On generic G-prevalent properties and a quantitative K-S theorem for C^r diffeomorphisms of the circle

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Abstract

We will consider a convex subset of a metric linear space and a certain group of actions G on this set, what allow us to define the notion of Haar zero measure on sets that has zero Haar measure for the translation (by adding) invariant HSY [9] prevalence theory. In this way we will be able to define the meaning of G-prevalent set according to the pioneering work of Christensen [2].

Our setting considers problems which take into account the convex structure and this is quite different from the previous results on prevalence which consider basically the linear additive structure.

In this setting we will show a kind of quantitative Kupka-Smale Theorem and also we generalize a result about rotation numbers which was first consider by J.-C. Yoccoz (and, also by M. Tsujii).

Among other thinks we present an estimation of the amount of hyperbolicity in a setting that we believe was not considered before.

1 Introduction

Generic properties in the sense of Baire consider only the metric structure (for instance, diffeomorphisms of the circle with the C^r-topology). The prevalent point of view aims to introduce some form of measure in the set of possible dynamics. In a vague way the "generic prevalent" element would be one which is outside a set of measure zero. There is not a unique way one can formalize these ideas.

In 1992 B. Hunt, T. Sauer and J. Yorke [9] introduced the idea of generic prevalent sets for a space of functions V (a complete metric vector space).

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This concept produces a meaning for saying that a certain property is true for a large set of functions (its complementary being small) in the measure theoretical sense. In the next years this idea was expanded in several different directions and settings [11], [12], [13], [14], [17], [18], [15] and [20]. The main goal was to introduce the idea of **shy** sets (see definition in the next section) , that are the equivalent of Christensen [2], Haar zero measure sets. These will play the role of the small sets. A set will be called **prevalent** if its the complement of a **shy** set. A set (of functions, vectors, etc, that has a certain property) will be said **generic**, if it contains some **prevalent** subset.

The results which are in the setting of [9] are called here the "Standard Prevalence Theory" (S. P. T. for short).

We will consider a convex boundaryless set and certain group of action of a group G on this set. This will substitute the translation (by adding) structure usually considered in the classical setting of prevalence (see Remark 1). In this way we will be able to define a new meaning for G-prevalent set.

We will use the underlying convex structure and this is quite different from the previous results on "Standard Prevalence Theory" which consider basically the linear additive structure. For instance, the probabilities there are over linear subspaces but not here. Note that for the study of prevalent diffeomorphisms on S^1 our approach seems more natural.

We need to elaborate a little further before explaining what we precisely mean. A measure μ over the Borel sigma algebra which is supported in a compact set is said to be transverse to a Borel set S, if $0 < \mu(U) < \infty$, for some compact set U, and $\mu(S + x) = 0$ for all $x \in V$. In [17], 2005, a prevalence generic theory is characterized by 5 fundamental axioms:

Axiom 1- A generic subset of V is dense;
Axiom 2- If $P \subset Q$ and P is generic then Q is generic;
Axiom 3- A countable intersection of generic subset of V is generic;
Axiom 4- Every translate of a generic subset is generic;
Axiom 5- A subset $P \subset \mathbb{R}^n$ is generic, if and only if, P has full Lebesgue measure.

For prevalence in nonlinear spaces M. Tsujii [20] has considered translation quasi-invariant measures from C^r sections of vector bundles in a compact m-dimensional smooth manifold, getting prevalent generic properties like Thom's transversality theorem. A particular setting is considered in that nice paper, but it's not clear how to extend the construction to other families of maps and other families of measures. V. Yu. Kaloshin in [11], [12] considered a hybrid idea of prevalence in manifolds, using topologically generic one-parameter families that has full probability on their parameter as being his approach to prevalence. This is quite interesting but it's a little bit different from the original idea of the S. P. T., since in [9] the prevalence should be related to a translation invariant measure. On the other hand, V. Yu. Kaloshin and B. Hunt, in [13], [14] proved prevalent bounds for the rate of growth of periodic orbits of diffeomorphisms in a smooth compact manifold M, by embedding M in $\mathbb{R}^N$, and making perturbations of diffeomorphisms in $C^r(U, \mathbb{R}^N)$, where U is tubular neighborhood of M in $\mathbb{R}^N$.

One of the main problems when considering prevalence on the space of diffeomorphisms is that several interesting subsets are shy, so we need an intrinsic theory of prevalence for a family of diffeomorphisms that has some additional convex structure, as, for instance, the liftings of C^r diffeomorphisms of $\mathbf{S}^1$. In this work we will introduce a different approach, replacing the group of translations (as HSY in [9] used in the original work) by a convenient group of reflections that acts transitively on these family. We notice that the prevalence on our setting is not equivalent to the translation invariant one, for instance, there exist a prevalent set in our setting such that has zero Haar measure in the S. P. T, see Remark 12. The probes (see definition later) in each theory are completely different.

We point out that in our setting (prevalence on nonlinear spaces) it is possible to generalize the result by M. Tsuji [20] in a quite natural way (see section 4). This kind of result does not fit into the S.P.T. In several kind of problems, as we will se later, our approach is the natural one.

We will present in section 5 a quantitative Kupka-Smale result which is completely new in the literature.

Among other thinks we present an estimation of hyperbolicity in a setting that we believe was not considered before.

In section 2 we present the main general definitions, results, and we show the consistency of this proposal. In each specific problem one have to find the natural probe to be considered. In section 3 we describe with full details the case of diffeomorphisms of S^1.

We do not have results for manifolds of higher dimension, but, in section 6 we present a list of other possible applications which will be consider in future works.

2 General theory for convex infinite dimensional subsets

2.1 A group action on a convex set

In this section we will consider a transitive group action on convex subset of a general metric linear space $\mathcal{V}$. In the beginning we will consider just an affine subspaces, however we can extend it to a general convex boundaryless subset that is not necessarily an affine subspace, by the use of *conditional* *prevalence*. This is a point point where (in our view) we can say that our approach has advantage over the others, because several problems concerning systems or spaces of functions are relevant just in a small convex neighborhood, like a bounded open cone, etc. This will become clear when we consider later applications of our general setting.

Let $(\mathcal{V}, d)$ be a complete metric linear space and $\mathcal{H} \subset \mathcal{V}$ an affine subspace, that is, for any $v, w \in \mathcal{H}$ the line $\{(1-\lambda)v + \lambda w \,|\, \lambda \in \mathbb{R}\}$ is contained in $\mathcal{H}$. Of course, if there exists at least three convex independent vectors $u, v, w \in \mathcal{H}$, then it contains the 2-dimensional convex subspace $\langle u, v, w\rangle = \{au + bv + cw \,|\, a + b + c = 1\}$, which means that $\mathcal{H}$ admits Lebesgue measures supported in finite k-dimensional affine subspaces.

We define a set of automorphisms of $\mathcal{H}$ by:

$$\mathcal{G} = \mathcal{G}_{\mathcal{H}} = \{\psi_{\lambda,w} : \mathcal{H} \to \mathcal{H} \,|\, \psi_{\lambda,w}(v) = (1-\lambda)v + \lambda w,\ \lambda \in \mathbb{R} - \{1\}, w \in \mathcal{H}\}.$$

Remark 1. *Note that the homeomorphisms $\psi_{\lambda,w}$ are not "translations" of $\mathcal{H}$. There is no sense to talk about translations in a convex space. On the other hand, the set of convex reflections $\mathcal{G}$ provides a set of actions that acts transitively in $\mathcal{H}$, and this will be enough to develop the machinery that one needs to built a reasonable prevalence theory. Later we will define a group of transformations G acting on $\mathcal{H}$.*

In the future, the property $\psi_{\lambda,w}(v) = \psi_{1-\lambda,v}(w) \in \mathcal{G}$, if $\lambda \neq 0$, will be very important.

Clearly $(\mathcal{G}, \circ)$ is not a group (because it is not closed under compositions). For example, if there exists at least three convex independent vectors $u, v, w \in \mathcal{H}$, then $\psi_{\frac{1}{2},w}\psi_{-1,u}(v) = v + \frac{1}{2}w - \frac{1}{2}u$, and this cannot be written as convex combination of two vectors in $\mathcal{H}$. In the next proposition we describe some of the main properties of this set:

Proposition 2. *Let $(\mathcal{G}, \circ)$ be the set defined above, then*

a) For any $\lambda \in \mathbb{R} - \{1\}$ and $w \in \mathcal{H}$, $\psi_{\lambda,w}$ is a homeomorphism of $\mathcal{H}$.

b) $\psi_{0,w} = id$, for any $w \in \mathcal{H}$ (identity property).

c) For any $\lambda, \sigma \in \mathbb{R} - \{1\}$ and $w, u \in \mathcal{H}$, $\psi_{\lambda,w} \circ \psi_{\sigma,u} = \psi_{\sigma,u} \circ \psi_{\lambda,w}$, iff, $\psi_{\sigma,u} = id$, or, $\psi_{\lambda,w} = id$ or $u = w$ (noncommutative property).

d) For any $\lambda \in \mathbb{R} - \{1\}$ and $w \in \mathcal{H}$, $\psi_{\lambda,w} \circ \psi_{\sigma,u} = id$, iff, $u = w$ and $\frac{1}{\lambda} + \frac{1}{\sigma} = 1$. In particular, $\psi^{-1}_{\lambda,w} = \psi_{\frac{\lambda}{\lambda-1},w}$ (inverse property). In particular also, $\psi_{\lambda,w} \circ \psi_{\sigma,u} \in \mathcal{G}$, iff, $\delta = \lambda + \sigma - \lambda\sigma \neq 0$ or $\frac{1}{\lambda} + \frac{1}{\sigma} = 1$, but with $u = w$.

e) For any $v \in \mathcal{H}$, there exists $\lambda \in \mathbb{R} - \{1\}$ and $\tilde{v}, w \in \mathcal{H}$, such that $\psi_{\lambda,w}\tilde{v} = v$ (transitivity).

Thus, $\mathcal{G}_{\mathcal{H}}$ is a noncommutative set of homeomorphisms which acts transitively on $\mathcal{H}$.

Proof.

a) Indeed, $(1-\lambda)\tilde{v} + \lambda w = (1-\lambda)v + \lambda w$, implies $\tilde{v} = v$, because $\lambda \in \mathbb{R} - \{1\}$. So, $\psi_{\lambda,w}$ is injective. On the other hand, the equation $\psi_{\lambda,w}v = \tilde{v}$, always have a solution in $\mathcal{H}$:

$$(1-\lambda)v + \lambda w = \tilde{v}$$
$$v = (1 - \frac{1}{(1-\lambda)})w + \frac{1}{(1-\lambda)}\tilde{v}.$$

b) It is trivial because $\psi_{0,w}v = (1-0)v + 0w = v$;

c) For any $\lambda \in \mathbb{R} - \{1\}$ and $w \in \mathcal{H}$,

$$\begin{aligned}\psi_{\lambda,w} \circ \psi_{\sigma,u}v &= (1-\lambda)\psi_{\sigma,u} + \lambda w \\ &= [1-\delta]v + (\delta - \lambda)u + \lambda w,\end{aligned}$$

where $\delta = \lambda + \sigma - \lambda\sigma$.

We observe that δ is a commutative combination of λ and σ, so

$$\psi_{\sigma,u} \circ \psi_{\lambda,w}v = [1-\delta]v + (\delta - \sigma)w + \sigma u.$$

Finally, $\psi_{\lambda,w} \circ \psi_{\sigma,u} = \psi_{\sigma,u} \circ \psi_{\lambda,w}$, iff, $(\delta - \lambda)u + \lambda w = (\delta - \sigma)w + \sigma u$, or, equivalently $\lambda\sigma(w-u) = 0$, what means that $\psi_{\sigma,u} = id$, or $\psi_{\lambda,w} = id$, or $u = w$.

d) We already know that $\psi_{\lambda,w} \circ \psi_{\sigma,u}v = [1-\delta]v + (\delta - \lambda)u + \lambda w$, where $\delta = \lambda + \sigma - \lambda\sigma$. So, $\psi_{\lambda,w} \circ \psi_{\sigma,u}v = v$, $\forall v$, implies that, $\delta = 0$ and $\lambda(w-u) = 0$. On the other hand, $\delta = 0$, is equivalent to $\frac{1}{\lambda} + \frac{1}{\sigma} = 1$, and $\lambda(w-u) = 0$ is equivalent to $w = u$, or $\lambda = 0$, which implies $\psi_{\lambda,w} = id$.

If $\delta \neq 0$, we have, $\psi_{\lambda,w} \circ \psi_{\sigma,u}v = [1-\delta]v + (\delta - \lambda)u + \lambda w = [1-\delta]v + \delta[\frac{(\delta-\lambda)}{\delta}u + \frac{\lambda}{\delta}w] \in \mathcal{G}$, because $\frac{(\delta-\lambda)}{\delta}u + \frac{\lambda}{\delta}w \in \mathcal{H}$.

e) Given $v \in \mathcal{H}$ we find $\lambda \in \mathbb{R} - \{1\}$ and $\tilde{v}, w \in \mathcal{H}$ such that $\psi_{\lambda,w}\tilde{v} = v$. Therefore,

$$\psi_{\lambda,w}\tilde{v} = v \;\Rightarrow\; w = \frac{1}{\lambda}v + (1 - \frac{1}{\lambda})\tilde{v},$$

so, for a fixed λ, we choose $w = \frac{1}{\lambda}v + (1 - \frac{1}{\lambda})\tilde{v}$, and, then $\psi_{\lambda,w}\tilde{v} = v$. ☐

We define now the group $(G, \circ)$ of the reflections of $\mathcal{H}$:

$$G = \{\psi_{\lambda_1,w_1} \circ \psi_{\lambda_2,w_2} \circ \cdots \circ \psi_{\lambda_n,w_n} \,|\psi_{\lambda_i,w_i} \in \mathcal{G}\}.$$

Some easy computations shows that G can be written as

$$G = \{\psi(v) = (1-\delta)v + \sum_{i=1}^{n} a_i w_i \ |\delta \in \mathbb{R} - \{1\},\ \sum_{i=1}^{n} a_i = \delta,\ w_i \in \mathcal{H}\}.$$

This group acts transitively in $\mathcal{H}$ and is topologically free:
Any $\psi \in G - \{Id\}$ has an unique fixed point given by the center of mass of the vectors w_i's that defines itself.

Indeed, $\psi(v) = v$ implies that $(1-\delta)v + \sum_{i=1}^{n} a_i w_i = v$. As $\psi \neq Id$ we get $\delta \neq 0$, so $v = \frac{1}{\delta}\sum_{i=1}^{n} a_i w_i = \frac{1}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i w_i$. In particular, the only fixed point of $\psi_{\lambda,w}$ is $v = w$.

2.2 Prevalence on $\mathcal{H}$

The goal of this section is to introduce our main definitions and to show the consistence of a theory which consider *reflection invariant* almost every properties.

Initially, we must define the notion of G- ***shy*** sets in $\mathcal{H}$, as being Haar (for the group of reflections) null sets for some measure in the Borel sets $\mathcal{H}$.

Definition 3. *Given a Borel subset S we will say that a measure μ is G-transverse to S, and denote $\mu \pitchfork S$, if:*
a) There exists a compact U, such that, $0 < \mu(U) < \infty$, (μ can be taken a compactly supported measure);
b) $\mu(\psi S) = 0$, for any $\psi \in G$.

In this case we say that S is G-shy Borel subset. The hypothesis (a) is natural from [17], and the second one ensures that the property is a *reflection invariant*. Analogously, a subset S in $\mathcal{H}$ is said to be G-**shy**, if S is contained in S', a G-**shy** Borel subset $\mathcal{H}$.

Finally, we call G-**prevalent** a subset P of $\mathcal{H}$, such that P^c is G-**shy**.

In order to simplify the notation, from now on we will drop the letter G.

Definition 4. *A set $P \subset \mathcal{H}$ it said to be **generic** (in the prevalent sense) if P contains some prevalent set.*

According to [17] the consistence of the category of the generic sets needs to satisfy the following set of axioms:

Axiom 1- A generic subset of $\mathcal{H}$ is dense;
Axiom 2- If $P \subset Q$ and P is generic then Q is generic;
Axiom 3- A countable intersection of generic subsets of $\mathcal{H}$ is generic;
Axiom 4- Every reflection of a generic subset is generic.

In order to prove Axiom 1 we need to consider the following lemma.

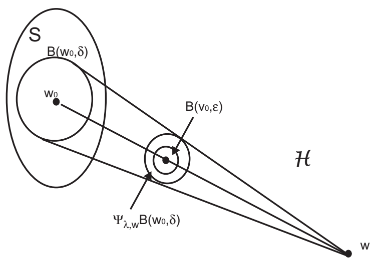
Lemma 5. *Given a Borel subset S and a measure μ transverse to S, there exists another compact supported measure ν such that:*
a) $\nu \pitchfork S$;
b) The support of ν is contained in a ball of arbitrarily small radius.

Proof. Given, $\varepsilon > 0$, we can take a cover of U, given by $\cup_{v\in U} B(v,\varepsilon)$. As U is compact, and $0 < \mu(U) < \infty$, we can choose $U' = \overline{B(v_0,\varepsilon)\cap U}$ with positive measure, and define $\nu(A) = \mu(U' \cap A)$. It is easy to see that ν satisfies the claim of the lemma. □

Now we are able to check Axiom 1.

Theorem 6. *Every generic subset is dense.*

Proof. If P is a generic set then its complement S must to be shy. We claim that S has no interior points. Indeed, if $w_0 \in \mathrm{int}S$, then there exists $\delta > 0$ such that $B(w_0,\delta) \subset S$. From transitivity of G we can find $w = \frac{1}{\lambda}v_0 + (1-\frac{1}{\lambda})w_0$, then $\psi_{\lambda,w}w_0 = v_0$. Using that relation $\psi_{\lambda,w}B(w_0,\delta) = B(\psi_{\lambda,w}w_0,(1-\lambda)\delta) = B(v_0,(1-\lambda)\delta)$, we choose $\lambda \in (0,1)$, then $B(v_0,\varepsilon) \subset B(v_0,(1-\lambda)\delta)$ for $\varepsilon < (1-\lambda)\delta$. Thus, $0 = \mu(\psi_{\lambda,w}S) \geq \mu(\psi_{\lambda,w}B(w_0,\delta)) = \mu(B(v_0,(1-\lambda)\delta)) \geq \mu(B(v_0,\varepsilon)) > 0$, and we get a contradiction.



Reflection of $B(w_0,\delta)$

□

The Axioms 2 and 4 are trivially obtained from the definition. Then, we start now the proof of Axiom 3 by proving first the simplest case:

Lemma 7. *Let S_1, S_2 be shy Borel subsets and $\mu \pitchfork S_1$, $\nu \pitchfork S_2$, then there exists another compact supported measure $\mu * \nu$ such that:*
$(\mu * \nu) \pitchfork S_1$, and $(\mu * \nu) \pitchfork S_2$.

Proof. In our proof we adapt, in part, the idea presented in [9], [17], (see [8] for convolutions on topological groups) where it is introduced the concept of a **convex convolution** of transversal measures in $\mathcal{H}$. However, in our setting we have to face several technical difficulties. Given $\lambda \in \mathbb{R}$, and a Borel subset $A \subseteq \mathcal{H}$, we define

$$(\mu * \nu)_\lambda(A) = \int_{\mathcal{H}} \int_{\mathcal{H}} \chi_{\{\psi_{\lambda,w} v \in A\}}(v,w) d\mu(v) d\nu(w).$$

In other words, for each Borel subset $A \subset \mathcal{H}$, we consider $A^\lambda = \{(v,w) \in \mathcal{H}^2 \,|\, \psi_{\lambda,w} v \in A\}$, and define

$$(\mu * \nu)_\lambda(A) = (\mu \times \nu)(A^\lambda).$$

As $\psi_{\lambda,w}(v) = \psi_{1-\lambda,v}(w) \in \mathcal{G}$, if $\lambda \neq 0$ then we can rewrite the convolution as,

$$(\mu * \nu)_\lambda(A) = \int_{\mathcal{H}} \mu(\psi_{\frac{\lambda}{\lambda-1},w}(A)) d\nu(w),$$

or

$$(\mu * \nu)_\lambda(A) = \int_{\mathcal{H}} \nu(\psi_{1-\frac{1}{\lambda},v}(A)) d\mu(v).$$

The equations above ensure the second part of the definition of shyness, because:

$$(\mu * \nu)_\lambda \psi_{\sigma,u}(S_1) = \int_{\mathcal{H}} \mu(\psi_{\frac{\lambda}{\lambda-1},w} \psi_{\sigma,u}(S_1)) d\nu(w) = 0,$$

and

$$(\mu * \nu)_\lambda \psi_{\sigma,u}(S_2) = \int_{\mathcal{H}} \nu(\psi_{1-\frac{1}{\lambda},v} \psi_{\sigma,u}(S_2)) d\mu(v) = 0,$$

for any $\psi_{\sigma,u} \in \mathcal{G}$.

As $\mu \pitchfork S_1$ and $\nu \pitchfork S_2$, there exists compact sets K_1, K_2, such that, $0 < \mu(K_1) < \infty$, and $0 < \nu(K_2) < \infty$. Let $K = (1-\lambda)K_1 + \lambda K_2$. We observe that the function $\Phi : \mathcal{H}^2 \to \mathcal{H}$, given by $\Phi(v,w) = \psi_{\lambda,w} v$, is continuous, so $K = \Phi(K_1 \times K_2)$ is compact because it is the image of the compact set $K_1 \times K_2$. Now we compute $(\mu * \nu)_\lambda(K)$:

$$(\mu * \nu)_\lambda(K) = (\mu \times \nu)(K^\lambda) = (\mu \times \nu)(K_1 \times K_2) = \mu(K_1)\nu(K_2).$$

□

The extension of this result for the case of a general finite union of shy sets is not a direct consequence of the previous one because G is noncommutative. We have to proceed in a combinatorial way:

First, we observe that $\psi_{\lambda,w}(v) = \psi_{1-\lambda,v}(w) \in \mathcal{G}$ implies the remarkable property:

$$\{\psi_{\lambda_{n-1},w_n} \psi_{\lambda_{n-2},w_{n-1}} ... \psi_{\lambda_{k-1},w_k} ... \psi_{\lambda_1,w_2}(w_1) \in A\} = \{w_k \in \psi A,\ \psi \in G\}.$$

In order to see this first we take,

$$\varphi = \psi_{\lambda_{n-1},w_n}...\psi_{\lambda_{k-2},w_{k-1}},\ \zeta = \psi_{\lambda_k,w_{k+1}}...\psi_{\lambda_1,w_2} \in G,$$

$$\begin{aligned}\psi_{\lambda_{n-1},w_n}\psi_{\lambda_{n-2},w_{n-1}}...\psi_{\lambda_{k-1},w_k}...\psi_{\lambda_1,w_2}(w_1) &= \varphi\psi_{\lambda_{k-1},w_k}\zeta(w_1) = z\\ \psi_{\lambda_{k-1},w_k}\zeta(w_1) &= \varphi^{-1}z\\ \psi_{1-\lambda_{k-1},\zeta(w_1)}(w_k) &= \varphi^{-1}z\\ w_k &= \psi^{-1}_{1-\lambda_{k-1},\zeta(w_1)}\psi^{-1}_{\varphi}z,\end{aligned}$$

then we take $\psi = \psi^{-1}_{1-\lambda_{k-1},\zeta}\psi^{-1}_{\varphi} \in G$.

This argument ensures that, if $\mu_i \pitchfork S_i$, $i = 1, 2, ..., n$, then

$$(\mu_n * \mu_{n-1} * ...\mu_1)(\psi S_i) = 0,\ \forall\psi \in G.$$

To prove that $(\mu_n * \mu_{n-1} * ...\mu_1)_\lambda(K) > 0$, for some compact K, the same argument will work, we just point out that the function $\Phi : \mathcal{H}^2 \to \mathcal{H}$ given by $\Phi(w_1, w_2, ..., w_n) = \psi_{\lambda_{n-1},w_n} \circ .. \circ \psi_{\lambda_1,w_2}w_1$ is continuous, so $K = \Phi(K_1 \times ... \times K_n)$ is compact as the image of the compact $K_1 \times ... \times K_n$, and if we compute $(\mu_n * \mu_{n-1} * ...\mu_1)_\lambda(K)$ we get

$$(\mu_n * \mu_{n-1} * ...\mu_1)_\lambda(K) = \mu_1(K_1)...\mu_n(K_n).$$

In order to prove that Axiom 3 is true, we need to understand how to control the process of countable convex convolution, and this will be done next.

Theorem 8. *If $\mu_i \pitchfork S_i$, $i = \{1, 2, ...\}$, then there exists a measure μ such that $\mu \pitchfork S_i$, $i = \{1, 2, ...\}$.*

Proof. In order to prove this, we introduce the **Countable Convex Convolution (CCC)** of $\{\mu_n\}$ (see [8] for tight sequences and infinite convolutions):

Consider $\mathcal{H}^\pi = \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times ....$, and let $K = K_1 \times K_2 \times ...$, where K_n is the support of each μ_n, by Tychonov theorem we know that K is compact; additionally, we suppose that each μ_n is a probability. We choose a sequence of $\psi_{\lambda_n,w} \in \mathcal{G}$ and define, for each measurable $A \subset \mathcal{H}$ the set:

$$A^\lambda = \{(w_1, w_2, ...) \in \mathcal{H}^\pi \,|\, ... \circ \psi_{\lambda_{n-1},w_n} \circ ... \circ \psi_{\lambda_1,w_2}(w_1) \in A\},$$

thus, the CCC probability of A is

$$(... * \mu_n * ... * \mu_1)_\lambda(A) = (\mu_1 \times \mu_2 \times ...)(A^\lambda).$$

On the other hand, $A^\lambda = \Phi^{-1}(A)$, where $\Phi : \mathcal{H}^\pi \to \mathcal{H}$ is given by:

$$\Phi(w_1, w_2, ...) = ... \circ \psi_{\lambda_{n-1},w_n} \circ ... \circ \psi_{\lambda_1,w_2}(w_1).$$

Now, we must show that Φ is well defined and continuous. Consider $\lambda_n = \frac{1}{\pi^2 n^2}$, $n \in \mathbb{N}$, then

$$\begin{aligned}\Phi(w_1, w_2, ...) &= ... \circ \psi_{\lambda_{n-1}, w_n} \circ ... \circ \psi_{\lambda_1, w_2}(w_1) \\ &= \lim_{n\to\infty} p_{n-1} w_1 + p_{n-1}\frac{\lambda_1}{p_1} w_2 + p_{n-1}\frac{\lambda_2}{p_2} w_3 + ... + p_{n-1}\frac{\lambda_{-1}}{p_{n-1}} w_n \\ &= \lim_{n\to\infty} p_{n-1}\left[w_1 + \sum_{i=1}^{n-1}\frac{\lambda_i}{p_i} w_{i+1}\right],\end{aligned}$$

where $p_n = \prod_{i=1}^{n}(1-\lambda_i)$. From Euler's theorem for infinite products, we know that

$$\sin(z) = z\prod_{n=1}^{\infty}(1 - \frac{z^2}{\pi^2 n^2}), \forall z \in \mathbb{C},$$

and, taking $z = 1$, we get $\lim_{n\to\infty} p_{n-1} = \sin(1)$, what means

$$\Phi(w_1, w_2, ...) = \sin(1) w_1 + \frac{\sin(1)}{\pi^2(1-\frac{1}{\pi^2})} w_2 + \frac{\sin(1)}{4\pi^2(1-\frac{1}{\pi^2})(1-\frac{1}{4\pi^2})} w_3 + ...$$

Φ is, obviously, a continuous map in the product topology of $\mathcal{H}^\pi$.

Finally, we observe that

$$\Phi(K) = \sin(1) K_1 + \frac{\sin(1)}{\pi^2(1-\frac{1}{\pi^2})} K_2 + \frac{\sin(1)}{4\pi^2(1-\frac{1}{\pi^2})(1-\frac{1}{4\pi^2})} K_3 + ...$$

is a compact set in $\mathcal{H}^\pi$, and $(...*\mu_n*...*\mu_1)_\lambda(K) = \mu_1(K_1)\mu_2(K_2)... = 1$. It's also easy, to see that for a fixed i, $(\mu_n*\mu_{n-1}*...\mu_1)(\psi S_i) = 0$, $\forall \psi \in G, \forall n \geq i$, so $(... * \mu_n * ... * \mu_1)_\lambda(\psi S_i) = 0$, $\forall \psi \in G$. Thus $(... * \mu_n * ... * \mu_1)_\lambda \pitchfork S_i$, $i = \{1, 2, ...\}$.

□

2.3 Conditional prevalence

We use the term conditional prevalence instead relative prevalence because this term is already used in the literature for prevalence trough translations by a prevalent set. For instance, the set $\mathcal{H}_0$ of all liftings of order preserving circle diffeomorphisms is defined by an open condition ($F' > 0$) in $\mathcal{H}$, the set of all liftings of degree 1 of $\mathbf{S}^1$, but it is easy to see that any subset $S \subset \mathcal{H}_0$ could not to be shy, since $\mathcal{H}_0$ is not invariant under the action of G (see for example, Remark 13).

In a future work we will analyze properties of maps on the circle, and potentials, so the general theory that we develop here will be useful in this task. However our focus is to study properties of diffeomorphisms so we introduce the idea of **Conditional prevalence**.

Given a set $\mathcal{H}_0 \subset \mathcal{H}$, we denote

$$G_0 = \{\psi_{\lambda_1,w_1} \circ \psi_{\lambda_2,w_2} \circ \cdots \circ \psi_{\lambda_n,w_n} |w_i \in \mathcal{H}_0\} \supset \mathcal{G}_0.$$

Definition 9. *Given a Borel subset $S \subset \mathcal{H}_0$ we will say that a measure μ is condional G-transverse to S in $\subset \mathcal{H}_0$, and denote $\mu \pitchfork_{\mathcal{H}_0} S$ if:*
a) There exists a compact $U \subset \mathcal{H}_0$, such that, $0 < \mu(U) < \infty$, (μ can be taken a compact supported measure in $\mathcal{H}_0$);
b) $\mu(\psi S) = 0$, for any $\psi \in G_0$.

In this case we say that S is a conditional G-shy Borel subset, and, we call conditional G-**prevalent** a subset P of $\mathcal{H}_0$, such that P^c is G-**shy**. In order to avoid complicated notations we will just say that S is shy in $\mathcal{H}_0$, or S is prevalent in $\mathcal{H}_0$.

It is easy to see that the above theory works well in this setting, even the CCC and, the density of conditional prevalent sets. Indeed in the proof of Axiom 3, we get, by contradiction, $B(w_0, \delta) \subset S \subset \mathcal{H}_0$ and, $\psi_{\lambda,w} B(w_0, \delta) \supset \mu(B(v_0, \varepsilon)) \supset \operatorname{supp} \mu$, what shows that $\operatorname{supp} \mu$ is stable under $\psi_{\lambda,w}^{-1}$, then the conclusion $\mu(\psi_{\lambda,w} B(w_0, \delta)) = 0$ is still valid. We observe that the conditional convex convolution $(\mu * \nu)_\lambda$ is well defined since the parameter λ can be chosen in such way that $\operatorname{supp}(\mu * \nu)_\lambda = K = \Phi(K_1 \times K_2) \subset \mathcal{H}_0$ for $\Phi : \mathcal{H}^2 \to \mathcal{H}$ given by $\Phi(v, w) = \psi_{\lambda,w} v$ (taking $\lambda \in (0, 1)$, for example), the same is true for CCC.

Another important fact, is that Axiom 1 implies that, if $\pi : \mathcal{H}_0 \to Diff^r(\mathbf{S}^1)$ given by $\pi(F) = F (\operatorname{mod} 1)$ is the canonical projection, then $\pi(P)$ is dense in $Diff^r(\mathbf{S}^1)$, for any prevalent $P \subset \mathcal{H}_0$.

Definition 10. *A set $\mathcal{O} \subset Diff^r(\mathbf{S}^1)$ is called **prevalent**, if $\pi^{-1}(\mathcal{O}) \subset \mathcal{H}_0$ is a prevalent subset of $\mathcal{H}_0$.*

2.4 Transversal invariant measures

The main goal in this section is to get probabilities on $\mathcal{H}$ that are "invariant" under the group action G.

The hard part to show that a certain given set S is transversal to a Borel measure μ is to show that $\mu(\psi S) = 0$, for any $\psi \in G$. So we can formulate the following question: "How one can obtain examples of G-invariant measures, that is, a Borel measure μ such that, for each Borel subset $S \subset \mathcal{H}$, with $\mu(S) = 0$, we have $\mu(\psi_{\lambda,w} S) = 0$, for any $\psi_{\lambda,w} \in \mathcal{G}$?"

Theorem 11. *Let $V = C^0([0, 1])$ the topological vector space where we consider the uniform topology. Then, for each $n \in \mathbb{N}$, there exists a Borel measure with compact support μ_n in V, that is, G^1-invariant measure (G is the group generated by $\mathcal{G}$ of reflections $\psi_{\lambda,w}$ with $\lambda \neq 1$). In particular $\mu_n \pitchfork S$ for every zero measure set S.*

Proof. Define for each $x \in [0,1]$ the linear functional $x : C^0 \to [0,1]$ given by

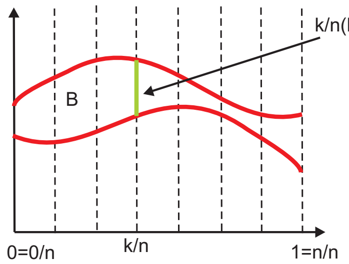
$$x(f) = f(x),$$

and consider the partition of $[0,1]$ given by $\{\frac{0}{n}, \frac{1}{n}, ...\frac{n}{n}\}$. Then, we can obtain the open map $\rho : C^0 \to \mathbb{R}^{n+1}$:

$$\rho(f) = (\frac{0}{n}(f), \frac{1}{n}(f), ...\frac{n}{n}(f)).$$

Now, we consider Leb the Lebesgue measure on $[0,1]$, that is, $\text{Leb}(A) = \int_{\mathbb{R}} \chi_A(x)dx$, and the product measure $\mu_0 = \text{Leb}^{n+1}$ on $[0,1]^{n+1}$. Thus we are able to define a Borel measure μ_n in C^0 by the push forward

$$\begin{aligned}\mu_n(B) &= \mu_0(\rho(B)) \\ &= \text{Leb}(\frac{0}{n}(B)) \cdot ... \cdot \text{Leb}(\frac{n-1}{n}(B)) \cdot \text{Leb}(\frac{n}{n}(B)) \\ &= \int_{\mathbb{R}} \chi_{\frac{0}{n}(B)}(x)dx \cdot ... \cdot \int_{\mathbb{R}} \chi_{\frac{n-1}{n}(B)}(x)dx \cdot \int_{\mathbb{R}} \chi_{\frac{n}{n}(B)}(x)dx.\end{aligned}$$



Push forward of the product measure.

Obviously, μ_n has compact support, we claim that μ_n is G-invariant. Indeed, suppose $\mu_n(A) = 0$, so there exists k such that, $\text{Leb}(\frac{k}{n}(A)) = 0$, and for any $\psi_{\lambda,h} \in \mathcal{G}$ we have

$$\begin{aligned}\text{Leb}(\frac{k}{n}(\psi_{\lambda,h}A)) &= \text{Leb}(\frac{k}{n}((1-\lambda)A + \lambda h)) \\ &= \text{Leb}((1-\lambda)\frac{k}{n}(A) + \lambda\frac{k}{n}(h))) \\ &= \int_{\mathbb{R}} \chi_{(1-\lambda)\frac{k}{n}(A)+\lambda h(k/n)}(x)dx \\ &= \int_{\mathbb{R}} \chi_{(1-\lambda)\frac{k}{n}(A)}(y)dy, \ y = x - \lambda h(k/n) \\ &= (1-\lambda)\int_{\mathbb{R}} \chi_{\frac{k}{n}(A)}(z)dz, \ (1-\lambda)z = y \\ &= (1-\lambda)\,\text{Leb}(\frac{k}{n}(A)) = 0,\end{aligned}$$

thus $\mu_n(\psi_{\lambda,h}A) = 0$. □

3 Applications

3.1 Liftings of C^r diffeomorphisms of $\mathbf{S}^1$:

Following the notation of [7], we denote $Diff_+^r(\mathbf{S}^1)$ the set of all C^r, preserving order, diffeomorphisms of $\mathbf{S}^1$, for $r \geq 1$ ($Diff_+^0(\mathbf{S}^1)$ means the set of all increasing homeomorphisms of $\mathbf{S}^1$). As $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$, we can consider the set of all liftings of order preserving diffeomorphisms, to the universal covering $\mathbb{R}$:

$$\mathcal{H}_0^+ = \{F \in Diff_+^r(\mathbb{R}) \,|\, F(x+1) = F(x) + 1, \, \forall n \in \mathbb{N}, F'(x) > 0\}.$$

This set is a representative subset of the set of all C^r maps commuting with the covering map, of degree 1:

$$\mathcal{H}^+ = \{F \in C^r(\mathbb{R}) \,|\, F(x+n) = F(x) + n, \, \forall n \in \mathbb{N}\}.$$

We can also define the set of liftings of order reversing diffeomorphisms $\mathcal{H}^-$. As $\mathcal{H}^- \cup \mathcal{H}^+$ is not connected by isotopy, we will restrict ourselves, w.l.o.g., to $\mathcal{H} = \mathcal{H}^+$ (one can get similar results for $\mathcal{H}^-$), unless we mention the contrary. Thus, from now on, we will denote $Diff^r$ instead $Diff_+^r$. Also we define, $\mathcal{V}$ be the linear topological space of all C^r function (diffeomorphisms) of $\mathbb{R}$ with the Whitney topology, so $\mathcal{H}$ is an infinite dimensional convex unbounded subset of $\mathcal{V}$. More than that, the topology induced by $\mathcal{V}$ on $\mathcal{H}$ agree with the uniform topology induced by the complete metric on $Diff^r(\mathbf{S}^1)$ given by,

$$d(F, G) = \sum_{i=0}^{n} \sup_{x \in \mathbf{S}^1} |F^{(i)}(x) - G^{(i)}(x)|.$$

Remark 12. *Using the natural structure of $\mathcal{V}$ as complete metric linear space we conclude from the S. P. T. that $\mathcal{H}$ is shy. So, for a generic C^r function (diffeomorphism) of $\mathbb{R}$, $F(x+n) \neq F(x) + n$, this means that F is not a lifting of any diffeomorphism (or homeomorphism) of $\mathbf{S}^1$. Of course, this is not the right direction of reasoning. Thus, we need to introduce an intrinsic prevalent theory for $\mathcal{H}$ in order to get the typical behavior of $\mathbf{S}^1$ diffeomorphisms.*

We would like to observe first that $\mathcal{H}_0$ is not in 1-1 correspondence with the diffeomorphisms (or homeomorphisms) of $\mathbf{S}^1$. This happens because two liftings of the same diffeomorphism differ by an integer constant. In other words, if $\pi : \mathcal{H}_0 \to Diff^r(\mathbf{S}^1)$ is given by

$$\pi(F) = F \,(\mathrm{mod}\, 1) = f,$$

(the canonical projection) then, $\pi(F) = \pi(G)$, implies $F = G + k$, $k \in \mathbb{Z}$. So defining the relation $F \sim G$ in $\mathcal{H}_0$, iff, $F = G + k$, $k \in \mathbb{Z}$, we get that $\mathcal{H}_0 / \sim$

is a module over $\mathbb{Z}$, with the natural operations in the equivalence classes, and, the induced map $\overline{\pi}$ is an 1-1 correspondence between $\mathcal{H}_0/\sim$, and the set of all order preserving diffeomorphisms (or homeomorphisms) of $\mathbf{S}^1$.

Remark 13. *Another important issue here is the geometry of $\pi(\mathcal{H}_0)$. If we take $F, F+k$, $k \in \mathbb{Z}$ in $\mathcal{H}_0$, the line $\{(1-\lambda)F + \lambda(F+k) \mid \lambda \in \mathbb{R}\}$ projects on*

$$\pi\{(1-\lambda)F + \lambda(F+k) \mid \lambda \in \mathbb{R}\} = \{(F+\sigma)\,(mod\,1) \mid \sigma = (\lambda - [\lambda]) \in \mathbf{S}^1\},$$

that is, this line projects on a closed path in $Diff^r(\mathbf{S}^1)$.

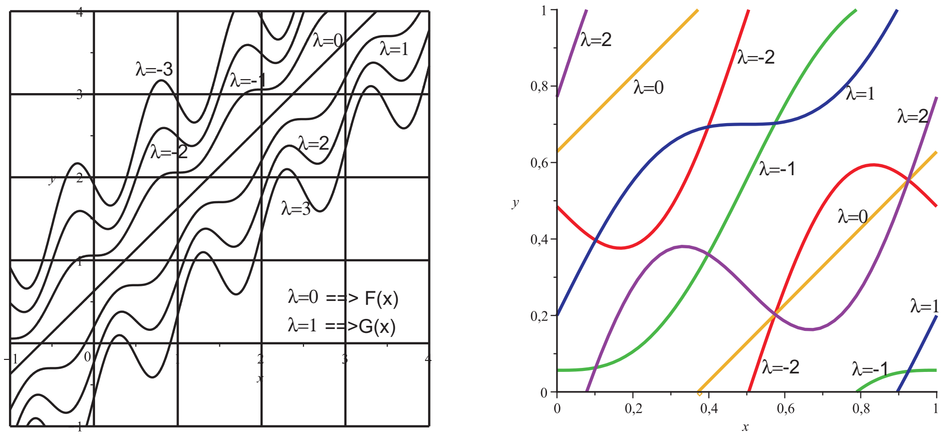
On the other hand, if we take arbitrary F, G in $\mathcal{H}_0$, then the line $\{(1-\lambda)F + \lambda G \mid \lambda \in \mathbb{R}\}$ projects on

$$\pi\{(1-\lambda)F + \lambda G \mid \lambda \in \mathbb{R}\},$$

which is, in general, a noncompact path not necessarily in $Diff^r(\mathbf{S}^1)$.

For example, take $F(x) = x + 0.2\pi$ and $G(x) = x + 0.2 + \frac{1}{2.1\pi}\sin(2\pi x)$, we get,

$$F_\lambda = ((1-\lambda)F + \lambda G)(x) = x + 0.2(\pi + \lambda(1-\pi)) + \frac{\lambda}{2.1\pi} sin(2\pi x).$$



Path of C^r maps F_λ (λ-foliation).

We note that if $\lambda \in [0,1]$ then all the F_λ are order preserving diffeomorphisms, but when the parameter λ grows the F_λ are now part of a family of liftings of continuous maps of S^1. However, there always exists some $\varepsilon > 0$, that depends of $\min\{\min F', \min G'\}$, such that, $(1-\lambda)F + \lambda G \in \mathcal{H}_0$, for $\lambda \in (-\varepsilon, 1+\varepsilon)$.

3.2 Evaluation maps for periodic points

In this section, and in the next one, we will consider

$$\mathcal{H}_0 = \text{Liftings}(Diff^r_+(\mathbf{S}^1)),$$

for $r \geq 2$, so the prevalence results are on this set. We need regularity $r \geq 2$ in order to apply evaluation technics. The main goal of this section is to provide the technical tools that we will need for applications, more specifically we will construct and analyze properties of one dimensional probes, that is, finitely dimensional spaces which support transversal measures.

For each fix $F, G \in \mathcal{H}_0$ we consider the path $\alpha_\lambda : \mathbb{R} \to \mathcal{H}$ given by

$$\alpha_\lambda(x) = (1-\lambda)F(x) + \lambda G(x).$$

We define the n**-evaluation map**, associated to α_λ, as the multivalued function:

$$\Delta_n(x) = \{\lambda \,|\, \alpha_\lambda^n(x) = x \text{ and } x \in S^1\},$$

which can be eventually empty.

We observe that the lifting property implies that $\Delta_n(x)$ can be extended as a time 1 periodic function in $\mathbb{R}$. Moreover, if $\alpha_\lambda \in \mathcal{H}_0$ then $\Delta_n(x) = \{\lambda\}$, what means that $\lambda = \Delta_n(x)$ is locally a C^r function. To see this, we define the C^r map $\varphi(x, \lambda) = \alpha_\lambda(x)$, and denote $\varphi^n(x, \lambda) = \varphi(\varphi(..., \lambda), \lambda)$, so $\alpha_\lambda^n(x) = \varphi^n(x, \lambda)$. Differentiating with respect to λ we get

$$\frac{\partial \varphi^n}{\partial \lambda}(x, \lambda) = \frac{1}{\frac{\partial \varphi}{\partial x}} \sum_{k=0}^{n-1} (G - F)(\varphi^k) \frac{\partial \varphi^{n-k}}{\partial x} \neq 0,$$

because $\alpha_\lambda \in \mathcal{H}_0$, and $G - F \neq 0$.

To establish the domain of $\lambda = \Delta_n(x)$ we need to assume that F has some fixed point, then we observe that $\alpha_\lambda \in \mathcal{H}_0$ for some interval $\lambda \in (a, b) \subset [0, 1]$ (for example, for $F(x) = x + 0.1 + \frac{1}{2.1\pi}\sin(2\pi x)$ and $G(x) = x + 0.2\pi$, we get, $(a, b) = (-0.05, 2.05)$ as the maximal interval for this property), then we define $D^n(F, G) = \Delta_n^{-1}(\Delta_n(S^1) \cap (a, b))$, because $\Delta_n(S^1) \cap (a, b) = \{\lambda \,|\, \alpha_\lambda(F) \in \mathcal{H}_0,$ and there exists x s.t. $\alpha_\lambda(F)^n(x) = x\}$. Thus $\lambda = \Delta_n : D^n(F, G) \to (a, b)$ represents all the periodic points of period n, for α_λ in the following sense:

"$p \in S^1$ is a periodic point of period n for the diffeomorphism α_λ with degenerate derivative $(\alpha_\lambda^n)'(p) = 1$ if, and only if, $\lambda = \Delta_n(p) \in (a, b)$ and $\Delta_n'(p) = 0$".

We like to observe that some particular cases will be useful in the next sections:

Type I - $F \in \mathcal{H}_0$ with fixed points, and $G = F + k$, $k \in (0, 1)$. In this case, $\alpha_\lambda(x) = (1-\lambda)F(x) + \lambda G(x) = F(x) + \lambda k \in \mathcal{H}_0$, $\forall \lambda \in \mathbb{R}$. Thus, the evaluation $\lambda = \Delta_n$ is fully defined and the study of the hyperbolic periodic points reduces to the study of the critical points of the evaluation. More than that, applying an element $\psi \in G_0$, we get

$$\psi(\alpha_\lambda)(x) = \alpha_{(1-\delta)\lambda}(x) = \psi(F)(x) + (1-\delta)\lambda k,$$

where, $\psi(F) = (1-\delta)F + \sum_{i=1}^{n} a_i G_i \mid \delta \in \mathbb{R} - \{1\},\ \sum_{i=1}^{n} a_i = \delta,\ G_i \in \mathcal{H}_0$. Thus, there are just two possibilities, $\psi(\alpha_\lambda) \subset \mathcal{H}_0$ if $\psi(F) \in \mathcal{H}_0$, or $\psi(\alpha_\lambda) \subset \mathcal{H} - \mathcal{H}_0$ if $\psi(F) \notin \mathcal{H}_0$.

Type II - $F, G \in \mathcal{H}_0$, F with fixed points, and $\min_{x\in S^1} |G(x) - F(x)| = \sigma > 0$. In this case, α_λ is a λ-foliation of $\mathbb{R}^2$ (see Remark13, for example). Let (a, b) be the maximal interval such that $\alpha_\lambda \in \mathcal{H}_0$ and $D^n(F, G)$ the domain of the evaluation map $\lambda = \Delta_n$, we need to compute the derivative $\frac{d\lambda}{dx}$. Using the notation $\varphi(x, \lambda) = \alpha_\lambda(x)$, and $\varphi^n(x, \lambda) = \varphi(\varphi(..., \lambda), \lambda)$, where $\lambda = \lambda(x)$, and $\alpha_\lambda^n(x) = x$, we get

$$1 = \frac{\partial \alpha_\lambda^n}{\partial x}(x, \lambda) = \frac{\partial \varphi^n}{\partial x}(x, \lambda) = \frac{d\varphi^n}{dx} + \lambda'(x)\frac{d\varphi^n}{dx}\sum_{k=0}^{n-1} \frac{(G-F)(\varphi^k)}{\frac{d\varphi^{k+1}}{dx}}.$$

Isolating λ', and using $\varphi(x, \lambda) = \alpha_\lambda(x)$, we get

$$\lambda'(x) = \frac{1 - \frac{d\alpha_\lambda^n}{dx}}{\frac{d\alpha_\lambda^n}{dx}\sum_{k=0}^{n-1}(\frac{d\alpha_\lambda^{k+1}}{dx})^{-1}(G-F)(\alpha_\lambda^k)}.$$

The main application of the evaluation map is to describe the behavior of the hyperbolic periodic points. Indeed, the formula above shows that for a diffeomorphism α_λ, one has $\frac{d\alpha_\lambda^n}{dx}(x) = 1$ only if $\lambda'(x) = 0$ and $\lambda = \Delta_n(x)$.

For type II probes the action of G is not trivial and the interval (c, d) where $\psi(\alpha_\lambda) \in \mathcal{H}_0$ can be different of (a, b). Anyway, the same conclusions are true in a possibly different domain. So the conditions $\psi(\alpha_\lambda)^n(p) = p$ and $\frac{d}{dx}\psi(\alpha_\lambda)^n(p) = 1$, are equivalent to $\lambda = \Delta_n(p)$ and $\Delta_n'(p) = 0$, because,

$$\frac{\partial}{\partial x}\psi(\alpha_{\lambda(x)})^n(x) + \frac{\partial}{\partial \lambda}\psi(\alpha_{\lambda(x)})^n(x)\lambda'(x) = 1,$$

thus,

$$\lambda'(p) = \Delta_n'(p) = \frac{1 - \frac{\partial}{\partial x}\psi(\alpha_{\lambda(p)})^n(p)}{\frac{\partial}{\partial \lambda}\psi(\alpha_{\lambda(p)})^n(p)}.$$

4 Results of Prevalence on $\mathbf{S}^1$

The first result we will get is about the non-degeneracy of the fixed points of a diffeomorphism of $\mathbf{S}^1$. This result is contained in the Kupka-Smale theorem, but we want to get this result in our setting. We present a direct proof here which is instructive, and, at the same time, shows a different choice of probe for getting the desired property.

We say that a fixed point p is non-degenerate (of order 1) if $F'(p) \neq 1$.

Proposition 14. *The order preserving diffeomorphisms of* $\mathbf{S}^1$ *are generically nondegenerate, in particular they are hyperbolic.*

Proof. Let $S = \{F \in \mathcal{H}_0 \,|\, \exists p \in Fix(F),\ \frac{d}{dx}F(p) = 1\}$, and we must show that S is shy, that is, the set of all diffeomorphisms without degenerate fixed point is prevalent.

The proof will be accomplish by choosing a special one dimensional probe of type II, in $\mathcal{H}_0$. We consider the one dimensional subspace generated by F and H,

$$\alpha(x, \lambda) = \psi_{\lambda,H}F = (1-\lambda)F + \lambda H,\ \lambda \in [0, 1],$$

and consider μ the Lebesgue measure supported on the trace of α, where $\mu(A)$ means $\mu(\{\lambda \,|\, \alpha_\lambda \in A\})$. We claim that $\mu(S \cap \alpha_\lambda) = 0$. We will not explain now this computations because it is a particular case of the next one ($\psi = Id$).

In order verify the condition of invariance we need to show that $\mu(\psi(S) \cap \alpha) = 0$, or equivalently, $\mu(S \cap \psi(\alpha)) = 0$, for all $\psi \in G$. Remember that $\psi(F) = (1-\delta)F + \sum_{i=1}^{n} a_i G_i \;|\delta \in \mathbb{R} - \{1\},\ \sum_{i=1}^{n} a_i = \delta,\ G_i \in \mathcal{H}_0$, so we have $\psi(\alpha)(x, \lambda) = (1-\delta)F(x) + \lambda(1-\delta)(H(x) - F(x)) + \displaystyle\sum_{i=1}^{n} a_i G_i(x)$. Now, $\psi(\alpha)(p, \lambda) = p$ implies that, $1 = (1-\delta)F'(p) + \lambda(1-\delta)(H'(p) - F'(p)) + \sum_{i=1}^{n} a_i G_i'(p)$. Thus,

$$\Delta(x) = \frac{x - F(x)}{H(x) - F(x)} + \frac{1}{H(x) - F(x)} \frac{\delta}{1-\delta} \left(\frac{1}{\delta} \sum_{i=1}^{n} a_i G_i(x) - x \right).$$

Substituting the two formulas above in $\Delta'(x)$ we conclude that $\Delta'(p) = 0$, and, proceeding as before, we get

$$\begin{aligned}\{\lambda \,|\, S \cap \psi(\alpha_\lambda)\} &= \{\lambda \in [0, 1] \,|\, \exists p \in Fix(\psi(\alpha_\lambda)(\cdot, \lambda)),\ \frac{d}{dx}\psi(\alpha_\lambda)(p, \lambda) = 1\} \\ &\subseteq \Delta(\{p \,|\, \Delta'(p) = 0\}),\end{aligned}$$

which has zero Lebesgue measure by Sard's theorem. So $\mu \pitchfork S$, that is S is shy. □

Applying thess technics we can prove a kind of Kupka-Smale theorem for non-degeneracy of the periodic points of a preserving order diffeomorphism of $\mathbf{S}^1$.

We remember that a periodic point p of order n ($F^n(p) = p$) is said to be hyperbolic, if $|\frac{d}{dx}F^n(p)| \neq 1$; for a order preserving transformation this is equivalent to say that every iterate F^n has no degenerate fixed points. A diffeomorphism F is said K-S (Kupka-Smale type) if all periodic points are hyperbolic.

Theorem 15. *The order preserving diffeomorphisms of $\mathbf{S}^1$ are generically K-S.*

Proof. Let $S_n = \{F \in \mathcal{H}_0 \,|\, \exists p \in Fix^n(F),\ \frac{d}{dx}F^n(p) = 1\}$ be the set of diffeomorphisms such that the iterate F^n has at least one degenerate fixed point. We must to show that S_n is shy, so

$$S = \bigcup_{n=1}^{\infty} S_n = \{F \in \mathcal{H}_0 \,|\, \text{has some degenerate periodic point.}\},$$

will be shy, that is, the set of all diffeomorphisms which preserve order and also without degenerate periodic point $\mathcal{H}_0 - S$, is prevalent.

The proof will follow easily by choosing a one dimensional probe of type I,

$$\alpha_\lambda(.) = \psi_{\lambda,H}F = (1-\lambda)F + \lambda H = F + \lambda k,\ \lambda \in [0,1],$$

and denote μ the Lebesgue measure supported on the trace of α, where $\mu(A)$ means $\mu(\{\lambda \,|\, \alpha_\lambda \in A\})$.

We claim that $\mu(\psi(S_n) \cap \alpha_\lambda) = 0$, or, equivalent,

$$\mu(\{\lambda | S_n \cap \psi(\alpha_\lambda) \neq \emptyset\}) = 0,$$

for all $\psi \in G$. In order to see this, remember that from Section 3.2 the conditions

$$\psi(\alpha_\lambda)^n(p) = p \text{ and } \frac{d}{dx}\psi(\alpha_\lambda)^n(p) = 1,$$

are equivalent to, $\lambda = \Delta_n(p)$ and $\Delta_n'(p) = 0$, thus,

$$\lambda'(p) = \Delta_n'(p) = \frac{1 - \frac{\partial}{\partial x}\psi(\alpha_{\lambda(p)})^n(p)}{\frac{\partial}{\partial \lambda}\psi(\alpha_{\lambda(p)})^n(p)}.$$

We have two possibilities, if $\psi(F) \notin \mathcal{H}_0$ then also $\psi(\alpha_\lambda) \notin \mathcal{H}_0$, so $\psi(\alpha_\lambda) \cap S_n$ has zero measure. Otherwise, if $\psi(F) \in \mathcal{H}_0$, then also $\psi(\alpha_\lambda) \in \mathcal{H}_0$.

Proceeding as before, we get

$$\{\lambda \,|\, S_n \cap \psi(\alpha_\lambda)\} = \Delta_n(\{p \,|\, \Delta_n'(p) = 0\}),$$

that has zero Lebesgue measure by Sard's theorem.

It's clear that is enough to show the result for a single $\psi \in G$. So $\mu \pitchfork S_n$, that is, S_n is shy. □

There are several nice papers in the last years about properties of one-parametric families of circle diffeomorphisms. Several results of this kind can be restated in our prevalence point of view. For example, M. Tsujii [21] has considered one-parametric families, $\{f_t = f + t \,|\, t \in S^1\}$ of orientation preserving circle diffeomorphisms, and he defines the following property: f satisfies $(*)_\beta$, if

$$\text{there exist finitely many } p, q \in \mathbb{Z}, \text{ such that } |\rho(f) - \frac{p}{q}| < 1/q^{2+\beta}.$$

J-C. Yoccoz [22] has proved that this condition implies that f is C^{r-2} conjugated to a rigid rotation $R_{\rho(f)}$, provided f is C^r, for $r \geq 3$, and, $0 < \beta < 1$.

M. Tsujii proved that the set

$$S_\beta = \{t \in S^1 \,|\, \rho(f_t) \notin \mathbb{Q} \text{ and } f_t \text{ not satisfy } (*)_\beta\},$$

has Lebesgue measure zero, since df is of bounded variation.

Using this property we can prove the next result. We note that this application is, in a certain way, similar to the spirit of the Remark 5 in [21], but, not exactly the same (in our setting).

The next result is an extension of a Theorem by M. Tsuji.

Theorem 16. *A C^r order preserving diffeomorphism of $\mathbf{S}^1$ is, generically, conjugated to a rigid rotation R_ρ, or, it has a rational rotation number, provided that $r \geq 3$.*

Proof. We just have to choose a particular one-dimensional probe of type I,

$$\alpha_\lambda(x) = \psi_{\lambda,H} F = (1-\lambda)F + \lambda H = F + \lambda k, \; \lambda \in \mathbb{R},$$

$0 < k < 1$ and, as before, we denote by μ the Lebesgue measure supported on the trace of α_λ, where $\mu(A)$ means $\mu(\{\lambda \,|\, \alpha_\lambda \in A\})$. Since the Lebesgue measure is sigma finite we get from Tsujii's theorem, that $\mu(S_\beta) = 0$, where

$$S_\beta = \{F \in \mathcal{H}_0 \,|\, \rho(\pi(F)) \notin \mathbb{Q} \text{ and } \pi(F) \text{ not satisfy } (*)_\beta\}.$$

We claim that $\mu(\psi(S_\beta) \cap \alpha_\lambda) = 0$, or, equivalent, $\mu(S_\beta \cap \psi(\alpha_\lambda)) = 0$, for all $\psi \in G_0$. But this is a simple consequence of the fact that the dilatation map is absolutely continuous, with respect to the Lebesgue measure. Remember that

$$\begin{aligned} \psi(\alpha_\lambda)(x, \lambda) &= (1-\delta)(F(x) + \lambda k) + \sum_{i=1}^{n} a_i G_i(x) \\ &= \psi(F) + \lambda(1-\delta)k. \end{aligned}$$

We assume that $\psi(F) = (1-\delta)F(x) + \sum_{i=1}^{n} a_i G_i(x) \in \mathcal{H}_0$ (otherwise the path has zero measure) is of bounded variation too. So we can use Tsujii's theorem again to get $\mu(\psi(S_\beta) \cap \alpha_\lambda) = 0$.

Thus, $\mu \pitchfork S_\beta$, what means that S_β is shy. □

Of course, the projection of the dense set S_β^c is dense in $Diff^r(S^1)$ (see Axiom 1). In [21], Tsujii obtains this result as a corollary of his main theorem.

5 A quantitative estimation of Hyperbolicity

In this section we will analyze a different problem, more precisely, we want to make a quantitative analysis of the amount of C^r diffeomorphism, $r \geq 2$, which are hyperbolic (we need $r \geq 2$ in order to get an C^1 evaluation map). In Theorem 15 we prove that the orientation preserving diffeomorphisms are prevalently K-S, through the use of an special parametric family. If we use the quantitative measure of hyperbolicity, as in [17] Definition 7.6,

$$E_{\gamma}^{n}(F) = \{x \in S^1 \,|\, x \in Fix(F^n), \|dF^n(x) - 1\| \leq \gamma\},$$

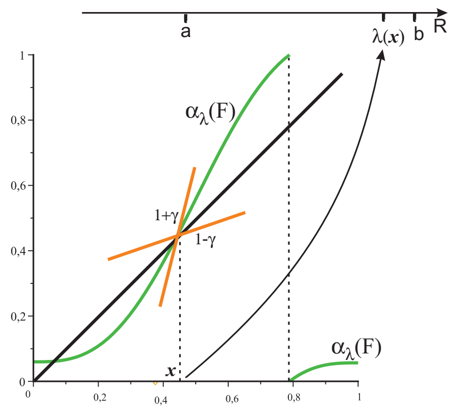
for $\gamma > 0$, then we can formulate the following problem:

Fix $F \in \mathcal{H}_0$ and choose $H \in A^F \subset \mathcal{H}_0$, given by $A^F = \{H \in \mathcal{H}_0 \,|\, H - F \neq 0\}$. Consider the path of liftings of circle maps of type II, $\alpha_\lambda(F)(x) = (\psi_{\lambda,H}F)(x) = F(x) + \lambda(H - F)(x),\ \lambda \in \mathbb{R}$.

We want to estimate the Lebesgue measure of

$$Z_{\gamma}^{n}(F) = \{\lambda(x) \,|\, x \in E_{\gamma}^{n}(\alpha_\lambda(F)) \text{ and } \alpha_\lambda(F) \in \mathcal{H}_0\}$$

where $\lambda(x) = \Delta_n(x) = \{\lambda \,|\, \alpha_\lambda(F)^n(x) = x\}$, *is the multi-valued map introduced in the Section3.2.*



The set $Z_{\gamma}^{n}(F)$.

We observe that the problem is well posed because the map $\Delta_n(x)$ is 1-periodic (otherwise such measure could be always ∞ or always 0), since F and H are liftings of circle diffeomorphisms. So this claim motivate the quantitative prevalent K-S theorem **(Q-K-S)**:

Theorem 17. *Fix* $F \in \mathcal{H}_0$ *and choose* $H \in A^F$, *where*

$$A^F = \{H \in \mathcal{H}_0 \,|\, \inf_{x \in S^1} |H - F| = \sigma > 0\}.$$

Consider the path of circle diffeomorphisms $\alpha_\lambda(F)(x)$, as above, and $\lambda(x) = \Delta_n(x) = \{\lambda \,|\, \alpha_\lambda(F)^n(x) = x\}$ the associated evaluation map. If m is the Lebesgue measure in $\mathbb{R}$ then

$$m(Z^n_\gamma(F)) \le c_n \frac{\gamma}{\sigma},$$

for some positive constant c_n which depends on $\min\limits_{x\in\bigcup E^n_\gamma(\alpha_\lambda(F))} \alpha_\lambda(F)'(x)$.

Proof. Let D_n be the domain of each evaluation and $(a,b) \subset [0,1]$ be the maximal interval such that $\alpha_\lambda(F) \in \mathcal{H}_0$. Then,

$$Z^n_\gamma(F) = \Delta_n\left(\bigcup_{\lambda\in(a,b)} E^n_\gamma(\alpha_\lambda(F))\right),$$

so Δ_n is a differentiable C^1 function on this set, because $E^n_\gamma(\alpha_\lambda(F)) \subset D_n$ and $\lambda \in (a,b)$.

From Section 3.2 we get the formula,

$$\Delta'_n(x) = \frac{1 - \frac{d\alpha_\lambda(F)^n}{dx}}{\frac{d\alpha_\lambda(F)^n}{dx}\sum_{k=0}^{n-1}(\frac{d\alpha_\lambda(F)^{k+1}}{dx})^{-1}(H-F)(\alpha_\lambda(F)^k)},$$

where $\lambda = \Delta_n(x)$.

From Schwartz [19], Chap. III, Thm. 3.1, we know that, if $D \subset \mathbb{R}^n$ is an open set, and $g : D \to \mathbb{R}^n$ is C^1, and $E \subseteq D$ is a mensurable subset, then, $g(E)$ is measurable, and $m(g(E)) \le \int_E |J_g|dm$, where J_g is the Jacobian determinant of g.

So, using $\frac{d\alpha_\lambda(F)^n}{dx}\cdot\big(\frac{d\alpha_\lambda(F)^{i+1}}{dx}\big)^{-1} = \frac{d\alpha_\lambda(F)^{n-i}}{dx}\circ\alpha_\lambda(F)^{i+1}$, and applying this property to our case we get,

$$\begin{aligned}
m(\{\lambda(x)\,|\,x\in E^n_\gamma(\alpha_\lambda(F)) \text{ and } \alpha_\lambda(F)\in\mathcal{H}_0\}) &= \\
m(\Delta_n(\bigcup_{\lambda\in(a,b)} E^n_\gamma(\alpha_\lambda(F)))) &\le \\
\int_{\bigcup E^n_\gamma(\alpha_\lambda(F))} |\Delta'_n(x)|dm &= \\
\int_{\bigcup E^n_\gamma(\alpha_\lambda(F))} \frac{\|1 - d_x\alpha_\lambda(F)^n(x)\|}{\sum_{i=0}^{n-1}\frac{d\alpha_\lambda(F)^{n-i}}{dx}\circ\alpha_\lambda(F)^{i+1}\cdot\|(H-F)(\alpha_\lambda(F)^i)\|}dm &\le \\
\frac{\gamma}{\sigma}\int_{\bigcup E^n_\gamma(\alpha_\lambda(F))}\frac{1}{\sum_{i=0}^{n-1}\frac{d\alpha_\lambda(F)^{n-i}}{dx}\circ\alpha_\lambda(F)^{i+1}}dm &= \frac{\gamma}{\sigma}I_n
\end{aligned}$$

where $I_n = \int_{\bigcup E^n_\gamma(\alpha_\lambda(F))}\frac{1}{\sum_{i=0}^{n-1}\frac{d\alpha_\lambda(F)^{n-i}}{dx}\circ\alpha_\lambda(F)^{i+1}}dm$.

Let, $u = \min_{x \in \bigcup E^n_\gamma(\alpha_\lambda(F))} \dfrac{d}{dx}\alpha_\lambda(F)(x)$ then

$$\sum_{i=0}^{n-1} \frac{d\alpha_\lambda(F)^{n-i}}{dx} \circ \alpha_\lambda(F)^{i+1} \geq \sum_{i=0}^{n-1} u^{n-i+1} = b_n.$$

Thus, $\frac{1}{\sum_{i=0}^{n-1} u^{n-i+1}} = \frac{1}{b_n} = c_n < \infty$. □

Finally, we would like to point out that if $u > 1$ on the above theorem, then $\sum \frac{1}{b_n}$ is finite, on the other hand, we always have $1 - \gamma \leq u \leq 1 + \gamma$, thus $\sum \frac{1}{b_n}$ may diverge.

6 Open problems where the present setting can be considered

We present some interesting problems where our approach can eventually work. Some of these open questions ((a) and (b)) are very close the ones we consider before, and will be the subject of our future work. In each case one have to find the exact probe.

a) Ergodic optimization

For a fixed Dynamical System T one can consider the problem of finding ν which maximize among T-invariant probabilities μ

$$\max \int_X A(x) d\mu(x),$$

where $T : X \to X$ is a discrete dynamical system, and $A : X \to \mathbb{R}$ is a potential with a certain regularity [4], [10]. If A is assumed to be Holder, it is natural to ask about properties of maximizing probabilities for a generic potential A in the Holder topology. It is easy to see that generically on A the maximizing probability is unique [4]. This approach does not consider any kind of probability in the set of possible A.

Alternatively, for a certain a convex set $\mathcal{H}$ of potentials, one can ask in the prevalent setting the questions:
1- The probability ν such that, $\int_X A(x)d\nu(x) = \max \int_X A(x)d\mu(x)$, is unique?
2- Is it supported in a periodic orbit of T?

b) Aubry-Mather theory

Given a Lagrangian $L : TM \to \mathbb{R}$ one can try to find a generic set (in Baire category sense) of potentials $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that for all $\varphi \in \mathcal{O}$ the Lagrangian $L + \varphi : TM \to \mathbb{R}$ has nice properties with respect to the

Mather measures [3] [1]. This approach is known as the study of generic properties in Mañé's sense and has been study for many author in the last years, [1], [5] and [16].

In the prevalent point of view we considered here, a version of these results can be obtained, not for translation invariant prevalent sets, but, by considering the affine subset $\mathcal{H}$ of $C^k(TM, \mathbb{R})$ (or $C^\infty(TM, \mathbb{R})$) and using our prevalence theory for getting generic properties in

$$\mathcal{H} = \{L + \varphi \mid \varphi \in C^\infty(M, \mathbb{R})\}.$$

We hope to prove in the future that the same strong properties of this theory, as uniqueness, characterization of the support as a periodic orbit, dimension of cohomology classes, etc.., [1] and [3], are also true in the G-prevalent sense.

We also observe that it will be necessary to consider conditional prevalent ideas, because if $d(\varphi, 0)$ is large, then the ergodic properties of the Mather measures of $L + \varphi$ and L could be very distinct (in the same energy level).

c) Torus $\mathbb{T}^n$

For the torus $\mathbb{T}^n$ we can consider the problem of studying dynamical properties identifying $\mathbb{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$ and maps $F : \mathbb{R}^n \to \mathbb{R}^n$ as liftings of $f : \mathbb{T}^n \to \mathbb{T}^n$, that is,

$$F(x_1 + k_1, ..., x_n + k_n) = F(x_1, ..., x_n) + (k_1, ..., k_n),$$

for any, $(k_1, ..., k_n) \in \mathbb{Z}^n$.

As in $\mathbb{S}^1$, this set of maps is shy for prevalent invariant theory but we can ask about G-prevalent properties like the K-S theorem.

In this case we should take

$$\mathcal{H} = \{F : \mathbb{R}^n \to \mathbb{R}^n | F(x_1 + k_1, ..., x_n + k_n) = F(x_1, ..., x_n) + (k_1, ..., k_n)\},$$

or, in a cone we impose some requirements concerning the derivative of F, as the control of the Lyapunov exponents, etc.

d) Twist maps

For nonconservative differentiable twist maps $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ (see [6] for references) one can consider liftings

$$F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R},$$

[1] Probabilities on TM which minimizes the integral $\int_{TM} L(v, v) d\mu(x, v)$ under a suitable holonomic condition.

such that

1) $F(x_1 + k, x_2) = F(x_1, x_2) + (k, 0)$, for any $k \in \mathbb{Z}$ (the lifting property);

2) $\exists\, c > 0$ such that $0 < c \leq \frac{\partial F_1}{\partial x_2} \leq \frac{1}{c}$ (the uniform twist condition).

The first assumption is a linear property, but for the second one we get just an open cone [2] of maps $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, so we get G-prevalent properties for this class of maps by considering

$$\mathcal{H} = \{F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} |\text{ such that 1 and 2 are true }\}.$$

We observe that for the important class of twist conservative maps this direct approach does not works, but one can try to find another group action G, in such way that $\det(DF) = 1$ is preserved.

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