On the Minimal Action Function of an Autonomous Lagrangian Associated with a Magnetic Phenomena

M. J. Dias Carneiro

Depto. de Matemática- ICEx - UFMG

Belo Horizonte - MG - Brazil

and

Artur Lopes

Inst. Mat. - UFRGS

Porto Alegre - RS - Brazil

Abstract: In this paper we show the existence of a plateau for the minimal action function associated with a model for a particle under the influence of a magnetic field (Hall effect). We will describe the structure of the Mather sets, that is, sets that are support of minimizing measures for the corresponding autonomous Lagrangian.

This description is obtained by constructing a twist map induced by the first return map associated with a certain transversal section on a fixed level of energy.

Supported by PRONEX-Sistemas Dinâmicos

0. Introduction

In [C] J.Mather's Theory about minimizing measures for the action of time periodic Lagrangians is developed for the autonomous case (time independent). Further developments were obtained by Contreras, Delgado, Iturriaga and Mañé in [CDI]

In this work we study a special Lagrangian on the two dimensional torus (two degrees of freedom and periodic on each spatial coordinate). In the model considered here there exists a non-trivial magnetic potential vector but there is no eletrostatic potential.This model appears in phenomena related to the Hall effect.

The objetive is to study the dynamical properties of the Euler Lagrange flow generated by the Lagrangian associated to a magnetic field. In $I\!R^3$ with coordinates (x_1, x_2, x_3) let us consider a C^∞ magnetic force $F = \dot{x} \times B$, $B = \nabla \times A$, associated to a Lagrangian on the two Torus T^2 defined by

$$L(x_1, x_2, v_1, v_2) = \frac{\|v\|^2}{2} + \langle A(x_1, x_2), v\rangle$$

where the metric $\| \quad \|$ is induced by the euclidean inner product and $A(x_1, x_2) = (a_1(x_1, x_2), a_2(x_1, x_2))$.

The Euler-Lagrange flow associated with this Lagrangian is generated by the vector field

$$X : \begin{cases} \dot{x} = v \\ \dot{v} = (\partial_2 a_1 - \partial_1 a_2) J v = v \times B \end{cases}$$

where

$$\partial_i = \frac{\partial}{\partial x_i},$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It follows immediately that the scalar velocity is constant along a solution of X and, by Stokes Theorem and the periodicity of b, that the locus of inflection points $\partial_1 a_2 - \partial_2 a_1 = 0$ is non-empty.

This set is relevant to the following problem: describe the minimizing measures of the action $A(\mu) = \int L d\mu$, among the probabilities with compact support invariant under the flow of X with a given rotation vector $\rho(\mu)$.

Let us explain the terms of this statement. First, we observe that the above Lagrangian is positive definite (that is for all $x \in T^2$, $L|T_x(T^2)$ has everywhere positive definite second derivative) and superlinear

$$\left(\lim_{\|v\|\to\infty} \frac{L(x_1,v)}{\|v\|} = \infty\right)$$

uniformly on T^2. Therefore the solutions of X are defined for all $t \in R$ (is complete) and L satisfies the hypothesis of Mather's Theory for autonomous Lagrangian . According to that theory, for a given invariant measure ν with compact support on the one point compactification of $T(T^2)$, we define the rotation vector or homological position, $\rho(\nu) = u \in H_1(M,\mathbf{R})$, the first real homology group of M, as the element $\rho(\nu)$ such that for any co-homology class $[w] \in H_1(M,\mathbf{R})^* = H^1(M,\mathbf{R})$,

$$< [w], \rho(\nu) >= \int w d\nu$$

In particular, if ν is ergodic, then

$$< [w], \rho(\nu) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} w_x(\dot{x}) dt,$$

where the trajectory $(x(t), \dot{x}(t)) \in \mathbf{R}^4$ (solution of the Euler-Lagrange equation)used on the right hand side integration is generic in the sense of Birkhoff's Theorem with respect to ν.

In the case of the two torus, T^2, $\rho(\nu) = (\alpha_1, \alpha_2) \in \mathbf{R^2} = H_1(M,\mathbf{R})$, means that the lifting $z(t) = (x_1(t), x_2(t))$ of a generic trajectory to the universal covering $\mathbf{R^2}$ is such that $x_1(t)$ has a mean value of inclination α_1, that is,

$$\lim_{t\to\infty} \frac{x_1(t) - x_1(0)}{t} = \alpha_1,$$

and $x_2(t)$ has mean value of inclination α_2 that is

$$\lim_{t\to\infty} \frac{x_1(t) - x_1(0)}{t} = \alpha_2.$$

This follows from the fact that dx_1 and dx_2 generates $H^1(M,\mathbf{R})$.

Whenever the above limits exist we say that the curve has asymptotic direction (α_1, α_2). For example, if there exists a vector $(m, n) \in \mathbf{Z}^2$ and a number T such that $z(t+T) = z(t) + (m,n)$ then it is easy to see that the associated homological position is equal to $\frac{1}{T}(m,n)$.

For a probability measure ν, the action is defined by $A(\nu) = \int L d\nu$.

Given a homological position u , we denote by $\beta(u) = \inf_{\rho(\mu)=u} A(\mu)$, (where μ is assumed to be invariant for the flow X), the minimal action function. A measure ν_u satisfying $A(\nu_u) = \beta(u)$ is called a minimizing measure (or a minimizing measure for u).

The minimal action function is convex and superlinear and many interesting properties of the Euler-Lagrange flow can be derived from its behaviour. For instance, if u is an extremal point for β, then there exists an ergodic minimal measure with rotation vector u. However, in general, β may have non trivial linear domains ("plateau"), which are convex sets such that the restriction of β is an affine function.

In the case of the torus it is easy to see, as [C] for example, that β can be non strictly convex only along closed intervals contained in one dimensional subspaces. Moreover, if the interval does not contain the origin, the subspace must have rational slope (rational homology).

It is well known that by adding a gradient vector field to the magnetic potential we do not change the Lagrangian, therefore, using the Fourier expansion and integration by parts, the magnetic potential can be written in the following form:

$A(x_1,x_2) = (a_1(x_2), a_2(x_1,x_2)))$, with $A_2(x_1,x_2) = \sum cos(2n\pi x_1)C_n(x_2) + sin(2n\pi x_1)D_n(x_2)$ and $n \geq 1$. We can now state our theorem:

Theorem A : Let us suppose that magnetic potential is vertical $A(x_1,x_2) = (0, b(x_1,x_2))$ and satisfies: (i) $b(x_1,x_2) = \sum \cos(2n\pi x_1)C_n(x_2)$ with n odd and $\sum \sin(2n\pi x_1)C_n(x_2) > 0$, $0 < x_1 < 1/2$. (ii) $4b_{min}\bar{b} > {b_{min}}^2 + \bar{b}^2$, where $b_{min} = \min\ b(x_1,x_2)$ and $\bar{b} = \int_0^1 b(\frac{1}{2}, x_2)dx_2$.

Then the minimal action function is not strictly convex, and there is a segment of the form $(0,I) \subset H_1(T^2,R)$, where $I = (\bar{b}, -\bar{b})$, such that if h belongs to the interior of I there is no ergodic minimizing measure μ such that $\rho(\mu) = (0,h)$.

Moreover, there is a positive number ζ such thatif $\|v\|^2 = E$ is a level set that contains the support of a minimizing measure, then E$\geq \zeta$.

Several examples satisfying the hypothesis of Theorem A are presented at the end of Theorem 3 in the next section.

In figure 1 we show the graph of $\beta(0,h)$ as a function of h.

As it was pointed out in the beginning of this introduction, the set of inflection points K (defined by $\partial_1 a_2 - \partial_2 a_1 = 0$) is always non-empty. Under the hypothesis of Theorem A it projects onto two closed curves $K_0 \cup K_{\frac{1}{2}}$.

Therefore any trajectory of X which projects on to a curve with non-zero asymptotic direction must intersect K transversally (or coincide with K). Therefore we have naturally associated a first return map $T : K \to K$.

Theorem B: Let $b(x_1, x_2)$ be a magnetic potential satisfying the hypotesis of Theorem A. Then there is a positive number E_0 such that if $E > E_0$ there is an open annulus $\bigwedge$ (E) and an area preserving twist map $B_E : \bigwedge(E) \to \bigwedge(E)$ such that the minimizing measure μ with supp μ contained in the level set E is described by orbits of B_E.

Moreover, there is a number $\alpha = \alpha(E) \in I\!R$ such that if μ is an ergodic minimizing measure with the slope of the rotation vector $\rho(\mu)$ bigger then α, then supp μ is not an invariant torus.

After Theorem 6 in section 2 we show examples where all these results apply.

In this work, we studied examples with a_1 constant equal 0. The situation in general is more complicated, however, we believe that these examples serve as model cases, in the following sense: the dynamics of the Euler-Lagrange flow in the level set is divided in two pieces one is described by the orbits of a twist map (or composition of twist maps) and the other, where invariant torus cannot exist, is similar to the dynamics near a homoclinic point, giving rise to horse-shoe type dynamics.

Theorem A will be analised in section 1 and Theorem B in section 2.

1. Existence of plateau for the minimal action function

We recall that the minimal action function β is convex in u and from

Theorem 1 [C], the total energy is constant in the support of any minimizing measure.

We will show that β has a plateau when restricted to vertical line $(0, h), h \in \mathbf{R}$.

We collect some elementary facts about the solutions of the particular L we consider.

It is easy to see that the total energy $L - L_v.v$ is constant on trajectories of the flow and is equal to

$$\frac{\|v\|^2}{2}.$$

Symmetry principle: it is also easy to see from the symmetry of the Lagrangean that if $z(t)$ is a solution then

$$\tilde{z}(t) = z(-t) + \Big(\frac{1}{2}, 0\Big)$$

is also a solution.

Proposition 1: The minimal-action function β associated to L is symmetric, $\beta(-u) = \beta(u)$ for all $u \in H_1(T^2, I\!R)$.

Proof: Suppose that $z(t)$ and $\tilde{z}(t)$ are solutions of the Euler-Lagrange flow such that

$$\tilde{z}(t) = z(-t) + \Big(\frac{1}{2}, 0\Big).$$

Then

$$\frac{A[\,\tilde{z}\,|_{-T}^{T}\,]}{2T} = 2E + \frac{1}{2T}\int_{-T}^{T} b(\tilde{z}(t))\dot{\tilde{x}}_2(t)dt =$$

$$= 2E + \frac{1}{2T}\int_{-T}^{T} -b\Big(z(-t) + \Big(\frac{1}{2}, 0\Big)\Big)\dot{x}_2(-t)dt =$$

$$= 2E + \frac{1}{2T}\int_{-T}^{T} b(z(-t))\dot{x}_2(-t)dt = 2E - \frac{1}{2T}\int_{T}^{-T} b(z(t))\dot{x}_2(t)dt$$

$$= 2E + \frac{1}{2T}\int_{-T}^{T} b(z(t))\dot{x}_2(t)dt = \frac{A[\,z\,|_{-T}^{T}\,]}{2T}.$$

Suppose that $z(t)$ is the projection of a generic solution which is contained in the support of an ergodic minimizing measure μ so that

$$A(\mu) = \beta(\rho(\mu)) = \lim_{T\to\infty} \frac{A[\, z\, |_{-T}^{T}\,]}{2T}.$$

One can define a new invariant measure $\tilde{\mu}$ on TM by

$$\int f(x,v)d\tilde{\mu} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f\Big(z(-t) + \Big(\frac{1}{2}, 0\Big), -\dot{z}(-t)\Big)dt.$$

Observe that the limit exists since

$$\int_{-T}^{T} f\Big(z(-t) + \Big(\frac{1}{2}, 0\Big), -\dot{z}(-t)\Big)dt = \int_{T}^{-T} f\Big(z + \frac{1}{2}, -\dot{z}(\mu)\Big) - du =$$

$$= \int_{-T}^{T} f\Big(z(u) + \frac{1}{2}, -\dot{z}(\mu)\Big)du = \int_{-T}^{T} g(z(t), \dot{z}(t))dt$$

where

$$g(x,v) = f\Big(x + \frac{1}{2}, -v\Big).$$

Considering $g(x,v) = w_x(v)$ where w is a 1-differential form, we can conclude that $\rho(\mu) = -\rho(\tilde{\mu})$.

It also easily follow that $A(\tilde{\mu}) = A(\mu)$ implies $\beta(-\rho(\mu)) \leq A(\tilde{\mu}) = \beta(\rho(\mu))$. Reversing the above construction we obtain the opposite inequality and we finally obtain $\beta(\rho(\mu)) = \beta(-\rho(\mu))$. □

Proposition 2: It follows from the symmetry of β that $\beta(0) = \min \beta \leq 0$.

Proof : If $\beta(u) = \min \beta$ then $\beta(-u) = \min \beta$ and by the convexity of β,

$$\beta(0) \leq \frac{1}{2}\beta(u) + \frac{1}{2}\beta(-u) = \min \beta.$$

Since $v = 0$, $x = x_0$ is a singularity of the Euler-lagrange vector field, and $L(x_0, 0) = 0$, then $\min \beta \leq 0$. □

In the case of the torus, if S is a supporting domain for the function β, then S is contained in a subspace of dimension 1 (Proposition 3 [C]).

Theorem A that will be proved bellow shows the existence of nontrivial support domains for the class of Lagrangians we consider here.

Theorem 3: Suppose the $b(x_1, x_2)$ satisfies the following hypothesis (equivalent to the ones stated in Theorem A of the Intoduction):

(i) $b(-x_1, x_2) = b(x_1, x_2)$

(ii) $b(x_1 + \frac{1}{2}), x_2) = -b(x_1, x_2)$

(iii) for each fixed x_2, $b(x_1, x_2)$ is monotone decreasing on the interval $(0, \frac{1}{2})$

(iv) $4b_{min} > {b_{min}}^2 + \bar{b}^2$ where b_{min} is the minimum of b and $0 > \bar{b} = \int_0^1 b(\frac{1}{2}, x_2)dx_2$,

then, there is a horizontal flat segment for the minimal action function β in the level set $\beta^{-1}(\beta(0))$.

For $(0, h)$, $h \in (\bar{b}, -\bar{b})$ there is no ergodic minimal measure with rotation vector $(0, h)$ and outside this set $\beta(0, h) = \beta(h)$ is strictly convex as a function of h.

Moreover, if μ is a minimizing measure, then the support of μ is contained on a level of Energy E such that $E \geq \frac{\bar{b}^2}{2}$.

Proof:

It follows from $b(x_1 + 1/2, x_2) = -b(x_1, x_2)$ and $b(-x_1, x_2) = b(x_1, x_2)$ that

$$\partial_1 b(0, x_2) = 0 = \partial_1 b\Big(\frac{1}{2}, x_2\Big).$$

Therefore

$$z_1 : t \mapsto (0, -\sqrt{2E}t)$$

and

$$z_2 : t \mapsto \Big(\frac{1}{2}, \sqrt{2E}t\Big)$$

are solutions of the Euler-Lagrange equation with the same mean action

$$A[z_1|_{-T}^T] = 2ET - \int_{-T}^T b(0, -\sqrt{2E}t)\sqrt{2E}dt = 2ET + \int_{\sqrt{2E}T}^{-\sqrt{2E}T} b(0, t)dt$$

and

$$A[z_2|_{-T}^{T}] = 2ET + \int_{-T}^{T} b\Big(\frac{1}{2}, \sqrt{2E}t\Big)\sqrt{2E}dt = 2ET + \int_{-\sqrt{2E}T}^{\sqrt{2E}T} b\Big(\frac{1}{2}, t\Big)dt,$$

then

$$A(z_1|_{-T}^{T}) = A(z_2|_{-T}^{T}) = 2ET - \int_{-\sqrt{2E}T}^{\sqrt{2E}T} b(0,t)dt.$$

As

$$z_1\Big(t - \frac{1}{\sqrt{2E}}\Big) = (0, -\sqrt{2E}t + 1) = z_1(t) + (0,1)$$

and

$$z_2\Big(t + \frac{1}{\sqrt{2E}}\Big) = (0, \sqrt{2E}t + 1) = z_2(t) + (0,1)$$

so

$$\rho(z_1) = -\sqrt{2E}(0,1)$$

$$\rho(z_2) = \sqrt{2E}(0,1)$$

and μ_1, μ_2 probabilities defined by

$$\int f(x,v)d\mu_i = \frac{1}{\delta_i}\int_0^{\delta_i} f(z_i(t), \dot{z}_i(t))dt, \quad \delta_1 = -\frac{1}{\sqrt{2E}}, \ i = \{1,2\}, \quad \delta_2 = \frac{1}{\sqrt{2E}}$$

are invariant under the Euler-Lagrange flow.

We have just seen that

$$A(\mu_1) = A(\mu_2) = E - \sqrt{2E}\int_0^1 b(0, x_2)dx_2,$$

and this implies that

$$\beta(0,h) \leq \min\Big\{\frac{h^2}{2} + h\bar{b}, \frac{h^2}{2} - h\bar{b}\Big\},$$

where $\bar{b} \neq 0$..

We now show that the measures μ_1 and μ_2 associated to the curves z_1 and z_2 with velocity $\sqrt{2E} = -\bar{b}$ are minimizing. In order to do that, let us evaluate

$$\int_0^\delta b(x_1, x_2)\dot{x_2}dt$$

for a solution of the Euler-Lagrange equation such that $x(\delta) = x(0) + (0, 1)$, with

$$\frac{1}{\delta} = -\bar{b} \qquad [\rho(x) = \rho(\mu_1)].$$

Partition the curve $x(t)$ into pieces $t_0 = 0 < t_1 < ... < t_k = \delta$ such that

$$\dot{x}_2|_{(t_i, t_{i+i})} \neq 0$$

so the above integral becomes

$$\sum \int_{t_i}^{t_{i+1}} b(x_2, x_2)\dot{x}_2 dt = \sum \int_{x_2(ti)}^{x_2(t_{i+1})} b[f_i(x_2), x_2]dx_2$$

where $f_i(x_2)$ is a C^1 function such that the image of x restrited to $[t_i, t_{i+1}]$ is contained in the graph of f_i. Of course $x_2(t_i) < x_2(t_{i+1})$, if $\dot{x}_2 > 0$, and $x_2(t_i) > x_2(t_{i+i})$, if $\dot{x}_2 < 0$.

By assumption

$$b(0, x_2) > b(x_1, x_2) > b\Big(\frac{1}{2}, x_2\Big) = -b(0, x_2),$$

so

$$\int_{x_2(t_i)}^{x_2(t_{i+1})} b(f_i(x_2), x_2)dx_2 > \int_{x_2(t_i)}^{x_2(t_{i+1})} b\Big(\frac{1}{2}, x_2\Big)dx_2,$$

if $x_2(t_i) < x_2(t_{i+1})$, otherwise,

$$\int_{x_2(t_i)}^{x_2(t_{i+1})} b(f_i(x_2), x_2)_d x_2 = -\int_{x_2(t_{i+1})}^{x_2(t_i)} b(f_i(x_2), x_2)dx_2 \ >$$

$$-\int_{x_2(t_{i+1})}^{x_2(t_i)} b(0, x_2)dx_2 = \int_{x_2(t_{i+1})}^{x_2(t_i)} b(\frac{1}{2}, x_2)dx_2.$$

That is

$$\int_0^\delta b(x_1, x_2)\dot{x}_2 dt > \sum_{l=1}^{k} \int_{x_2^l}^{x_2^{l+1}} b(\frac{1}{2}, x_2)dx_2 \qquad (*)$$

where $[x_2^i, x_2^{i+1}] = [x_2(t_i), x_2(t_{i+1})]$, if $\dot{x}_2 > 0$ on (t_i, t_{i+1})

and $[x_2^i, x_2^{i+1}] = [x_2(t_{i+1}), x_2(t_i)]$, if $\dot{x}_2 < 0$ on (t_i, t_{i+1}).

Let

$$b_{min} = \min b(x_1, x_2) = \min b\Big(\frac{1}{2}, x_2\Big) \quad \text{and} \quad \bar{b} = \int_0^1 b(\frac{1}{2}, x_2)dx_2.$$

Observe that $\bar{b} < 0$.

Since x is a solution of the Euler-Lagrange equation, it is contained in a energy level, say E, and from the hypothesis $x(t+\delta) = x(t) + (0,1)$ it follows that

$$\sqrt{2E}\delta = \text{ lenght } (\, x\Big|_{0<t<\delta}) > \sum_{l=0}^{k-1}(x_2^{l+1} - x_2^l) > 1.$$

However due to the convexity of x in the strips

$$0 < x_1 < \frac{1}{2}$$

or

$$\frac{1}{2} < x_1 < 1$$

this bound can be improved.

Before doing that, let us suppose that the number of points with horizontal direction is equal to 5 (as in figure 2) .The case with fewer critical points are treated similarly.

By the above construction the curve $x|_{[0,\delta]}$ is subdivided into 4 pieces on each of one $\dot{x}_2 \neq 0$, with the corresponding points labeled as follows: $x_2^0 < x_2^2 < x_2^1 < x_2^4 < x_2^3$ and where $x_2^4 = x_2^0 + 1$ ($x_2^0 = x_2(0)$ is the smallest local minimum)

For instance, in the picture: min < max < min < max < min, therefore alternating minimum and maximum.

The integral in (*) is

$$\int_{x_2^0}^{x_2^1} b(\frac{1}{2}, x_2)dx_2 + \int_{x_2^2}^{x_2^1} b\Big(\frac{1}{2}, x_2\Big)dx + \int_{x_2^2}^{x_2^3} b\Big(\frac{1}{2}, x_2\Big)dx_2 + \int_{x_2^4}^{x_2^3} b\Big(\frac{1}{2}, x_2\Big)dx_2 =$$

$$\int_{x_2^0}^{x_2^1} b\Big(\frac{1}{2}, x_2\Big)dx_2 + \int_{x_2^1}^{x_2^4} b\Big(\frac{1}{2}, x_2\Big)dx_2 + 2\int_{x_2^2}^{x_2^1} b\Big(\frac{1}{2}, x_2\Big)dx_2 + 2\int_{x_2^4}^{x_2^3} b\Big(\frac{1}{2}, x_2\Big)dx_2 =$$

$$\int_0^1 b\Big(\frac{1}{2}, x_2)\Big)dx_2 + 2\int_{x_2^2}^{x_2^1} b\Big(\frac{1}{2}, x_2\Big)dx_2 + 2\int_{x_2^4}^{x_2^3} b\Big(\frac{1}{2}, x_2\Big)dx_2 \geq$$

$$\bar{b} + 2b_{min}[x_2^1 - x_2^2 + x_2^3 - x_2^4] = \bar{b} + 2b_{min}[x_2^1 - x_2^2 + x_2^3 - x_2^0 - 1]$$

or

$$\frac{1}{\delta}\int_0^\delta b\dot{x}_2 dt > \frac{\bar{b}}{\delta} + \frac{2b_{\min}}{\delta}[x_2^1 - x_2^2 + x_2^3 - x_2^0 - 1] =$$

$$= \frac{\bar{b}}{\delta} - \frac{2b_{\min}}{\delta} + \frac{2b_{\min}}{\delta}[x_2^1 - x_2^0 + x_2^3 - x_2^2].$$

Since

$$\rho(x) = \frac{1}{\delta}(0,1) = -\bar{b}(0,1),$$

we obtain $\frac{1}{\delta} = -\bar{b}$.

Denote $M = 2[x_2^1 - x_2^0 + x_2^3 - x_2^2]$.

Then,

$$\frac{1}{\delta}\int_0^\delta b\dot{x}_2 dt > -\bar{b}^2 + 2b_{\min}\bar{b} - b_{\min}\bar{b}M.$$

However, $\sqrt{2E}\delta$ =lengh ($x|_0^\delta$) $> 1 + M$, or

$$E > \frac{\bar{b}^2}{2}(1 + 2M + M^2).$$

So we obtain the following estimate for A[x]:

$$A[x] > \frac{\bar{b}^2}{2}(1 + 2M + M^2) - \bar{b}^2 + 2b_{\min}\bar{b} - b_{\min}\bar{b}M$$

that is,

$$A(x) > \frac{-\bar{b}^2}{2} + 2b_{\min}\bar{b} + \bar{b}^2 M + \frac{\bar{b}^2}{2}M^2 - b_{\min}\bar{b}M,$$

The right hand side of this inequality, as a function of M has minimum value for

$$M = \frac{b_{\min} - \bar{b}}{\bar{b}},$$

therefore

$$A[x] > \frac{-\bar{b}^2}{2} + 2b_{\min}\bar{b} + \bar{b}(b_{\min} - \bar{b}) + \frac{(b_{\min} - \bar{b})^2}{2} - b_{\min}(b_{\min} - \bar{b}),$$

that is,

$$A[x] > \frac{-\bar{b}^2}{2} - \frac{(b_{\min} - \bar{b})^2}{2} + 2b_{\min}\bar{b}.$$

Therefore, if $-\bar{b}^2 - b\ {b_{min}}^2 + 4b_{min}\bar{b} > 0$, then $A[x] > A[\mu_1]$.

The same procedure also works if there are more critical points. If $x(t) = [x_1(t), x_2(t)] \in I\!R^2$, $0 \le t \le \delta$, we denote by $x_2^0 = x_2(0)$ the smallest local minimum for $x_2(t)$.

Let $x_2^0 < x_2^1 < ... < x_2^k$ be the image of the critical points of $\pi_2(x(t))$, $0 \le t \le \delta$, where $\pi_2(x_1, x_2) = x_2$ is the canonical projection. The lines $x_2 = x_2^j$ determine a partition of $x(t)$ in the interval $0 \ge t \ge \delta$. Observe that $x_2^l = x_2^0 + 1$ for some $l < k$, then using this partition we obtain

$$\int_0^\delta b(x_1, x_2)\dot{x}_2 dt \ge \int_0^1 b(\bar{x}_1, x_2)dx - 2b_{\min} + b_{\min} \sum (x_j - 1)(x_2^j - x_2^{j-1}) =$$

$$= \bar{b} - 2b_{\min} + b_{\min} \sum (n_j - 1)(x_2^j - x_2^{j-1}),$$

where n_j is the number of components of the intersectionn of x(t) with the strip $x_2^j < x_2 < x_2^{j-1}$.

However, $\sqrt{2E}\delta = \text{lenght}(x) \ge 1 + \sum(n_j - 1)(x_2^j - x_2^{j-1})$

Therefore, using the above estimate, and denoting by $M = \sum(n_j - 1)(x_2^j - x_2^{j-1}$ we obtain

$A[x] = E + \frac{1}{\delta}\int_0^\delta b(x_1(t)), x_2(t))\dot{x}_2(t)dt \ge \frac{(1+M)^2}{2\delta^2} + \frac{\bar{b}-2b_{min}}{\delta} + \frac{b_{min}M}{\delta}$.

Since $\frac{1}{\delta} = -\bar{b}$, we get

$$A[x] \ge \frac{(1+M)^2}{2}\bar{b}^2 - \bar{b}(\bar{b} - 2\,b_{\min}) - \bar{b}\,b_{\min}M$$

or

$$A[x] \ge -\frac{\bar{b}^2}{2} + M^2\frac{\bar{b}^2}{2} + M\bar{b}^2 + 2b_{min}\bar{b} - \bar{b}b_{min}M.$$

As before,

$$A[x] \ge \frac{-\bar{b}^2}{2} + \frac{(b_{min} - \bar{b})^2}{2} + (b_{min} - \bar{b})\bar{b} + 2b_{min}\bar{b} - b_{min}(b_{min} - \bar{b}),$$

or

$$A[x] \ge \frac{-\bar{b}^2}{2} + \frac{b_{min}^2}{2} - \bar{b}b_{min} + \frac{\bar{b}^2}{2} + \bar{b}b_{min} - \bar{b}^2 +$$

$$+2\bar{b}b_{min} - b_{min}^2 + \bar{b}\, b_{min}$$

or

$$A[x] \geq -\frac{\bar{b}^2}{2} - \frac{b_{min}^2}{2} +$$

$$+4\bar{b}b_{min} - \frac{\bar{b}^2}{2} - \frac{\bar{b}^2}{2} - \frac{(b_{min} - \bar{b})^2}{2} + 2b_{min}\bar{b}$$

as before.

Therefore, if $-\bar{b}^2 - {b_{min}}^2 + 4\bar{b}b_{min} > 0$, then $A[x] > A[\mu_1]$.

This shows that the vertical solutions $(0, \bar{b}t)$ and $(0, -\bar{b}t)$ are minimizers and $\beta(0, \bar{b}) = \beta(0, -\bar{b}) = -\frac{\bar{b}^2}{2}$. Therefore, the interval $I = \{h \,|\, \bar{b} \leq h \leq -\bar{b}\}$ is a non-trivial linear domain for the minimal action funcion.

This also implies that there are no ergodic minimizing measures with rotation vector $(0, h)$ with $h \neq 0$, inside the interval I. In fact, the graph property of $\Lambda(I)$ (=the closure of the union of the support of all minimizing measures with rotation vector inside I) implies that any solution contained in $\Lambda(I)$ does not intersect the lines $x_1 = 0$, $x_1 = \frac{1}{2}$. This means that the projection must be a convex curve (nonzero curvature), but this contradicts the assumption that the rotation vector is multiple of $(0, 1)$.

Also using the above estimate one can prove that the value of the action on a curve with vertical rotation vector $(0, h)$ with $h \in I$ and $h \neq 0$ is bigger that $-\frac{\bar{b}^2}{2}$. Now from Corollary 2 in [C], the minimum energy level that contains a minimizing measure is $E = -\frac{\bar{b}^2}{2}$.

We consider now the case $h = 0$.

If there is an ergodic minimizing measure μ with $\rho(\mu) = 0$, then the lift of the projection of supp μ to $\mathbf{R}^2$ is a closed convex curve homotopically trivial. Also using the ideas of [C] we get that such curve is parametrized with constant speed $|\bar{b}|$. By the graph property, if such curve comes from a minimizing action measure, then it can not intersect the lines $x_1 = 0, x_1 = \frac{1}{2}, x_1 = 1$ (that are in the support of minimizing measures).

We can assume without lost of generality the case where the solutions are on the strip $0 < x_1 < \frac{1}{2}$.

Suppose that γ_1 and γ_2 are closed convex curves homotopically trivial contained in the strip $0 < x_1 < \frac{1}{2}$ with γ_1 contained in the interior of the region bounded by γ_2, then length $\gamma_1 <$ length γ_2, so the respective periods satisfy $\tau_1 < \tau_2$.

Hence

$$A[\gamma_2]-A[\gamma_1] = \frac{1}{\tau_2}\int_0^{\tau_2} b(\gamma_2)\dot{\gamma_2} - \frac{1}{\tau_1}\int_0^{\tau_1} b(\gamma_1)\dot{\gamma_1} < \frac{1}{\tau_2}(\int_0^{\tau_2} b(\gamma_2)\dot{\gamma_2} - \int_0^{\tau_1} b(\gamma_1)\dot{\gamma_1})$$

$$= \frac{1}{\tau_2}(\int_{\gamma_2} b\,dx_2 - \int_{\gamma_1} b\,dx_2) = \frac{1}{\tau_2}\int_{\gamma_2-\gamma_1} b\,dx_2 = \frac{1}{\tau_2}\int\int_R \partial_1 b\,dx_1 dx_2,$$

where R is the annulus bounded by $\gamma_2 - \gamma_1$. Since R is contained in strip $0 < x_1 < \frac{1}{2}$, where $\partial_1 b$ is negative we obtaion

$$A[\gamma_2] < A[\gamma_1].$$

This shows that there are no minimizing curve which is homotopically trivial because the action can always be decreased for homotopically trivial extremals.

Finally for $(0,h)$ outside the interval $(0,I)$, estimates analogous to the one used in the previous case show that the solutions $(0,ht)$ and $(\frac{1}{2},ht)$ are global minimizers.

For h not in the set I, it is easy to see from the above that $\beta(0,h) = \frac{h^2}{2} - h\bar{b}$ for $h > -\bar{b}$ and $\beta(0,h) = \frac{h^2}{2} + h\bar{b}$ for $h < \bar{b}$.

This shows that the graph of $\beta(0,h) = \beta(h)$ as a function of h has the shape of figure 1.

This is the end of Theorem 3. □

Now we will show some examples:

1) when $b = b_\lambda = \cos 2\pi x_1(1 + \lambda \sin 2\pi x_2)$, where λ is a constant small enough: $\bar{b} = -1$ and $b_{\text{min}} = -(1+\lambda)$, so the condition is $-1 - (1+\lambda)^2 + 4(1+\lambda) > 0$ or: $1-\sqrt{2} < \lambda < 1+\sqrt{2}$, and since we are assuming $0 < \lambda < 1$, we always have $A[x] > A[\mu] = \frac{-1}{2}$.

2) when $b(x_1,x_2) = cos 2\pi x_1(1 + \lambda \sin \pi x_2)$, then $\bar{b} = [-1 + \frac{\lambda}{\pi}]$ and $b_{min} = -[1+\lambda]$

3) in general whem b is of the form $b(x_1)c(x_2)$, then $\bar{b} = b(\bar{x}_1)\int_0^1 c(x_2)dx_2 = b(\bar{x})\bar{c}$ and $b_{\text{min}} = b(\bar{x}_1)c(\bar{x}_2)$ (if $b(\bar{x}_1) < 0$ then $b_{\text{min}} = b(\bar{x}_1)\max c(x_2)$) and

the above condition becomes:

$$-b(\bar{x}_1)^2\bar{c}^2 - b(\bar{x}_1)^2c(\bar{x}_2)^2 + 4b(\bar{x}_1)^2c(\bar{x}_2) > 0$$

or

$$-\bar{c}^2 + c(\bar{x}_2)^2 + 4c(\bar{x}_2)\bar{c} > 0$$

(condition only on the perturbation term).

2. The twist map

In this section we show Theorem B, that is, the existence of a twist map defined by the first return map associated with a certain tranversal section.

We will need first the following proposition:

Proposition 4: Suppose that $z : I\!R \to I\!R^2$ is a minimizer for a Lagrangian satisfying the hypothesis of Theorem 3 with non-vertical homological mean position i.e, $\rho(z)$ is not a multiple of (0,1).

Then the map $t \mapsto \pi_1 \circ z(t) = x_1(t)$ is injective.

Proof: If $\dot{x}_1(t_0) = 0$, since $|\dot{z}(t)| = \sqrt{2E}$, we have $\dot{z}(t_0) = (0, \pm\sqrt{2E})$.

By uniqueness of O.D.E.

$$x_1(t_0) \neq \frac{1}{2}, 0$$

because we are assuming that the homological mean position of $z(t)$ is non-vertical.

Let us suppose without lost of generality (otherwise use the symmetry principle) that

$$x_1(t_0) \in \left(\frac{1}{2}, 1\right).$$

By the convexity of $z(t)$ in the strip

$$x_1 \in \left(\frac{1}{2}, 1\right)$$

and non-verticality of the homological position, there exist two points $t_1 < t_0 < t_2$ such that

$$x_1(t_1) = x_1(t_2) = \frac{1}{2}.$$

or

$$x(t_1) = x(t_2) = 1.$$

Without lost of generality suppose the first case happens (otherwise apply the symmetry principle).

Observe that $\dot{x}_2(t_0) > 0$, otherwise, by convexity of $z(t)$ it will never hit the side $x_1 = \frac{1}{2}$.

Therefore there are two values c, d such that $c < t_0 < d$ with $x_1(c) = x_1(d)$.

From this it follows that

$$A[z|_c^d] = E(d-c) + \int_c^d b(z(t))\dot{x}_2(t)dt \geq E(d-c) + \int_c^d b\Big(x_1(c), x_2(t)\Big)\dot{x}_2(t)dt.$$

The right hand side is the action of the curve $(x_1(c), x_2(t))$ with the same end point condition. Therefore z is not a global minimizer. $\square$

Now we will show that under appropriate conditions and using certain variables there exists a twist map induced by the first return on the torus to $x_1 = 0$. First we will show that under these assumptions a trajectory beginning in $x_1 = 0$ will hit $x_1 = 1/2$. The same reasoning, after that, will produce a sucessive hitting in $x_1 = 1$.

This procedure will induce a first return map that we will show later is a twist map. It will be necessary that the solution $z(t)$ has a large value of energy in order it can cross from $x_1 = 0$ to $x_1 = 1$.

First we will need the next theorem.

Theorem 5: Let $\varphi(t)$ be the angle (with the horizontal line) of a trajectory $z(t)$ of the Euler-Lagrange flow on $\mathbb{R}^2$, $z(t) = (x_1(t), x_2(t))$. Suppose that $x_1(0) = 0$, $x_2(0) = x_2^0$. There are positive numbers E_0 and θ_0 such that if the energy $E > E_0$ and the initial condition $(x_1(0), x_2(0), x_1^{'}(0), x_2^{'}(0))$, $\tan\varphi_0 = \frac{x_2^{'}(0)}{x_1^{'}(0)}$ is such that $-\theta_0 < \varphi_0 < \theta_0$, then $\exists t_0$ such that

$$x_1(t_0) = \frac{1}{2}.$$

Proof: The proof is by contradiction.

Start with some initial condition

$$-\frac{\pi}{2} < \varphi_0 < \frac{\pi}{2}.$$

Suppose $\dot{x}_1(t) \neq 0$ for t in some interval $(0, \delta)$, then there is a function $y(x)$ such that $x_2(t) = y(x_1(t))$.

Let

$$\lambda(t) = \int_0^{x_1(t)} \partial_1 b(x_1, y(x_1))dx_1 - \sqrt{2E}\sin\varphi(t)$$

for $0 \leq t \leq \delta$.

As $\dot{x_1} = \sqrt{2E}\cos\varphi$ and $\partial_1 b = \dot{\varphi}$, $\lambda'(t) = \partial_1 b(x_1(t), y(x_1(t)))\dot{x}_1(t) - \sqrt{2E}\cos\varphi(t)\dot{\varphi}(t) = 0$.

Therefore, λ is constant along the trajectory $z(t)$.

Suppose by contradiction that there is no t_0 as asserted and let t_1 be the first value such that $\varphi(t_1) = \frac{\pi}{2}$.

Denote $x_1(t_1) = x_1 < 1/2$.

As $\lambda(t)$ is constant

$$\lambda(t_1) = \int_0^{x_1} \partial_1 b(x_1, y(x_1))dx_1 - \sqrt{2E} = -\sqrt{2E}\sin\varphi_0,$$

or

$$\int_0^{x_1} \partial_1 b(x_1, y(x_1))dx_1 = \sqrt{2E}(1 - \sin\varphi_0).$$

Since

$$\int_0^{x_1} \partial_1 b(x_1, y(x_1))dx_1$$

is bounded above by some V (depending only on b), then

$$\frac{V}{\sqrt{2E}} \geq (1 - \sin\varphi_0).$$

If E is large and $\sin\varphi_0$ bounded away from 1 the last expression is not possible. Therefore, $z(t)$ has to cross $x_1 = 1/2$, otherwise the solution is always in a region of negative curvature and will bend until $\varphi(t)$ attains the value $\pi/2$.

This shows the Theorem. $\square$

Remark 1: After the hiting of the line $x_1 = \frac{1}{2}$ the trajectory will hit the line $x_1 = 1$ by the same argument (symetry principle). This shows the existence of a first return map of trajectories (with large enough value of energy) on the torus beginning in $x_1 = 0$ to itself. The domain of definition of such map is all $0 \leq x_2^0 \leq 1$ and φ_0 on a uniform neighbourhood of 0.

For a better geometrical understanding of the domain of the returning map we describe the phase space of the Lagrangian flow in the example 1 : $b(x_1, x_2) = \cos 2\pi x_1(1 + \lambda \sin 2\pi x_2)$.

For $\lambda = 0$ and E fixed it is easy to see that $H(x_1, \varphi) = \cos(2\pi x_1) + \sqrt{2E}\sin\varphi$ is a first integral.

This follows from

$$\frac{dx_1}{d\varphi} = \frac{\sqrt{2E}\cos\varphi}{-2\pi\sin(2\pi x_1)}.$$

The critical points of $H(x_1, \varphi)$ are

$(0, \pi/2)$ maximum

$(0, -\pi/2)$ saddle

$(1/2, \pi/2)$ saddle

and $(1/2, -\pi/2)$ minimum.

Depending of the level of energy, the separatrix of the saddle points prevent or not the trajectories to cross from $x_1 = 0$ to $x_1 = 1$. This property can be seen in figures 3 (parameter E=0.1) and 4 (parameter (E=20).

A necessary condition for existing trajectories with non-vertical rotation vectors is $E > \frac{1}{2}$.

In fact, since the equation for the separatrices are

$$\sqrt{2E}(1 - \sin\varphi) = 1 + \cos 2\pi x_1$$

and

$$\sqrt{2E}(1 + \sin\varphi) = 1 - \cos 2\pi x_1,$$

if $E \leq 1/2$, then both curves intersect the axis $\varphi = 0$ and therefore the saddle conection will be among saddle points that are in the same vertical line ($x = 0$ and $x = 1/2$).

This property prevents any trajectory of going from $x = 0$ to $x = 1/2$. In the case $E > 1/2$, the saddle connection will be between saddle points located in the same horizontal line.

The analysis of the dynamics of the returning map T in the case of small λ is obtained by continuity properties of the perturbation of the case $\lambda = 0$ described above. Note that the domain of definition of the perturbed case is a subset of the domain of definiton of the unperturbed case.

A geometrical picture that may help the reader is shown in figure 5 and 6. In fig 5 we show the unperturbed case $\lambda = 0$ and fig 6 shows the case of $\lambda \neq 0$ but small enough.

Now we will show that under suitable change of coordinates the map defined above is a twist map.

Fix a value E of energy such that there exist minimal solutions

$$(x_1(t), x_2(t), \dot{x_1}(t), \dot{x_2}(t)) = (x_1(t), x_2(t), v_1(t), v_2(t))$$

with $x_1(t)$ coming from 0 to 1.

Consider $\tan\varphi = \frac{v_2}{v_1}$. The Euler Lagrange equation can be written

$\dot{x_1} = v_1 = \sqrt{2E}\cos\varphi$

$\dot{x_2} = v_2 = \sqrt{2E}\sin\varphi$

$\dot{v_1} = -\sqrt{2E}\sin\varphi\,\dot{\varphi}$

$\dot{v_2} = \sqrt{2E}\cos\varphi\,\dot{\varphi}$

$\dot{\varphi} = K(x_1, x_2) = \partial_1 b(x_1, x_2)$,

because $v_1^2 + v_2^2 = 2E$.

Expressing the last two equations in terms of $x_1(\varphi)$, and $x_2(\varphi)$ we obtain

$\frac{dx_1}{d\varphi} = \frac{\sqrt{2E}\cos\varphi}{K(x_1,x_2)}$,

$\frac{dx_2}{d\varphi} = \frac{\sqrt{2E}\sin\varphi}{K(x_1,x_2)}$,

Let the variable w be $\sqrt{2E}\sin\varphi$.

The last two equations in terms of w can be read as (remember that x_2 is a function of x_1)

$\frac{dw}{dx_1} = \sqrt{2E}\,\cos\varphi\,\frac{d\varphi}{dx_1} = \frac{\sqrt{2E}\cos\varphi K(x_1,x_2)}{\sqrt{2E}\cos\varphi} = K(x_1, x_2) = \partial_1 b(x_1, x_2)$.

$\frac{dx_2}{dx_1} = \tan\varphi = \frac{w}{\sqrt{2E}\cos\varphi} = \frac{w}{\sqrt{2E-w^2}}$.

The transformation T should be seen as a first hitting map in the variable (x_1, x_2) of the trajectory beginning in the line $(x_1, x_2^0) = (0, x_2^0)$ to the line $(x_1, x_2^1) = (1, x_2^1)$.

The domain of definition of T is the set of (x_2^0, w^0) obtained in Theorem 5 and remark 1. Note that in this case, the solution $z(t)$ of the Euler-Lagrange equation , $z(t) = (x_1(t), x_2(t))$, with initial condition $(0, x_2^0, w^0)$ should satisfy the condition $v_1(t) = x_1^{'}(t) \neq 0$ for all t.

The map $T(x_2, w)$ is formally defined by taking the time one of the flow ψ_{x_1} generated by this (time-dependent) vector-field.

If b is a function of x_1 only, as in the above example, the vector-field is integrated explicitly and the return map becomes

$$T(x_2, w) = (x_2 + \int_0^1 \frac{(b(x_1) - b(0) + w)}{\sqrt{2E - (b(x_1) - b(0) + w)^2}} dx_1, w).$$

In this case we call T integrable.

Such T is clearly a twist map and therefore, for small λ, the map $T = T_\lambda$ is also a twist map defined on an open annulus.

This is also valid in the general case but the region where T is twist will depend of the particular form of b.

Now we will show Theorem B.

Theorem 6: Lef $b(x_1, x_2)$ be a magnetic potential satisfying the hypotesis of Theorem A. Then there is a positive number E_0 such that if $E > E_0$ there is an open annulus $\bigwedge$ (E) and an area preserving twist map $B_E : \bigwedge(E) \to \bigwedge(E)$ such that the minimizing measure μ with supp μ contained in the level set E is described by orbits of B_E.

Moreover, there is a number $\alpha = \alpha(E) \in \mathbb{R}$ such that if μ is an ergodic minimizing measure with the slope of the rotation vector $\rho(\mu)$ bigger then α, then supp μ is not an invariant torus.

Proof:

First we observe that the local maximum of the slope of any solution occurs at $x_1 = 0$ and the minimum at $x_1 = \frac{1}{2}$ and by the graph property, if there is an invariant torus in the tangent bundle contained in some energy level E and foliated by minimizers, then it is a Lipschitz graph of the form $\varphi = \varphi(x_1, x_2)$.

Let $\bigwedge(E)$ be the domain of the twist map as described in Theorem 5, then there are two C^1 functions φ_1, φ_2 such that $\bigwedge(E) = \{\varphi_1(x_2) < w < \varphi_2(x_2)\}$.

If $T : \bigwedge(E) \to \sum$ denotes the return map, then $\cap_{j \in \mathbf{Z}} T^j(\bigwedge(E))$ is an annulus bounded by the graph of two Lipchitz functions $\alpha_1^E(x_2), \alpha_2^E(x_2)$.

Let $\beta_+(E) = \sup \alpha_1^E(x_2)$ and $\beta_-(E) = \inf \alpha_2^E(x_2)$ and

$$\alpha_+(E) = \frac{\beta_+(E)}{\sqrt{2E - \beta_+(E)^2}}$$

and

$$\alpha_-(E) = \frac{\beta_-(E)}{\sqrt{2E - \beta_-(E)^2}}.$$

If $S_{p/q}$ is an invariant torus contained in the level set E and with the associated rotation vector a multiple of p/q, then there is a point (x_1^0, x_2^0) belonging to the projection of $S_{p/q}$ on the torus T^2 such that $\tan \varphi(x_1^0, x_2^0) =$ p/q.

It follows from the invariance of $S_{p/q}$ that $\alpha_-(E) < p/q < \alpha_+(E)$.

On the other hand if S_α is an invariant torus with associated rotation vector with irrational slope, then there is a sequence of rational numbers p_n/q_n converging to α and a sequence of points (x_1^n, x_2^n) in T^2 such that $\tan \varphi(x_1^n, x_2^n) = p_n/q_n$. Therefore, from the invariance of S_α we obtain

$$\alpha_-(E) < \alpha < \alpha_+(E)$$

We conclude that if $\frac{\rho_2}{\rho_1} > \alpha_+(E)$ then there is not an invariant torus with rotation vector $\rho = (\rho_1, \rho_2)$. $\square$

To obtain the twist property we use the fact that minimizers with non-vertical homological positions are graphs $(x_1, y(x_1))$ and proceed as in [B].

References

[B] V. Bangert, Mather Sets for twist maps and geodesic tori, Dyn. Reported (1994) Vol 1, 1-56

[C] M. J. Dias Carneiro, On minimizing measures of the action of autonomous Lagrangians, Nonlinearity 8 (1995) 1077-1085

[CDI] G. Contreras, J. Delgado and R. Iturriaga, Lagrangian flows: the dynamics of globally minimizing orbits II.

[D] J. Delgado, Vertices of the action function of a Lagrangian system, preprint PUC-Rio, 1993

[M] J. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, Math Z. 207 (1991) 269- 290

[Ma1] R. Mañé, On the minimizing measures of Lagrangian dynamical systems, Nonlinearity 5, (1992) 623-638

[Ma2] R. Mañé, Generic properties and problems of minimizing measures of Lagrangian dynamical systems (preprint IMPA 1993)

[Ma3] R. Mañé, Ergodic Theory and differentiable dynamics, Springer Verlag