Semiclassical limits, Lagrangian states and coboundary equations

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Abstract

Assume that f is a continuous transformation $f : S^1 \to S^1$. We consider here the cases where f is either the transformation $f(z) = z^2$ or f is a smooth diffeomorphism of the circle S^1.

Consider a fixed continuous potential $\tau : S^1 = [0,1) \to \mathbb{R}$, $\nu \in \mathbb{R}$ and $\varphi : S^1 \to \mathbb{C}$ (a quantum state). The transformation $\hat{F}_\nu$ acting on $\varphi : S^1 \to \mathbb{C}$, $\hat{F}_\nu(\varphi) = \phi$, defined by $\phi(z) = \hat{F}_\nu(\varphi(z)) = \varphi(f(z))e^{i\nu\tau(z)}$ describes a discrete time dynamical evolution of the quantum state φ.

Given $S : \mathbb{R} \to \mathbb{R}$ we define the Lagrangian state

$$\varphi_x^S(z) = \sum_{k\in\mathbb{Z}} e^{\frac{iS(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}}.$$

In this case $\hat{F}_\nu(\varphi_x^S(z)) = \sum_{k\in\mathbb{Z}} e^{\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)}$.

Under suitable conditions on S the micro-support of $\varphi_x^S(z)$, when $\hbar \to 0$, is $(x, S'(x))$. One of meanings of the semiclassical limit in Quantum Mechanics is to take $\nu = \frac{1}{\hbar}$ and $\hbar \to 0$. We address the question of finding S such that φ_x^S satisfies the property: $\forall x$, we have that $\hat{F}_\nu(\varphi_x^S)$ has micro-support on the graph of $y \to S'(y)$ (which is the micro-support of φ_x^S). In other words: which S is such that $\hat{F}_\nu$ leaves the micro-support of φ_x^S invariant? This is related to a coboundary equation for τ, twist conditions and the boundary of the fat attractor.

1 Introduction

The Gaussian wavepacket for $x \in \mathbb{R}, \xi \in \mathbb{R}$ is the function

$$y \to \tilde{\varphi}_{x,\xi}(y) = e^{\frac{i\xi y}{\hbar}} e^{-\frac{(y-x)^2}{4a^2}}.$$

This quantum state describes a quantum particle located at x and moment ξ. The variance of the position is a and the mean is x (see [16] or [4]).

We assume the variance of the position is $a = \sqrt{\hbar}$. Then we get

$$\tilde{\varphi}_{x,\xi}\,(y) = \tilde{\varphi}_{x,\xi,\hbar}\,(y) = \; e^{\frac{i\xi\, y}{\hbar}} \;\; e^{-\frac{(y-x)^2}{4\,\hbar}}. \tag{1}$$

One can also consider the case where the variance of the position is $a = \sqrt{\hbar/m}$ and similar results for the setting we consider here can also be obtained, when $m \to \infty$ and $\hbar$ is fixed. That is, $\hbar \to 0$ for the present setting is equivalent to $m \to \infty$ in this new setting.

In any case the Gaussian wavepacket minimizes the Heisenberg uncertain principle for states with mean position x and mean moment $\xi = m\,v$ (see [16]).

When $\hbar \to 0$ the distribution of the position of of $\tilde{\varphi}_{x,\xi}$ will be more and more concentrated in x. The natural way to describe this property which also contemplates the momentum is the concept of microsupport (see [15] or [12]).

As we are on a setting of functions which are defined on S^1 we will have to consider the periodic Gaussian wavepacket (see [9]):

$$\varphi_{x,\xi}(z) = \sum_{k\in\mathbb{Z}} \tilde{\varphi}_{x,\xi}(z-k) = \sum_{k\in\mathbb{Z}} e^{\frac{i\xi(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}}. \tag{2}$$

We say that a function $z(\hbar)$ is $O(\hbar^\infty)$ if for each $N > 0$ there exists C_N and $\delta > 0$ such that

$$|z(\hbar)| \leq C_N\, \hbar^N, \quad \text{for} \;\; |\hbar| \leq \delta.$$

Definition 1. *Given a family of functions $\phi_\hbar$, $\hbar \sim 0$, in $\mathcal{L}^2(S^1)$, we say that it is micro-locally small near (x_0, ξ_0) if*

$$|<\varphi_{x,\xi}, \phi_\hbar>|$$

is $O(\hbar^\infty)$ uniformly in a neighbourhood of (x_0, ξ_0). The complementary of such points (x_0, ξ_0) is called the micro-support of the family $\phi_\hbar$.

Consider fixed (x, ξ) and the corresponding periodic Gaussian wavepacket $\phi_\hbar = \varphi_{x,\xi}$ (according to (2)) depending on $\hbar$. For any $(y, \eta) \neq (x, \xi)$ the modulus of the inner product

$$|<\varphi_{y,\eta}\,,\,\varphi_{x,\xi}>| = |<\varphi_{y,\eta}\,,\,\phi_\hbar>|$$

is $O(\hbar^\infty)$.

Therefore, the micro-support of $\varphi_{x,\xi}$ is (x,ξ). See the proof in chapter 3 in [15] for the non-periodic case. The proof in the periodic case is similar (see also [9]).

We consider a fixed continuous potential $\tau : S^1 = [0,1) \to \mathbb{R}$ and $\nu \in \mathbb{R}$. We also consider a general dynamical system described by a continuous transformation $f : S^1 \to S^1$. One particular case is the transformation $f(z) = z^2$ (section 2). Another possibility we will consider here is when f is a smooth diffeomorphism of the circle (section 3).

Then, for a given $z \in [0,1)$ and function φ taking complex values we denote the transformation $\hat{F}_\nu$, acting on φ, by

$$\hat{F}_\nu(\varphi(z)) = \phi(z) = \varphi(f(z))\, e^{i\,\nu\,\tau(z)}.$$

One can consider the above expression as a kind of discrete time dynamical evolution of a quantum state φ. The transformation $\hat{F}_\nu$ was considered in [9] and we borrow the notation of that paper.

Note that if $\varphi = \varphi_{x,\xi}$ is the Gaussian wavepacket, we have

$$\hat{F}(\varphi_{x,\xi}\,(z)\,) = \sum_{k\in\mathbb{Z}} e^{\frac{i\,\xi(f(z)-k)}{\hbar}}\, e^{-\frac{(f(z)-k-x)^2}{4\hbar}}\, e^{i\nu\tau(z)}.$$

Given $S : \mathbb{R} \to \mathbb{R}$ a Lagrangian state is an expression of the form

$$a(x)\, e^{\frac{i}{\hbar}\, S(x)},$$

where $a : \mathbb{R} \to \mathbb{R}$ is such that $\int a(x)^2 dx = 1$. One example is $\frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-x^2}{4\sigma^2}}\, e^{\frac{i}{\hbar}\, S(x)}$.

We consider here a periodic setting: given $S : S^1 \to \mathbb{R}$ the associated Lagrangian state is the function

$$\varphi_x^S(z) = \sum_{k\in\mathbb{Z}} e^{\frac{iS(z-k)}{\hbar}}\, e^{-\frac{(z-k-x)^2}{4\hbar}}.$$

We are interested in the microsupport of $\hat{F}_\nu(\varphi_x^S(z))$ in the semiclassical limit, $\nu = \frac{1}{\hbar}, \hbar \to 0$. The analysis of the problem in the case of the transformation $f(z) = z^2$ is related to a certain coboundary equation. It is interesting that this question is associated to the study of the boundary of a certain attractor.

The general study of the boundary of attractors is the object of several papers in dynamics (see [6] and [13])

We would like to thanks D. Smania for helpful conversations related to section 3 of the present paper. We would like to thanks J. Hounie for providing us with some nice references.

2 The transformation $f(x) = 2\,x$ (mod 1)

Consider a fixed continuous function $A : S^1 = [0,1) \to \mathbb{R}$. Remember that $f(z) = z^2$ (or, in an equivalent way $f(x) = 2\,x$ (mod 1)).

Consider $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by

$$F(z,s) = (z^2, \lambda\, s + A(z)) = (f(z), \lambda\, s + A(z)),$$

where $0 < \lambda \leq 1$. The term λ is called the discounted term. In [13] the ergodic analysis of this problem when λ is fixed was considered and was also analyzed the limit problem when $\lambda \to 1$. Here we consider the case when $\lambda = \frac{1}{2}$.

We denote by G_1 and G_2 the two inverse branches of F, which are respectively

$$G_1(y,r) = \left(\frac{y}{2}, \frac{r - A(\frac{y}{2})}{\lambda}\right) \text{ and } G_2(y,r) = \left(\frac{y}{2} + \frac{1}{2}, \frac{r - A(\frac{y}{2} + \frac{1}{2})}{\lambda}\right)$$

Among others possible choices are $\lambda = \frac{1}{2}$ (preserves volume), or $\lambda = 1$.

In Proposition 6 in [9] the case when $A = 0$ and $\lambda = \frac{1}{2}$ is considered.

Theorem 2. *(see [9]) Suppose $f(z) = 2\,z$ (mod 1). As $\hbar \to 0$,*

$$\phi_\hbar(z) = \hat{F}_\nu(\varphi_{x,\xi}\,(z)) = \varphi_{x,\xi}\,(f(z))\, e^{i\,\nu\tau(z)}$$

has micro-support in

$$\{G_1(x,\xi), G_2(x,\xi)\} \subset T^*\, S^1,$$

where G_1 and G_2 are the inverse branches for

$$F(z,s) = (z^2, \lambda\, s) = (f(z), \frac{1}{2}\, s).$$

Proof:

For fixed (x,ξ) and variable (y,η), $x, y \in S^1$

$$< \varphi_{y,\eta}\,,\, \hat{F}_\nu \varphi_{x,\xi} >=< \varphi_{y,\eta}\,, \varphi_{x,\xi}\,(f)\, e^{i\,\nu\tau} >=$$

$$\int_{S^1} \overline{\varphi_{y,\eta}\,(z)} \sum_{k\in\mathbb{Z}} e^{\frac{i\,\xi(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)}\, dz =$$

$$\int_{S^1} \left(\sum_{q\in\mathbb{Z}} e^{-\frac{i\eta(z-q)}{\hbar}} e^{-\frac{(z-q-y)^2}{4\hbar}}\right) \left(\sum_{k\in\mathbb{Z}} e^{\frac{i\,\xi(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)}\right) dz.$$

Now if we define $S(z) = \xi z$, then we can apply proposition 12 (see appendix), with a slight modification, because $e^{i\nu\tau(z)}$ does not depend on $\hbar$, to obtain that

$$< \varphi_{y,\eta}\,,\, \hat{F}_\nu \varphi_{x,\xi} >= O(\hbar^\infty)$$

if $f(y) \neq x$ or if $f(y) = x$ and $\eta \neq \xi f'(y) = 2\xi$. Then the micro-support requires $f(y) = x$ and $\eta = 2\,\xi$. That is, (y, η) is contained in $\{G_1(x, \xi), G_2(x, \xi)\}$. □

Now we assume that ν and $\hbar$ are coupled by $\hbar = \frac{1}{\nu}$ and $A = -\frac{1}{2}\tau'$.

Theorem 3. *(see [9]) Suppose $f(z) = 2\,z$ (mod 1). As $\hbar = \frac{1}{\nu} \to 0$*

$$\phi_\hbar(z) = \hat{F}_\nu(\varphi_{x,\xi}\,(z)) = \varphi_{x,\xi}\,(f(z))\, e^{i\,\nu\tau(z)} = \varphi_{x,\xi}\,(f(z))\, e^{\frac{i\,\tau(z)}{\hbar}}$$

has micro-support in

$$\{G_1(x, \xi), G_2(x, \xi)\} \subset T^*\, S^1,$$

where G_1 and G_2 are the inverse branches for

$$F(z, s) = (z^2, \frac{1}{2}\, s - \frac{1}{2}\tau'(z)) = (f(z), \frac{1}{2}\, s - \frac{1}{2}\tau'(z)).$$

Proof:

For fixed (x, ξ) and variable (y, η), $x, y \in S^1$

$$< \varphi_{y,\eta}\,,\, \hat{F}_\nu \varphi_{x,\xi} >=< \varphi_{y,\eta}\,, \varphi_{x,\xi}\,(f)\, e^{i\,\nu\tau} >=$$

$$= \int_{S^1} \left(\sum_{q\in\mathbb{Z}} e^{-\frac{i\eta(z-q)}{\hbar}} e^{-\frac{(z-q-y)^2}{4\hbar}} \right) \left(\sum_{k\in\mathbb{Z}} e^{\frac{i\,\xi(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{\frac{i\tau(z)}{\hbar}} \right) dz.$$

Again, if we define $S(z) = \xi z$, then we can apply proposition 12 (see appendix) to obtain that

$$< \varphi_{y,\eta}\,,\, \hat{F}_\nu \varphi_{x,\xi} >= O(\hbar^\infty)$$

if $f(y) \neq x$ or if $f(y) = x$ and $\eta \neq \xi f'(y) + \tau'(y) = 2\xi + \tau'(y)$. Then the micro-support requires $f(y) = x$ and $\eta = 2\,\xi + \tau'(y)$. That is, (y, η) is contained in $\{G_1(x, \xi), G_2(x, \xi)\}$. □

Now we consider a different kind of problem.

If $S : \mathbb{R} \to \mathbb{R}$ we define the Lagrangian state as the periodic function

$$\varphi_x^S(z) = \sum_{k\in\mathbb{Z}} e^{\frac{iS(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}}.$$

It is a well known fact that:

Proposition 4. *The Lagragian state φ_x^S, associated to S, has microsupport on $(x, S'(x))$, when $\hbar \to 0$.*

Proof: For fixed x and variable y, η we have

$$< \varphi_{y,\eta}, \varphi_x^S >= \int_{S^1} \overline{\sum_{q\in\mathbb{Z}} e^{\frac{i\eta(z-q)}{\hbar}} e^{-\frac{(z-q-y)^2}{4\hbar}}} \sum_{k\in\mathbb{Z}} e^{\frac{iS(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}} \, dz.$$

It follows by the same kind of arguments used in Proposition 12, that the micro-support requires $y = x$ and $\eta = S'(x)$, i.e., φ_x^S has microsupport on the graph $(x, S'(x))$, when $\hbar \to 0$, as we claimed.

□

Lagrangian states of the form $u_\hbar(x) = a(x)e^{\frac{iS(x)}{\hbar}}$ were consider in Example 2.3 in page 32 in [1], where a is differentiable and positive, in the setting of Aubry-Mather Theory.

Now, if we consider $f : S^1 \to S^1$ we define

$$\hat{F}_\nu(\varphi_x^S(z)) = \varphi_x^S(f(z))e^{i\nu\tau(z)} = \sum_{k\in\mathbb{Z}} e^{\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)}.$$

Definition 5. *Given a continuous function $A : S^1 \to \mathbb{R}$ and $\lambda \in (0, 1)$, we say that a continuous function $b = b_\lambda : [0, 1) \to \mathbb{R}$* **is a λ-calibrated plus-subaction for** *A, if for all $z \in S^1$, $b(z) = \max_{f(y)=z} \{\lambda\, b(y) + A(y)\}$, where $f(z) = 2\,z$ (mod 1).*

This problem is related to λ-maximizing probabilities (see [13]).

We will consider here the case where $\lambda = \frac{1}{2}$ and $A = -\frac{\tau'}{2}$.

Denote by u the $\frac{1}{2}$-subaction for $A(z) = -\frac{1}{2}\tau'(z)$. The graph $(z, u(z))$ is the upper boundary of the fat attractor and invariant for F, where

$$F(z, s) = \Big(z^2, \frac{1}{2}\; s - \frac{1}{2}\tau'(z)\Big) = \Big(f(z), \frac{1}{2}\; s - \frac{1}{2}\tau'(z)\Big).$$

Then we have $(y, u(y)) = F(z, u(z)) = \Big(f(z), \frac{1}{2}\, u(z) - \frac{1}{2}\tau'(z)\Big)$, this implies $y = f(z)$ and $\frac{1}{2}u(z) - \frac{1}{2}\tau'(z) = u(y) = u(f(z))$, in this way for any z

$$u(f(z)) = \frac{1}{2}u(z) - \frac{1}{2}\tau'(z). \tag{3}$$

We call the equation $\frac{1}{2}\tau'(z) = \frac{1}{2}u(z) - u(f(z))$ a **$\frac{1}{2}$-coboundary equation** for u and τ'.

In several examples u is C^∞ up to some finite number of points (this kind of property is the main issue in [13]). On these points there exists the left and right limit of the derivative (see [13]) which are not zero nor infinite. This happens, for instance, if the potential A is C^∞ and $A = \tau'$ satisfies the twist condition to be defined later (see [13]).

In the appendix in the estimates of the asymptotic $\hbar \to 0$ it will be important for technical reasons that u is piecewise smooth C^∞ (left and right finite non zero derivatives in a finite number of singular points).

From now on we assume τ is C^∞ and then can write locally $u = S'$ for some $S : [0,1) \to \mathbb{R}$ which is C^∞ by parts.

Then, from (3) we get that for any z:

$$S'(f(z)) = \frac{1}{2} S'(z) - \frac{1}{2}\tau'(z),$$

or

$$2\, S'(f(z)) = S'(z) - \tau'(z). \tag{4}$$

We would like to consider different possible $S : [0,1) \to \mathbb{R}$, compare the corresponding S-Lagrangian state and look for the S such the associated micro-support (the graph of S') is invariant under the action of $\hat{F}_\nu$, when $\hbar = \frac{1}{\nu} \to 0$. Here $\hat{F}_\nu$ is such that $\hat{F}_\nu(\phi\,(z)) = \phi\,(f(z))\, e^{i\,\nu \tau(z)}$.

The S such that $S' = u$, where u satisfies (3), is the one we have to look for as we will show now.

Remember that for any x we have that φ_x^S has micro-support on the graph $(x, S'(x))$.

Theorem 6. *Consider*

$$\varphi_x^S\,(z) = \sum_{k \in \mathbb{Z}} e^{\frac{iS(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}}$$

where $S' = u$ is the $\frac{1}{2}$-plus-calibrated subaction for $-\frac{\tau'}{2}$ (see equation 3). Assume that S is piecewise smooth C^∞ (left and right finite non zero derivatives in a finite number of singular points), and also that τ is C^∞.

When $\hbar = 1/\nu \to 0$, we get that

$$\hat{F}_\nu(\varphi_x^S\,(z)) = \varphi_x^S\,(f(z))\, e^{i\,\nu \tau(z)} = \varphi_x^S\,(f(z))\, e^{\frac{i\tau(z)}{\hbar}}$$

has micro-support on the graph of S' (locally defined).

The underlying dynamics is

$$F(z,s) = \Big(f(z), \frac{1}{2}\, s - \frac{1}{2}\tau'(z)\Big).$$

Proof: For fixed x and variable η, y, where $x, y \in S^1$, first we apply proposition 12 (see appendix) to obtain that

$$< \varphi_{y,\eta} \, , \, \hat{F}_\nu \varphi_x^S >= O(\hbar^\infty)$$

if $f(y) \neq x$ or if $f(y) = x$ and $\eta \neq S'(f(y))f'(y) + \tau'(y) = 2S'(f(y)) + \tau'(y)$. Then the micro-support requires $f(y) = x$ and $\eta = 2\,S'(f(y)) + \tau'(y)$. As we assume that S' is $\frac{1}{2}-$subaction for $-\frac{1}{2}\tau'$, then, from (4) we get that $\eta = S'(y) = u(y)$. Therefore micro-support is on the graph $(y, u(y)) = (f^{-1}(x), u(f^{-1}(x)))$ of $u = S'$.

□

Now we will present sufficient conditions on τ in order the function S is under the hypothesis of last theorem.

Denote $A = -\frac{1}{2}\tau'$. The inverses branches of f are denoted by τ_1 and τ_2. We assume that $\tau_1(0,1) = (0, 0.5)$ and $\tau_2(0,1) = (0.5, 1)$.

We also denote $a = (a_1, a_2, ...)$ a generic element in $\Omega = \{1,2\}^{\mathbb{N}}$.

Consider (as Tsujii in [18]) the function $s : (S^1, \Omega) \to \mathbb{R}$, where $\Omega = \{1,2\}^{\mathbb{N}}$, given by

$$s(x,a) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k A(\,(\tau_{a_k} \circ \tau_{a_{k-1}} \circ \; ... \; \circ \tau_{a_0})\,(x)\,),$$

and, $a = (a_0, a_1, a_2, ...)$.

Note that for a fixed a the function $s(x,a)$ is not a periodic function on $x \in [0,1]$.

We define $\pi(x) = i$, if x is in the image of $\tau_i(S^1)$, $i \in \{1,2\}$.

Note that $(x,a) \to (f(x), \pi(x)a)$ defines a skew product on $S^1 \times \Omega$.

Note also [18] that

$$s(f(x), \pi(x)a) = A(x) + \frac{1}{2}\, s(x,a).$$

For a fixed $b \in \Omega$ consider

$$\varphi_x^{S(\cdot,b)}\,(z) = \; e^{\frac{iS(z,b)}{\hbar}} \quad e^{-\frac{(z-x)^2}{4\,\hbar}}.$$

If we suppose $s(z,b) = \frac{\partial S}{\partial z}(z,b)$, then we can show in a similar way as in last result that

$$\hat{F}_\nu(\varphi_x^{S(\cdot,b)}\,(z))$$

has support on the graph $(\, y, s(y,a)\,)$ where $f(y) = x$ and $\pi(y)\,a = b$.

From [13] one gets that the plus-calibrated subaction u satisfies

$$u(x) = \sup_{c \in \{1,2\}^{\mathbb{N}}} s(x,c).$$

This property resembles the classical method of obtaining solutions of the Hamilton-Jacobi equation via envelopes (Huygens's generation of the wave front) as described for instance in [8] Chapter I.9.

Definition 7. *Consider a fixed $\overline{x} \in S^1$ and variable $x \in S^1$, $a \in \{1,2\}^{\mathbb{N}}$, then we define*

$$W(x,a) = s(x,a) - s(\overline{x},a). \tag{5}$$

We call such W the $\frac{1}{2}$-involution kernel for $A = -\frac{1}{2}\tau'$.

For a fixed $W(x,a)$ is smooth on $x \in (0,1)$.

Below we consider the lexicographic order in $\{1,2\}^{\mathbb{N}}$.

Definition 8. *We say that A satisfies the twist condition, if an (then, any) associated involution kernel W, satisfies the property: for any $a < b$, we have*

$$\frac{\partial W}{\partial x}(x,a) - \frac{\partial W}{\partial x}(x,b) > 0.$$

It is equivalent to state the above relation for s or for W.

If $A = -\frac{1}{2}\tau'$ satisfies the twist condition, then, u will be piecewise smooth (see Corollary 15 in [13]) and Theorem 6 can be applied.

3 Diffemorphisms of the circle $f : S^1 \to S^1$

Suppose f is an orientation preserving diffeomorphism of the circle S^1 of class C^k and $\tau : S^1 \to \mathbb{R}$ is also of class C^k. The value of k may depend of f but will require at least $k \geq 4$.

We are interest in condition on f and τ such that there exists $u : S^1 \to \mathbb{R}$ which is differentiable and satisfies

$$-\tau'(z) = u(f(z))f'(z) - u(z). \tag{6}$$

Suppose f as above and $A : S^1 \to \mathbb{R}$ is also of class C^k. If there exists differentiable $w : S^1 \to \mathbb{R}$ such that

$$A(z) = w(f(z)) - w(z), \tag{7}$$

then, taking derivative on both sides, denoting $u = w'$ and $\tau = -A$, we get (6).

Given a smooth A, the problem of finding a smooth w is considered in section 2.2 in [2] and [11].

Given a real number α we say that it satisfies the Diophantine condition if:

$$\left|\alpha - \frac{p}{q}\right| > \frac{K}{q^{2+\beta}} \text{ for all } \frac{p}{q} \in \mathbb{Q},$$

where K and β are positive constants.

In the case $f(z) = R_\alpha(z) = z + \lambda$, where λ is diophantine and $A = -\tau$ is $C^{3+\alpha}$, there exists a differentiable function w such that

$$-\tau(z) = w(f(z)) - w(z), \tag{8}$$

as before, taking $u = w'$ we get a solution for equation (6).

Now let f be a C^k circle diffeomorphism, $k \geq 3$ and suppose the rotation number α of f satisfies the Diophantine condition:

$$\left|\alpha - \frac{p}{q}\right| > \frac{K}{q^{2+\beta}} \text{ for all } \frac{p}{q} \in \mathbb{Q},$$

where K and β are positive constants. Suppose also that $k > 2\beta + 1$, then, given $A = -\tau$ of class C^k there exists w of class C^k satisfying equation (7), again if $u = w'$ we get a solution for equation (6).

Theorem 9. *Suppose f is a C^∞ diffeomorphism of the circle, $\tau : S^1 \to \mathbb{R}$ is smooth C^∞ and assume that for any z the coboundary equation*

$$u(z) = u(f(z))\, f'(z) + \tau'(z) \tag{9}$$

is true (u is C^∞).

Suppose also that $S' = u$. Then, the micro-support of the Lagrangian state $\varphi_x^S(y) = e^{\frac{iS(y)}{\hbar}} e^{-\frac{(y-x)^2}{4\hbar}}$ is invariant for the transformation $\hat{F}_\nu(\varphi_x^S(z)) = \varphi_x^S(f(z))e^{\frac{i\tau(z)}{\hbar}}$, when $\hbar \to 0$.

The underlying dynamics is given by $F(z,s) = \left(f(z), \frac{s-\tau'(z)}{f'(z)}\right)$, $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ and the graph of u is invariant by F.

Proof:

As before, for fixed x and variable η, y, where $x, y \in S^1$, first we apply proposition 12 (see appendix) to obtain that

$$< \varphi_{y,\eta}\,,\, \hat{F}_\nu \varphi_x^S >= O(\hbar^\infty)$$

if $f(y) \neq x$ or if $f(y) = x$ and $\eta \neq S'(f(y))f'(y) + \tau'(y)$.

Then the micro-support requires $f(y) = x$ and $\eta = S'(f(y))f'(y) + \tau'(y)$. And, as we suppose that $S' = u$, using equation (6), we obtain

$$\eta = S'(f(y))f'(y) + \tau'(y) = u'(f(y))f'(y) + \tau'(y) = u(y),$$

Therefore, we get that the microsupport of $\hat{F}_\nu(\varphi_x^S)$ is on the graph $(y, u(y)) = (f^{-1}(x), u(f^{-1}(x)))$.

Finally, let us show that the graph of u is invariant for F. In fact, note that if $y = f(z)$ then by (6) we get $u(y) = u(f(z)) = \frac{u(z)-\tau'(z)}{f'(z)}$ and

$$F(z, u(z)) = \left(f(z), \frac{u(z) - \tau'(z)}{f'(z)} \right) = (y, u(y)).$$

□

4 Appendix

Let $(x, \xi) \in \mathbb{R}^2$, then for each $z \in \mathbb{R}$ we define before

$$\tilde{\varphi}_{x,\xi}(z) = e^{\frac{i\xi z}{\hbar}} e^{-\frac{(z-x)^2}{4\hbar}},$$

and if $z \in S^1$ we define the periodic function

$$\varphi_{x,\xi}(z) = \sum_{k\in\mathbb{Z}} \tilde{\varphi}_{x,\xi}(z-k) = \sum_{k\in\mathbb{Z}} e^{\frac{i\xi(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}}.$$

For each $x \in \mathbb{R}$, $S : \mathbb{R} \to \mathbb{R}$ the Lagrangian package is defined, for each $z \in \mathbb{R}$, by

$$\tilde{\varphi}_x^S(z) = e^{\frac{iS(z)}{\hbar}} e^{-\frac{(z-x)^2}{4\hbar}}.$$

Remember that for $S : \mathbb{R} \to \mathbb{R}$ we define the periodic function

$$\varphi_x^S(z) = \sum_{k\in\mathbb{Z}} \tilde{\varphi}_x^S(z-k) = \sum_{k\in\mathbb{Z}} e^{\frac{iS(z-k)}{\hbar}} e^{-\frac{(z-k-x)^2}{4\hbar}}.$$

And, if we take $f : S^1 \to S^1$ we define

$$\hat{F}_\nu(\varphi_x^S(z)) = \varphi_x^S(f(z))e^{i\nu\tau(z)} = \sum_{k\in\mathbb{Z}} e^{\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)}$$

Therefore, if $z \in S^1$

$$< \varphi_{y,\eta}(z), \hat{F}_\nu(\varphi_x^S(z)) >=$$
$$\int_{S^1} \sum_{j\in\mathbb{Z}} e^{\frac{i\eta(z-j)}{\hbar}} e^{-\frac{(z-j-y)^2}{4\hbar}} \sum_{k\in\mathbb{Z}} e^{-\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)} dz$$

Proposition 10. *Let us fix $x, y \in \mathbb{R}$. Then, there exist constants $\bar{k}$ and $\bar{j}$ such that*

$$< \varphi_{y,\eta}(z), \hat{F}_\nu(\varphi_x^S(z)) >\approx \int_{S^1} e^{\frac{i\eta(z+\bar{j})}{\hbar}} e^{-\frac{(z+\bar{j}-y)^2}{4\hbar}} e^{-\frac{iS(f(z)+\bar{k})}{\hbar}} e^{-\frac{(f(z)+\bar{k}-x)^2}{4\hbar}} e^{i\nu\tau(z)} dz,$$

Proof: We choose the constants $\bar{k}, \bar{l}$ such that $x \in (\bar{k}, \bar{k}+1)$ and $y \in (\bar{j}, \bar{j}+1)$.

We will show that the other terms in the integral rapidly decrease to zero, when $\hbar \to 0$. Indeed,

$$< \varphi_{y,\eta}(z), \hat{F}_\nu(\varphi_x^S(z)) >=$$
$$= \int_{S^1} \sum_{j\in\mathbb{Z}} e^{\frac{i\eta(z-j)}{\hbar}} e^{-\frac{(z-j-y)^2}{4\hbar}} \sum_{k\in\mathbb{Z}} e^{-\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)} dz =$$
$$= \int_{S^1} e^{\frac{i\eta(z+\bar{j})}{\hbar}} e^{-\frac{(z+\bar{j}-y)^2}{4\hbar}} e^{-\frac{iS(f(z)+\bar{k})}{\hbar}} e^{-\frac{(f(z)+\bar{k}-x)^2}{4\hbar}} e^{i\nu\tau(z)} dz +$$
$$+ \int_{S^1} e^{-\frac{iS(f(z)+\bar{k})}{\hbar}} e^{-\frac{(f(z)+\bar{k}-x)^2}{4\hbar}} e^{i\nu\tau(z)} \sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{\frac{i\eta(z-j)}{\hbar}} e^{-\frac{(z-j-y)^2}{4\hbar}} dz +$$
$$+ \int_{S^1} e^{\frac{i\eta(z+\bar{j})}{\hbar}} e^{-\frac{(z+\bar{j}-y)^2}{4\hbar}} e^{i\nu\tau(z)} \sum_{k\in\mathbb{Z}, k\neq -\bar{k}} e^{-\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} dz +$$
$$+ \int_{S^1} \sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{\frac{i\eta(z-j)}{\hbar}} e^{-\frac{(z-j-y)^2}{4\hbar}} \sum_{k\in\mathbb{Z}, k\neq -\bar{k}} e^{-\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)} dz.$$

We define the positive constant

$$C = \min\{d(y, \bar{j}), d(y, \bar{j}+1), d(x, \bar{k}), d(x, \bar{k}+1)\}.$$

As $z \in S^1$, we obtain

$$\sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{-\frac{(z-j-y)^2}{4\hbar}} = e^{-\frac{(z+\bar{j}-1-y)^2}{4\hbar}} + e^{-\frac{(z+\bar{j}+1-y)^2}{4\hbar}} + e^{-\frac{(z+\bar{j}-2-y)^2}{4\hbar}} + e^{-\frac{(z+\bar{j}+2-y)^2}{4\hbar}} + \ldots$$

$$\leq 2(e^{-\frac{C^2}{4\hbar}} + e^{-\frac{(C+1)^2}{4\hbar}} + e^{-\frac{(C+2)^2}{4\hbar}} + \ldots) \leq 2\Big(e^{-\frac{C^2}{4\hbar}} + \int_C^\infty e^{-\frac{x^2}{4\hbar}} dx\Big) = O\Big(e^{-\frac{C^2}{4\hbar}}\Big).$$

Therefore, the second term above can be bounded by

$$\Big|\int_{S^1} e^{-\frac{iS(f(z)+\bar{k})}{\hbar}} e^{-\frac{(f(z)+\bar{k}-x)^2}{4\hbar}} e^{i\nu\tau(z)} \sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{\frac{i\eta(z-j)}{\hbar}} e^{-\frac{(z-j-y)^2}{4\hbar}} dz\Big| \leq$$

$$\leq \int_{S^1} \sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{-\frac{(z-j-y)^2}{4\hbar}} dz = O\Big(e^{-\frac{C^2}{4\hbar}}\Big)$$

The third term above can be bounded by the same arguments.

Note that if $\hbar$ small enough we have that $\sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{-\frac{(z-j-y)^2}{4\hbar}} \leq 1$. Therefore, the last term above can be bounded by

$$\Big|\int_{S^1} \sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{\frac{i\eta(z-j)}{\hbar}} e^{-\frac{(z-j-y)^2}{4\hbar}} \sum_{k\in\mathbb{Z}, k\neq -\bar{k}} e^{-\frac{iS(f(z)-k)}{\hbar}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} e^{i\nu\tau(z)} dz\Big| \leq$$

$$\leq \int_{S^1} \sum_{j\in\mathbb{Z}, j\neq -\bar{j}} e^{-\frac{(z-j-y)^2}{4\hbar}} \sum_{k\in\mathbb{Z}, k\neq -\bar{k}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} dz \leq$$

$$\leq \int_{S^1} \sum_{k\in\mathbb{Z}, k\neq -\bar{k}} e^{-\frac{(f(z)-k-x)^2}{4\hbar}} dz = O\Big(e^{-\frac{C^2}{4\hbar}}\Big).$$

□

Now, if $x, y \in S^1$ and $\nu = \frac{1}{\hbar}$, then we get

$$< \varphi_{y,\eta}(z), \hat{F}_\nu(\varphi_x^S(z)) > \approx \int_{S^1} e^{-\frac{i\eta z}{\hbar}} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{iS(f(z))}{\hbar}} e^{-\frac{(f(z)-x)^2}{4\hbar}} e^{\frac{i\tau(z)}{\hbar}} dz.$$

Lemma 11. *Let us fix a constant a, then*

$$\int_{y-a}^{y+a} \Big|\frac{d^{2n}}{dz^{2n}} e^{-\frac{(z-y)^2}{4\hbar}}\Big| dz$$

decay fast to zero as $\hbar \to 0$, for each $n \geq 1$.

Moreover, for each $n \geq 0$, we get

$$\int_{y-a}^{y+a} \Big|\frac{d^{2n+1}}{dz^{2n+1}} e^{-\frac{(z-y)^2}{4\hbar}}\Big| dz = O\Big(\frac{1}{\hbar^n}\Big).$$

Proposition 12. *Assume that S is piecewise smooth C^∞ (left and right finite non zero derivatives in a finite number of singular points). Suppose f is C^∞, $\nu = \frac{1}{\hbar}$ and $\tau : S^1 \to \mathbb{R}$ is smooth C^∞. Let us fix $x, y \in S^1$. If $f(y) \neq x$ or $f(y) = x$ and $\eta \neq S'(f(y))f'(y) + \tau'(y)$, then*

$$< \varphi_{y,\eta}(z), \hat{F}_\nu(\varphi_x^S(z)) >= O(\hbar^\infty).$$

Proof: Define $\phi(z) = S(f(z)) - \eta z + \tau(z)$, then by the Proposition 10 it remains to analyze the following integral.

$$\int_{S^1} e^{-\frac{(z-y)^2+(f(z)-x)^2}{4\hbar}} e^{\frac{i(S(f(z))-\eta z+\tau(z))}{\hbar}} dz = \int_{S^1} e^{-\frac{(z-y)^2+(f(z)-x)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz$$

Case 1: We suppose that $f(y) \neq x$, then we can obtain constants C and ϵ such that $(f(z) - x)^2 > C^2$ for all $z \in (y - \epsilon, y + \epsilon)$, also if $z \in (y + \epsilon, 1)$ or $z \in (0, y - \epsilon)$ we have that $(z - y)^2 \geq \epsilon^2$. Therefore, if we denote by $\delta = \min\{C, \epsilon\}$ we get

$$\left| \int_{S^1} e^{-\frac{(z-y)^2+(f(z)-x)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz \right| \leq$$

$$\int_{S^1} e^{-\frac{(z-y)^2}{4\hbar}} e^{-\frac{(f(z)-x)^2}{4\hbar}} dz \leq e^{-\frac{\epsilon^2}{4\hbar}} \int_0^{y-\epsilon} e^{-\frac{(f(z)-x)^2}{4\hbar}} dz +$$

$$+ e^{-\frac{C^2}{4\hbar}} \int_{y-\epsilon}^{y+\epsilon} e^{-\frac{(z-y)^2}{4\hbar}} dz + e^{-\frac{\epsilon^2}{4\hbar}} \int_{y+\epsilon}^{1} e^{-\frac{(f(z)-x)^2}{4\hbar}} dz = O\Big(e^{-\frac{\delta^2}{4\hbar}}\Big).$$

Case 2: Suppose that $f(y) = x$ and $\eta \neq S'(f(y))f'(y) + \tau'(y)$, i.e., $\phi'(y) = C \neq 0$, then we can find a constant a such that $\phi'(z) \geq \frac{C}{2}$ for $z \in (y - a, y + a)$ Lebesgue a.e.w. Hence,

$$\int_{S^1} e^{-\frac{(z-y)^2+(f(z)-x)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz = \int_{S^1} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz =$$

$$= \int_0^{y-a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz + \int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz + \int_{y+a}^{1} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz.$$

Note that, $\int_0^{y-a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz = O\Big(e^{-\frac{a^2}{4\hbar}}\Big)$, $\int_{y+a}^{1} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz = O\Big(e^{-\frac{a^2}{4\hbar}}\Big)$.

Therefore, it remains to analyze the following integral

$$\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz. \tag{10}$$

We define the operator $L := \frac{\hbar}{i}\frac{1}{\phi'(z)}\frac{d}{dz}$ and we note that $L^n(e^{\frac{i\phi}{\hbar}}) = e^{\frac{i\phi}{\hbar}}$ for all $n \geq 1$. Then, integrating by parts we get

$$\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz =$$

$$\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} L(e^{\frac{i\phi(z)}{\hbar}}) dz = \int_{y-a}^{y+a} \frac{\hbar}{i}\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)}\frac{d}{dz}(e^{\frac{i\phi(z)}{\hbar}}) dz =$$

$$= \frac{\hbar}{i}\left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} e^{\frac{i\phi(z)}{\hbar}} \right]_{y-a}^{y+a} - \int_{y-a}^{y+a} \frac{d}{dz}\left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)}\right) e^{\frac{i\phi(z)}{\hbar}} dz \Bigg) =$$

$$= \frac{\hbar}{i}\Bigg[e^{-\frac{a^2}{4\hbar}} \left(\frac{e^{\frac{i\phi(y+a)}{\hbar}}}{\phi'(y+a)} - \frac{e^{\frac{i\phi(y-a)}{\hbar}}}{\phi'(y-a)} \right)$$
$$- \int_{y-a}^{y+a} \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{1}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{\phi''(z)}{\phi'(z)^2} \right) e^{\frac{i\phi(z)}{\hbar}} dz \Bigg]$$

In order to estimate the last integral, we will define the following functions $l_1^1(z) = \frac{1}{\phi'(z)}$ and $l_2^1(z) = \frac{\phi''(z)}{\phi'(z)^2}$. Note that these functions do not depend on $\hbar$, hence we can find a constant C_1, such that $|l_1^1(z)| \leq C_1$ and $|l_2^1(z)| \leq C_1$, if $z \in (y-a, y+a)$. Now, using the property $\left| \int_I u(z)v(z)dz \right| \leq \int_I |u(z)||v(z)|dz \leq \max_{z\in I}|v(z)| \int_I |u(z)|dz$ and Lemma 11 we obtain

$$\left| \int_{y-a}^{y+a} \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} l_1^1(z) - e^{-\frac{(z-y)^2}{4\hbar}} l_2^1(z) \right) e^{\frac{i\phi(z)}{\hbar}} dz \right| \leq$$
$$\leq C_1 \int_{y-a}^{y+a} \left| \frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz + C_1 \int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} dz = O(1)$$

And finally, the integral in (10) can be estimated by

$$\left| \int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz \right| \leq \hbar \left[\tilde{C}_1 e^{-\frac{a^2}{4\hbar}} + O(1) \right] = O(\hbar).$$

We want to show that the integral in (10) is $O(\hbar^N)$ for each $N \in \mathbb{N}$, and in order to obtain that we will integrate by parts several times.

Using the operator L^2 we get the following estimate

$$\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz =$$

$$\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} L^2(e^{\frac{i\phi(z)}{\hbar}}) dz = \frac{\hbar^2}{i^2} \int_{y-a}^{y+a} \frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} \frac{d}{dz}\Big(\frac{1}{\phi'(z)} \frac{d}{dz} e^{\frac{i\phi(z)}{\hbar}} \Big) dz =$$

$$= \frac{\hbar^2}{i^2} \frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} \Big(\frac{1}{\phi'(z)} \frac{d}{dz} e^{\frac{i\phi(z)}{\hbar}} \Big) \Big]_{y-a}^{y+a} - \frac{\hbar^2}{i^2} \int_{y-a}^{y+a} \frac{1}{\phi'(z)} \frac{d}{dz} \left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} \right) \frac{d}{dz} e^{\frac{i\phi(z)}{\hbar}} dz =$$

$$= \frac{\hbar}{i} \frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} e^{\frac{i\phi(z)}{\hbar}} \Big]_{y-a}^{y+a} - \frac{\hbar^2}{i^2} \int_{y-a}^{y+a} \frac{1}{\phi'(z)} \frac{d}{dz} \left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} \right) \frac{d}{dz} e^{\frac{i\phi(z)}{\hbar}} dz \qquad (11)$$

Now we will estimate the integral in (11)

$$\int_{y-a}^{y+a} \frac{1}{\phi'(z)} \frac{d}{dz} \left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)} \right) \frac{d}{dz} e^{\frac{i\phi(z)}{\hbar}} dz =$$

$$= \int_{y-a}^{y+a} \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_1^1(z)}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_2^1(z)}{\phi'(z)} \right) \frac{d}{dz} e^{\frac{i\phi(z)}{\hbar}} dz =$$

$$= \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_1^1(z)}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_2^1(z)}{\phi'(z)} \right) e^{\frac{i\phi(z)}{\hbar}} \Big]_{y-a}^{y+a} -$$

$$- \int_{y-a}^{y+a} \frac{d}{dz} \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_1^1(z)}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_2^1(z)}{\phi'(z)} \right) e^{\frac{i\phi(z)}{\hbar}} dz \qquad (12)$$

and by calculating the derivative that appears in the equation (12) we have

$$\frac{d}{dz} \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_1^1(z)}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_2^1(z)}{\phi'(z)} \right) =$$

$$= \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_1^1(z)}{\phi'(z)} + \frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{d}{dz} \frac{l_1^1(z)}{\phi'(z)} - \frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_2^1(z)}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{d}{dz} \frac{l_2^1(z)}{\phi'(z)} =$$

$$= \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} l_1^2(z) + \frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} l_2^2 - e^{-\frac{(z-y)^2}{4\hbar}} l_3^2, \qquad (13)$$

where $l_1^2 = \frac{1}{\phi'^2}$, $l_2^2 = -\frac{3\phi''}{\phi'^3}$, $l_3^2(z) = \frac{d}{dz} \frac{\phi''}{\phi'^3} = \frac{\phi'''\phi' - 3\phi''^2}{\phi'^4}$ do not depend on $\hbar$ and are bounded by a constant C_2 in $(y-a, y+a)$ Lebesgue a.e.w.

Hence we can estimate the integral in (10) by substituting the equation (13) in (12) and the equation (12) in (11) in order to get

$$\left| \int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz \right| \leq$$

$$\leq \hbar [\tilde{C}_1 e^{-\frac{a^2}{4\hbar}}] + \hbar^2 \left| \left(\frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_1^1(z)}{\phi'(z)} - e^{-\frac{(z-y)^2}{4\hbar}} \frac{l_2^1(z)}{\phi'(z)} \right) e^{\frac{i\phi(z)}{\hbar}} \Big]_{y-a}^{y+a} \right| +$$

$$+ \hbar^2 C_2 \left[\int_{y-a}^{y+a} \left| \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz + \int_{y-a}^{y+a} \left| \frac{d}{dz} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz + \int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} dz \right] \leq$$

$$\leq \hbar[\tilde{C}_1 e^{-\frac{a^2}{4\hbar}}] + \hbar^2 e^{-\frac{a^2}{4\hbar}} C\left(\frac{a}{\hbar}+1\right) + \hbar^2 C_2[O(1)] = O(\hbar^2).$$

Note that in the last inequality we have used the Lemma 11.

When we apply the operator L^3, we integrate by parts three times and hence $\hbar^3$ appears multiplying all terms. The only term we need to be concern is

$$\int_{y-a}^{y+a} \frac{d}{dz}\left[\frac{1}{\phi'(z)}\left(\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}} l_1^2(z) + \frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}} l_2^2 - e^{-\frac{(z-y)^2}{4\hbar}} l_3^2\right)\right] e^{\frac{i\phi(z)}{\hbar}} dz,$$

because all the other terms, as we see when we apply L^2, will have a factor $e^{-\frac{a^2}{4\hbar}}$ multiplying it. Then, computing the derivative that appears in the integrand we get

$$\int_{y-a}^{y+a} \left[\frac{d^3}{dz^3}e^{-\frac{(z-y)^2}{4\hbar}} \frac{1}{\phi'(z)^3} - \frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}} \frac{6\phi''}{\phi'^4} - \frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}} \frac{4\phi'''\phi' - 15\phi''^2}{\phi'^5} + \right.$$

$$\left. - e^{-\frac{(z-y)^2}{4\hbar}} \frac{\phi^{(4)}\phi'^2 - 10\phi'\phi''\phi''' + 15\phi''^2}{\phi'^6}\right] e^{\frac{i\phi(z)}{\hbar}} dz.$$

As before, we define $l_1^3 = \dfrac{1}{\phi'(z)^3}$, $l_2^3 = \dfrac{6\phi''}{\phi'^4}$, $l_3^3 = \dfrac{4\phi'''\phi' - 15\phi''^2}{\phi'^5}$, $l_4^3 = \dfrac{\phi^{(4)}\phi'^2 - 10\phi'\phi''\phi''' + 15\phi''^2}{\phi'^6}$ and we see that these functions are bounded by a constant C_3 that is independent of $\hbar$.

Now using Lemma 11, we obtain

$$\left|\int_{y-a}^{y+a} \frac{d}{dz}\left[\frac{1}{\phi'(z)}\left(\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}} l_1^2(z) + \frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}} l_2^2 - e^{-\frac{(z-y)^2}{4\hbar}} l_3^2\right)\right] e^{\frac{i\phi(z)}{\hbar}} dz\right| \leq$$

$$\leq C_3\left(\int_{y-a}^{y+a} \left|\frac{d^3}{dz^3}e^{-\frac{(z-y)^2}{4\hbar}}\right| dz + \int_{y-a}^{y+a} \left|\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}}\right| dz + \right.$$

$$\left. + \int_{y-a}^{y+a} \left|\frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}\right| dz + \int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} dz\right) = O\left(\frac{1}{\hbar}\right).$$

Therefore, because of the multiplying factor $\hbar^3$, we get

$$\left|\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}} e^{\frac{i\phi(z)}{\hbar}} dz\right| = O(\hbar^2).$$

Analogously, when we apply L^4 it will appear the multiplicative term $\hbar^4$, and now using Lemma 11 it can be shown that

$$\int_{y-a}^{y+a} \frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)}\right)\right)\right)\right)e^{\frac{i\phi(z)}{\hbar}}dz = O\left(\frac{1}{\hbar}\right),$$

hence

$$\left|\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}}e^{\frac{i\phi(z)}{\hbar}}dz\right| = O(\hbar^3).$$

Analogously, when we apply L^5 it will appear the multiplicative term $\hbar^5$, and now using Lemma 11 it can be shown that

$$\int_{y-a}^{y+a} \frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{1}{\phi'(z)}\frac{d}{dz}\left(\frac{e^{-\frac{(z-y)^2}{4\hbar}}}{\phi'(z)}\right)\right)\right)\right)\right)e^{\frac{i\phi(z)}{\hbar}}dz =$$

$$O\left(\frac{1}{\hbar^2}\right),$$

hence

$$\left|\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}}e^{\frac{i\phi(z)}{\hbar}}dz\right| = O(\hbar^3).$$

Finally, it can be shown that if we apply L^{2n} or L^{2n+1} we obtain

$$\left|\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}}e^{\frac{i\phi(z)}{\hbar}}dz\right| = O(\hbar^n),$$

and this implies

$$\left|\int_{y-a}^{y+a} e^{-\frac{(z-y)^2}{4\hbar}}e^{\frac{i\phi(z)}{\hbar}}dz\right| = O(\hbar^\infty).$$

□

Proof (of the Lemma 11): Let us first calculate the derivatives of the function $e^{-\frac{(z-y)^2}{4\hbar}}$:

$$\frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}} = e^{-\frac{(z-y)^2}{4\hbar}}\frac{(y-z)}{2\hbar},$$

$$\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}} = e^{-\frac{(z-y)^2}{4\hbar}}\left(\frac{(y-z)^2-2\hbar}{4\hbar^2}\right),$$

$$\frac{d^3}{dz^3}e^{-\frac{(z-y)^2}{4\hbar}} = e^{-\frac{(z-y)^2}{4\hbar}}\left(\frac{(y-z)^3-6\hbar(y-z)}{8\hbar^3}\right),$$

$$\frac{d^4}{dz^4}e^{-\frac{(z-y)^2}{4\hbar}} = e^{-\frac{(z-y)^2}{4\hbar}}\left(\frac{(y-z)^4 - 12\hbar(y-z)^2 + 12\hbar^2}{16\hbar^4}\right),$$

$$\frac{d^5}{dz^5}e^{-\frac{(z-y)^2}{4\hbar}} = e^{-\frac{(z-y)^2}{4\hbar}}\left(\frac{(y-z)^5 - 20\hbar(y-z)^3 + 60\hbar^2(y-z)}{32\hbar^5}\right),$$

$$\frac{d^6}{dz^6}e^{-\frac{(z-y)^2}{4\hbar}} = e^{-\frac{(z-y)^2}{4\hbar}}\left(\frac{(y-z)^6 - 30\hbar(y-z)^4 + 180\hbar^2(y-z)^2 - 120\hbar^3}{64\hbar^6}\right),$$

It is enough to see that $\frac{d^{2n+1}}{dz^{2n+1}}e^{-\frac{(z-y)^2}{4\hbar}}|_{z=y} = 0$ and $\frac{d^{2n}}{dz^{2n}}e^{-\frac{(z-y)^2}{4\hbar}}|_{z=y} = \frac{C}{\hbar^n}$, where $C \neq 0$ is a constant.

Now we will estimate $\displaystyle\int_{y-a}^{y+a}\left|\frac{d^n}{dz^n}e^{-\frac{(z-y)^2}{4\hbar}}\right|dz$, for $n \geq 1$.

Note that $\frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}$ only change the signal in $z = y$, then

$$\int_{y-a}^{y+a}\left|\frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}\right|dz = \int_{y-a}^{y}\frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}dz - \int_{y}^{y+a}\frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}dz =$$

$$= e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{y-a}^{y} - e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{y}^{y+a} = 2 - 2e^{-\frac{a^2}{4\hbar}} = O(1).$$

Observe that $\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{z=y} = -\frac{1}{\hbar}$ and it can change of signal at most twice in $(y-a, y+a)$. Let us suppose that this happens, i.e., there exist $r_1 < y < r_2$ points in $(y-a, y+a)$ such that the second derivative is positive in $(y-a, r_1)$ and in $(r_2, y+a)$ and negative in (r_1, r_2), then we obtain

$$\int_{y-a}^{y+a}\left|\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}}\right|dz =$$

$$= \int_{y-a}^{r_1}\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}}dz - \int_{r_1}^{r_2}\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}}dz + \int_{r_2}^{y+a}\frac{d^2}{dz^2}e^{-\frac{(z-y)^2}{4\hbar}}dz =$$

$$= \frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{y-a}^{r_1} - \frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{r_1}^{r_2} + \frac{d}{dz}e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{r_2}^{y+a} =$$

$$= 2\left(e^{-\frac{(r_1-y)^2}{4\hbar}}\left(\frac{y-r_1}{\hbar}\right) - e^{-\frac{a^2}{4\hbar}}\left(\frac{a}{\hbar}\right) - e^{-\frac{(r_2-y)^2}{4\hbar}}\left(\frac{y-r_2}{\hbar}\right)\right),$$

and this decay fast to zero as $\hbar \to 0$.

Note that $\frac{d^3}{dz^3}e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{z=y} = 0$ and it can change of signal at most three times in $(y-a, y+a)$. Let us suppose that this happens, i.e., there exist $r_1 < y < r_2$ points in $(y-a, y+a)$ such that the derivative of order 3 is positive in $(y-a, r_1)$ and in (y, r_2) and negative in (r_1, y) and in $(r_2, y+a)$, then, we obtain

$$\int_{y-a}^{y+a} \left| \frac{d^3}{dz^3} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz =$$

$$= \int_{y-a}^{r_1} \frac{d^3}{dz^3} e^{-\frac{(z-y)^2}{4\hbar}} dz - \int_{r_1}^{y} \frac{d^3}{dz^3} e^{-\frac{(z-y)^2}{4\hbar}} dz +$$

$$+ \int_{y}^{r_2} \frac{d^3}{dz^3} e^{-\frac{(z-y)^2}{4\hbar}} dz - \int_{r_2}^{y+a} \frac{d^3}{dz^3} e^{-\frac{(z-y)^2}{4\hbar}} dz =$$

$$= \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} \Big|_{y-a}^{r_1} - \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} \Big|_{r_1}^{y} + \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} \Big|_{y}^{r_2} - \frac{d^2}{dz^2} e^{-\frac{(z-y)^2}{4\hbar}} \Big|_{r_2}^{y+a} =$$

$$= 2e^{-\frac{(r_1-y)^2}{4\hbar}} \left(\frac{(y-r_1)^2 - \hbar}{\hbar^2} \right) - 2e^{-\frac{a^2}{4\hbar}} \left(\frac{a^2 - \hbar}{\hbar^2} \right) +$$

$$+ \frac{2}{\hbar} + 2e^{-\frac{(r_2-y)^2}{4\hbar}} \left(\frac{(y-r_2)^2 - \hbar}{\hbar^2} \right) = O\left(\frac{1}{\hbar}\right).$$

It became clear from these calculations that, $\int_{y-a}^{y+a} \left| \frac{d^{2n}}{dz^{2n}} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz$ decay fast to zero as $\hbar \to 0$, because $\frac{d^{2n}}{dz^{2n}} e^{-\frac{(z-y)^2}{4\hbar}}$ does not change of signal in $z = y$. Moreover, the only term in $\int_{y-a}^{y+a} \left| \frac{d^{2n+1}}{dz^{2n+1}} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz$ that does not decay fast to zero is $\frac{d^{2n}}{dz^{2n}} e^{-\frac{(z-y)^2}{4\hbar}}\Big|_{z=y} = \frac{C}{\hbar^n}$, and this implies

$$\int_{y-a}^{y+a} \left| \frac{d^{2n+1}}{dz^{2n+1}} e^{-\frac{(z-y)^2}{4\hbar}} \right| dz = O\left(\frac{1}{\hbar^n}\right).$$

□

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