C^*- Algebras and Thermodynamic Formalism

Ruy Exel (*) and Artur O. Lopes (**)

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*** Departamento de Matemática, UFSC, Florianópolis, Brasil.**
and
**** Instituto de Matemática, UFRGS, Porto Alegre, Brasil.**

Abstract

We show a relation of the KMS state of a certain C^*-Algebra $\mathcal{U}$ with the Gibbs state of Thermodynamic Formalism. More precisely, we consider here the shift $T : X \to X$ acting on the Bernoulli space $X = \{1, 2, ..., k\}^{\mathbf{N}}$ and μ a Gibbs (equilibrium) state defined by a Holder continuous normalized potential $p : X \to \mathbf{R}$, and $\mathcal{L}^2(\mu)$ the associated Hilbert space.

Consider the C^*-Algebra $\mathcal{U} = \mathcal{U}(\mu)$, which is a sub-$C^*$-Algebra of the C^*-Algebra of linear operators in $\mathcal{L}^2(\mu)$ which will be precisely defined later. We call μ the reference measure. Consider a fixed Holder potential $H > 0$ and the C^*-dynamical system defined by the associated homomorphism σ_t. We are interested in describe for such system the KMS states ψ_β for all $\beta \in \mathbf{R}$.

We show a relation of a new Gibbs (eigenprobability of a Ruelle operator) probability ν_β to a KMS state $\psi_{\nu_\beta} = \psi_\beta$, in the C^*-Algebra $\mathcal{U} = \mathcal{U}(\mu)$, for every value $\beta \in \mathbf{R}$, where β is the parameter that defines the time evolution associated to a homomorphism $\sigma_t = \sigma_{\beta i}$ defined by the potential H. We show that for each real β the KMS state is unique and we explicit it. The probability ν_β is the eigenprobability of the dual of the Ruelle operator of the non-normalized potential $-\beta \log H$. The purpose of the present work is to explain (for an audience which is more oriented to Dynamical System Theory) part of the content of a previous paper written by the authors.

Introduction

In this paper we show a relation of the KMS state of a certain C^*-Algebra $\mathcal{U}$ [BR] [P] [EL2] with the Gibbs state of Thermodynamic Formalism [PP] [Bo]

[R3]. The purpose of this work is to explain for an audience which is more oriented to Dynamical System Theory part the content of the paper [EL3]. See also [Re1],[Re2] for related material.

R. Bowen, D. Ruelle and Y. Sinai are the founders of what is called in our days Thermodynamic Formalism Theory (see [PP] [R3]).

We will present initially the precise definitions we are going to consider.

We point out that we show here only the uniqueness part of the results in [EL3]. The existence is based on the paper [W] which is of Functional Analysis nature.

We refer the reader to [CL1] for a detailed analysis of the different meanings of the concept of Gibbs state from the point of view of Thermodynamic Formalism.

We consider here an expanding transformation $T : X \to X$ (to simplify ideas one can consider the particular case where T is the shift acting on the Bernoulli space $X = \{1, 2, ..., k\}^{\mathbf{N}}$). Consider μ the Gibbs state defined by a normalized Holder continuous potential $p : X \to \mathbf{R}$, and $\mathcal{L}^2(\mu)$ the associated Hilbert space. The function p is sometimes called the Jacobian of μ.

Consider the C^*-Algebra $\mathcal{U} = \mathcal{U}(\mu)$, which is a sub-$C^*$-Algebra of the C^*-Algebra of linear operators in $\mathcal{L}^2(\mu)$ which will be precisely defined later.

We call μ the reference measure.

Consider a fixed Holder potential $H > 0$ and the C^*-dynamical system defined by the associated σ_z. We are interested in describe for such system the KMS states ψ_β for all $\beta \in \mathbf{R}$.

We show a relation of a new Gibbs probability ν_β to a KMS state $\psi_{\nu_\beta} = \psi_\beta$, in the C^*-Algebra $\mathcal{U} = \mathcal{U}(\mu)$, for every value $\beta \in \mathbf{R}$, where β is the parameter that defines the time evolution associated to a homorphism $\sigma_t = \sigma_{\beta i}$ defined by the potential H. We show that for each real β the KMS state is unique in Theorem 2.2. We present the explicit expression of ψ_β.

The probability ν_β is the Gibbs state (eigenprobability) for the potential $-\beta \log H$ (which is not normalized).

Given a potential H, we say the potential $\tilde{H}$ is cohomologous to H, if there is V such that $\log \tilde{H} = \log H - V + V \circ T$.

After we present our main results for Holder potentials in section 1 and 2 in section 3 we consider a non-Holder potential H and we will make an analysis of phase transition nature (which do not occur at C^* Algebra level, in this case) associated to the KMS problem in a case where H can attain the value 1 (and where there is phase transition at Thermodynamic Formalism level).

Ground states in C^*-Algebras are also consider in the paper [EL3]. These corresponds to limits of the KMS state ψ_β when $\beta \to \infty$.

An important contribution in the relation of C^* algebras and Thermodynamic Formalism appears in chapters I.3 e II.5 in J. Renault Phd thesis [Re] (se also [R0])

We refer the reader to [L5] and [CL1] for a detailed description of phase transition in the sense of Thermodynamic Formalism.

Section 1 - KMS and Gibbs states

We denote $C(X)$ the space of continuous functions on X taking values on the complex numbers where (X, d) is a compact metric space.

Consider the Borel sigma-algebra $\mathcal{B}$ over X and a continuous transformation $T : X \to X$. Denote by $\mathcal{M}(T)$ the set of invariant probabilities for T. We assume that T is an expanding map.

We refer the reader to [Bo] [R1] [R2] [R3] [L4] for general definitions and properties of Thermodynamic Formalism and expanding maps.

Tipical examples of such transformations (for which that are a lot of nice results [R2]) are the shift in the Bernoully space and also $C^{1+\alpha}$-tranformations of the circle such that $|T'(x)| > c > 1$, where $|\ |$ is the usual norm (one can associate the circle to the interval $[0, 1)$ in a standard way) and c is a constan.

The geodesic flow in compact constant negative curvature surfaces induces in the boundary of Poincaré disk a Markov transformation G such that for some n, we have $G^n = T$, and where T is continuous expanding and acts on the circle (see [BS]). Our results can be applied for such T.

We denote by $\mathcal{H} = \mathcal{H}_\alpha$ the set of α-Holder functions taking complex values, where α is fixed $0 < \alpha \leq 1$.

For each $\nu \in \mathcal{M}(T)$, the real non-negative value $h(\nu)$ denotes the the Shanon-Kolmogorov entropy of ν and $h(T) = \sup\{h(\nu) | \nu \in \mathcal{M}(T)\}$. $h(T)$ is called the topological entropy of T.

Given a continuous function $A : X \to \mathbb{R}$ we denote the Ruelle operator by $\mathcal{L}_A$ (which acts on continuous function f). More precisely if $g = \mathcal{L}_A(f)$, then $g(x) = \mathcal{L}_A(f)(x) = \sum_{T(z)=x} e^A(z) f(z)$.

We say that the potential A is normalized if $\mathcal{L}_A(1) = 1$.

Given A, the dual operator $\mathcal{L}_A^*$ acts on probabilities on $\mathcal{M}(X)$.

We say that $\mathcal{L}_A^*(\nu) = \rho$ if for any continuous function f

$$\int f \, \mathcal{L}_A^*(\nu) \; = \; \int f \rho = \int \mathcal{L}_A(f) \, d\nu.$$

We denote by μ a fixed Gibbs state for a real Holder potential $\log p : X \to \mathbf{R}$. We suppose $\log p$ is already normalized [Bo][R3], in the sense that, if $\mathcal{L}_{\log p}$ (for short $\mathcal{L}_p$) denotes the Ruelle-Perron-Frobenius operator for $\log p$, that is for any $f : X \to \mathbf{C}$, and all $x \in X$, we have $(\mathcal{L}_p(f))(x) = \sum_{T(z)=x} p(z) f(z)$, then we assume that $\mathcal{L}_p(1)(x) = \sum_{T(z)=x} p(z) = 1$ and $\mathcal{L}_p^*(\mu) = \mu$.

We will show later that the index $\lambda(x) = p(x)^{-1}$ for the C^*-algebra associated to μ.

As an interesting example we mention the case where T has degree k, that is, for each $x \in X$ there exists exactly k different solutions z for $T(z) = x$. We call each such z a pre-image of x.

If T has degree k and in the particular case where μ is the maximal entropy measure (that is, $h(\mu) = h(T) = \log k$), then $p = 1/k$.

In order to simplify the arguments in our proofs we will assume from now on that T has degree k.

One can consider alternatively in Thermodynamic Formalism $\mathcal{L}_p$ acting on $C(X)$ or on $\mathcal{H}_\alpha$. Different spectral properties for $\mathcal{L}_p$ ocurr in each one of these two cases (see[Bo][R2]).

We will consider in the sequel a fixed real Holder-continuous positive potential $H : X \to \mathbf{R}$ and $\mathcal{L}_{H,\beta},\ \beta \in \mathbf{R}$ the Ruelle-Perron-Frobenius operator for $-\beta \log H$, that is, for each continuous f we have by definition

$$\mathcal{L}_{-\beta\ \log H}(f)(x) = \mathcal{L}_{H,\beta}(f)(x) = \sum_{T(z)=x} H(z)^{-\beta} f(z).$$

We denote by $\lambda_{H,\beta} \in \mathbf{R}$ the largest eigenvalue of $\mathcal{L}_{H,\beta}$. We also denote $\nu_{H,\beta}$ the unique probability such that $\mathcal{L}^*_{H,\beta}(\nu_{H,\beta}) = \lambda_{H,\beta}\nu_{H,\beta}$, and $h_{H,\beta}$ the unique function $h \in C(X)$ such that $\int h d\nu_{H,\beta} = 1$ and $\mathcal{L}_{H,\beta}(h) = \lambda_{H,\beta} h$.

As H is fixed for good in order to simplify the notation we will sometimes write $\mathcal{L}_\beta, \mathcal{L}^*_\beta, \lambda_\beta, \nu_\beta,\ h_\beta$.

h_β is a real positive Holder function.

The hypothesis about H and p being Holder in the Statistical Mechanics setting means that in the Bernoulli space the interactions between spins in neighborhoods positions decrease very fast [L2] [L3]. In section 2.3 we will consider a non-Holder potential H where in this case it will appear a phase-transition phenomena. This model is known as the Fisher-Felderhof model [FF], [L2], [L3], [FL]. In this case the interactions do not decrease so fast.

We return now to the Holder case.

It is well known the variational principle for such potential $-\beta \log H$,

$$P_H(\beta) = \log \lambda_{H,\beta} = \sup\{h(\nu) + \int (-\beta \log H) d\nu | \nu \in \mathcal{M}(T)\}.$$

The probability $\mu_{H,\beta} = h_{H,\beta}\nu_{H,\beta} \in \mathcal{M}(T)$ and satisfies

$$\sup\{h(\nu) + \int (-\beta \log H) d\nu | \nu \in \mathcal{M}(T)\} =$$

$$h(\mu_{H,\beta}) + \int (-\beta \log H) d\mu_{H,\beta}.$$

Definition 1.1: The probability $\mu_\beta = h_{H,\beta}\nu_{H,\beta}$ is called equilibrium state for the function $-\beta \log H$ where β and H are fixed.

Definition 1.2: The probability $\nu_{H,\beta}$ is called eigenmeasure or Gibbs state for the function $-\beta \log H$ where β and H are fixed. It satisfies

$$\mathcal{L}^*_{H,\beta}(\nu_{H,\beta}) = \lambda_{H,\beta}\nu_{H,\beta}.$$

The probability $\mu_{H,\beta}$ is unique for the variational problem and $\nu_{H,\beta}$ is unique for the the eigenmeasure problem associated to the value $\lambda_{H,\beta}$, if p and H are Holder. If we do not assume p and H Holder then there exist counterexamples for uniqueness in both cases [L2] [L3]. We will return to this point later.

For some reason the eigen-probabilities have a distinguished role here, but not the equilibrium states.

$P_H(\beta)$ is called the pressure of $-\beta \log H$ (or sometimes Free-Energy) and is a convex analytic function of β.

If T has degree k and in the particular case where μ is the maximal entropy measure (that is, $h(\mu) = h(T) = \log k$), then $p = 1/k$.

We consider the C^*-Algebra $L(\mathcal{L}^2(\mu))$ of bounded linear operators acting on $\mathcal{L}^2(\mu)$ with the strong norm. The operation $*$ on operators is the one induced from the inner product on $\mathcal{L}^2(\mu)$.

Definition 1.3: Denote by $S : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu)$ the Koopman operator where for $\eta \in \mathcal{L}^2(\mu)$ we define $(S\eta)(x) = \eta(T(x))$. Such S defines a linear bounded operator in $\mathcal{L}^2(\mu)$.

In Thermodynamic Formalism it is usual to consider the Koopman operator acting on $\mathcal{L}^2(\mu)$ (the space of complex square integrable functions over $\mathcal{L}^2(\mu)$), and it is well known that its adjoint (over $\mathcal{L}^2(\mu)$) is the operator $\mathcal{L}_p = S^*$ acting on $\mathcal{L}^2(\mu)$.

As we assume X is compact, any continuous function f is in $\mathcal{L}^2(\mu)$.

Definition 1.4: Another important class of linear operators is $M_f : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu)$, for a given fixed $f \in C(X)$, and defined by $M_f(\eta)(x) = f(x)\eta(x)$, for any η in $\mathcal{L}^2(\mu)$.

In order to simplify the notation, sometimes we denote by f the linear operator M_f.

Note that for M_f and M_g, $f, g \in C(X)$, the product operation satisfies $M_f \circ M_g = M_{f.g}$, where . means multiplication over the complex field $\mathbf{C}$.

Note that the $*$ operation applied on M_f, $f \in C(X)$, is given by $M_f^* = M_{\overline{f}}$, where $\overline{z}$ is the complex conjugated of $z \in \mathbf{C}$. In this sense, M_f^* is the adjoint operator of M_f over $\mathcal{L}^2(\mu)$.

The main point for our choice of μ as eigen-probability for $\mathcal{L}_p^*$, is that in $\mathcal{L}^2(\mu)$, the dual of the Koopman operator S is the operator $\mathcal{L}_p = S^*$ acting on $\mathcal{L}^2(\mu)$. Indeed, for any f, g we have

$$\int f\,(g \circ T)\, d\mu \;=\; \int f\,(g \circ T)\, d\mathcal{L}_p^*(\mu) = \int \mathcal{L}_p(\, f\,(g \circ T)\,)\, d\mu = \int \mathcal{L}_p(\, f\,)\, g\, d\mu.$$

It is important not confuse the dual of the Ruelle operator $\mathcal{L}_p$ in the Hilbert structure sense with the dual of $\mathcal{L}_p$ as a linear functional on continuous functions.

$L(\mathcal{L}^2(\mu))$, the set of linear operators over $\mathcal{L}^2(\mu)$, is a very important C^*-Algebra. We will analyze here a sub-C^*-Algebra of such C^*-Algebra (defined with the above operations . and $*$), more precisely the C^*-Algebra $\mathcal{U}$.

Definition 1.5: We denote by $\alpha : C(X) \to C(X)$ the linear operator such that for any f, we have $\alpha(f) = f \circ T$.

We have to show how the operators S and M_f acting on $\mathcal{L}^2(\mu)$ interact with the operators $\mathcal{L}_p$ and α acting on $C(X)$.

One can easily see that $\alpha(M_f) = M_{f \circ T}$. This is the first relation.

In the simplified notation (we identify M_f with f), one can read last expression as $\alpha(f) = f \circ T$.

In this way $\alpha^n(f) = f \circ T^n$.

If $\mathcal{B}$ is the Borel sigma-algebra then we denote by $\mathcal{F}_n$ the Sigma-algebra $T^{-n}\mathcal{B}$.

It is know that if we consider the probability μ, then the conditional expected value

$$E(f \,|\, \mathcal{F}_n) = E_\mu(f \,|\, \mathcal{F}_n) = \alpha^n(\mathcal{L}_p^n(f)).$$

More precisely

$$E(f \,|\, \mathcal{F}_n)(x) = \mathcal{L}^n(f)\,(\sigma^n(x)). \tag{1}$$

As $\mathcal{F}_m \subset \mathcal{F}_n$ for $m \geq n$, we have

$$E_\mu(\,(\,E_\mu\,(f|\,\mathcal{F}_m\,)\,)\,|\,\mathcal{F}_n\,) = E_\mu\,(f|\,\mathcal{F}_m\,),$$

and

$$E_\mu(\,(\,E_\mu\,(f|\,\mathcal{F}_n\,)\,)\,|\,\mathcal{F}_m\,) = E_\mu\,(f|\,\mathcal{F}_m\,).$$

Definition 1.6: Consider the C^*-Algebra contained in the set of bounded operators $L(\mathcal{L}^2(\mu))$ generated by the elements of the form $M_f S^n (S^*)^n M_g$, where $n \in \mathbf{N}$ and $f, g \in C(X)$. We denote such C^*-Algebra by $\mathcal{U} = \mathcal{U}(\mu, T)$. We call $\mathcal{U}$ the C^*-Algebra associated to μ.

Each element a in $\mathcal{U}$ is the limit of finite sums $\sum_i M_{f_i} S^{n_i} (S^*)^{n_i} M_{g_i}$.

$C(X)$ is contained in $\mathcal{U}$, via M_f, where f is any continuous function $f : X \to \mathbb{R}$.

Note that $f \to M_f$ defines a linear injective function of $C(X)$ on $\mathcal{U}$.

We denote $e_n = S^n\,(S^n)^* = E(f \,|\, \mathcal{F}_n) \in \mathcal{U}$.

Important Properties:

We have basic relations in such C^*-Algebra $\mathcal{U}$:

a) $(S^*)^n S^n = 1$, for all $n \in \mathbf{N}$ (it follows from $S^* S = 1$) .
proof: for any $\eta \in \mathcal{L}^2(\mu)$, we have

$$S^*\,S(\eta)(x) = \mathcal{L}_p(\eta(T(.))(x) = \sum_{T(y)=x} p(y)\,\eta(T(y)) = \sum_{T(y)=x} p(y)\,\eta(x) \;=\; \eta(x).$$

That is, $(S^*)^n S^n$ is the identity operator.

b) $(S^*)^n M_f S^n = M_{\mathcal{L}_p^n(f)}$, for all $n \in \mathbf{N}$, $f \in C(X)$ (it follows from $S^* M_f S = M_{\mathcal{L}_p(f)}$).
proof: for any $\eta \in \mathcal{L}^2(\mu)$, we have

$$S^*\,M_f\,S\,(\eta)\,(x) = \mathcal{L}_p(f\,\eta(T(.)))(x) = \mathcal{L}_p(f)\,(x)\,\eta(x).$$

c) $SM_f = \alpha(f) S$ for any continuous f, that is, for any $\eta \in \mathcal{L}^2(\mu)$, $SM_f(\eta) = f \circ T.\eta \circ T = \alpha(f).S(\eta)$.

d) $[e^n\,M_f](\eta) = [\,S^n\,(S^*)^n M_f\,](\eta) = E_\mu(\,(f\,\eta)\,|\,\mathcal{F}_n\,)$.

e) $e^n(\eta) = \; S^n\,(S^*)^n(\eta) = E_\mu(\;\eta\,|\,\mathcal{F}_n\,)$.

f) $M_f\, e^n(\eta) = [\, M_f\, S^n\, (S^*)^n](\eta) = f\;\; E_\mu(\,\eta\,|\mathcal{F}_n\,)$.
g) $M_f\, e^n\, M_g(\eta) =$

$$[\, M_f\, S^n\, (S^*)^n\, M_g](\eta) = f\;\; E_\mu(g\,\eta\,|\mathcal{F}_n\,) = f\,[\,\mathcal{L}_p^n(g\,\eta\,)\,(\sigma^n\,)\,]. \tag{2}$$

h) $S^n\,(S^*)^n\, M_g S^n\,(S^*)^n \;=\; E_\mu(g\;|\mathcal{F}_n\,)\,(\,S^n\,(S^*)^n\,)\,E_\mu(g\;|\mathcal{F}_n\,)\,e^n$ because

$$[S^n\,(S^*)^n\, M_g S^n\,(S^*)^n](\eta) = E_\mu(g\;E_\mu(\eta\;|\mathcal{F}_n\,)\,|\,\mathcal{F}_n\,) = E_\mu(g\;|\mathcal{F}_n\,)\;\,E_\mu(\eta\;|\mathcal{F}_n\,).$$

i) If $n \leq m$ we have

$$[M_f\, e^n\, M_g e^m\, M_h] = [M_f\, S^n\,(S^*)^n\, M_g S^m\,(S^*)^m\, M_h] =$$

$$[M_f\,(\,S^n\,(S^*)^n\, M_g S^m\,(S^*)^m\,)\,M_h] = M_f\, E_\mu(g\;|\mathcal{F}_n\,)\,(\,S^m\,(S^*)^m\,)\,M_h =$$

$$M_f\, E_\mu(g\;|\mathcal{F}_n\,)\,e^m\, M_h.$$

j) If $n \geq m$ we have

$$[M_f\, e^n\, M_g e^m\, M_h] = [M_f\, S^n\,(S^*)^n\, M_g S^m\,(S^*)^m\, M_h] = M_f\, e^n\, E_\mu(g\;|\mathcal{F}_m\,)\;\,M_h$$

proof: note first that taking adjoint with respect to the $\mathcal{L}^2(\mu)$ structure

$$(M_f\, S^n\,(S^*)^n\, M_g\,)^* = (M_g\, S^n\,(S^*)^n\, M_f\,).$$

Then,

$$[M_f\, S^n\,(S^*)^n\, M_g S^m\,(S^*)^m\, M_h]^* = M_h\, S^m\,(S^*)^m\, M_g S^n\,(S^*)^n\, M_f$$

and we can apply item i) to get

$$[M_f\, S^n\,(S^*)^n\, M_g S^m\,(S^*)^m\, M_h]^* = M_h\, E_\mu(g\;|\mathcal{F}_m\,)\,e^n\, M_f.$$

Now taking adjoint once more we get

$$[M_f\, S^n\,(S^*)^n\, M_g S^m\,(S^*)^m\, M_h] = M_f\, e^n\, E_\mu(g\;|\mathcal{F}_m\,)\;\,M_h.$$

Example 1:

$$[\, M_f\, e^3 M_g\, e^4 M_h\,](\eta)\,(x) =$$

$$[\, M_f\, S^3\,(S^*)^3 M_g\, S^4\,(S^*)^4 M_h\,](\eta)\,(x) =$$

$$M_f\, S^3\,(S^*)^3 M_g\,[\,E_\mu(\,(f\,\eta)\,|\,\mathcal{F}_4\,)(x)] =$$

$$M_f\, S^3\,(S^*)^3\,[\,g(x)\,E_\mu(\,(f\,\eta)\,|\,\mathcal{F}_4\,)(x)] =$$

$$f(x)\,E_\mu(\;\; g(x)\;\;\; E_\mu(\,(f\,\eta)\,|\,\mathcal{F}_4\,)(x)\;\;\,|\,\mathcal{F}_3\,)\,(x).$$

If u is $\mathcal{F}_m$ measurable and $m > n$, then u is $\mathcal{F}_n$ measurable.
Then,

$$[\, M_f\, S^3\,(S^*)^3 M_g\, S^4\,(S^*)^4 M_h\,](\eta)\,(x) =$$

$$f(x)\, E_\mu(\, (f\,\eta)\,|\,\mathcal{F}_4\,)(x)\;\; E_\mu(\, g(x)\;|\,\mathcal{F}_3\,)\,(x).$$

By the other hand

$$[\, M_f\, e^4 M_g\, e^3 M_h\,](\eta)\,(x) =$$
$$[\, M_f\, S^4\,(S^*)^4 M_g\, S^3\,(S^*)^3 M_h\,](\eta)\,(x) =$$
$$f(x)\, E_\mu(\;\; g(x)\;\; E_\mu(\,(f\,\eta)\,|\,\mathcal{F}_3\,)(x)\;\;|\,\mathcal{F}_4\,)\,(x).$$

Remark 0: If we consider the C^*-algebra generated $M_f\, S^m (S^*)^n\, M_g$, where $n, m \in \mathbf{N}$ and $f, g \in C(X)$, we have a different setting (which is usually called a Vershik C^*-algebra) which was consider in another paper by R. Exel [E3]. In this case, the KMS state exists only for one value of β.

We now return to our setting.

An extremely important result will be shown in expression (*1) and (*2) in Lemma 2.1 which claims that there exists functions u_i, $i \in \{1, 2, .., k\}$, such that

$$\sum_{i=1}^{k} M_{u_i} S S^* M_{u_i} = 1.$$

A bijective linear transformation $K : \mathcal{U} \to \mathcal{U}$ which preserves the composition and the $*$ operation is called an automorphism of $\mathcal{U}$

We denote by Aut($\mathcal{U}$) the set of automorphism of the C^*-Algebra $\mathcal{U}$.

Definition 1.7: Given a positive function H we define the group homomorphism σ_t, where for each $t \in \mathbf{R}$ we have $\sigma_t \in$ Aut($\mathcal{U}$) [BP] [P], is defined by:

a) for each fixed $t \in \mathbf{R}$ and any M_f, we have $\sigma_t(M_f) = M_f$,

b) for each fixed $t \in \mathbf{R}$, we have $\sigma_t(S) = M_{H^{it}} \circ S$, , in the sense that $(\sigma_t(S)(\eta))(x) = H^{it}(x)\eta(T(x)) \in \mathcal{L}^2(\mu)$, for any $\eta \in \mathcal{L}^2(\mu)$.

The value t above is related to temperature and not time, more precisely we are going to consider bellow $t = \beta i$ where β is related to the inverse of temperature in Thermodynamic Formalism (or Statistical Mechanics).

It can be shown that for each t fixed, we just have to define σ_t over the generators of $\mathcal{U}$ in order to define σ_t uniquely on $\mathcal{U}$. In this way a) and b) above define σ_t.

We will assume in this section from now on that H is Holder in order we can use the strong results of Thermodynamic Formalism.

Remark 1: Note that for $\eta \in \mathcal{L}^2(\mu)$, we have

$$(\sigma_t(S^2)\eta)(x) = \sigma_t(M_{H^{it}}(\eta \circ T))(x) =$$

$$M_{H^{it}} M_{H^{it}\circ T}(\eta \circ T^2)(x),$$

therefore $\sigma_t(S^2) = H^{ti}(H \circ T)^{ti} S^2$. It follows easily by induction that

$$\sigma_t(S^n) = \Pi_{j=0}^{n-1}(H \circ T^j)^{ti} S^n.$$

Taking dual in both sides of the above expression we get other important relation

$$\sigma_t((S^*)^n) = (S^*)^n \Pi_{j=0}^{n-1} (H \circ T^j)^{-ti}.$$

Finally,

$$\sigma_t(M_{f_2} S^m (S^*)^m M_{g_2}) = M_{f_2} H^{ti\,[m]} S^m (S^*)^m H^{-t\,i\,[m]}\, M_{g_2}, \qquad (*5)$$

where $H^{t\,i\,[m]}(x) = \Pi_{i=0}^{m-1} H(T^i(x))^{t\,i}$.

From f) above we get for $t = i$

$$\sigma_i(M_{f_2} S^m (S^*)^m M_{g_2})(\eta)(x) =$$

$$[\, M_{f_2} H^{ii\,[m]} S^m (S^*)^m H^{-\,i\,i[m]}\, M_{g_2}]\,(\eta)(x) =$$

$$[\, M_{f_2} H^{-\,[m]} S^m (S^*)^m H^{[m]}\, M_{g_2}\,]\,(\eta)(x) =$$

$$f_2(x)\, H^{\,-[m]}(x)\, E_\mu(H^{[m]}\, g_2\, \eta\, | \mathcal{F}_m\,)\,(x).$$

In terms of the formalism of C^*-dynamical systems, the positive function H defines the dynamics of the evolution with time $t \in \mathbf{R}$ of a C^*-dynamical system. Our purpose is to analyze such system for each pair (H, β).

Definition 1.8: An element a in a C^*-Algebra is positive, if it is of the form $a = b\, b^*$ with b in the C^*-Algebra.

Definition 1.9: By definition a "C^*-dynamical system state" is a linear functional $\psi : \mathcal{U} \to \mathbf{C}$ such that

a) $\psi(M_1) = 1$

b) $\psi(a)$ is a positive real number for each positive element a on the C^*-Algebra $\mathcal{U}$.

A "C^*-dynamical system state" ψ in C^*-dynamical systems plays the role of a probability ν in Thermodynamic Formalism. For a fixed H, we have a dynamic temporal evolution defined by σ_t where $t \in \mathbf{R}$.

Definition 1.10: An element $a \in \mathcal{U}$ is called analytic for σ if $\sigma_t(a)$ has an analytic extension from $t \in \mathbf{R}$ to all $t \in \mathbf{C}$.

Definition 1.11: For a fixed $\beta \in \mathbf{R}$ and H, by definition, ψ is a **KMS state associated to** H **and** β in the C^*-Algebra $\mathcal{U}(\mu, T)$, if ψ is a C^*-dynamical system state, such that for any $b \in \mathcal{U}$ and any analytic $a \in \mathcal{U}$ we have

$$\psi(a.b) = \psi(b.\sigma_{\beta i}(a)).$$

For H and β fixed, we denote a KMS state by $\psi_{H,\beta} = \psi_\beta$ and we leave ψ for a general C^*-dynamical system state.

It is easy to see that for H and β fixed, the condition

$$\psi(a.b) = \psi(b.\sigma_{\beta i}(a)),$$

is equivalent to $\forall \tau \in \mathbf{C}$,

$$\psi(\sigma_\tau(a).b) = \psi(b.\sigma_{\tau+\beta i}(a)).$$

It follows from section 8.12 in [P] that if ψ_β is a KMS state for H, β, then for any analytic $a \in \mathcal{U}$, we have that $\tau \to \psi_\beta(\sigma_\tau(a))$ is a bounded entire function and therefore constant. In this sense ψ is stationary for the continuous time evolution defined by the flow σ_t.

Note that the KMS state, in principle, could depend of the initially chossen μ because we are considering $\mathcal{L}^2(\mu)$ when defining $\mathcal{U}$, but in the end it will be defined by a measure that depends only in β and H

We point out that it can be shown that in order to characterize ψ as a KMS state we just have to check the condition $\psi(a.b) = \psi(b.\sigma_{\beta i}(a))$ for a, b the linear generators of $\mathcal{U}$, that is, a of the form $M_{f_1} S^n (S^*)^n M_{g_1}$ and b of the form $M_{f_2} S^m (S^*)^m M_{g_2}$.

A natural question is: for a given β and H, when the KMS state $\psi_{H,\beta}$ exist and when it is unique?

We are interested mainly in uniqueness and explicitly. We will explain this point more carefully later.

Remark 2: Note that when ψ is a KMS state, $\psi(f\,.a\,.g) = \psi(\sigma_{\beta i}(g)\, f\, a) = \psi(g.f\, a) = \psi(f.g\, a)$, for any $f, g \in C(X)$ and $a \in \mathcal{U}$.

Our purpose here is to show how to associate in a unique way each KMS state $\psi_{H,\beta} = \psi_\beta$ to the eigenmeasure $\nu_{H,\beta} = \nu_\beta$ defined before.

Remember that over $\mathcal{L}^2(\mu)$ the operator $\mathcal{L}_p = S^*$ is adjoint of the operator $f \to S(f) = f \circ T$.

We call $\lambda(x) = p(x)^{-1}$ the index and we denote by

$$\lambda^{[n]}(x) = (p(x)p(T(x))...p(T^{n-1}(x)))^{-1}.$$

We denote $H^{\beta\,[n]}(x) = \Pi_{i=0}^{n-1} H(T^i(x))^\beta$ and $\Lambda_n = H^{-\beta\,[n]}\,\lambda^{[n]}$.

From this follows that for any continuous function f we have $\mathcal{L}_\beta^n(f) = \mathcal{L}_p^n(\Lambda_n\, f)$.

Remember that for any continuous function k we have $\mathcal{L}_p^n(k \circ T^n)\,(x) = k(x)$ because $\mathcal{L}_p^n(1) = 1$.

Lemma 1.1 For any any β and continuous function f

$$\int f\, d\nu_\beta = \int (\Lambda_n)^{-1} E_\mu(\Lambda_n\, f \,|\, \mathcal{F}_n\,)\, d\nu_\beta. \qquad (3)$$

Proof:

Note that

$$\int \mathcal{L}_p^n(\Lambda_n\, f)\, d\nu_\beta = \int \mathcal{L}_\beta^n(\, f) d\nu_\beta = \lambda_\beta^n \int f d\nu_\beta.$$

Now taking $f = (\Lambda_n)^{-1}\alpha^n(g) = (\Lambda_n)^{-1}\,(g \circ T^n)$ we get from above

$$\int g\, d\nu_\beta = \int \mathcal{L}_p^n(\Lambda_n\,(\Lambda_n)^{-1}\,(g \circ T^n))\, d\nu_\beta = \lambda_\beta^n \int (\Lambda_n)^{-1}\,(g \circ T^n) d\nu_\beta.$$

Now taking $g = \mathcal{L}_p^n(\Lambda_n\, f)$ we get

$$\int (\Lambda_n)^{-1} E_\mu(\Lambda_n\, f \,|\, \mathcal{F}_n\,)\, d\nu_\beta = \int (\Lambda_n)^{-1} \alpha^n (\mathcal{L}_p^n(\Lambda_n\, f))\, d\nu_\beta =$$

$$\lambda_\beta^{-n} \int \mathcal{L}_p^n(\Lambda_n\, f)\, d\nu_\beta = \lambda_\beta^{-n} \int \mathcal{L}_\beta^n(\,f)\, d\nu_\beta = \int f\, d\nu_\beta.$$

□

Section 2 - The main result

We define $G : \mathcal{U} \to C(X)$ by $G(M_f e_n M_g) = f \lambda^{-[n]}\, g$ where $e_n = S^n (S^*)^n$.

Moreover, $G(M_f M_g) = f\, g$

Note that we define G in the elements of the form $M_f e_n M_g$, $n \geq 0$, and then we define G in $\mathcal{U}$ by linear combinations and limits.

Suppose $\phi = \phi_\nu : C(X) \to \mathbb{C}$ is of the form $\phi(f) = \int f d\nu$ where ν is a probability on X.

There is a canonical way to define a C^*-dynamical system state $\psi_\nu : \mathcal{U} \to \mathbb{C}$ by

$$\psi_\nu(M_f e_n M_g) = \phi_\nu(G(M_f e_n M_g)) \,=\, \int f \lambda^{-[n]}\, g\, d\nu.$$

In this way if $n \leq m$ (by item i))

$$\psi_\nu(\, M_f\, S^n\, (S^*)^n\, M_g S^m\, (S^*)^m\, M_h\,) =$$

$$\psi_\nu(M_f\, E_\mu(g\,|\mathcal{F}_n\,)\,(\, S^m\, (S^*)^m\,)\, M_h) =$$

$$\psi_\nu(M_f\, E_\mu(g\,|\mathcal{F}_n\,)\, e^m\, M_h) = \int\, E_\mu(g\,|\mathcal{F}_n\,)\, f\, h\, \lambda^{-[m]}\; d\nu.$$

In this way if $n \geq m$ (by item j))

$$\psi_\nu(\, M_f\, S^n\, (S^*)^n\, M_g S^m\, (S^*)^m\, M_h\,) =$$

$$\psi_\nu(M_f\, e^n\, E_\mu(g\,|\mathcal{F}_m\,)\; M_h\,) = \int\, E_\mu(g\,|\mathcal{F}_m\,)\, f\, h\, \lambda^{-[n]}\; d\nu.$$

Theorem 2.1: Given ϕ_ν and $\psi_\nu = \phi_\nu \circ G$ we get that ψ_ν is KMS for temperature β, if and only if, ϕ_ν satisfies

$$\phi_\nu(f\,) = \phi_\nu(\,(\Lambda_n)^{-1} E_\mu(\Lambda_n\, f \,|\, \mathcal{F}_n\,)\,),$$

which is the same that to say that ν satisfies

$$\int f\, d\nu = \int (\Lambda_n)^{-1} E_\mu(\Lambda_n\, f \,|\, \mathcal{F}_n\,)\, d\nu.$$

Proof:

In order to simplify the notation we call $E_n(f) = E_\mu(f \,|\, \mathcal{F}_n)$.

Suppose that ψ is a KMS state. Then for all $a, b, c, d \in C(X)$ and all n we have

$$\psi((ae_n b)\sigma_{i\beta}(ce_n d)) = \psi((ce_n d)(ae_n b)). \quad (*3)$$

The left hand side is equals to

$$\psi(a\, e_n\, b\; cH^{-\beta[n]}e_n H^{\beta[n]}d) = \psi(aS^n\, (S^*)^n\, b\; cH^{-\beta[n]}e_n H^{\beta[n]}d) =$$

$$\psi(aE_n(bcH^{-\beta[n]})\, e_n\, H^{\beta[n]}d) = \phi(aE_n(bcH^{-\beta[n]})\, \lambda^{-[n]}\, H^{\beta[n]}d).$$

The right hand side of (*3) is equals to

$$\psi(cE_n(da)e_n b) = \phi(cE_n(da)\lambda^{-[n]}b).$$

Now take $b = 1$, $c = H^{\beta[n]}$, and $d = H^{-\beta[n]}\lambda^{[n]}$ and from (*3) we get

$$\phi(a) = \phi(H^{\beta[n]}E_n(H^{-\beta[n]}\lambda^{[n]}a)\lambda^{-[n]}) = \phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}a)).$$

Now, we want to prove the other implication.

Note that $\phi_\nu(ab) = \phi_\nu(ba)$ for continuous functions a and b.

We would like to prove that

$$\psi((ae_n b)\sigma_{i\beta}(ce_m d)) = \psi((ce_m d)(ae_n b)), \quad (*4)$$

for all $a, b, c, d \in A$ and $n, m \in \mathbb{N}$.

Suppose first the case $n \leq m$.

By the important property i) we get that the left hand side of (*4) is equals to

$$\psi(ae_n bcH^{-\beta[m]}e_m H^{\beta[m]}d) = \psi(aE_n(bcH^{-\beta[m]})e_m H^{\beta[m]}d) =$$

$$\phi(aE_n(bcH^{-\beta[m]})\lambda^{-[m]}H^{\beta[m]}d) = \phi(E_n(bcH^{-\beta[m]})H^{\beta[m]}\lambda^{-[m]}da).$$

Observe that $H^{\beta[m]}(x) = H^{\beta[n]}(x)\, H^{\beta[m-n]}(T^n(x))$ so the above is equals to

$$\phi(E_n(bcH^{-\beta[n]}H^{-\beta[m-n]}(T^n\,))\, H^{\beta[m-n]}(T^n)\, H^{\beta[n]}\lambda^{-[m]}da) =$$

$$\phi(E_n(bcH^{-\beta[n]})H^{\beta[n]}\lambda^{-[m]}da) =$$

$$\phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}E_n(bcH^{-\beta[n]})H^{\beta[n]}\lambda^{-[m]}da)) =$$

$$\phi(\Lambda^{-[n]}E_n(bcH^{-\beta[n]})E_n(\lambda^{[n]}\lambda^{-[m]}da)) =$$

$$\phi(\Lambda^{-[n]}E_n(bcH^{\beta[n]})\, \lambda^{[n]}\lambda^{-[m]}E_n(da)).$$

where in the last equality we use the fact that $\lambda^{-[m]}\lambda^{[n]} = \lambda^{-[m-n]}(T^n)$.

By the other hand the right hand side of (*4) is equals to

$$\psi(ce_m E_n(da)b) = \phi(c\lambda^{-[m]}E_n(da)b) = \phi(bc\lambda^{-[m]}E_n(da)) =$$

$$\phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}bc\lambda^{-[m]}E_n(da))).$$
$$\phi(\Lambda^{-[n]}E_n(bc\lambda^{-[m]}\lambda^{[n]}H^{-\beta[n]})E_n(da)) =$$
$$\phi(\Lambda^{-[n]}E_n(bcH^{-\beta[n]})\,\lambda^{-[m]}\lambda^{[n]}\,E_n(da))$$

where in the last equality we use once more the fact that $\lambda^{-[m]}\lambda^{[n]} = \lambda^{-[m-n]}(T^n)$.

In this way we showed the KMS condition in the case $n \leq m$.

For the case $n \geq m$, using the important property j) we note that the left hand side of (*4) is

$$\psi(ae_nb \quad c\,H^{-\beta[m]}e_mH^{\beta[m]}\,d) =$$
$$\psi(ae_nE_m(bcH^{-\beta[m]})H^{\beta[m]}d) =$$
$$\phi(a\lambda^{-[n]}E_m(bcH^{-\beta[m]})H^{\beta[m]}d) =$$
$$\phi(\Lambda^{-[m]}E_m(\Lambda^{[m]}\lambda^{-[n]}E_m(bcH^{-\beta[m]})H^{\beta[m]}da)) =$$
$$\phi(\Lambda^{-[m]}E_m(bcH^{-\beta[m]})E_m(H^{\beta[m]}da\Lambda^{[m]}\lambda^{-[n]})) =$$
$$\phi(\Lambda^{-[m]}E_m(bcH^{-\beta[m]})E_m(da\lambda^{[m]}\lambda^{-[n]})).$$

The right hand side of (*4) equals

$$\psi(cE_m(da)e_nb) = \phi(cE_m(da)\lambda^{-[n]}b) =$$
$$\phi(\lambda^{-[n]}bcE_m(da)) = \phi(\Lambda^{-[m]}E_m(\Lambda^{[m]}\lambda^{-[n]}bcE_m(da))) =$$
$$\phi(\Lambda^{-[m]}E_m(bcH^{-\beta[m]}\lambda^{[m]}\lambda^{-[n]})E_m(da)).$$

The conclusion follows at once because $\lambda^{[m]}\lambda^{-[n]} \in \mathcal{F}_m$.

□

Corollary 2.1. Suppose ν_β is an eigenprobability for the Ruelle operator of the potential $-\beta\log H$. If the C^*-dynamical system state $\psi_{\nu_\beta} : \mathcal{U} \to \mathbb{C}$ is defined by

$$\psi_{\nu_\beta}(M_fe_nM_g) = \phi_{\nu_\beta}(G(M_fe_nM_g)) \;=\; \int f\lambda^{-[n]}\,g\,d\nu_\beta,$$

then, ψ_{ν_β} is a KMS state for temperature β.

Proof: This follows from last theorem and Lemma 1.1

□

Note that when H is constant then μ is an eigenprobability for the associated Ruelle operator for any $\beta > 0$. From expression (*5) we can see that σ_t in this case is the identity for any t. Moreover, by the KMS relation $\psi_\mu(a\,b) = \psi_\mu(b\,a)$.

We can ask about uniqueness of the KMS state. To address this question is the purpose of the next results.

Our main theorem says:

Theorem 2.2: If H is Holder positive and μ is a Gibbs state for p Holder, then for any given $\beta \in \mathbf{R}$, a KMS state ψ in $\mathcal{U}(\mu)$ exists, it is unique and of the form

$$\psi_\beta(b) = \int \frac{f\,g}{\lambda^{[n]}} d\nu_{H,\beta}, \;\; \forall b = M_f e^n M_g \in \mathcal{U},$$

where ν_β is the eigenmeasure for $\mathcal{L}^*_{-\beta \log H}$.

Proof of Theorem 2.2:

The existence of a KMS follows from the results from above. We fixed β.

Now we want to show precisely how one can associate a Gibbs measure to a KMS state. We denote such KMS state by ψ. We will denote ψ_β the KMS state obtained from ν_β.

Suppose ψ is a KMS state, where the H is fixed and defines the semigroup σ_t.

Given the KMS state ψ, then $\psi(M_f) = \psi(f)$ defines a continuous positive linear functional over $C(X)$ such that $\psi(M_1) = 1$. Therefore by Riesz Theorem, there exists a probability ν such that for any $f \in C(X)$ we have $\psi(f) = \int f d\nu = \int f S^0 (S^*)^0 d\nu$.

The above definition takes in account just $n = 0$ in a) above. Remains the question: what conditions are imposed on ν (defined from ψ as above) due to the fact that ψ is a KMS state for H, β?

This ν is our candidate to be the one associated to ψ via $\psi = \psi_\nu = \phi_\nu \circ G$ where hopefully ν satisfies

$$\int f\, d\nu = \int (\Lambda_n)^{-1} E_\mu(\Lambda_n\, f \,|\, \mathcal{F}_n\,)\, d\nu.$$

for all continuous f, and also

$$\psi_\nu(M_f e_n M_g) = \phi_\nu(G(M_f e_n M_g)) \; = \; \int f \lambda^{-[n]}\, g\, d\nu.$$

Now we will show a recurrence relation which do not assume any KMS state condition for ψ.

Lemma 2.1: Suppose that the C^*-state ψ is such that $\psi(f) = \int f\, d\,\nu$, for any $f \in C(X)$.

Then, for any $f \in C(X)$ and $n \in \mathbb{N}$

$$\psi(f\, e^n) = \psi(f\, S^n (S^*)^n) = \psi(S^n (S^*)^n\, f) = \int f \lambda^{-[n]} d\nu\,. \tag{4}$$

In other words, if $G(f\, S^n (S^*)^n) = G(M_f e^n\,) \, = \, f \lambda^{-[n]}$ for any continuous function f, and $\phi_\nu(f) = \int f d\nu = \psi(f)$, then

$$\psi(f\, e^n) = \phi_\nu(G(f\, e^n)).$$

Proof:

The first claim of the lemma will follow from

$$\psi(f S^n (S^*)^n) = \psi(f\,(\lambda\circ T^n)\, S^{n+1} (S^*)^{n+1}).$$

Indeed, for instance we get $\psi(f) = \psi(f\,\lambda\, S(S^*))$, and, $\psi(f S\,(S^*)) = \psi(f\,(\lambda\circ T)\, S^2 (S^*)^2)$, and so on.

We need a preliminary estimate before proving the lemma.

For the transformation T, consider a partition $A_1, ..., A_k$ of X such that T is injective in each A_i. Our proof bellow is for the shift in the Bernouilli space. In the case of the Bernoulli space with k symbols A_i is the cylinder $\bar{i}$ with first coordinate i. Now we consider a partition of unity given by k non-negative functions $v_1, ..., v_k$ such that each $v_i(x) = I_{\bar{i}}(x)$ (the indicator function of the cilinder $\bar{i}$) which has support on A_i and $\sum_{i=1}^k v_i(x) = 1$ for all $x \in X$. In the case X is the unitary circle and T is expansive, using a conjugacy with the shift, we obtain similar results.

Denote now the functions u_i given by: if x is in the cylinder $\bar{i}$

$$u_i(x) = (\lambda(x))^{1/2} = (p(x)^{-1})^{1/2}, \qquad (*1)$$

for each $i \in \{1, ..., k\}$, so, $\sum_{i=1}^k u_i^2(x) = \lambda(x) = p^{-1}(x)$, for any $x \in X$.

An easy computation shows that $\sum_{i=1}^k M_{u_i} S S^* M_{u_i} = 1$. More precisely, if x is in the cylinder $\bar{i}$, then given η, we have

$$[\, M_{u_i} S S^* M_{u_i}(\eta)\,](x) = u_i(x) \sum_{\{z \,|\, \sigma(z)=\sigma(x)\}} p(z)\, u_i(z)\, \eta(z) \;=\; \eta(x). \qquad (*2)$$

Indeed, for x in the cylinder $\bar{i}$, take $u_i(x) = p^{-1/2}(x)$. This is so because for $x = (j, x_2, x_3, ...)$ we get

$$\sum_{i=1}^k M_{u_i} S S^* M_{u_i}(\xi)(x) = M_{u_j} S S^* M_{u_j}(x) = p^{-1/2}(x)[\mathcal{L}_p(\xi\, u_j)](\sigma(x)) =$$

$$p^{-1/2}(x) \sum_{i=1}^d p(i, x_2, x_3, ..) u_j(i, x_2, x_3, ..) \xi(i, x_2, x_3, ..) =$$

$$p^{-1/2}(x) p(j, x_2, x_3, ..) u_j(j, x_2, x_3, ..) \xi(j, x_2, x_3, ..) =$$

$$p^{-1/2}(x) p(x) p^{-1/2}(x) \xi(x) = \xi(x).$$

Now, we continue the argument:

$$S^n (S^*)^n = S^n 1 (S^*)^n = S^n [\sum_{i=1}^k (M_{u_i} S S^* M_{u_i})\,] (S^*)^n =$$

$$\sum_{i=1}^k (S^n\,(M_{u_i} S S^* M_{u_i})\,(S^*)^n).$$

Now we use the relations $S^n M_g = M_{\alpha^n(g)} S^n$ and $M_g (S^*)^n = (S^*)^n M_{\alpha^n(g)}$ in last expression and we get

$$S^n (S^*)^n = \sum_{i=1}^{k} M_{\alpha^n(u_i)} S^n \, S S^* \, (S^*)^n M_{\alpha^n(u_i)} = \sum_{i=1}^{k} M_{\alpha^n(u_i)} S^{n+1} (S^*)^{n+1} M_{\alpha^n(u_i)}$$

Now we will prove the lemma.

Using last expression and then Remark 2 for $g = \alpha^n(u_i) \in C(X)$ and $a = S^{n+1}(S^*)^{n+1}$ we get

$$\psi(M_f S^n (S^*)^n) = \psi(M_f \sum_{i=1}^{k} M_{\alpha^n(u_i)} S^{n+1} (S^*)^{n+1} M_{\alpha^n(u_i)}) =$$

$$\psi(M_f \sum_{i=1}^{k} M_{\alpha^n(u_i)} M_{\alpha^n(u_i)} S^{n+1} (S^*)^{n+1}) =$$

$$\psi(M_f \sum_{i=1}^{k} M_{\alpha^n(u_i)^2} S^{n+1} (S^*)^{n+1}) =$$

$$\psi(M_f \, M_{\alpha^n(\sum_{i=1}^{k} (u_i)^2)} S^{n+1} (S^*)^{n+1}). =$$

$$\psi(M_f \, (\lambda \circ T^n) S^{n+1} (S^*)^{n+1})$$

This shows the claim of the lemma.

□

We denote $E_m(f) = E_\mu(f \,|\, \mathcal{F}_m)$.

Corollary 2.1 If ψ is Gibbs for H at temperature zero, and ν is such that for any continuous function f we have $\psi(f) = \int f d\nu$, then

$$\phi_\nu(f\,) = \phi_\nu(\, (\Lambda_n)^{-1} E_\mu(\Lambda_n \, f \,|\, \mathcal{F}_n \,) \,),$$

which is the same that to say that ν satisfies

$$\int f \, d\nu = \int (\Lambda_n)^{-1} E_\mu(\Lambda_n \, f \,|\, \mathcal{F}_n \,) \, d\nu. \quad (*6)$$

Proof: We get from last lemma that $\psi(f\, e^n) = \phi_\nu(G(f\, e^n))$ where $\phi_\nu(f) = \int f d\nu = \psi(f)$. Now, from Theorem 2.1 we get that (*6) is true.

□

Now we will show the uniqueness of the KMS state:

Theorem 2.3: Given any KMS ψ, then $\psi = \psi_\beta$ where ψ_β is the KMS state associated to the Gibbs probability ν_β.

Proof:

In order to do that we will show that any possible ν as defined above from the KMS ψ is equal to ν_β.

Take ν a probability associated to ψ, then for each n, and $f \in C(X)$ we have

$$\int f d\nu = \int E_n(f\Lambda_n^{-1})\Lambda_n d\nu = \int \alpha^n(\mathcal{L}_\beta^n(f))\Lambda_n d\nu. \qquad (*7)$$

We claim that

$$\lim_{n\to\infty} \int E_n(f\Lambda_n^{-1})\Lambda_n d\nu = \int f d\nu_\beta,$$

and this shows that $\nu = \nu_\beta$, and therefore $\psi = \psi_\beta$.

Now we show the claim. Note that

$$\int f d\nu = \int E_n(f\Lambda_n^{-1})\Lambda_n d\nu = \int \alpha^n(\mathcal{L}_\beta^n(f))\Lambda_n d\nu = \int \alpha^n(\frac{\mathcal{L}_\beta^n(f)}{\lambda_\beta^n})\Lambda_n \lambda_\beta^n d\nu,$$

where λ_β is the eigenvalue associated to $\mathcal{L}_\beta$.

Applying the above expression to $f = h_\beta$ (we can assume h_β is such that $\int h_\beta d\nu_\beta = 1$) and using the fact that $\mathcal{L}_\beta^n(h_\beta) = \lambda_\beta^n h_\beta$ we get

$$0 < d = \int h_\beta d\nu = \int \alpha^n(h_\beta)\Lambda_n \lambda_\beta^n d\nu.$$

As h_β is continuous and positive, there exists $c > 0$ such for all $x \in X$ we have $h_\beta(x) > c$.

From this follow that

$$d = \int \alpha^n(h_\beta)\Lambda_n \lambda_\beta^n d\nu > c \int \lambda_\beta^n \Lambda_n d\nu.$$

Therefore,

$$\int \lambda_\beta^n \Lambda_n d\nu < d/c$$

Denote $I = \int f d\nu_\beta$.

It is known (see [Bo]) that uniformly in $z \in X$, we have

$$\lim_{n\to\infty} \frac{\mathcal{L}_\beta^n(f)(z)}{\lambda_\beta^n} = h_\beta(z) I = h_\beta(z) \int f d\nu_\beta.$$

Therefore, given $\epsilon > 0$, we can find $N > 0$ such that for all $n > N$ we have for all $z \in X$

$$|\frac{\mathcal{L}_\beta^n(f)(z)}{\lambda_\beta^n} - I h_\beta(z)| \leq \epsilon.$$

Then, for $n > N$

$$|\int \frac{\alpha^n(\mathcal{L}_\beta^n(f))}{\lambda_\beta^n}\Lambda_n \Lambda_\beta^n d\nu - \int I\alpha^n(h_\beta)\Lambda_n \Lambda_\beta^n d\nu \,| \leq$$

$$\int |\frac{\alpha^n(\mathcal{L}_\beta^n(f))}{\lambda_\beta^n}(y) - I\alpha^n(h_\beta)(y)|\Lambda_n(y)\Lambda_\beta^n(y)d\nu =$$

$$\int |\frac{\mathcal{L}_\beta^n(f)}{\lambda_\beta^n}(T^n(y)) - Ih_\beta(T^n(y))|\Lambda_n(y)\lambda_\beta^n(y)d\nu \leq \frac{\epsilon d}{c}$$

The conclusion from (*7) is that for any $f \in C(X)$

$$\lim_{n\to\infty} I \int \alpha^n(h_\beta)\Lambda_n\lambda_\beta^n d\nu = \int f d\nu.$$

Consider now $f = 1$ and we get

$$\lim_{n\to\infty} \int \alpha^n(h_\beta)\Lambda_n\lambda_\beta^n d\nu = 1.$$

From this we conclude that $\int f d\nu = I = \int f d\nu_\beta$ for all $f \in C(X)$.

This shows the uniqueness and that $\nu = \nu_\beta$.

□

The final conclusion is that any KMS ψ for H, β is equal to the ψ_β associated to ν_β.

Section 3 - no phase transitions

We consider here an interesting example of a KMS state associated with the reference measure μ given by the maximal entropy measure for the shift in 2 symbols $\{0, 1\}$. In this case $p = 1/2$ is contant. We will define a special potential H and we will consider specifically the special value $\beta = 1$

We refer the reader to [H] [L2] [L3] [FL] [L5] [CL1] [Y] [L] for references and results about the topics discussed in this section.

We are going to introduce the Fisher-Fedenhorf model of Statistical Mechanics in the therminology of Bernoulii spaces and Thermodynamic Formalism [H].

We define Σ^+ to be the shift space $\Sigma^+ = \Pi_0^\infty\{0, 1\}$ and denote by $T : \Sigma^+ \to \Sigma^+$ the left shift map. We write $z = (z_0z_1 \ldots)$ for a point in Σ^+ and $[w_0w_1 \ldots w_k] = \{z : \; z_0 = w_0, z_1 = w_1, \ldots z_k = w_k\}$ for a cylinder set of Σ^+.

We denote by $M_k \subset \Sigma^+$, for $k > 1$, the cylinder set $[\underbrace{111 \ldots 11}_{k}0]$ and by M_0 the cylinder set $[0]$. The ordered collection $(M_k)_{k=0}^\infty$ is a partition of Σ^+; in other words these sets are disjoint and their union is the whole space (minus the point $(11 \ldots)$). Note that T maps M_k bijectively onto M_{k-1} for $k \geq 1$, and onto Σ^+ for $k = 0$.

The point (1111...) is fixed for T.

For $\gamma > 1$ a fixed real constant, we consider the potential $g(x)$ such that $g(111111 \ldots.) = 0$,

$$g(x) = a_k = -\gamma \log\left(\frac{k+1}{k}\right),$$

for $x \in M_k$, for $k \neq 0$, and

$$a_0 = -\log(\zeta(\gamma)),$$

for $x \in M_0$, where ζ is the Riemann zeta function.

By definition,

$$\zeta(\gamma) = (1^{-\gamma} + 2^{-\gamma} + \dots)$$

and so the reason for defining a_0 in such way is that, if we define $s_k = a_0 + a_1 + \dots + a_k$, then $\Sigma e^{s_k} = 1$.

From now on we assume $\gamma > 2$, otherwise we have to consider sigma-finite measures and not probabilities in our problem.

The potential $1 < (\frac{k+1}{k})^\gamma = H(x) = e^{-g(x)}$, for $x \in M_k$, is not Hölder and in fact is not of summable variation. Note that $H(1111...) = 1$, The pressure $P(-\log H) = P(g) = P(\log p + \log 2 - 1 \log H) = 0$ and one can show that there exist two equilibrium states for such a potential g (in the sense of minimizing measures for the variational problem): a point mass (the Dirac delta $\delta(111...)$) at $(1111\dots)$, and a second measure which we shall denote by $\tilde{\mu}$ (see [H])

The existence of two probabilities $\tilde{\mu}$ and $\delta_{(1111...)}$ for the variational problem of pressure defines what is called a phase transition in the sense of Statistical Mechanics [H] [L3].

We will describe bellow how to define this measure $\tilde{\mu}$.

Consider as in [H] $\mathcal{L}_g^*$, the dual of the Ruelle-Perron-Frobenius operator $\mathcal{L}_g$ associated to g, where the action of $\mathcal{L}_g$ on continuous functions is given by

$$\mathcal{L}_{\beta=1}(\psi)(y) = \sum_{T(x)=y} e^{g(x)} \psi(x).$$

The function $P(-\beta \log H) = P(\beta g)$ is strictly monotone for $\beta < 1$ and constant equal zero for $\beta > 1$ [H].

We claim that there is a unique probability measure ν on Σ^+ which satisfies $\mathcal{L}_g^* \nu = \nu$ [FL] [H]. To prove this, note first that ν cannot have any mass at $(11\dots)$; it follows that M_0 has positive mass, and the stipulation that ν be an eigenmeasure then gives a recurrence relation for the masses of M_k. Since $T(M_k) = M_{k-1}$ for $k \geq 1$, we have that the masses of the sets in this partition are

$$\nu(k) = \nu(M_k) = e^{s_k} = \frac{(k+1)^{-\gamma}}{\zeta(\gamma)},\ k \geq 0;$$

in particular,

$$\nu(0) = \nu(M_0) = e^{s_0} = e^{a_0} = \frac{1}{\zeta(\gamma)}.$$

By the same reasoning, ν is determined on all higher cylinder sets for the partition $(M_k)_{k=0}^\infty$. Hence ν exists and is unique.

The measure ν defined above is the unique eigenmeasure for $\mathcal{L}_{\beta=1}^*$ and denoted by ν_1.

The measure defined by the delta-Dirac on (111...) is invariant but is not a fixed eigenmeasure for $\mathcal{L}_g^*$.

This measure ν_1 defines a KMS state ψ_{ν_1} for such H, $\beta = 1$ and $\mathcal{U}(\mu)$.

We conjecture that there is another KMS state ψ different from ψ_{ν_1} but not associated to a measure. Note that such H assumes the value 1 in just one point.

We define $\tilde{h}(x)$ for $x \in M_t$ by

$$\tilde{h}_t = \tilde{h}(x) = \nu(t)^{-1} \sum_{i=t}^{\infty} \nu(i).$$

The function $\tilde{h}$ satisfies $\mathcal{L}_g(\tilde{h}) = \tilde{h}$.

The integral $\int \tilde{h}(x) d\nu_1(x)$ is finite if and only if $\gamma > 2$. One can normalize $\tilde{h}$, multiplying by a constant u to get $h = u\tilde{h}$ with $\int h d\nu_1 = 1$.

This constant is

$$u = \frac{1}{\sum_{t=1}^{\infty} t\nu(t-1)} = \frac{\zeta(\gamma)}{\sum_{t=1}^{\infty} t^{1-\gamma}} = \frac{\zeta(\gamma)}{\zeta(\gamma-1)}.$$

The probability $\tilde{\mu}$ has positive entropy and its support is all Σ^+ (see [H] or [L3] [FL]).

Consider now the invariant probability measure $\tilde{\mu} = h\nu_1$. It is known that $\tilde{\mu}$ is an equilibrium state for $-\log H$ in the variational sense ($\beta = 1$) [H]. It is easy to see (because $-\log H(11111..) = -\log 1 = 0$) that the Dirac-delta measure $\delta_{(11111...)}$ is also an equilibrium state for $-\log H$ in the variational sense ($\beta = 1$).

The probability $\tilde{\mu}$ has positive entropy and its support is all Σ^+ (see [H] or [L3] [FL]).

We can conclude from the above considerations that not always an equilibrium probability ρ for the pressure is associated to a KMS state ψ_ρ whitout the hypothesis of H and p been Holder. In the present example, this happen because $\rho = \delta_{(1111...)}$ is not an eigenmeasure of the dual of the Ruelle-Perron-Frobenius operator $\mathcal{L}_\beta$ but it is an equlibrium measure for $\beta = 1$.

In [L2] and [L3] the lack of differentiability of the Free energy is analyzed and in [L3] [Fl] [Y] it is shown that such systems present polynomial decay of correlation. In [L1] it is presented a dynamical model with three equilibrium states.

Bibliography

[Bi] P. Billingsley *Probability and Measure*, Wiley, New York, (1968)

[Bo] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lecture Notes in Mathematics* 470, Springer Verlag, (1975)

[BS] R. Bowen and C. Series, Markov maps associated with a Fuchsian group, *IHES Publ Math*, N 50, pp 153-170 (1979)

[BR] O. Bratelli and W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer Verlag, (1994)

[GL] G. Castro and A. Lopes, *KMS States, Entropy and a Variational Principle for Pressure*, Real Analysis Exchange, Vol 34 , Issue 2, 333-346 (2009)

[CL1] L. Cioletti and A. O. Lopes, Interactions, Specifications, DLR probabilities and the Ruelle Operator in the One-Dimensional Lattice, Discrete and Cont. Dyn. Syst. - Series A, Vol 37, Number 12, 6139-6152 (2017)

[L5] L. Cioletti and A. Lopes, Phase Transitions in One-dimensional Translation Invariant Systems: a Ruelle Operator Approach, Journ. of Statistical Physics, 159 - Issue 6, 1421-1455 (2015)

[E1] R. Exel, Crossed-Products by finite Index Endomorphisms and KMS states, *J. Funct. Anal.* 199, no. 1, pp 153-188 (2003).

[E2] R. Exel, A new look at the Crossed-Products of a C^*-Algebra by endomorphism, *Erg. Theo. Dyn. Syst.* 23, no. 6, pp 1733–1750 (2003)

[E3] R. Exel, KMS states for generalized Gauge actions on Cuntz-Krieger Algebras, *Bull. Braz. Math. Soc.* 35, n. 1, pp 1-12, (2004)

[EL1] R. Exel and M. Lacca, Cuntz-Krieger Algebras for infinite matrices, *J. reine angew. Math*, 512, pp 119-172, (1999)

[EL2] R. Exel and M. Lacca, Partial Dynamical Systems and the KMS condition, *Comm. Math. Phys.* 232, no. 2, pp 223-277 (2003).

[EL3] R. Exel and A. Lopes, C^* Algebras, approximately proper equivalence relations and Thermodynamic Formalism, *Erg. Theo. and Dyn. Syst.*, Vol 24, pp 1051-1082 (2004)

[E0] R. Exel, Inverse semigroups and combinatorial C^*-Algebras, Arxiv (2008)

[FL] A. Fisher and A. Lopes, Exact bounds for the polynomial decay of correlation, 1/f noise and the central limit theorem for a non-Holder Potential, *Nonlinearity*, 14, pp 1071-1104 (2001).

[FF] B. Felderhof and M. Fisher, four articles, *Ann. Phys. 58*, pp 176-281 (1970)

[H] F. Hofbauer, Examples for the non-uniqueness of the equilibrium states, *Trans. AMS*, 228, pp 133-149 (1977)

[J] V. Jones, Index for subfactors, *Invent Math*, 72, pp 1-25, (1983)

[K] H. Kosaki, Extensions of Jones' Theory on index to arbitrary subfactors, *J. Funct Anal*, 66, pp 123-140, (1986)

[L] M. Lacca, Semigroups of *-endomorphisms, Dirichlet Series and phase transitions, *J. Funct Anal*, 152, pp 330-378, (1998)

[L1] A. Lopes, Dynamics of Real Polynomials on the Plane and Triple Point Phase Transition, *Mathematical and Computer Modelling*, Vol. 13, Nº 9, pp. 17-32, 1990.

[L2] A. Lopes, A First-Order Level-2 Phase Transition in Thermodynamic Formalism, *Journal of Statistical Physics*, Vol. 60, Nº 3/4, pp. 395-411, 1990.

[L3] A. Lopes, The Zeta Function, Non-Differentiability of Pressure and The Critical Exponent of Transition", *Advances in Mathematics*, Vol. 101, pp. 133-167, 1993.

[L4] A. Lopes, Entropy, Pressure and Large Deviation, *Cellular Automata, Dynamical Systems and Neural Networks*, E. Goles e S. Martinez (eds.), Kluwer, Massachusets, pp. 79-146, 1994.

[OP] D. Olesen and G. K. Pedersen, Some C^*-dynamical systems with a single KMS state, *Math Scand*, 42, pp 111- 118, (1978)

[PP] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Asterisque*, Vol 187-188, pp 1-268 (1990)

[P] G. K. Pedersen, *C^*-Algebras and and their automorphism groups*, Acad Press, (1979)

[Re1] J. Renault, AF-equivalence relations and their cocycles, Operator algebras and mathematical physics (Constança, 2001), Theta, Bucharest, pp 365-377, (2003).

[Re2] J. Renault, KMS states on C^*-Algebras associated to expansive maps, *Proc. Amer. Math. Soc.* 134, no. 7, pp 2067-2078, (2006)

[Re] J. Renault, The grupoid approach to C^*-Algebras, Lecture notes in Math 793, Springer Verlag (1980)

[R1] D. Ruelle, Statistical Mechanics of a one-dimensional lattice gas, *Comm in Math Phys*, 9, pp 267-278, (1968)

[R2] D. Ruelle, The thermodynamic Formalism for Expanding maps, *Comm in Mat Phys*, 125, pp 239-262, (1989)

[R3] D. Ruelle, *Thermodynamic Formalism*, Addison Wesley (1978).

[W] Y. Watatani, Index for C^*-subalgebras, *Mem. Am. Math. Soc.* 424, pp 1-117 (1990).

[Y] L.-S. Young, Recurrence times and rates of mixing, *Israel Jour of Math*, 110, pp 153-188, (1997)

[OP] D. Olesen and G. K. Pedersen, Some C^*-dynamical systems with a single KMS state, *Math Scand*, 42, pp 111- 118, (1978)

[PP] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Asterisque*, Vol 187-188, pp 1-268 (1990)

[P] G. K. Pedersen, *C^*-Algebras and and their automorphism groups*, Acad Press, (1979)

[Re1] J. Renault, AF-equivalence relations and their cocycles, Operator algebras and mathematical physics (Constança, 2001), Theta, Bucharest, pp 365-377, (2003).

[Re2] J. Renault, KMS states on C^*-Algebras associated to expansive maps, *Proc. Amer. Math. Soc.* 134, no. 7, pp 2067-2078, (2006)

[Re] J. Renault, The grupoid approach to C^*-Algebras, Lecture notes in Math 793, Springer Verlag (1980)

[R1] D. Ruelle, Statistical Mechanics of a one-dimensional lattice gas, *Comm in Math Phys*, 9, pp 267-278, (1968)

[R2] D. Ruelle, The thermodynamic Formalism for Expanding maps, *Comm in Mat Phys*, 125, pp 239-262, (1989)

[R3] D. Ruelle, *Thermodynamic Formalism*, Addison Wesley (1978).

[W] Y. Watatani, Index for C^*-subalgebras, *Mem. Am. Math. Soc.* 424, pp 1-117 (1990).

[Y] L.-S. Young, Recurrence times and rates of mixing, *Israel Jour of Math*, 110, pp 153-188, (1997)