{K} = \mathbb{C}1 \subseteq \mathcal{C}(\Omega)$, which determines η up to sign. Take it so $\eta(1) = 1$.

This means that we could apply [26, Proposition 1.4], which says our distance induces a topology finer than weak-$*$. Additionally, we will see in Example 21 that we do have states for which the distance is $+\infty$, which means it induces a *strictly* finer topology than the weak-$*$ topology in $\mathcal{S}(\mathcal{C}(\Omega))$. Notice this topology has many nontrivial bounded components.

We now pass to the study of the bounded components of the Connes distance. This means we want to discriminate between the pairs of states for which it is finite and the pairs of states for which it is not. First, we will restrict our attention to pure states. We begin with a simple example:

Example 15. *Let us estimate the Connes distance d_p for a pair of Dirac deltas $\delta_x, \delta_{\sigma(x)} \in \mathcal{S}(\mathcal{C}(\Omega))$. Proposition 13 implies, setting $\lambda = \max\{p, p'\}$:*

$$\begin{aligned} d_p(\delta_x, \delta_{\sigma(x)}) &= \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|[\mathcal{D}_p, \pi(M_f)]\| \le 1}} |f(x) - f \circ \sigma(x)| \\ &\le \sqrt[\lambda]{2} \\ &< +\infty. \end{aligned}$$

Example 16. *Consequently, if $x \in \Omega$ and $y \in \Omega$ are two points such that there exists two numbers $m, n \in \mathbb{N}$ for which the respective orbits meet at $\sigma^m(x) = \sigma^n(y)$, then:*

$$\begin{aligned} d_p(\delta_x, \delta_y) &\le d_p(\delta_x, \delta_{\sigma(x)}) + d_p(\delta_{\sigma(x)}, \delta_y) \\ &\le d_p(\delta_x, \delta_{\sigma(x)}) + d_p(\delta_{\sigma(x)}, \delta_{\sigma^2(x)}) + d_p(\delta_{\sigma^2(x)}, \delta_y) \\ &\le d_p(\delta_x, \delta_{\sigma(x)}) + \cdots + d_p(\delta_{\sigma^{m-1}(x)}, \delta_{\sigma^m(x)}) + d_p(\delta_{\sigma^m(x)}, \delta_y) \end{aligned}$$

$$
\begin{aligned}
&= d_p(\delta_x, \delta_{\sigma(x)}) + \cdots + d_p(\delta_{\sigma^{m-1}(x)}, \delta_{\sigma^m(x)}) + d_p(\delta_{\sigma^n(y)}, \delta_y) \\
&\leq \begin{array}{l} p(\delta_x, \delta_{\sigma(x)}) + \cdots + d_p(\delta_{\sigma^{m-1}(x)}, \delta_{\sigma^m(x)}) + \\ \quad +d_p(\delta_{\sigma^n(y)}, \delta_{\sigma^{n-1}(y)}) + \cdots + d_p(\delta_{\sigma^1(y)}, \delta_y) \end{array} \\
&\leq \sqrt[\lambda]{2}\,(m+n) \\
&< +\infty.
\end{aligned}
$$

Now let us calculate more examples. We are looking for a function of arbitrary variation and "Lipschitz constant = 1".

Example 17. *If* $f := 2\chi_{01} + 4\chi_{11} + 2\chi_{10}$, *then:*

$$
\begin{aligned}
f \circ \sigma - f = &\begin{array}{l} (\chi_{001} + \chi_{101} + \chi_{010} + \chi_{110}) + 4\,(\chi_{011} + \chi_{111}) + \\ \quad -2\,(\chi_{010} + \chi_{011} + \chi_{100} + \chi_{101}) - 4\,(\chi_{110} + \chi_{111}) \end{array} \\
&= 2\,(\chi_{001} - \chi_{100} - \chi_{011} + \chi_{110}) + 4\,(\chi_{011} - \chi_{110}) \\
&= 2\,(\chi_{001} - \chi_{100} + \chi_{011} - \chi_{110})\,.
\end{aligned}
$$

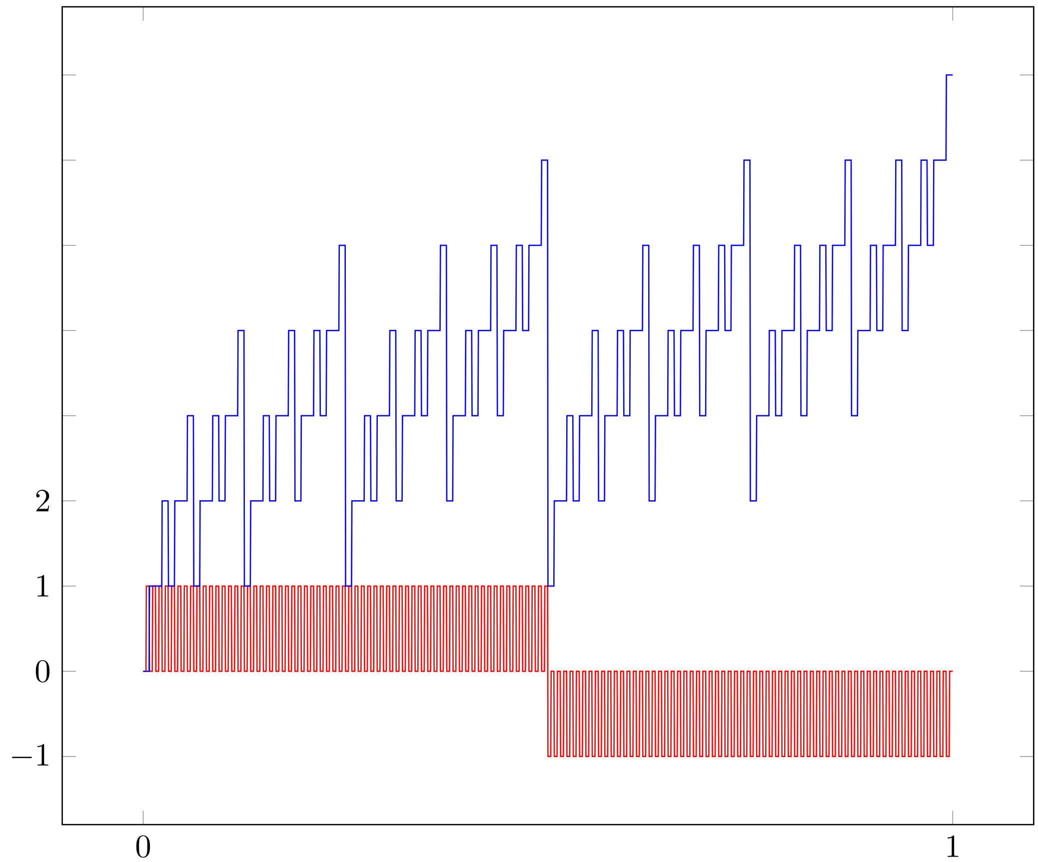
Example 18. *If* $f := 2\chi_{001} + 4\chi_{011} + 6\chi_{111} + 4\chi_{110} + 2\chi_{100}$, *then:*

$$
\begin{aligned}
f \circ \sigma - f = &\begin{array}{l} 2\,(\chi_{0001} + \chi_{1001} + \chi_{0100} + \chi_{1100}) \\ \quad +4\,(\chi_{0011} + \chi_{1011} + \chi_{0110} + \chi_{1110}) \\ \qquad +6\,(\chi_{0111} + \chi_{1111}) \\ \quad -2\,(\chi_{0010} + \chi_{0011} + \chi_{1000} + \chi_{1001}) \\ \qquad -4\,(\chi_{0110} + \chi_{0111} + \chi_{1100} + \chi_{1101}) \\ \qquad\quad -6\,(\chi_{1110} + \chi_{1111}) \end{array} \\
= &\begin{array}{l} 2\,(\chi_{0001} + \chi_{0100} + \chi_{1100} - \chi_{0010} - \chi_{0011} - \chi_{1000}) \\ \quad +4\,(\chi_{0011} + \chi_{1011} + \chi_{1110} - \chi_{0111} - \chi_{1100} - \chi_{1101}) \\ \qquad +6\,(\chi_{0111} - \chi_{1110}) \end{array} \\
= &\begin{array}{l} 2\,(\chi_{0001} + \chi_{0100} - \chi_{1100} + \chi_{0111} - \chi_{0010} + \chi_{0011} - \chi_{1000} - \chi_{1110}) \\ \quad +4\,(\chi_{1011} - \chi_{1101}) \end{array}.
\end{aligned}
$$

Example 19. *If* $f := 2\,(\chi_{001} + \chi_{010} + \chi_{100}) + 4\,(\chi_{011} + \chi_{101} + \chi_{110}) + 6\chi_{111}$, *then:*

$$
f \circ \sigma - f = \begin{array}{l} 2\,(\chi_{0001} + \chi_{1001} + \chi_{0010} + \chi_{1010} + \chi_{0100} + \chi_{1100}) \\ \quad +4\,(\chi_{0011} + \chi_{1011} + \chi_{0101} + \chi_{1101} + \chi_{0110} + \chi_{1110}) \\ \qquad +6\,(\chi_{0111} + \chi_{1111}) \\ \quad -2\,(\chi_{0010} + \chi_{0011} + \chi_{0100} + \chi_{0101} + \chi_{1000} + \chi_{1001}) \\ \qquad -4\,(\chi_{0110} + \chi_{0111} + \chi_{1010} + \chi_{1011} + \chi_{1100} + \chi_{1101}) \\ \qquad\quad -6\,(\chi_{1110} + \chi_{1111}) \end{array}
$$

$$
\begin{aligned}
= \quad & 2\left(\chi_{0001}+\chi_{1010}+\chi_{1100}-\chi_{0101}-\chi_{0011}-\chi_{1000}\right) \\
& +4\left(\chi_{0011}+\chi_{0101}+\chi_{0110}+\chi_{1110}-\chi_{0110}-\chi_{0111}-\chi_{1010}-\chi_{1100}\right) \\
& +6\left(\chi_{0111}-\chi_{1110}\right) \\
= \quad & 2\left(\chi_{0001}+\chi_{1010}+\chi_{1100}-\chi_{0101}-\chi_{0011}-\chi_{1000}+\chi_{0111}-\chi_{1110}\right) \\
& +4\left(\chi_{0011}+\chi_{0101}+\chi_{0110}-\chi_{0110}-\chi_{1010}-\chi_{1100}\right) \\
= \; & 2\left(\chi_{0001}-\chi_{1010}+\chi_{1100}+\chi_{0101}-\chi_{0011}-\chi_{1000}+\chi_{0111}-\chi_{1110}\right).
\end{aligned}
$$



f_7^γ (blue) and $f_7^\gamma \circ \sigma - f_7^\gamma$ (red). In this picture, the sequence $x \in \Omega$ corresponds to the real number $\sum x_i 2^{-i} \in [0,1]$.

Proposition 20. *If $x, y \in \Omega$ are two sequences such that:*

$$\sup_{n\in\mathbb{N}} \left|\#\left\{i \le n \mid x_i = 1\right\} - \#\left\{i \le n \mid y_i = 1\right\}\right| \ge N,$$

then $d_p(\delta_x, \delta_y) \ge N$. In particular, $d_p(\delta_{0^\infty}, \delta_{1^\infty}) = +\infty$.

Proof. Consider the following family of continuous functions of Ω:

$$f_k^\gamma := \sum_{w \in \hat{W}_k} \# \{i \mid w_i = 1\} \chi_w.$$

Notice that:

$$\begin{aligned}\|[\mathcal{D}_p, \pi(f_k^\gamma)]\| &\leq |a(f_k^\gamma \circ \sigma - f_k^\gamma)|_\infty \\ &\leq |f_k^\gamma \circ \sigma - f_k^\gamma|_\infty \\ &\leq 1,\end{aligned}$$

so that this family gives us a lower bound for the Connes distance. That is:

$$\begin{aligned}d_p(\delta_x, \delta_y) &= \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|\left[\mathcal{D}_p, \pi(M_f)\right]\| \leq 1}} |f(x) - f(y)| \\ &\geq \sup_k |f_k^\gamma(x) - f_k^\gamma(y)| \\ &\geq N.\end{aligned}$$

□

Example 21. *In particular,* $d_p(\delta_{0^\infty}, \delta_{1^\infty}) = +\infty$.

Proposition 22. *If* f_k *is a function of the form* $\sum_{w \in \hat{W}_k} \theta_w \chi_w$ *such that* $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$, *then* $|f_k(x) - f_k(y)| \leq \sqrt[\lambda]{2}k$.

Proof. Consider the point $z = (x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k) \in \Omega$. It is clear that $f_k(z) = f_k(x)$ and $f_k \circ \sigma^k(z) = f_k(y)$. Now the values $f_k(x)$ and $f_k(y)$ are telescopically related as:

$$\begin{aligned}|f_k(x) - f_k(y)| &= \left|f_k(z) - f_k \circ \sigma^k(z)\right| \\ &= \left|f_k(z) - f_k \circ \sigma(z) + f_k \circ \sigma(z) - f_k \circ \sigma^k(z)\right| \\ &= \left|\begin{array}{l} f_k(z) - f_k \circ \sigma(z) + \\ \quad + f_k \circ \sigma(z) - f_k \circ \sigma^2(z) + \\ \qquad + \cdots + \\ \qquad\quad + f_k \circ \sigma^{k-1}(z) - f_k \circ \sigma^n(z) \end{array}\right| \\ &\leq \begin{array}{l} |f_k(z) - f_k \circ \sigma(z)| + \\ \quad + |f_k(\sigma(z)) - f_k \circ \sigma(\sigma(z))| + \\ \qquad + \cdots + \\ \qquad\quad + \left|f_k(\sigma^{k-1}(z)) - f_k \circ \sigma(\sigma^{k-1}(z))\right| \end{array}\end{aligned}$$

$$\leq \sqrt[\lambda]{2}k,$$

where the last inequality follows from Proposition 13. □

We will now consider words of finite length on the alphabet $\{0, 1\}$. For a given $k \in \mathbb{N}$, W_k is the set of words $w = [w_1, w_2, ..., w_s]$, $s \leq k$, of length at most k and $\hat{W}_k \cong \{0,1\}^k$ is the set of words $w = [w_1, w_2, ..., w_s]$ of length exactly k. By abuse of language we say that $\sigma([x_1, x_2, .., x_k]) = [x_2, .., x_k]$, and words $[w_1, w_2, ..., w_s]$ can also represent cylinder sets $[w_1, w_2, ..., w_s] \subset \{0, 1\}^{\mathbb{N}}$.

Given $x = (x_1, x_2, \cdots, x_n, \cdots)$, we denote by $x|_k$ the word $[x_1, x_2, \cdots, x_k]$ of length k.

It will be appropriate to define a metric d_k which is a graph distance in $\hat{W}_k$.

Proposition 23. *Consider the graph* (V_k, E_k) *given by:*

$$V_k = \hat{W}_k \text{ and } E_k = \left\{(u, v) \in \hat{W}_k \times \hat{W}_k \mid (\sigma([u]) \cap [v]) \cup ([u] \cap \sigma([v])) \neq \emptyset\right\}.$$

Let d_k *denote the graph distance in* $V_k = \hat{W}_k$*. Then,*

$$\sup_k d_k(x|_k, y|_k) \leq d_p(\delta_x, \delta_y) \leq \sqrt[\lambda]{2} \sup_k d_k(x|_k, y|_k).$$

Proof. Let $P = ((w_1 = x|_k, w_2), (w_2, w_3), \cdots, (w_{n-1}, w_n = y|_k))$ be a path joining $x|_k$ and $y|_k$ along E_k. Also, let f_k^θ be a function that only depends on the first k coordinates. If f_k^θ satisfies $\|\left[\mathcal{D}_p, \pi(f_k^\theta)\right]\| \leq 1$, then by definition of E_k, we have that $\left|\theta_{w_i} - \theta_{w_{i+1}}\right| \leq \sqrt[\lambda]{2}$. Also notice the family of all f_k^θ is dense in $\mathcal{C}(\Omega)$. From the above we get:

$$\begin{aligned} d_p(\delta_x, \delta_y) &= \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|\left[\mathcal{D}_p, \pi(M_f)\right]\| \leq 1}} |f(x) - f(y)| \\ &= \sup_k \sup_{\substack{\theta \in \mathbb{R}^{\hat{W}_k} \\ \|\left[\mathcal{D}_p, \pi(f_k^\theta)\right]\| \leq 1}} \left|f_k^\theta(x) - f_k^\theta(y)\right| \\ &= \sup_{k,\theta} \left|f_k^\theta(x) - f_k^\theta(y)\right| \\ &= \sup_{k,\theta} \left|\theta_{x|_k} - \theta_{y|_k}\right| \\ &\leq \sqrt[\lambda]{2} \sup_k d_k(x|_k, y|_k). \end{aligned}$$

Now define f_k^γ by:

$$\gamma_w = d_k(w, y|_k).$$

It is clear from the definition that:

$$\| [\mathcal{D}_p, \pi(f_k^\theta)] \| \leq 1 \text{ and } |f_k^\gamma(x) - f_k^\gamma(y)| = d_k(x|_k, y|_k),$$

which gives the other inequality. □

Remark 24. *It can happen that:*

$$\sup_{n\in\mathbb{N}} |\#\{i \leq n \mid x_i = 1\} - \#\{i \leq n \mid y_i = 1\}| < +\infty,$$

and yet $d_p(\delta x, \delta y) = +\infty$. For instance, consider the sequences:

$$x = (0,1,0,1,0,1,0,1,\cdots) \text{ and } y = (0,0,1,1,0,0,1,1,\cdots).$$

Their incidences of 1's up to length n differ by at most $1 \ll +\infty$, and yet the distance between the truncations of this two points on the graph (V_k, E_k) is of the same order of magnitude as k. Then Proposition 23 gives $d(\delta_x, \delta_y) = +\infty$.

Proposition 25. *Consider the length ℓ of the longest common subword c of u and v, which are words of length k; that is, there exists $m, n \in \mathbb{N}$ such that:*

$$\begin{array}{ccccc} c_1 & = & u_m & = & v_n, \\ c_2 & = & u_{m+1} & = & v_{n+1}, \\ c_3 & = & u_{m+2} & = & v_{n+2}, \\ & & \vdots & & \\ c_\ell & = & u_{m+\ell} & = & v_{n+\ell}. \end{array}$$

Then the k^{th} graph distance from $u \in \hat{W}_k$ to $v \in \hat{W}_k$ satisfies

$$d_k(u,v) = \min\{k, k-\ell+m+2n, k-\ell+n+2m\}.$$

Proof. From the definition of E_k it follows that two vertices $w, w' \in V_k = \hat{W} = \{0,1\}^k$ are "connected" if and only if they are of the form:

$$\begin{array}{ccc} w_1 & = & w'_2 \\ w_2 & = & w'_3 \\ w_3 & = & w'_4 \\ & \vdots & \\ w_{k-1} & = & w'_k \end{array} \quad \text{, or:} \quad \begin{array}{ccc} w_2 & = & w'_1 \\ w_3 & = & w'_2 \\ w_4 & = & w'_3 \\ & \vdots & \\ w_k & = & w'_{k-1} \end{array} .$$

In particular, each vertex of w has four neighbours w', uniquely characterized by one of the alternatives: $w'_1 = 0$, $w'_1 = 1$, $w'_k = 0$, or $w'_k = 1$.

We say that c is *on the same side* in u and v if $m \vee n \leq \frac{(k-\ell)}{2}$ or if $m \wedge n \geq \frac{(k-\ell)}{2}$. Otherwise, we say that c is *on opposite sides* in u and v.

Of course, the diameter of the graph is k. Now let us exhibit a shorter path from u to v when their maximal common word c has length $\ell \geq \frac{k}{2}$ and is on the same side in u and v. We are going to do the case $m < n \leq \frac{(k-\ell)}{2}$; the other possibilities are analogous.

1. Starting at $w^1 = u$, take $m - 1$ steps to the neighbour w^{i+1} such that $w_k^{i+1} = 0$.
2. This will get us to $w^m = [w, u|^{m+\ell+1}, 0^{m-1}]$.
3. Take $k - \ell - 1$ steps to the neighbour w^{i+1} such that $w_1^{i+1} = v_{n+m-i}$.
4. This will get us to $w^{m+k-\ell} = [v|_{n-1}, w, u|_{k-n+m}^{k-\ell-n+1}]$.
5. Take $n - 1$ steps to the neighbour w^{i+1} such that $w_1^{i+1} = 0$.
6. This will get us to $w^{m+k-\ell+n} = [0^{k-\ell-n+1}, v|_{n-1}, w]$.
7. Take $n - 1$ steps to the neighbour w^{i+1} such that $w_1^{i+1} = v_{k-\ell+2n+m-i}$.
8. This will get us to $w^{k-\ell+m+2n} = v$.

□

Remark 26. *In particular,* $d_k(u, v) \geq k - \ell$, $u, v \in \hat{W}_k$.

Proposition 27. *If x and y are two points such that $d_p(\delta_x, \delta_y) < +\infty$, then there exist $m, n \in \mathbb{N}$ such that $\sigma^m(x) = \sigma^n(y)$.*

Proof. Let $\ell(k)$ denote the size of the largest common word between $x|_k$ and $y|_k$. By Proposition 23 we have that $d(\delta_x, \delta_y) < +\infty \implies \sup_k k - \ell(k) < +\infty$. But this implies that there exists a k_0 such that $k - \ell(k) \equiv r = r(x, y)$ for every $k \geq k_0$. This means that x and y only differ for finitely many terms, so there exist $m, n \in \mathbb{N}$ such that $\sigma^m(x) = \sigma^n(y)$. □

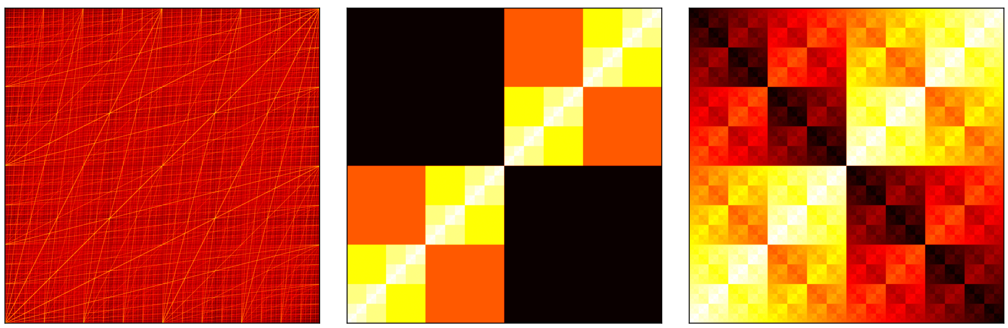
By combining Proposition 27 with Example 16 we have just proved:

Theorem 28. *For any pair of points $x, y \in \Omega$, $p \geq 1$,*

$$d_p(\delta_x, \delta_y) < +\infty \iff \text{there exist } m, n \in \mathbb{N} \text{ such that } \sigma^m(x) = \sigma^n(y).$$

□

The properties mentioned above in Theorem 28 are somehow related to the so-called *homoclinic equivalence relation* as defined in Section 6 in [13]; the particular case where $\sigma^m(x) = \sigma^m(y)$, $m \in \mathbb{N}$ (see also [17, 18]). In this case we get $d_p(\delta_x, \delta_y) < +\infty$. For the case $m \neq n$ see Section 9 in [19] (and also [22]).



(V_k, E_k)-graph, truncated and cumulative distances on $\{0,1\}^{12}$. Or $d_k(u,v)$, $2^{-N(u,v)}$ and $\sum_{u_i=v_i} 2^{-i}$ respectively, $N(u,v)$ being the smallest index for which u differs from v. Here, we identified the word $u \in \hat{W}_k$ with the real number $\sum u_i 2^{-i} \in [0,1]$. The distance from u to v is plotted lighter if it is close to zero and darker if it is close to the diameter of Ω (1 for truncated and cumulative distances and k for d_k). This figure shows $k = 12$.

6 General States - $d_p(\mu, \nu)$

Now we pass to the question of computing and estimating the Connes distance $d_p(\mu, \nu)$, for two general states $\mu, \nu \in \mathcal{S}(\mathcal{C}(\Omega)) = \mathcal{P}(\Omega)$. First, we will exhibit an analog of Example 15:

Proposition 29. *If $\sigma_\sharp$ denotes the push-forward through σ, then for any given state $\mu \in \mathcal{P}(\Omega)$, Proposition 13 implies (recall we have set $\lambda = \max\{p, p'\}$):*

$$d_p(\mu, \sigma_\sharp(\mu)) = \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1}} \left| \int f \, d\mu - \int f \, d\sigma_\sharp(\mu) \right|$$

$$= \sup_{\substack{f\in\mathcal{C}(\Omega)\\ \|[\mathcal{D}_p,\pi(M_f)]\|\leq 1}} \left|\int f\circ\sigma - f\,\mathrm{d}\mu\right|$$
$$\leq \sqrt[\lambda]{2}.$$

□

For the next definitions the compact metric space $(X,\tilde{d})$ will represent either the set $\{0,1\}^{\mathbb{N}}$, or the set $V_k=\{0,1\}^k$, $k\geq 1$.

Given a metric $\tilde{d}$ the Wasserstein distance between the probabilities μ and ν on X is (see [30])

$$W_{\tilde{d}}(\mu,\nu) = \sup_{\substack{f\in\mathcal{C}(X)\\ |f(x)-f(y)|\leq\tilde{d}(x,y)}} \left|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu\right|. \tag{16}$$

Next, an analog of Proposition 23 will consider the case where $\tilde{d}=d_k$ and $X=V_k$. Later we will introduce a metric $\tilde{d}=d^\infty$ on $X=\Omega$.

Proposition 30. *Let $\mu,\nu\in\mathcal{P}(\Omega)$ be any two states. Then, given $x,y\in\Omega=\{0,1\}^{\mathbb{N}}$, and recalling that $\lambda=\max\{p,p'\}$:*

$$\sup_{\substack{k\in\mathbb{N}\\ y\in\Omega}}\left|\int d_k(x|_k,y|_k)\,\mathrm{d}(\mu-\nu)\right| \leq d_p(\mu,\nu)\leq \sqrt[\lambda]{2}\sup_{\substack{k\in\mathbb{N}\\ y\in\Omega}}\left|\int d_k(x|_k,y|_k)\,\mathrm{d}(\mu-\nu)\right|.$$

Or:

$$\sup_{k\in\mathbb{N}} W_{d_k}(\mu,\nu)\leq d_p(\mu,\nu)\leq \sqrt[\lambda]{2}\sup_{k\in\mathbb{N}} W_{d_k}(\mu,\nu),$$

Proof. Analogous to the proof of Theorem 31. Observe the Wasserstein distance is equal to the supremum in y for each respective k. It is also increasing in k so that the suprema are actually limits. □

Finally, we have that:

Theorem 31. *Let $\mu,\nu\in\mathcal{P}(\Omega)$ be any two states. Then, for $p\geq 1$ and $\lambda=\max\{p,p'\}$*

$$W_{d^\infty}(\mu,\nu)\leq d_p(\mu,\nu)\leq\sqrt[\lambda]{2}W_{d^\infty}(\mu,\nu),$$

where d^∞ is given by:

$$d^\infty(x,y) := \min_{\substack{m,n\in\mathbb{N}\\ \sigma^m(x)=\sigma^n(y)}} m+n.$$

Proof. Consider a function $f \in \mathcal{C}(\Omega)$ such that $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$. Proposition 13 shows that $|f \circ \sigma(x) - f(x)| \leq \sqrt[\lambda]{2}$. Now, if $m, n \in \mathbb{N}$ are such that $\sigma^m(x) = \sigma^n(y)$, then:

$$\begin{aligned}
|f(x) - f(y)| &= \left| \begin{array}{l} f(x) - f\circ\sigma(x) + \\ \quad + f\circ\sigma(x) - f(y) \end{array} \right| \\
&= \left| \begin{array}{l} f(x) - f\circ\sigma(x) + \\ \quad + f\circ\sigma(x) - f\circ\sigma^2(x) + \\ \qquad + f\circ\sigma^2(x) - f(y) \end{array} \right| \\
&= \left| \begin{array}{l} f(x) - f\circ\sigma(x) + \\ \quad + \cdots + \\ \qquad + f\circ\sigma^{m-1}(x) - f\circ\sigma^m(x) + \\ \qquad\quad + f\circ\sigma^n(y) - f\circ\sigma^{n-1}(y) + \\ \qquad\qquad + \cdots + \\ \qquad\qquad\quad + f\circ\sigma(y) - f(y) \end{array} \right| \\
&\leq \sqrt[\lambda]{2}\,(m+n)\,,
\end{aligned}$$

which shows that $|f(x) - f(y)| \leq \sqrt[\lambda]{2} d_\infty(x,y)$. On the other hand, if $f \in \mathcal{C}(\Omega)$ is such that $|f(x) - f(y)| \leq d_\infty(x,y)$, then $|f \circ \sigma(x) - f(x)| \leq 1$. Therefore:

$$\begin{aligned}
\|[\mathcal{D}_p, \pi(M_f)]\| &= \sqrt[\lambda]{\begin{array}{l} \frac{|f(x)-f(0x)|^\lambda}{2} + \\ \quad + \frac{|f(x)-f(1x)|^\lambda}{2} \end{array}} \\
&\leq \sqrt[\lambda]{\frac{1}{2} + \frac{1}{2}} \\
&= 1.
\end{aligned}$$

□

Note that the distance d^∞ does not produce the same topology as the one obtained from the usual metric on Ω.

Corollary 32. *The Connes distance between two states $\mu, \nu \in \mathcal{P}(\Omega)$ is finite if and only if they give the same weight to each equivalence class of the relation given by $x\mathcal{R}y \iff \exists m, n \in \mathbb{N} : \sigma^m(x) = \sigma^n(y)$, that is: if $\mu(\bar{x}) = \nu(\bar{x})$ for any $x \in \Omega$, $\bar{x}$ the equivalence class of x. Each of these equivalence classes is the bounded component of each of its elements with respect to distance d^∞.*

Taking the limits p in Theorem 31 gives $d_{\mathcal{D}_1} = d_{\mathcal{D}_\infty} = W_{d^\infty}$. Notice, $d_{\mathcal{D}_p} = \sqrt[\lambda]{2} d_\infty$. This shows the family $d_{\mathcal{D}_p}$ interpolates between the Connes and Wasserstein distances; the Wasserstein distance corresponds to the cases $p = 1, +\infty$. We have numerical evidence for the inequalities in Theorem 31 being strict.

Appendix A

The material of this section was taken from [7]. The noncommutative generalization of the optimal transport problem in [7] is a version of the Monge-Kantorovich optimal transport problem (on compact spaces with positive continuous cost) according to the following dictionary:

Real functions $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$	Self-adjoint elements $a \in \mathcal{A}$, $b \in \mathcal{B}$
Probability measures $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$	C*-algebra states $\eta \in \mathcal{S}(\mathcal{A})$, $\xi \in \mathcal{S}(\mathcal{B})$
Cost function $c \in \mathcal{C}(X \times Y)$, $c \geq 0$	Cost element $c \in \mathcal{A} \otimes \mathcal{B}$, $c \geq 0$
Coupling probabilities $\rho \in \mathcal{P}(X \times Y)$ $\int f(x) + g(y)\, \mathrm{d}\rho = \int f(x)\, \mathrm{d}\mu + \int g(y)\, \mathrm{d}\nu$	Coupling states $\omega \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ $\omega(a \otimes 1 + 1 \otimes b) = \eta(a) + \xi(b)$
$W_c(\mu, \nu) := \inf_\rho \int c\, \mathrm{d}\rho$	$\mathcal{W}_c(\eta, \xi) := \inf_\omega \omega(c)$

Therefore, it is natural to pursue the following reasoning:
Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C*-algebras.
Denote by $\mathcal{A} \otimes \mathcal{B}$ the maximal tensor product between $\mathcal{A}$ and $\mathcal{B}$.
Let $c \in (\mathcal{A} \otimes \mathcal{B})^+$ be a positive element (henceforth called *cost element*).
Let $\eta \in \mathcal{S}(\mathcal{A})$ and $\xi \in \mathcal{S}(\mathcal{B})$ be two given C*-algebra states.
Denote by $\Gamma(\eta, \xi)$ the set of all states $\omega \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ such that:

$$\omega(a \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes b) = \eta(a) + \xi(b) \quad \forall\, a \in \mathcal{A}, b \in \mathcal{B}.$$

These states are called the *admissible couplings of* η *and* ξ.

Denote by $\tilde{\Gamma}(c)$ the set of all pairs of self-adjoint elements $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that:

$$a \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes b \leq c$$

These pairs are called the *admissible pairings for the cost c*.

The noncommutative optimal transport problem in [7] consists of minimizing the evaluation of the cost element among all admissible couplings of η and ξ. We call this value the minimum optimal transport cost from η to ξ. A value c is fixed according to convenience in each problem. Thus, we write:

$$\mathcal{W}_c(\eta, \xi) := \inf_{\omega \in \Gamma(\eta,\xi)} \omega(c) \tag{17}$$

It is possible to prove that minimizers for (17) exist by employing the direct method of the calculus of variations (see [7]). Furthermore, and also done in [7], it is possible to prove a formula analogous to the Kantorovich duality formula for (17), as we will see. Notice this recovers the existence of minimizers as a corollary.

Theorem 33. *Let $\mathcal{A}$, $\mathcal{B}$, $c \in (\mathcal{A} \otimes \mathcal{B})^+$, $\eta \in \mathcal{S}(\mathcal{A})$, and $\xi \in \mathcal{S}(\mathcal{B})$ be as above, and consider the aforementioned definitions of $\Gamma(\eta, \xi)$ and $\tilde{\Gamma}(c)$. Then:*

$$\mathcal{W}_c(\eta, \xi) := \inf_{\omega \in \Gamma(\eta,\xi)} \omega(c) = \sup_{\substack{a \in A \\ b \in \mathcal{B} \\ a \otimes 1 + 1 \otimes b \leq c}} \eta(a) + \xi(b).$$

Proof. We very closely follow [30], which amounts to employing the Fenchel-Rockafellar duality theorem. In the notation therein, our normed vector space E is the real vector space of self-adjoint elements of $\mathcal{A} \otimes \mathcal{B}$, and our convex functions $\Theta : E \to \mathbb{R}$ and $\Xi : E \to \mathbb{R}$ are given by:

$$\Theta(x) := \begin{cases} 0 & \text{if } x \geq -c, \\ +\infty & \text{otherwise.} \end{cases}$$

And:

$$\Xi(x) := \begin{cases} \eta(a) + \xi(b) & \text{if } x = a \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes b, \\ +\infty & \text{otherwise.} \end{cases}$$

The point $x_0 = 1_{\mathcal{A} \otimes \mathcal{B}}$ lies in the intersection of the effective domains of both functions (that is, $\Theta(1_{\mathcal{A} \otimes \mathcal{B}}) < +\infty$ and $\Xi(1_{\mathcal{A} \otimes \mathcal{B}}) < +\infty$), because $1_{\mathcal{A} \otimes \mathcal{B}} \geq 0 \geq -c$ and:

$$\Xi(1_{\mathcal{A} \otimes \mathcal{B}}) = \Xi(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})$$

$$
\begin{aligned}
&= \Xi\left(\frac{1}{2}1_{\mathcal{A}} \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes \frac{1}{2}1_{\mathcal{B}}\right) \\
&= \eta\left(\frac{1}{2}1_{\mathcal{A}}\right) + \xi\left(\frac{1}{2}1_{\mathcal{B}}\right) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}
$$

The function Θ is continuous at the point $x_0 = 1_{\mathcal{A}\otimes\mathcal{B}}$ because this point lies in the interior of its effective domain. For example, the set $\mathcal{A}\otimes\mathcal{B}^{+^\circ}$ of all strictly positive elements of $\mathcal{A}\otimes\mathcal{B}$ is an open set entirely contained in the effective domain; and $1_{\mathcal{A}\otimes\mathcal{B}}$ pertains to such set.

Applying the Fenchel-Rockafellar duality, we conclude:

$$
\inf_{x\in E} \Theta(x) + \Xi(x) = \max_{\chi\in E^\star} -\Theta^\star(-\chi) - \Xi^\star(\chi). \tag{18}
$$

Now we pass to the issue of computing both sides of (18). On the left side, we have:

$$
\begin{aligned}
\inf_{\substack{x=a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b \\ x\geq -c}} \eta(a) + \xi(b) &= - \sup_{\substack{x=a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b \\ x\leq c}} \eta(a) + \xi(b) \\
&= - \sup_{(a,b)\in\tilde{\Gamma}(c)} \eta(a) + \xi(b).
\end{aligned}
$$

On the right side, the Legendre transform of Θ is:

$$
\Theta^\star(-\chi) = \sup_{x\in E} -\chi x - \Theta x = \sup_{x\geq -c} -\chi x.
$$

If $\chi \not\geq 0$, there must be some $x \geq 0 \geq -c$ for which $\chi(x) < 0$. Given such x, the family of positive elements nx ensures that the supremum be $+\infty$. If otherwise $\chi \geq 0$, then we can compare the evaluation of χ at $-c$ with the evaluation of χ at any other x:

$$
-\chi(-c) - [-\chi(x)] = -\chi(-c-x) = \chi(c+x) \geq 0,
$$

and see that the supremum must be $\chi(c)$.

Still on the right side of (18), the Legendre transform of Ξ is:

$$
\Xi^\star(\chi) = \sup_{x\in E} \chi x - \Xi x
$$

$$= \sup_{x = a\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes b} \chi(a\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes b) - (\eta(a) + \xi(b)).$$

If, for any of these x, the quantity $\chi(a\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes b) - (\eta(a) + \xi(b))$ is not zero, then either the family nx or $-nx$ ensures the supremum be $+\infty$. In the absence of such x, $\Xi^\star(\chi) = 0$. Synthetically:

$$\Theta^\star(-\chi) = \begin{cases} \chi(c) \text{ if } \chi \geq 0, \\ +\infty \text{ otherwise.} \end{cases}$$

And:

$$\Xi^\star(\chi) = \begin{cases} 0 \text{ if } \chi(a\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes b) = \eta(a) + \xi(b), \\ +\infty \text{ otherwise.} \end{cases}$$

Conveniently, the intersection of the effective domains of such functions is precisely $\Gamma(\eta, \xi)$.

Finally, we rewrite Fenchel-Rockafellar duality in terms of the previous observations:

$$\inf_{a\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes b \geq -c} \eta(a) + \xi(b) = \max_{\substack{\chi \geq 0 \\ \chi(a\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes b) = \eta(a) + \xi(b)}} -\chi(c),$$

and exchange signs, to obtain:

$$\sup_{(a,b)\in\tilde{\Gamma}(c)} \eta(a) + \xi(b) = \min_{\omega\in\Gamma(\eta,\xi)} \omega(c).$$

□

When $\mathcal{A} = \mathcal{B} = \mathcal{C}(X)$ and the cost $c = d$ is a metric, we recover the following form of the Kantorovich duality formula:

$$\mathcal{W}_d(\mu, \nu) = \sup_{\substack{f\in\mathcal{C}(X) \\ |f(x) - f(y)| \leq d(x,y)}} \left| \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu \right|.$$

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