Notes on Thermodynamic Formalism

Ricardo Mañé

February 3, 2023

Contents

Preface

These notes were written by Ricardo Mañé in draft form 30 years ago. At that time, Marcos Craizer prepared - from the above mentioned manuscript and with his agreement - what is basically the current latex-pdf version of the text. For several reasons the latex version of these notes were lost for some time. They have been briefly revised in our days by Artur O. Lopes, but with the utmost care to be faithful to the original version. Ricardo passed away in 1995. We believe this short and elegant presentation of basic results in Thermodynamic Formalism is still very valuable today. For the benefit of the reader, a few references have been added at the end of the text citing some more recent books on the subject.

Artur O. Lopes

Porto Alegre, February 3, 2023

1 Introduction

In this chapter, we will state several results related to the Ruelle operator and to the topological pressure of expanding maps. We point out that the Bernoulli shift is a very important case where the results we will present here can be applied. The proofs of the main results will be presented in the next chapters.

Given a compact metric space (X, d), we denote by $B(X)$ the Borel sigma-algebra on X, $C(X)$ the set of real continuous function $\phi : X \to \mathbf{R}$ and $\mathcal{M}(X)$ denotes the set of Borel probabilities on X.

Recall the definition:

Definition 1.1. *A continuous map T from a compact metric space (X, d) to itself is expanding if there exist $c > 0$ and $\lambda > 1$ such that, for any $x, y \in X$ and $n \in \mathbb{N}$, $d(T^n(x), T^n(y)) > c\lambda^n d(x, y)$.*

$\mathcal{M}(T)$ denotes the set of T-invariant probabilities.

When $X = \Omega = \{1, 2, ..., m\}^{\mathbb{N}}$, the shift transformation $\sigma : \Omega \to \Omega$ given by $\sigma(x_0, x_1, x_2, x_3, ...) = (x_1, x_2, x_3, x_4, ...)$ is an example of a continuous expanding transformation acting in a compact metric space, when considering the metric described by (5).

Sometimes, we will assume the map T is differentiable and we will be able to give a more precise description of the results using the derivative of the map. An equivalent definition of expanding map in this case is:

Definition 1.2. *Let M be a compact manifold without boundary. A map $T : M \hookleftarrow$ is expanding if it is of class C^1 and there exists $0 < \lambda < 1$ such that $\| D_xT \cdot v \| \geq \lambda^{-1} \| v \|$ for any $x \in M$, $v \in T_xM$.*

We shall prove results concerning the existence of some invariant measures which have significant properties from the dynamic point of view. The first of these results shows that every expanding differentiable map has one and only one invariant measure equivalent to the Lebesgue measure obtained as the limit of iteration of the Lebesgue measure by the map.

A map $T : M \hookleftarrow \in C^{-1}$ is Hölder-C^1 if $det\, D_xT$ is Hölder-continuous. Denote by $C^{1+\gamma}$ the space of Hölder-C^1-maps with Hölder-constant γ.

Definition 1.3. *A probability $\mu \in \mathcal{M}(X)$ is exact with respect to T if for any $A \in \cap_{n\geq 0} T^{-n}(B(X))$, then $\mu(A) = 0$ or $\mu(A) = 1$. Here $B(X)$ denotes the Borel σ-algebra of X.*

If $\mu \in \mathcal{M}(T)$ is **exact** then it is **ergodic** (see [11] , [10] or [6]).

Notation We will use the following notation, for $\phi \in C(X)$ and $\nu \in \mathcal{M}(X)$: we denote $\langle \phi, \nu \rangle$ the value $\int \phi(x) d\nu(x)$.

Definition 1.4. *For a given operator $\mathcal{L}$ from $C(X)$ to itself, the dual of $\mathcal{L}$ is the operator $\mathcal{L}^*$ defined from the dual space $C(X)^* = \mathcal{S}(X)$ (the space of signed-measures) to itself defined in the following way: $\mathcal{L}^*$ is the only operator from $\mathcal{S}(X)$ to itself such that for any $\phi \in C(X)$ and $\nu \in \mathcal{S}(X)$*

$$\langle \mathcal{L}(\phi), \nu \rangle = \langle \phi, \mathcal{L}^*(\nu) \rangle.$$

Remark 1.5. *Such $\mathcal{L}^*$ operator is well defined by the Riesz Theorem. This is so because for a given fixed $\nu \in \mathcal{S}(X)$ the operator $\mathcal{H}$ from $C(X)$ to $\mathbb{R}$ given by $\mathcal{H}(\phi) = \langle \mathcal{L}(\phi), \nu \rangle = \int \mathcal{L}\phi(x) d\nu(x)$ satisfies the hypothesis of Riesz Theorem, therefore there exists a signed-measure μ such that $\int \mathcal{L}\phi(x) d\nu(x) = \mathcal{H}(\phi) = \int \phi(x) d\mu(x) = \langle \phi, \mu \rangle$. Hence, by definition $\mathcal{L}^*(\nu) = \mu$.*

Theorem 1.6. *Let $T : M \hookleftarrow$ be a Hölder-C^1 expanding map. There exists a unique $\mu \in \mathcal{M}(T)$ absolutely continuous with respect to the Lebesgue measure-m. Moreover, μ satisfies:*

(1) $\frac{d\mu}{dm} \in C^\gamma(M)$ and is strictly positive
(2) μ is exact
(3) $h(\mu) = \int_M \log \mid det\, T' \mid d\mu$
(4) $h(\eta) < \int_M \log \mid det\, T' \mid d\eta$, for any $\eta \in \mathcal{M}(T)$, $\eta \neq \mu$.
(5) $\underset{n\to\infty}{\frac{m(T^{-n}(A))}{m(M)}} = \mu(A)$ for any Borel set A.

Consider now the question of studying the asymptotic distribution of the pre-images of a point x by T^n, when $n \to \infty$.

Definition 1.7. *Define $\mu_n(x) \in \mathcal{M}(M)$ by*

$$\mu_n(x) = \frac{1}{d^n} \sum_{T^n(y)=x} \delta_y$$

where $d = \# f^{-1}(a)$ independs on a.

Theorem 1.8. *Let $T : M \hookleftarrow$ be an expanding map. There exists $\mu \in \mathcal{M}(T)$ such that $\mu = \lim_{n\to\infty} \mu_n(x)$ for any $x \in M$. Moreover μ satisfies:*

(1) μ is exact and positive on open sets
(2) $h(\mu) = \log d$
(3) $h(\eta) < \log d$ for any $\eta \in \mathcal{M}(T)$, $\eta \neq \mu$.

Definition 1.9. *The above defined measure μ is called the maximal measure*

Definition 1.10. *Suppose that $T : M \hookleftarrow$ is a continuous map and $\psi : M \to \mathbf{R}$ is a continuous function. Remember that we denote by $C(M)$ the space of continuous functions on M. Define $\mathcal{L}_\psi : C(M) \hookleftarrow$ by*

$$\mathcal{L}_\psi \phi(x) = \sum_{y \in T^{-1}x} e^{\psi(y)} \phi(y) \tag{1}$$

for any $\phi \in C(M)$ and $x \in M$. We call this operator the Ruelle-Perron-Frobenius Operator (Ruelle Operator for short)

The function $\psi : M \to \mathbf{R}$ is usually called the potential (a terminology originated in Statistical Mechanics).

It is quite easy to see that:

$$\mathcal{L}^n_\psi \phi(x) = \sum_{y \in T^n(x)} e^{\psi(y)+\psi(T(y))+\psi(T^2(y))+\ldots+\psi(T^{n-1}(y))} \phi(y) \tag{2}$$

Theorem 1.11. *Ruelle Theorem - Let $T : M \hookleftarrow$ be an expanding map and $\psi : M \to \mathbb{R}$ be Hölder-continuous. Then there exist $h : X \to \mathbb{R}$ Hölder-continuous and strictly positive, $\nu \in \mathcal{M}(X)$ and $\lambda > 0$ such that:*

(1) $\int h d\nu = 1$
(2) $\mathcal{L}_\psi h = \lambda h$
(3) $\mathcal{L}^*_\psi \nu = \lambda \nu$
(4) $\| \lambda^{-n} \mathcal{L}^n_\psi \phi - h \int \phi d\nu \|_{C(X)} \to 0$ for any $\phi \in C(X)$.
(5) h is the unique positive eigenfunction of $\mathcal{L}_\psi$, except for multiplication by scalars.

(6) The probability $\mu = \mu_\psi = h\nu$ is T-invariant (that is, $\mu \in \mathcal{M}(T)$), exact, positive on open sets and satisfies

$$\log \lambda = h(\mu) + \int \psi d\mu.$$

(7) For any $\eta \in \mathcal{M}(T)$, $\eta \neq \mu$

$$\log \lambda > h(\eta) + \int \psi d\eta$$

(8) For any probability $w \in \mathcal{M}(X)$,

$$\lim_{n\to\infty} \frac{\mathcal{L}_\psi^{n*} w}{\lambda^n} = \nu$$

Definition 1.12. *Given a continuous potential $\psi : X \to \mathbf{R}$, the value*

$$P(\psi) = \sup_{\eta\in\mathcal{M}(T)} \{h(\eta) + \int \psi d\eta\},$$

is called the Topological Pressure of ψ. A probability μ_ψ attaining the maximal value $P(\psi)$ will be called an equilibrium probability for ψ.

Corolary 1.13. *If the potential ψ is Hölder-continuous, then the equilibrium probability μ_ψ for ψ is unique and satisfies $\mu_\psi = h\nu$. Moreover, $P(\psi) = \log \lambda$.*

The proof follows from (6) and (7) of Theorem 1.11

Theorem 1.14. *Let $T : X \hookleftarrow$ be an expanding map and $\psi : X \to \mathbb{R}$, $\overline{\psi} : X \to \mathbb{R}$ be Hölder-continuous. Then the following properties are equivalent:*

(1) $\mu_\psi << \mu_{\overline{\psi}}$
(2) $\mu_{\overline{\psi}} << \mu_\psi$
(3) $\mu_{\overline{\psi}} = \mu_\psi$
(4) For any $n > 0$ and $x \in X$ with $T^n(x) = x$ it holds that

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi(T^j x) - \frac{1}{n}\sum_{j=0}^{n-1} \overline{\psi}(T^j x) = \log \lambda_\psi - \log \lambda_{\overline{\psi}}.$$

(5) There exists $u \in C(X)$ such that

$$u \circ T - u = \log \lambda_\psi - \log \lambda_{\overline{\psi}} + (\overline{\psi} - \psi).$$

Observe that condition (4) is extremely strong and “in general” different “potentials” ψ induce mutually singular probabilities μ_ψ.

Let us see now how Theorems 1.6 and 1.8 follow from Theorem 1.11.

Let us prove first Theorem 1.6. Let $\psi(x) = -\log \mid det\, T'(x) \mid$ which is Hölder-continuous because T is Hölder-C^1. If $\varphi \in C^\circ(M)$

$$\int \varphi d(\mathcal{L}_\psi^* m) = \int \mathcal{L}_\psi \varphi dm = \int \left(\sum_{y \in T^{-1}x} \mid det\, T'(y) \mid^{-1} \varphi(y) \right) dm(x).$$

Take a collection of disjoint open sets $A_1, ..., A_n$ which cover M except for a set of Lebesgue measure zero and such that $T^{-1}(A_j)$ consisting of a finite number of disjoint open sets restricted to which T is a diffeomorphism. This can be done using the compactness of M and the fact that $det\, T'(x)$ does not vanish. Then

$$\begin{aligned} \int \varphi d(\mathcal{L}_\psi^* m) = & \sum_j \int_{A_j} \left(\sum_{y \in T^{-1}x} \mid det\, T'(y) \mid^{-1} \varphi(y) \right) dm(x) \\ = & \sum_j \int_{T^{-1}(A_j)} \varphi dm = \int \varphi dm. \end{aligned}$$

Therefore $\mathcal{L}_\psi^* m = m$. Let λ, h and ν be given by Theorem 1.11. Then

$$\frac{1}{\lambda^n} \mathcal{L}_\psi^n \varphi \overset{C(X)}{\longrightarrow} h \int \varphi d\nu$$

for any $\varphi \in C(X)$. Integrating with respect to m

$$\frac{1}{\lambda^n} \int \varphi d(\mathcal{L}_\psi^{*^n} m) \longrightarrow \int hdm \int \varphi d\nu.$$

Hence

$$\frac{1}{\lambda^n} \int \varphi dm \longrightarrow \int hdm \int \varphi d\nu.$$

Taking $\varphi \equiv 1$ and since $h > 0$, the relation above shows that $\lambda = 1$ and $\int hdm = m(X)$. Therefore $\int \varphi dm = m(M) \int \varphi d\nu$ for any $\varphi \in C()$. It results that $\nu = m/m(M)$.

The probability $\mu = h\nu$, satisfies then the conditions (1), (2), (3) and (4) of the Theorem 1.6. Observe that condition (1) actually implies the equivalence between μ and m. The claim concerning the uniqueness of μ follows then from the fact that if $\mu_1 \in \mathcal{M}(T)$, $\mu_1 << \mu$ then $\mu_1 = \mu$ since μ is ergodic. Property (5) follows from the fact that μ is mixing.

Let us prove now Theorem 1.8. Take $\psi \equiv 0$ and let λ, h and ν be given by Theorem 1.11. Then

$$\mathcal{L}_\psi 1(x) = \sum_{y \in T^{-1}x} 1(y) = d.$$

Because of the part (5) of Theorem 1.14, $d = \lambda$ and $h \equiv 1$. Also, part (4) shows that

$$\frac{1}{d^n} \sum_{y \in T^{-n}x} \varphi(y) \longrightarrow \int \varphi d\nu$$

for any $\varphi \in C^\circ(M)$. This proves Theorem 1.8.

Definition 1.15. *A continuous function $J : X \to \mathbb{R}$ is called the Jacobian of $T : X \to X$ with respect to $\mu \in \mathcal{M}(X)$, if*

$$\mu(f(A)) = \int_A J d\mu,$$

for any Borel set A such that $T\mid_A$ is injective.

It is easy to prove that if such a J exists it is unique. Some ergodic properties of μ can be analyzed through J.

Theorem 1.16. *Suppose that J is Hölder-continuous and strictly positive. Then*

(a) $h(\mu) = \int \log J d\mu$
(b) μ *is exact.*

Consider now the question of finding a T-invariant probability with Jacobian $J > 1$ given. It is easy to prove that every function $J > 1$ that is Jacobian of T with respect to some T-invariant probability must satisfy

$$\sum_{T(x)=y} \frac{1}{\mid J(x) \mid} = 1 \tag{3}$$

for any $y \in X$. This condition is also sufficient. In fact the the following result is true:

Theorem 1.17. *Let $T : X \hookleftarrow$ be an expanding map and $J : X \to \mathbb{R}$ strictly positive and Hölder-continuous. Let λ, h, ν be given by Theorem 1.11 with $\psi = -\log J$. Then the Jacobian of T with respect to $\mu = h\nu$ is*

$$J_\mu T = \lambda J \frac{h \circ T}{h}.$$

Condition (2) Theorem 1.11 implies that $h \equiv 1$ and $\lambda = 1$ in the last theorem. Hence $P(-\log J) = 0$.

Theorem 1.18. *Suppose ψ is Holder continuous, μ_ψ is the equilibrium state associated with ψ, h is the eigenfunction associated with λ in Theorem 1.11 then the Jacobian J_ψ of the probability μ_ψ is given by:*

$$J_\psi(x) = \lambda e^{-\psi(x)} \frac{h \circ T(x)}{h(x)} \tag{4}$$

In section 5 we will present results about the differentiability of the pressure and in section 7 we will present results about the differentiability of the Hausdorff dimension of the Julia set of rational maps.

2 Properties of expanding maps

Let (K, d) be a compact metric space.

Definition 2.1. *$T : K \hookleftarrow$ continuous is said to be an expanding map if there exist $r > 0$, $0 < \lambda < 1$ and $c > 0$ such that:*

(a) $x \neq y$ and $T(x) = T(y) \Longrightarrow d(x, y) > c$.

(b) $\forall\, x \in K$ and $a \in T^{-1}(x)$ there exists $\varphi : B_r(x) \to K$ such that $\varphi(x) = a$ and

$$T(\varphi(y)) = y, \forall\, y \in B_r(x)$$
$$d(\varphi(z), \varphi(w)) \leq \lambda d(z, w), \forall\, z, w \in B_r(x)$$

Examples:

(a) Let M be a compact manifold without boundary and $T : M \hookleftarrow$ a C^{-1}-map. Then T is an expanding map by the definition above iff T is an expanding map by the first definition, that is, $\exists\, 0 < \lambda < 1$ such that

$$\| D_x T \cdot v \| \geq \frac{1}{\lambda} \| v \|,$$

$\forall\, x \in M$, $\forall\, v \in T_x M$. For verifying this, suppose first that T is an expanding map by the first definition.

Given $x \in M$, $\exists$ neighborhood V of x such that $T^{-1}(V)$ consists of a finite number of open sets $W_1, ..., W_n$ such that $f\, |_{W_j}$ is a diffeomorfism. Cover M with neighborhoods V of this type and let r_0 be the Lebesgue number of this covering.

Let $r_1 > 0$ be such that if z, $w \in M$ and $d(z, w) < r_1$, then $\exists$ a geodesic $\beta : [0, 1] \to M$ with $\beta(0) = z$, $\beta(1) = w$ and $d(z, w) = l(\beta)$ (= length of β). Let $r = \min(r_0, r_1)$.

Then if $\varphi : B_r(x) \to M$ is a branch of T^{-1},

$$\begin{aligned} d(\varphi(z), \varphi(w)) \leq & \quad l(\varphi\beta) = \int_0^1 \| \varphi'(\beta(t)) \cdot \beta'(t) \| \, dt \leq \\ \leq & \quad \frac{1}{\lambda} \int_0^1 \| \beta'(t) \| \, dt = \frac{1}{\lambda} d(z, w) \end{aligned}$$

This verifies condition (b) of the above definition. Condition (a) is very easy to verify.

Reciprocally, let $x \in M$. Take $y \in \varphi(B_r(f(x))$, where φ is such that $\varphi(T(x)) = x$, and sufficiently near x in such a way that x and y can be joined by a geodesic β with $d(x,y) = l(\beta)$. Then

$$l(\beta) = d(x,y) \leq \lambda d(T(x), T(y)) \leq \lambda l(T \circ \beta)$$

$$\Rightarrow \int_0^1 \parallel \beta'(t) \parallel dt \leq \int_0^1 \lambda \parallel T'(\beta(t))\beta'(t) \parallel dt.$$

If y tends to x through the geodesic β with $\beta'(0) = v$ it results that

$$\parallel v \parallel \leq \lambda \parallel T'(x)v \parallel .$$

This proves the claim.

Below we will list a series of interesting cases related to the topics covered here.

Definition 2.2. *Let M be a manifold, $T : M \hookleftarrow$ a C^{-1}-map and $\wedge \subset M$ a completely invariant compact set. We say that $\wedge$ is* **expanding** *if*

(1) $\wedge$ is isolated, that is, there exists a neighborhood U of $\wedge$ such that

$$\cap_{n \geq 0} T^{-n} U = \wedge.$$

(2) There exists $0 < \lambda < 1$ such that

$$\parallel D_x T \cdot v \parallel \geq \frac{1}{\lambda} \parallel v \parallel$$

for any $x \in \wedge$ and $v \in T_x M$.

It is easy to verify that $T |_\wedge$ is an expanding map.

This example occurs when T is an Axiom A rational map of the Riemann sphere $\overline{\mathbf{C}}$ and $\wedge$ is its Julia set (see [5] or [9]) and also Section 7.

I) If $T : K \hookleftarrow$ is an expanding map and $\wedge$ an invariant compact set is not true that necessarily $T |_\wedge$ is also an expanding map.

For example, let p be a fixed point in K and $\{q_i\}_{i \in \mathbb{Z}}$ an orbit of T with the following properties:

(1) $q_{N+m} = q_N$ for some $N \in \mathbf{N}$ and some $m > 0$.

(2) $\lim_{i\to -\infty} q_i = p$.

Let p_0 be a pre-image of p different from p, and define

$$\wedge = \{p_o\} \cup \{p\} \cup \{q_i\}_{i\in \mathbf{N}}.$$

Then $T(\wedge) = \wedge$, but it is easy to see that $T \mid_{\wedge}$ cannot be an expansive map.

II) Let $A = (a_{ij})$ be a $m \times m$ matrix of 0's and 1's. Define the operator σ in $B^+(A) = \{(x_0, x_1, ...) \mid x_i \in \{1, ..., m\}$ and $a_{x_i, x_{i+1}} = 1\}$ by $\sigma(x_0, x_1, ...) = (x_1, x_2, ...)$. The pair $(\sigma, B^+(A))$ is called a unilateral subshift of finite type. Define a metric in $B^+(A)$ by

$$d(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{1}{2^n} \mid \alpha(n) - \beta(n) \tag{5}$$

where $\alpha = (\alpha(0), \alpha(1), ...)$ and $\beta = (\beta(0), \beta(1), ...)$.

With this metric σ is an expanding map, with $r = 1$, $\lambda = 1/2$ and $c < 1$. Because if $\alpha \neq \beta$ and $\sigma(\alpha) = \sigma(\beta)$ then $\alpha(0) \neq \beta(0)$ and so $d(\alpha, \beta) \geq 1 > c$. This verifies (a). Also if α and β satisfy $d(\alpha, \beta) < 1$ then $\alpha(0) = \beta(0)$. Hence the pre-images by σ of α and β are (x_j, α) and (x_j, β) where $x_j \in \{1, ..., m\}$ and $a_{x_j, \alpha(0)} = 1$, and $d((x_j, \alpha), (x_j, \beta)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mid \alpha(n-1) - \beta(n-1) \mid = \frac{1}{2}\, d(\alpha, \beta)$. this verifies (b). The dynamics in this case is called a shift of finite type.

III) Let $T : \mathcal{S}^1 \hookleftarrow$ be a C^2-map with degree greater than one and such that $T'(x) \neq 0, \forall x$. We define $\sum(T) = (\cup$ basin of attractors). Then if all periodic points of T are hyperbolic (this is a generic property) $T \mid_{\sum(T)}$: $\sum(T) \hookleftarrow$ is an expanding map.

Definition 2.3. *Let $T : K \hookleftarrow$ be an expanding map and $\mathcal{S} \subset K$.*

Then $g : \mathcal{S} \to K$ is a contractive branch of T^{-n} if $T^n g(x) = x$, $\forall x \in \mathcal{S}$ and

$$d(T^j g(x), T^j g(y)) \leq \lambda^{n-j} d(x, y)$$

for any x, $y \in \mathcal{S}$, $0 \leq j \leq n$.

It is easy to see that given $x \in K$ and $a \in T^{-n}(x)$ there exists $g : B_r(x) \to K$ contractive branch of T^{-n} such that $g(x) = a$.

Lemma 2.4. *Let $B(n,\xi,x) = \{z \in K \mid d(T^j z, T^j x) \leq \xi$, for $0 \leq j \leq n\}$. Then there exists $\xi_0 > 0$ such that if $0 < \xi < \xi_0$; then*

(a) For $n \geq 1$, let $g : B_r(T^n(x)) \to K$ be a contractive branch of T^{-n} such that $g(T^n(x)) = x$. Then $B(n,\xi,x) = g(B_\xi(T^n(x))$.

(b) If $d(T^n z, T^n w) \leq \xi$, $\forall\, n \geq 0 \Rightarrow z = w$.

Proof. Suppose $n = 1$ and let $\xi_0 < \min\left(r, \frac{C}{1+\lambda}\right)$, where $r > 0$, $0 < \lambda < 1$, $c > 0$ are given by Definition 2.1. If $z \in B(1,\xi,x)$ then $d(z,x) \leq \xi$ and $d(T(z),T(x)) \leq \xi$ and so $d(g(T(z)),x) \leq \lambda\xi$. By the triangular desiguality, $d(z, g(T(z))) < C$. It follows that $z = g \circ f(z)$ and so $z \in g(B_\xi(T(x)))$. This proves (a) for $n = 1$.

With analogous arguments we complete the proof of (a) by induction.

If $d(T^n(z), T^n(w)) \leq \xi$, $\forall\, n \geq 0$, by the first item $d(z,w) \leq \lambda^n \xi$, $\forall\, n$. Hence $z = w$, and this proves (b). □

Remark 2.5. *ξ_0 is called an expansivity constant for f.*

Lemma 2.6. *For any $\xi > 0$, there exists $\delta > 0$ such that if a sequence $\{x_n \mid n \geq 0\}$ satisfies $d(T(x_n), x_{n+1}) < \delta$, $\forall\, n \geq 0$, then there exists $x \in K$ satisfying $d(T^n(x), x_n) < \xi$, $\forall\, n \geq 0$.*

Proof. Let $\varphi_n : B_r(x_n) \to K$ be contractive branches of T^{-1} with $\varphi_n(T(x_{n-1})) = x_{n-1}$. Take $\delta < \min\left(\frac{1-\lambda}{\lambda} \cdot \xi, \frac{r}{\lambda} - r\right)$.

If $z \in \overline{B_r(x_n)}$ then $d(z, Tx_{n-1}) \leq r+\delta$ and so $d(\varphi_n z, x_{n-1}) \leq \lambda(r+\delta) < r$. It results that $\varphi_n(\overline{B_r(x_n)}) \subset \overline{B_r(x_{n-1}), \forall\, n} \geq 1$.

Consider the sequence $\{\varphi_1, ..., \varphi_n(\overline{B_r(x_n)})\}_{n\geq 1}$. It is a decrescent sequence of compact sets whose diameters tend to zero. Hence $\cap_{n\geq 1} \varphi_1 ... \varphi_n(\overline{B_r(x_n)})$ consists of a unique point that we shall call x.

Let $l \in \mathbf{N}$. Then

$$
\begin{aligned}
d(T^l x, x_l) \leq & \quad \lambda d(T^{l+1}x, Tx_l) \leq \lambda d(T^{l+1}x, x_{l+1}) + \lambda d(Tx_l, x_{l+1}) \leq \\
\leq & \quad \lambda d(Tx_l, x_{l+1}) + \lambda^2 d(Tx_{l+1}, x_{l+2}) + ... + \lambda^k d(Tx_{l+k-1}, x_{l+k}) + \\
+ & \quad \lambda^k d(T^{l+k}x, x_{l+k}). \qquad (*)
\end{aligned}
$$

Making $h \to \infty$

$$d(f^l x, x_l) \leq \frac{\delta\lambda}{1-\lambda} < \xi.$$

$\square$

Lemma 2.7. *In last Lemma, if the sequence $\{x_n, n \geq 0\}$ is periodic of period N, then x is periodic of period N, if it is assumed that $2\xi < \xi_0$, where ξ_0 is the expansivity constant given by Lemma 2.4.*

Proof. Consider the orbits $(x, T(x), T^2(x), ...)$ and $(T^N(x), T^{N+1}(x), ...)$. Since they are 2ξ close one to each other, using the Lemma 2.7, (b), it results that $x = T^N(x)$. $\square$

Remarks:

(1) We shall use in the future the following refinement of previous Lemmas. In case that $x_0, x_1, ...$ is periodic of period N and $x_{j+1} = fx_j, j = 0, ..., N-2$ and $d(T(x_{N-1}), x_0) < \delta$, it results from (*) that

$$\begin{aligned} d(T^j(x), T^j(x_0)) = d(T^j(x), x_j) \leq & \quad \lambda^{N-j} d(T^N(x), T(x_{N-1})) \\ = & \quad \lambda^{N-j} d(x, T(x_{N-1})) \end{aligned}$$

for $0 \leq j \leq N-1$.

(2) In the following Lemma, we shall use (*) with $l = 0$, i.e.,

$$d(x, x_0) \leq \sum_0^\infty \lambda^{n+1} d(fx_n, x_{n+1}).$$

Lemma 2.8. *Given $\xi_0 > 0$, there exists $\delta_0 > 0$ such that for all $\xi > 0$, there exist $N \in \mathbf{N}$ and $\delta_1 > 0$ such that if a sequence $\{x_n \mid n \geq 0\}$ satisfies*

$$\begin{aligned} d(T(x_n), x_{n+1}) \leq \delta_0, & \qquad \forall\, n \leq 0 \\ T(x_n) = x_{n+1}, & \qquad \forall\, 1 \leq n \leq N \\ d(T(x_0), x_1) < \delta_1 & \end{aligned}$$

then there exists $x \in K$ such that

$$d(T^n(x), x_n) \leq \xi_0, \quad \forall\, n \geq 0$$

$$d(x, x_0) < \xi.$$

Proof. Given $\xi_0 > 0$, take δ_0 as in a previous Lemma. By item (2) in the remark,

$$\begin{aligned} d(x, x_0) \leq \sum_{0}^{\infty} \lambda^{n+1} d(T(x_n), x_{n+1}) \leq & \quad \lambda\delta_1 + \sum_{N+1}^{\infty} \lambda^{n+1} d(T(x_n), x_{n+1}) \\ \leq & \quad \lambda\delta_1 + \delta_0 \frac{\lambda^{N+2}}{1-\lambda}. \end{aligned}$$

If N is sufficiently large and δ_1 sufficiently small it results that $d(x, x_0) < \xi$. □

Definition 2.9. *A sequence $\{x_n \mid n \geq 0\}$ is a pre-orbit of x if $x = x_0$ and $T(x_{n+1}) = x_n$.*

Lemma 2.10. *If $d(x, y) < r$ and $\{x_n \mid n \geq 0\}$ is a pre-orbit of x, then there exists a pre-orbit of y, $\{y_n \mid n \geq 0\}$ such that $d(x_n, y_n) \leq \lambda^n d(x_0, y_0)$.*

Proof. Consider $g : B_r(x) \to K$ a contractive branch of T^{-n} with $g(x) = x_n$ and define $y_n = g(y)$. □

Lemma 2.11. *Denote by $Per\, T = \{$ periodic points of $T\}$ and $\wedge = \overline{Per\, T}$. Then $T\mid_{\wedge} : \wedge \hookleftarrow$ is an expansive map.*

Proof. Let $r > 0$, $0 < \lambda < 1$ and $c > 0$ be given by Definition 2.1. Let δ_0 be given by Lemma 2.4 with ξ_0 an expansivity constant and define $r_1 = \min(r, \delta_0)$. For proving that $T |_{\wedge}$ is an expanding map it is sufficient to prove that $\varphi(B_{r_1}(x) \cap \wedge) \subset \wedge$, if $x \in \wedge$ and $\varphi : B_r(x) \to K$ is a contractive branch of T^{-1} with $\varphi(x) = a \in \wedge$. Let $z \in B_{r_1}(x) \cap \wedge$. We must verify that $\varphi(z) \in \wedge$. Without loss of generality, we can assume that z and a (and therefore x) are periodic. Let $s =$period of $a =$period of x, and $t =$period of z.

Let $w = \varphi(z)$. Take a pre-orbit $\{w_n \mid n \geq 0\}$ of w asymptotic to the periodic pre-orbit of a and a pre-orbit $\{x_n \mid n \geq 0\}$ of x asymptotic to the periodic pre-orbit of z, as in a previous Lemma.

Given $\xi > 0$, take N large as above and consider the periodic δ_0-pseudo-orbit $w, x_{Nt}, ..., x_1, w_{s-1}, ..., w_1, w, ...$

By Lemma 2.6, there exists p such that $d(p, w) < \xi$ and such that its orbit ξ_0-shadows the δ_0-pseudo-orbit above.

By a previous Lemma, p is periodic and therefore $w \in \wedge$.

□

Theorem 2.12. $K = \cup_{n \geq 0} T^{-n}(\overline{Per\, T})$.

Proof. Let $x \in K$ and let $w(x)$ be the w-limit set of the orbit of x, that is,

$$\{y \mid \text{ there exists a sequence } \{n_k\}_k \subset \mathbf{N}, n_k \to \infty \text{ such that } T^{n_k}(x) \to y\}.$$

If $y \in w(x)$, and $\xi > 0$ is arbitrary, let $\delta > 0$ be given by Lemma 2.10. Take l and N such that $d(T^l(x), y) < \delta/2$ and $d(T^{l+N}x, y) < \delta/2$. So

$$T^l(x), T^{l+1}(x), ..., T^{l+N-1}(x), T^l(x), ...$$

is a periodic δ-pseudo-orbit.

By a previous, it can be ξ-shadowed by a periodic orbit. Therefore $y \in \overline{Per\, T}$.

Consider the map $T |_{\wedge}$, which is expanding. Fixing $\xi > 0$, let $\delta > 0$ be given by Lemma 2.11 applied to $T |_{\wedge}$. Let $\delta' < \delta/2$ be such that $d(z, w) < \delta'$ implies $d(T(z), T(w)) < \delta/2$ for any z, $w \in K$.

Let $x \in K$. Since $w(x) \subset \wedge$, there exist $N \in \mathbf{N}$ and $\{x_n\}_{n\geq 0} \subset \wedge$ such that $d(T^{N+n}(x), (x_n)) < \delta'$, for each $n \geq 0$. Hence

$$\begin{aligned} d(T(x_n), x_{n+1}) \leq & \quad d(T(x_n), T^{N+n+1}(x)) + d(T^{N+n+1}(x), x_{n+1}) \\ \leq & \quad \delta/2 + \delta/2 = \delta \end{aligned}$$

for each $n \geq 0$. By Lemma 2.11, there exists $z \in \wedge$ such that $d(T^n(z), x_n) \leq \xi$, for each $n \geq 0$. Hence

$$d(T^n(z), T^n(T^N(x))) \leq \delta + \xi$$

for each $n \geq 0$. But by the expansiveness of T, if ξ and δ are small, $T^N(x) = z$. Therefore $x \in T^{-N}(\wedge)$. $\square$

Theorem 2.13. *There exist unique disjoint compact sets* $\wedge_i^{(m)}$, $i = 1, ..., n_m$, $m = 1, ..., M$ *such that*

(a) $T(\wedge_i^{(m)}) = \wedge_{i+1}^{(m)}, \quad 1 \leq i < n_m, \quad 1 \leq m \leq M$

$\quad T(\wedge_{n_m}^{(m)}) = \wedge_1^{(m)}, \quad 1 \leq m \leq M.$

(b) $\cup_{i,m} \wedge_i^{(m)} = \wedge (= \overline{Per\, T})$.

(c) $T^{n_m} |_{\wedge_i^{(m)}} \colon \wedge_i^{(m)} \hookleftarrow$ *is topologically mixing.*

More over, the following properties are valid:

(d) $T^{n_m} |_{\wedge_i^{(m)}} \colon \wedge_i^{(m)}$ *is an expanding map.*

(e) For each open set $V \subset \wedge_i^{(m)}$, *there exists* $N > 0$ *such that*

$$(T^{n_m})^N(V) = \wedge_i^{(m)}.$$

Proof. Let p and q be periodic points, and $\{p_n \mid n \geq 0\}$ and $\{q_n \mid n \geq 0\}$ their periodic pre-orbits. Define $p \sim q$ if there exist pre-orbits $\{p'_n \mid n \geq 0\}$ of p and $\{q'_n \mid n \geq 0\}$ of q such that $d(q'_n, p_n) \to 0$ and $d(p'_n, q_n) \to 0$.

Let us verify that $\sim$ is an equivalence relation in $Per\, T$. Clearly $\sim$ is reflexive and simetric. If $p \sim q$ and $q \sim r$, then there exist pre-orbits

$\{q'_n \mid n \geq 0\}$ of q and $\{r'_n \mid n \geq 0\}$ of r asymptotic to the periodic pre-orbits $\{p_n \mid n \geq 0\}$ of p and $\{q_n \mid n \geq 0\}$ of q respectively. Let $n_0 > 0$ be such that $d(r'_n, q_n) < r$ if $n > n_0$, and let $m_0 > n_0$ be a multiple of the periods of p and q. Then $d(r'_{m_0}, q) < r$.

By a previous Lemma, there exists a pre-orbit $\{r''_n \mid n \geq 0\}$ of r'_{m_0} asymptotic to $\{p_n \mid n \geq 0\}$. Since m_0 is a multiple of the period of this orbit, the pre-orbit $r'_0, r'_1, ..., r'_{m_0-1}, r''_0, r''_1, ...$ is asymptotic to $\{p_n \mid n \geq 0\}$. By simmetry, $r \sim p$.

Therefore $\sim$ is actually an equivalence relation em $Per\, T$. If $d(p,q) < r$, then $p \sim q$. This implies that there are only a finite number of classes of equivalence $X_1, ..., X_n$ and that they are open and closed in $Per\, T$. Besides, f transform classes of equivalence in classes of equivalence. Therefore, we can denote the classes by $X_i^{(m)}$, $1 \leq i \leq n_m$, $1 \leq m \leq M$ satisfying

$$T(X_i^{(m)}) = X_{i+1}^{(m)},\ 1 \leq i < n_m, 1 \leq m \leq M$$
$$T(X_{n_m}^{(m)}) = X_1^{(m)}, 1 \leq m \leq M.$$

Define $\wedge_i^{(m)} = \overline{X_i^{(m)}}$. Then conditions (a) and (b) are obviously satisfied.

Let us prove (d). As $f\mid_\wedge$ is an expanding map (Lemma II.9), $T^{n_m}\mid_\wedge$ is an expanding map. It follows then from $T^{-n_m}(\wedge_i^{(m)}) = \wedge_i^{(m)}$ that $T^{n_m}\mid_{\wedge_i^{(m)}}$ is an expanding map.

Let us prove now (e). Denote by g the map $T^{n_m}\mid_{\wedge_i^{(m)}}$ and let V be an open set of $\wedge_i^{(m)}$. Let $p \in V$ be periodic of g-period n_0 and let $\{p_n \mid n \geq 0\}$ be its g-periodic-pre-orbit. Let q be periodic in $\wedge_i^{(m)}$ and $\{q_n^{(j)} \mid n \geq 0\}$, $0 \leq j \leq n_0 - 1$, g-pre-orbits of q such that $d(q_n^{(j)}, p_{j+n}) \to 0$ when $n \to \infty$.

Let $\delta > 0$ be such that $B_\delta(p) \subset V$ e $N > 0$ be such that $d(q_n^{(j)}, p_{j+n}) < \delta$, when $n \geq N$, $0 \leq j \leq n_0 - 1$. Then if $n \geq N$, take $0 \leq j \leq n_0 - 1$ such that $j + n$ is a multiple of n_0 and so we obtain $q_n^{(j)} \in B_\delta(p) \subset V$ and $g^n(q_n^{(j)}) = q$.

In reality, since g is an expanding map, $g^n(B_\delta(p)) \supset B_r(q)$ if $\lambda^n < \delta$. Taking a finite number of $B_r(q)$ covering $\wedge_i^{(m)}$ we obtain (e).

The assertion (c) is corollary of (e).

Suppose now that there are a decomposition of $\wedge$ in disjoint compact sets $\wedge_i(m)$, $1 \leq i \leq n_m$, $1 \leq m \leq M$ satisfying (a) and (c). Let p and q be periodic points. From (a) it follows that if $p \sim q$, then p and q are in the same $\wedge_i^{(m)}$. From (c) it follows that if p and q are in the same $\wedge_i^{(m)}$, then $p \sim q$. Therefore this decomposition is well determined by (a) and (c). $\square$

Theorem 2.14. *If K is connected and $T : K \hookleftarrow$ is an expanding map, then T is topologically mixing.*

Proof. Let $r > 0$, $c > 0$ and $0 < \lambda < 1$ be given by Definition 2.1 applied to $f\ |_\wedge \colon \wedge \hookleftarrow$, which by Theorem 2.13 is an expanding map. Let $\xi < \min(r, \frac{c}{1+\lambda})$ e $\delta < \xi$ be such that if $d(z,w) < \delta$ then $d(fz, fw) < \xi$.

Claim: If $z \in f^{-1}(\wedge)$ and $d(z, \wedge) < \delta$, then $z \in \wedge$.

Proof of the Claim. In fact, there exists $w \in \wedge$ such that $d(z,w) < \delta$. Hence $d(T(z), T(w)) < \xi$. Considering the contractive branch of T^{-1}, $\varphi : B_r(T(w)) \cap \wedge \to \wedge$ such that $\varphi T(w) = w$ we obtain that $d(\varphi f z, z) < \delta + \lambda\xi < C$. Hence $\varphi T(z) = z$ and therefore $z \in \wedge$, proving the claim. $\square$

It follows from the claim $S = T^{-1}(\wedge) \backslash \wedge$ is closed. From the fact that $\wedge$ is invariant it follows that $\wedge$, S, $T^{-1}S$, $T^{-2}S$, ... is a disjoint collection of closed sets. By Theorem 2.13, such a collection covers K. If follows then by the Baire Theorem that one of them has non-empty interior. As T is an open map, this implies that $\wedge$ has non-empty interior.

Consider the decomposition of $\wedge$ given by Theorem 2.14 at least one of the $\wedge_i^{(m)}$ contains an open set V, and since $(T^{n_m})^N V = \wedge_i^{(m)}$ for some $N > 0$, it follows that $\wedge_i^{(m)}$ is open in K. But since K is connected, it results that $\wedge_i^{(m)} = K$. This proves the Theorem. $\square$

Suppose that f is an expanding topologically mixing map.

Definition 2.15. *ψ and φ in $C(K)$ are homologous if there exists $u \in C(K)$ such that $\psi = \varphi + u \circ T - u$. Denote it by $\psi \sim \varphi$.*

Theorem 2.16. *Suppose that ψ is γ-Hölder-continuous. Then*

$$\psi \sim 0 \Longleftrightarrow \Big[T^n(x) = x \Longrightarrow S_n\psi(x) \overset{def}{=} \sum_{j=0}^{n-1} \psi(T^j(x)) = 0 \Big].$$

In adition, the funtion u that satisfies $\psi = u \circ T - u$ is γ-Hölder-continuous.

Proof. If $\psi \sim 0$, $\psi = u \cdot f - u$, for some $u \in C(K)$. It results that $S_n\psi(x) = u(T^n(x)) - u(x)$. Therefore, if $T^n(x) = x$, $S_n\psi(x) = 0$.

Reciprocally, suppose that $S_n\psi(x) = 0$, for any x such that $T^n(x) = x$. Let $a \in K$ be transitive for f (i.e., $\{T^n(a)\}_{n\geq 0}$ is dense in K) and define u in the orbit of a by

$$u(T^n(a)) = u(a) + S_n\psi(a)$$

where $u(a)$ is arbitrarily defined.

Claim: u is γ-Hölder-continuous in the orbit of a.

Proof of the Claim. For $x_i > 0$ satisfying $(1-\lambda)x_i < r - \lambda$, take $\delta = \frac{1-\lambda}{\lambda} \cdot x_i$. Then by Lemma 2.4, if $d(T^m(a), T^{m+n}(a)) < \delta$, the δ-pseudo-orbit $T^m(a)$, $T^{m+1}(a)$, ...,$T^{m+n-1}(a)$, $T^m(a)$, ... can be x_i-shadowed by the orbit of a periodic point x of period n. In addition, this orbit satisfies $d(T^jx), T^{m+j}(a)) \leq \lambda^{n-j}d(x, T^{m+n}(a))$, for $0 \leq j \leq n-1$ according to Remark 2.5 following Lemma 2.4. Then

$$\begin{aligned}
\mid u(T^{m+n}(a)) - u(T^n(a)) \mid &= \mid S_{m+n}\psi(a) - S_n\psi(a) \mid = \mid S_n\psi(T^m(a)) \mid \\
&= \mid S_n\psi(f^m a) - S_n\psi(x) \mid \\
&\leq \sum_0^{n-1} \mid \psi(T^{m+j}(a)) - \psi(T^j(x)) \mid \\
&\leq C \sum_0^{n-1} d(T^{m+j}(a), T^j(x))^\gamma \leq C \sum_0^{n-1} (\lambda^{n-j}\xi)^\gamma \leq C' \cdot \delta^\gamma
\end{aligned}$$

where $C' = C\left(\frac{\lambda^\gamma}{1-\lambda^\gamma}\right)^2$. This proves the claim. □

It follows from the claim that u can be extended to a γ-Hölder-continuous function in K. Let $y \in K$, $y = \lim T^{n_j}a$. Then

$$\begin{aligned}
u(T(y)) - u(y) &= \lim_{j\to\infty} u(T^{n_j+1}(a)) - \lim_{j\to\infty} u(T^{n_j}(a)) \\
&= \lim_{j\to\infty} S_{n_j+1}\psi(a) - S_{n_j}\psi(a) \\
&= \lim_{j\to\infty} \psi(T^{n_j}(a)) = \psi(y).
\end{aligned}$$

Therefore $\psi = u \circ T - u$. □

Corolary 2.17. *Let ψ and φ be γ-Hölder-continuous functions. Then*

$$\psi \sim \varphi \Longleftrightarrow \big[T^n(x) = x \Longrightarrow S_n\psi(x) = S_n\varphi(x)\big].$$

In adition the function u that satisfies $\psi = \varphi + u \circ T - u$ is γ-Hölder-continuous.

3 Jacobians and Entropy

Let $T : K \hookleftarrow$ be a continuous and locally injective map and $\mu \in \mathcal{M}(T)$.

A continuous function $F : K \to \mathbf{R}$ is the Jacobian of T if

$$\mu(T(A)) = \int_A F d\mu$$

for any Borel set A such that $T\mid_A$ is injective. The Jacobian, if it exists, is unique *a.e.* and it is denoted by $J_\mu f$. If $\psi \in C^\circ(K)$ define $\mathcal{L}_\psi : C^\circ(K) \hookleftarrow$ by

$$(\mathcal{L}_\psi \varphi)(x) = \sum_{y \in T^{-1}(x)} e^{\psi(y)} \varphi(y)$$

for any $\varphi \in C^\circ(K)$.

Lemma 3.1. *Let $\nu \in \mathcal{M}(K)$ satisfying $\mathcal{L}_\psi^* \nu = \lambda\nu$, $\lambda > 0$.*
Then

$$J_\nu T = \lambda e^{-\psi} \ .$$

Let h be continuous and strictly positive and $\mu = h\nu$. Then

$$J_\mu T = \lambda e^{-\psi} \frac{h \circ T}{h}.$$

Proof. Let A be a Borel set such that $f\mid_A$ is injective.

Take a sequence $\{h_n\}_{n\geq 1}$ in $C^\circ(K)$ such that $h_n \longrightarrow \mathcal{X}_A \quad a.e. \quad [\nu]$ and $\| h_n \|_{C^\circ} \leq 2$, $\forall\, n \geq 1$. Then

$$\mathcal{L}_\psi(e^{-\psi} h_n)(x) = \sum_{y \in T^{-1} x} e^{\psi(y)} e^{-\psi(y)} h_n(y) = \sum_{y \in T^{-1}(x)} h_n(y).$$

This last expression converges to $X_{T(A)}(x) \quad a.e. \quad [\nu]$ and so, by the dominated convergence Theorem

$$\int \lambda e^{-\psi} h_n d\nu = \int \mathcal{L}_\psi(e^{-\psi} h_n) d\nu \to \nu(f(A)).$$

Hence

$$\int_A \lambda e^{-\psi} d\nu = \nu(T(A)).$$

Also

$$\int \lambda e^{-\psi}\frac{h\circ T}{h}h_n d\mu = \int \mathcal{L}_\psi(e^{-\psi}h\circ T h_n)d\nu$$
$$= \int \mathcal{L}_\psi(e^{-\psi}h_n)d\mu.$$

Since μ and ν are equivalent, this last expression converges to $\mu(T(A))$ and so

$$\int_A \lambda e^{-\psi}\frac{h\circ T}{h}d\mu = \mu(f(A)).$$

$\square$

From now on, we will suppose that T is a topologically mixing expanding map. If $\mu \in \mathcal{M}(K)$, define the support of μ as

$$supp(\mu) = \overline{\{x \in K \mid \text{ any neighborhood } V \text{ of } x, \mu(V) > 0\}}.$$

Lemma 3.2. *If $\mu \in \mathcal{M}(K)$ admits a Jacobian $J_\mu f$, then $supp(\mu) = K$.*

Proof. Suppose that there exists an open set V with $\mu(V) = 0$.

Cover V with Borel sets $A \subset V$ such that $T\mid_A$ is injective. Then

$$\mu(T(A)) = \int_A J_\mu(T)d\mu = 0.$$

Hence $\mu(T(V)) = 0$. Inductively, $\mu(T^n(V)) = 0$. But since there exists $n \in \mathbb{N}$ such that $T^n(V) = K$, this is a contradiction. $\square$

Lemma 3.3. *If $J_\mu T$ is strictly positive and Hölder-continuous, there exists $A > 0$ such that $\forall n$, if $g : S \to K$ is a contractive branch of T^{-n}, then*

$$\frac{J_\mu T^n(x)}{J_\mu T^n(y)} \leq A$$

for any x, $y \in g(S)$.

Proof. Since g is a contractive branch of T^{-n}, and x, $y \in g(S)$

$$d(T^j(x), T^j(y)) \leq \lambda^{n-j} d(T^n(x), T^n(y)) \leq \lambda^{n-j} d$$

for $0 \leq j \leq n$, where d=diameter(S). Then

$$\frac{J_\mu T^n(x)}{J_\mu T^n(y)} = \prod_{j=o}^{n-1} \frac{J_\mu T(T^j x)}{J_\mu T(T^j(y))} \leq \prod_{j=o}^{n-1} \frac{\mid J_\mu T(T^j x) - J_\mu T(T^j y) \mid}{J_\mu T(T^j y)} + 1$$

$$\leq \prod_{j=o}^{n-1} 1 + \frac{1}{c} \mid J_\mu T(T^j(x)) - J_\mu T(T^j(y)) \mid$$

where $c = \inf_{x \in K} J_\mu T(x) > 0$. Hence

$$\begin{aligned} \frac{J_\mu T^n(x)}{J_\mu T^n(y)} \leq & \quad \prod_{j=o}^{n-1} 1 + \frac{C}{c} d(T^j(x), T^j(y))^\gamma \leq \prod_{j=o}^{n-1} 1 + \frac{C}{c}(\lambda^{n-j})^\gamma d \\ \leq & \quad \prod_{j=o}^{\infty} 1 + \frac{C}{c}(\lambda^\gamma)^j d \stackrel{def}{=} A. \end{aligned}$$

□

Corolary 3.4 (Distortion Lemma)**.** *If $J_\mu T$ is strictly positive and Hölder-continuous, then, there exists $B > 0$ such that for any S_1, $S_2 \subset S$*

$$\frac{1}{B} \frac{\mu(g(S_1))}{\mu(g(S_2))} \leq \frac{\mu(S_1)}{\mu(S_2)} \leq B \frac{\mu(g(S_1))}{\mu(g(S_2))}.$$

Proof. Fix $x_0 \in g(S)$. Then

$$\begin{aligned} \mu(S_1) = & \quad \int_{g(S_1)} J_\mu T^n d\mu \leq A J_\mu T^n(x_0) \mu(g(S_1)) \\ \mu(S_2) = & \quad \int_{g(S_2)} J_\mu T^n d\mu \geq \frac{1}{A} J_\mu T^n(x_0) \mu(g(S_2)). \end{aligned}$$

Then

$$\frac{1}{A^2} \frac{\mu(S_1)}{\mu(S_2)} \leq \frac{\mu(g(S_1))}{\mu(g(S_2))}.$$

Inverting the roles of S_1 and S_2 we obtain the other inequality. □

Corolary 3.5. *Let ξ_0 be an expansivity constant for T given by former Lemma, and $0 < \xi \leq \xi_0$. Then there exists $C_\xi > 0$ such that*

$$\frac{1}{C_\xi} \leq \mu(B(n, \xi, x)) \cdot J_\mu T^n(x) \leq C_\xi$$

$\forall n \geq 0$, $\forall x \in K$.

Proof. By a former Lemma, $B(n, \xi, x) = g(B_\xi(f^n x))$, where $g : B_r(T^n(x)) \to K$ is a contractive branch of T^{-n}.

Cover K with balls $B_1 ... B_l$ of radius $\xi/3$ and let $\delta_\xi = \min_{1 \leq i \leq l} \mu(B_i)$. δ_ξ is strictly positive, because by a former Lemma, μ is positive on open sets. Also, if $y \in K$,

$$\mu(B_\xi(y)) \geq \delta_\xi.$$

Hence

$$\delta_\xi \leq \mu(B_\xi(T^n(x))) = \int_{g(B_\xi(T^n(x)))} J_\mu T^n d\mu \leq A J_\mu T^n(x) \mu(B(n, \xi, x)).$$

It follows that

$$\mu(B(n, \xi, x)) \cdot J_\mu T^n(x) \geq \frac{\delta_\xi}{A} \stackrel{def}{=} \frac{1}{C_x i}.$$

Also

$$1 \geq \mu(B_\xi(T^n x)) \geq \frac{J_\mu T^n(x)}{A} \mu(B(n, \xi, x)) \geq \frac{\mu(B(n, \xi, x)) J_\mu T^n(x)}{C_\xi}.$$

Hence

$$\mu(B(n, \xi, x)) J_\mu T^n(x) \leq C_\xi.$$

□

Corolary 3.6. *Suppose that μ is f-invariant and ergodic. Then*

$$h_\mu(f) = \int \log J_\mu f d\mu.$$

Proof. The Theorem of Brin-Katok [4] claims that

$$h_\mu(T) = \lim_{\xi \downarrow 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B(n,\xi,x))$$

a.e. $x \in K$. From Corollary 3.5 it follows that

$$\limsup_{n\to\infty} -\frac{1}{n} \log \mu(B(n,\xi,x)) = \limsup_{n\to\infty} \frac{1}{n} \log J_\mu T^n(x)$$

if $0 < \xi < \xi_0$. From Birkhoff's Theorem it results that

$$\frac{1}{n} \log J_\mu T^n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \log J_\mu T(T^j x) \to \int \log J_\mu T d\mu$$

a.e. $x \in K$. Hence

$$h_\mu(T) = \int \log J_\mu T d\mu.$$

□

Lemma 3.7. *Let K be a compact metric space, $\mu \in \mathcal{M}(K)$ and $\mathcal{P}$ a Borelian partition of K, $\mathcal{P} = \{P_1, ..., P_n\}$. Let $\{C_m\}_{m\geq 1}$ be a sequence of partitions with $diam\, C_m = \max_{C\in C_m}(diam\, C)$ converging to zero, when $m \to \infty$. Then there exist partitions $\{E_1^{(m)}, ..., E_n^{(m)}\}$ such that*

(1) Each $E_i^{(m)}$ is a union of atoms of C_m.

(2) $\lim_{m\to\infty} \mu(E_i^{(m)} \Delta P_i) = 0$, $\forall i$.

Proof. Let $K_1, ..., K_n$ be compact sets with $K_i \subset P_i$ and $\mu(P_i \setminus K_i) < \xi$. Let $\delta = \inf_{i\neq j} d(K_i, K_j) > 0$ and consider $m > 0$ such that $diam\, C_m < \delta/2$. Divide the elements $C \in C_m$ in groups whose unions we call $E_1^{(m)}, ..., E_n^{(m)}$ in the following way: $C \subset E_i^{(m)}$ if $c \cap K_i \neq \emptyset$. An element $c \in C_m$ can intersect at most one of the K_i's and in case that it doesn't intersect anyone include it arbitrarily in any of the $E_i^{(m)}$'s. Then

$$\begin{aligned} \mu(E_i^{(m)} \Delta P_i) = & \quad \mu(P_i \setminus E_i^{(m)}) + \mu(E_i^{(m)} \setminus P_i) \\ \leq & \quad \mu(P_i \setminus K_i) + \mu(K \setminus \cup_{i=1}^n K_i) \leq (n+1)\xi. \end{aligned}$$

As ξ was arbitrarily chosen, the result follows. □

Theorem 3.8. *If $J_\mu T$ is strictly positive and Hölder-continuous, then μ is exact.*

Proof. Let $\mathcal{P}_0 = \{P_1^0, ..., P_{l_0}^0\}$ be a partition of K, with $diam\,\mathcal{P}_0 < r$ and $int(P_i^0) \neq$, for $1 \leq i \leq l_0$. Define contractive branches $g_{ij}^n : P_i^0 \to K$ of f^{n-1}, for $0 \leq j \leq n_i$, $1 \leq i \leq l_0$. Let $P_{ij}^n = g_{ij}^n(P_i^0)$. Then $\mathcal{P}_n = \{P_{ij}^n, 0 \leq j \leq n_i, 1 \leq i \leq l_0\}$ is a partition of K and $diam\,\mathcal{P}_n \longrightarrow 0$.

Suppose by contradiction that there exists $A \in \cap_{j\geq 0} T^{-j}(B(K))$ such that $\mu(A) > 0$ and $\mu(A^c) > 0$. Applying a former Lemma, it follows that $\forall \xi > 0$, $\exists\, N(\xi) > 0$ such that if $n \geq N(\xi)$, $\exists P_{ij}^n \in \mathcal{P}_n$ such that

$$\frac{\mu(A \cap P_{ij}^n)}{\mu(P_{ij}^n)} \geq 1 - \xi.$$

Since $A \in \cap_{j\geq 0} f^{-j}(\mathcal{B}(K))$, $A = T^{-n}(A_n)$, for a certain

$$A_n \in \cap_{j\geq 0} T^{-j}(\mathcal{B}(K))$$

and so $g_{ij}^n(A_n \cap P_i^0) = A \cap P_{ij}^n$. By the distortion Lemma

$$\frac{\mu(A_n \cap P_i^0)}{\mu(P_i^0)} \geq 1 - \xi B.$$

In an analogous way, $\exists 1 \leq k \leq l_0$ such that

$$\frac{\mu(A_n^c \cap P_k^0)}{\mu(P_k^0)} \geq 1 - \xi B.$$

Since T is topologically mixing, there exists $N > 0$ such that $\forall 1 \leq i \leq l_0$ there exists $j(i)$ such that the contractive branch $g_{1,j(i)}^N$ of T^{-N} satisfies $P_{1,j(i)}^N \subset P_i^0$.

For simplicity, we shall denote $P_{1,j(i)}^N$ by Q_i and $g_{1,j(i)}^N$ by g_i.

Let $c = \min_i \mu(Q_i)$, c is strictly positive because Q_i has non-empty interior. Let us take $\xi > 0$ such that

$$\frac{\xi B \sup_i \mu(P_i^0)}{c} < \xi$$

where $\delta < \frac{1}{2B}$. So we have

$$\frac{\mu(A_n \cap Q_i)}{\mu(Q_i)} \geq 1 - \delta$$

$$\frac{\mu(A_n^c \cap Q_k)}{\mu(Q_k)} \geq 1 - \delta.$$

Observing that $A_n = T^{-N} B_N$ and $A_n^c = T^{-N} B_N^c$ and applying again the distortion Lemma

$$\frac{\mu(B_N \cap P_1^0)}{\mu(P_1^0)} > 1 - \delta B$$

$$\frac{\mu(B_N^c \cap P_1^0)}{\mu(P_1^0)} > 1 - \delta B.$$

Summing the terms leads to

$$1 > 2 - 2\delta B$$

which is a contradiction. $\square$

4 Proof of the Ruelle Theorem

In this section we will prove the main result.

Theorem 4.1. *Let $T : K \hookleftarrow$ to be an expanding map and topologically mixing; and ψ Hölder-continuous. Then, there exist $h : K \to \mathbb{R}$ Hölder-continuous and strictly positive, $\gamma \in \mathcal{M}(K)$ and $\lambda > 0$ such that*

1. $\int h d\nu = 1$
2. $\mathcal{L}_\psi \cdot h = \lambda h$
3. $\mathcal{L}_\psi^* \nu = \lambda \nu$
4. $\| \lambda^{-n} \mathcal{L}_\psi^n \varphi - h \int \varphi d\nu \|_{C^0} \to 0$, $\forall \varphi \in C^0(K)$.
5. *h is the unique positive eigen-function of $\mathcal{L}_\psi$, except for multiplication by scalars.*
6. *The probability $\mu \stackrel{def}{=} h\nu$ is invariant, exact and*
$$\log \lambda = h_\mu T + \int \psi d\mu.$$
7. *For any $\hat{\mu} \in \mathcal{M}(T)$, $\hat{\mu} \neq \mu$*
$$\log \lambda > h_{\hat{\mu}} f + \int \psi d\hat{\mu}.$$

Proof. (1) Consider $\mathcal{G} : \mathcal{M}(K) \hookleftarrow$ given by $\mathcal{G}(\mu) = \mathcal{L}^*\mu / \mathcal{L}^*\mu(1)$.

$\mathcal{G}$ is well defined since $\mathcal{L}(1) > 0$ and therefore $\mathcal{L}^*\mu(1) = \int \mathcal{L}(1) d\mu > 0$. Also, as $\mathcal{L}$ is positive, $\mathcal{G}(\mu)$ is actually a probability. By the Theorem of Tychonoff-Schauder we get that $\mathcal{G}$ has a fixed point ν. Define $\lambda = \mathcal{L}^*\nu(1) > 1$. Then
$$\mathcal{L}^*\nu = \lambda\nu.$$

For simplifying the notation we are using $\mathcal{L}$ in place of $\mathcal{L}_\psi$. $\| \varphi \|$ will denote the C^0-norm of φ. □

(2) There exists $A > 0$ such that $\mathcal{L}^m(1)(x)/\mathcal{L}^m(1)(y) < A$, if x, $y \in K$.

Proof. We shall prove first the following Lemma.

Lemma 4.2. *There exists $\delta > 0$ such that if x, $y \in K$, with $d(x,y) < \delta$ and $n > 0$ then $T^{-n}(x) = \{x_1, ..., x_k\}$, $T^{-n}(y) = \{y_1, ..., y_k\}$ satisfying $d(T^j x_i, T^j y_i) \leq \lambda^{n-j} d(x,y)$ for all $0 \leq j \leq n$, $1 \leq i \leq k$.*

Proof. Let $\delta = \min(r, c/2\lambda)$ and let $g^{(i)}$ be the contractive branch of T^{-1} such that $g^{(i)}x = x_i$, where $T^{-1}(x) = \{x_1, ..., x_k\}$.

Define $y_i = g^{(i)}y$. Then if $y_i = y_j$

$$d(x_i, x_j) \leq d(x_i, y_i) + d(x_j, y_j) \leq 2\lambda\delta \leq c$$

a contradiction. Hence $x_i = x_j$. Therefore to each x_i there correspond distinct y_i's such that $d(x_i, y_i) \leq \lambda d(x,y)$. Inverting the roles of x and y we see that this correspondence is one-to-one. This proves the Lemma for $n = 1$. By induction, using analogous reasoning, we complete the proof of the Lemma. □

Let us prove then (2). Suppose at first that $d(x,y) < \delta$. Then $T^{-m}(x) = \{x_1, ..., x_k\}$ and $T^{-m}(y) = \{y_1, ..., y_k\}$ as in the above Lemma.

$$\begin{aligned} | S_m\psi(x_i) - S_m\psi(y_i) | \leq & \sum_{j=0}^{m-1} | \psi(T^j x_i) - \psi(f^j y_i) | \leq \\ \leq & \sum_{j=0}^{m-1} C d(T^j x_i, T^j y_i)^\gamma \\ \leq & \sum_{j=0}^{m-1} C (\lambda^{m-j})^\gamma d(x,y)^\gamma \leq A'. \end{aligned}$$

Hence

$$\mathcal{L}^m(1)(x) = \sum_{i=1}^{k} e^{S_m\psi(x_i)} \leq \sum_{i=1}^{k} e^{S_m\psi(y_i)} e^{A'} = A\mathcal{L}^m(1)(y).$$

For the general case cover K with balls of diameter δ; $B_1, ..., B_l$ and take $N > 0$ such that $T^N(B_i) = K$, for all $1 \leq i \leq l$.

$$\mathcal{L}^{m+N}(1)(x) = \sum_{z\in f^{-m-N}(x)} e^{S_{m+N}\psi(z)}$$
$$= \sum_{z:f^m z\in f^{-N}(x)} e^{S_N\psi(f^m z)} e^{S_m\psi(z)}.$$

Write

$$T^{-N}(x) = \{x_1^{(1)}, ..., x_{k_1}^{(1)}, ..., x_1^{(l)}, ..., x_{k_l}^{(l)}\}$$

$$T^{-N}(y) = \{y_1^{(1)}, ..., y_{S_1}^{(1)}, ..., y_1^{(l)}, ..., y_{S_l}^{(l)}\}$$

where the index above means that $x_i^{(j)}(y_i^{(j)})$ is in B_j. The condition on N implies that $k_1, ..., k_l, S_1, ..., S_l$ are strictly positive.

$$\mathcal{L}^{m+N}(1)(x) \leq e^{N\|\psi\|} \sum_{j=1}^{l} \sum_{z:T^m z\in\{x_i^{(j)},...,x_{k_i}^{(j)}\}} e^{S_m\psi(z)}$$
$$= e^{N\|\psi\|} \sum_{j=1}^{l} \sum_{i=1}^{k_j} \mathcal{L}^m(1)\cdot(x_i^{(j)}).$$

We know that

$$\mathcal{L}^m(1)(x_i^{(j)}) \leq A\mathcal{L}^m(1)(y_r^{(j)})$$

for all $i = 1, ..., k_j \cdot r = 1, ..., S_j$. Therefore

$$\mathcal{L}^{m+N}(1)(y) \geq e^{-N\|\psi\|} \sum_{j=1}^{l} \sum_{r=1}^{S_j} \mathcal{L}^m(1)(y_r^{(j)})$$
$$\geq \frac{e^{-2N\|\psi\|}}{A} \frac{1}{\max\limits_{1\leq j\leq l} k_j} e^{N\|\psi\|} \sum_{j=1}^{l} \sum_{i=1}^{k_j} \mathcal{L}^m(1)\cdot(x_i^{(j)})$$
$$= A' \frac{1}{\max\limits_{1\leq j\leq l} k_j} \mathcal{L}^{m+N}(1)(x).$$

But as $k_j(x)$, $x \in K$ is bounded, (2) is proved. $\square$

Proof. (3)

(a) $\sup_n \parallel \lambda^{-n}\mathcal{L}^n(1) \parallel< \infty$

(b) $\inf_n \inf_x \mid \lambda^{-n}\mathcal{L}^n(1)(x) \mid> 0$.

From (1) it follows that

$$\int \lambda^{-n}\mathcal{L}^n(1)d\nu = 1$$

for all $n \in \mathbf{N}$. It follows that there exist s_n, $t_n \in K$ such that

$$\lambda^{-n}\mathcal{L}^n(1)(S_n) \leq 1$$

and

$$\lambda^{-n}\mathcal{L}^n(1)(t_n) \geq 1.$$

Using (2) it results that

$$\lambda^{-n}\mathcal{L}^n(1)(x) < A$$

and

$$\lambda^{-n}\mathcal{L}^n(1)(x) > \frac{1}{A}$$

for all $x \in K$ and $n \in \mathbf{N}$.

Let $C^\gamma(K)$ be the space of real valued Hölder-continuous functions with Hölder constant γ endowed with the norm

$$\parallel \varphi \parallel_\gamma=\parallel \varphi \parallel_0 + \sup_{x\neq y} \frac{\mid \varphi(x) - \varphi(y) \mid}{d(x,y)^\gamma}.$$

Given $a > 0$ define the seminorm $\mid \cdot \mid_{a,\gamma}$ on $C^\gamma(K, \mathbf{R})$ by

$$\mid \varphi \mid_{a,\gamma}= \sup_{x\neq y_{d(x,y)<a}} \frac{\mid \varphi(x) - \varphi(y) \mid}{d(x,y)^\gamma}$$

and the norm

$$\parallel \varphi \parallel_{a,\gamma}=\parallel \varphi \parallel_0 + \mid \varphi \mid_{a,\gamma} .$$

This norm is equivalent to $\| \cdot \|_\gamma$ because obviously $\| \varphi \|_\gamma \geq \| \varphi \|_{a,\gamma}$ and on the other hand

$$\begin{aligned}
\sup_{x\neq y} \frac{| \varphi(x) - \varphi(y) |}{d(x,y)^\gamma} \leq & \qquad | \varphi |_{a,\gamma} + \sup_{d(x,y)>a} \frac{| \varphi(x) - \varphi(y) |}{d(x,y)^\gamma} \leq \\
\leq & \qquad | \varphi |_{a,\gamma} + \frac{2 \| \varphi \|_0}{a^\gamma}.
\end{aligned}$$

Hence

$$\| \varphi \|_\gamma \leq \left(1 + \frac{2}{a^\gamma}\right) \| \varphi \|_{a,\gamma} .$$

Analogous considerations remains valid for the space $C^\gamma(K, \mathbf{C})$ of complex Hölder-continuous functions with Hölder constant γ. □

Proof. (4) Suppose that γ is a Hölder-constant for ψ and that $a = \delta$ given by Lemma 4.2. Then there exists $C > 0$ such that

$$| \mathcal{L}^n \varphi |_{a,\gamma} \leq \left((\lambda^\gamma)^n \mid \varphi |_{a,\gamma} + C \| \varphi \|_0\right) \| \mathcal{L}^n 1 \|_0$$

for all $\varphi \in C^\gamma(K, \mathbf{C})$. It results that

$$\mathcal{L}(C^\gamma(K, \mathbf{C})) \subset C^\gamma(K, \mathbf{C})$$

and

$$\sup_n \| \lambda^{-n} \mathcal{L}^n \|_{\gamma,a} < \infty$$

where $\| \lambda^{-n} \mathcal{L}^n \|_{\gamma,a}$ is the operator norm of $\lambda^{-n}\mathcal{L}^n$ in the space $C^\gamma(K, \mathbf{C})$.

Take a as in Lemma 4.2. If x, $y \in K$ with $d(x,y) < a$ then $T^{-n}(x) = \{x_1, ..., x_k\}$ and $T^{-n}(y) = \{y_1, ..., y_k\}$ satisfying

$$d(T^j x_i, T^j y_i) \leq \lambda^{n-j} d(x,y)$$

for all $n > 0$, $0 \leq j \leq n$ and $1 \leq i \leq k$. Then

$$\begin{aligned}
| \mathcal{L}^n \varphi(x) - \mathcal{L}^n \varphi(y) | = & \quad | \sum_{i=1}^k \varphi(x_i) \exp S_n\psi(x_i) - \varphi(y_i) \exp S_n\psi(y_i) | \\
\leq & \qquad \sum_{i=1}^k | \varphi(x_i) - \varphi(y_i) | \exp S_n\psi(x_i) +
\end{aligned}$$

$$+\sum_{i=1}^{k} \mid \varphi(y_i) \mid \exp S_n\psi(y_i) \mid 1-\exp(S_n\psi(x_i)-S_n\psi(y_i)) \mid .$$

Also

$$\begin{aligned} \mid S_n\psi(x_i)-S_n\psi(y_i) \mid \leq & \sum_{j=0}^{n-1} \mid \psi(f^j x_i)-\psi(f^j y_i) \mid \\ \leq & \mid \psi \mid_{a,\gamma} \sum_{j=0}^{n-1} (\lambda^{n-j})^{\gamma} d(x,y)^{\gamma} \\ \leq & \mid \psi \mid_{a,\gamma} \frac{\lambda^{\gamma}}{1-\lambda^{\gamma}} d(x,y)^{\gamma}. \end{aligned}$$

Therefore there exists $C>0$ such that

$$\mid 1-\exp(S_n\psi(x_i)-S_n\psi(y_i)) \mid \leq C d(x,y)^{\gamma}.$$

Then

$$\begin{aligned} \mid \mathcal{L}^n\varphi(x)-\mathcal{L}^n\varphi(y) \mid \leq & \sum_{i=1}^{k} \mid \varphi \mid_{a,\gamma} d(x_i,y_i)^{\gamma} \exp S_n\psi(x_i) + \\ + & \sum_{i=1}^{k} \mid \varphi(y_i) \mid \exp S_n\psi(y_i) C d(x,y)^{\gamma} \\ \leq & \Big[(\lambda^{\gamma})^n \mid \varphi \mid_{a,\gamma} \| \mathcal{L}^n 1 \|_0 + C \| \varphi \|_0 \| \mathcal{L}^n 1 \|_0\Big] d(x,y)^{\gamma}. \end{aligned}$$

This proves the required estimate. For concluding the proof of (4) observe that by (3) (a) $\lambda^{-n} \| \mathcal{L}^n 1 \|_0$ is bounded above. It follows that

$$\mid \lambda^{-n}\mathcal{L}^n\varphi \mid_{a,\gamma} \leq C \| \varphi \|_{a,\gamma} .$$

Using (3) (a) again

$$\| \lambda^{-n}\mathcal{L}^n\varphi \|_0 \leq \| \lambda^{-n}\mathcal{L}^n 1 \|_0 \| \varphi \|_0 \leq A \| \varphi \|_0 .$$

From this the claim (4) follows easily. □

Proof. (5) There exists $h \in C^\gamma(K)$ strictly positive such that $\mathcal{L}h = \lambda h$ and $\int h d\nu = 1$.

Let us consider the sequence $\{g_n\}_n$ given by

$$g_n = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j}\mathcal{L}^j 1.$$

By (4), $\sup_n \| g_n \|_\gamma < \infty$. We can therefore use the Arzelá-Ascoli Theorem and find a subsequence $\{g_{n_k}\}_k$ such that $g_{n_k} \to h$ in the norm C^0. It follows that $h \in C^\gamma(K)$ and

$$\begin{aligned}
\mathcal{L}h &= \lim_{k\to\infty} \frac{\lambda}{n_k}\sum_{j=1}^{n_k} \lambda^{-j}\mathcal{L}^j 1 \\
&= \lim_{k\to\infty} \frac{\lambda}{n_k}\left(\sum_{j=0}^{n_k-1} \lambda^{-j}\mathcal{L}^j 1 - 1 + \lambda^{-n_k}\mathcal{L}^{n_k} 1\right) \\
&= \lambda h
\end{aligned}$$

because $\sup_n \| \lambda^{-n}\mathcal{L}^n 1 \| < \infty$. Also, since $\int g_{n_k} d\nu = 1$, it results that $\int h d\nu = 1$. From (3) (b) it follows that h is strictly positive. □

Proof. (6) Let $\mu = h\nu$. Then μ is invariant and exact.

Let $\varphi \in C^0(K)$. Then

$$\begin{aligned}
\int \varphi \circ f \, d\mu &= \int \varphi \circ f \cdot h d\nu = \lambda^{-1} \int \mathcal{L}(h \cdot \varphi \circ f) d\nu \\
&= \lambda^{-1} \int \mathcal{L}h \cdot \varphi d\nu = \int \varphi h d\nu = \int \varphi d\mu.
\end{aligned}$$

It follows that μ is invariant. In Lemma 3.1 we have seen that $J_\nu T = \lambda e^{-\psi}$ and therefore is Hölder-continuous and strictly positive. It results from Theorem 3.8 that ν is exact. As μ is equivalent to ν, μ is also exact. □

Proof. (7) let u and φ be in $C^0(K)$. Then

$$\begin{aligned}\int \varphi(\lambda^n \mathcal{L}^n u)d\nu = & \int \lambda^{-n}\mathcal{L}^n(\varphi \circ T^n \cdot u)d\nu \\ = & \int \varphi \circ T^n \cdot ud\nu \\ = & \int \varphi \circ T^n u \cdot \frac{d\mu}{h}.\end{aligned}$$

This last expression converges to $\int \varphi d\mu \cdot \int \frac{u}{h} d\mu$ because μ is mixing. Hence

$$\int \varphi(\lambda^{-n}\mathcal{L}^n u)d\nu \longrightarrow \int \varphi(h \int ud\nu)d\nu$$

for all φ and u in $C^0(K)$. We shall now prove the following Lemma. $\square$

Lemma 4.3. *Let $\nu \in \mathcal{M}(K)$ positive on open sets and $\{\psi_n\}_n \subset C^0(K, \mathbf{C})$ an equicontinuous and bounded sequence such that there exists $\psi \in C^0(K)$ satisfying*

$$\int \varphi\psi_n d\nu \longrightarrow \int \varphi\psi d\nu$$

for all $\varphi \in C^0(K)$. Then $\psi_n \to \psi$ in C^0.

Proof. Let ψ_0 be an accumulation point of $\{\psi_n\}_{n\geq 0}$ in the C^0-topology. Then, if $\psi_{nj} \to \psi_0$

$$\int \varphi\psi_{nj} d\nu \to \int \varphi\psi_0 d\nu$$

for all $\varphi \in C^0(K)$. Hence

$$\int \varphi(\psi - \psi_0)d\nu = 0$$

for all $\varphi \in C^0(K)$. Suppose by contradiction that $\psi \neq \psi_0$. Then there exists an open set V where $\psi - \psi_0 > \delta > 0$ (or $\psi_0 - \psi > \delta > 0$).

Choose $\varphi \in C^0(K)$ strictly positive and with support in V. It results that

$$\int \varphi(\psi - \psi_0)d\nu > \int_V \delta\varphi d\nu > 0.$$

Which is a contradiction. Hence $\psi = \psi_0$ and the unique accumulation point of $\{\psi_n\}_{n\geq 0}$ in C^0 is ψ. Since $\{\psi_n\}_{n\geq 0}$ is relatively compact in C^0, $\psi_n \to \psi$ in C^0. $\square$

From a previous Lemma we conclude that $\lambda^{-n}\mathcal{L}^n u \to h \int u d\nu$ in C^0, for all $u \in C^\gamma(K)$. Observing that $C^\gamma(K)$ is dense in $C^0(K)$ and

$$\sup_n \parallel \lambda^{-n}\mathcal{L}^n \parallel < \infty,$$

it results that

$$\lambda^{-n}\mathcal{L}^n u \to h \int u d\nu$$

in C^0 for all $u \in C^0(K)$.

(8) Suppose $\hat{h} \geq 0$ in $C^0(K)$ such that $\mathcal{L}\hat{h} = \hat{\lambda}\hat{h}$. Then

$$\lambda^{-n}\mathcal{L}^n\hat{h} = \left(\frac{\hat{\lambda}}{\lambda}\right)^n \cdot \hat{h} \to h \int \hat{h} d\nu.$$

Since ν is positive on open sets and $\hat{h} \geq 0$, $\hat{h} \neq 0$, it follows that $\int \hat{h} d\nu > 0$. Hence $\hat{\lambda} = \lambda$. It results that $\hat{h} = h \int \hat{h} d\nu$.

(9) From a previous Lemma, we have

$$J_\mu T = \lambda e^{-\psi} \frac{h \circ T}{h}.$$

Hence by Corollary 3.6

$$h_\mu(T) = \int \log J_\mu T \, d\mu = \log \lambda - \int \psi d\mu$$

because μ is invariant.

The items (1) to (9) prove the Theorem of Ruelle, except for (7).

(10) Let us prove (7).

Let $\mathcal{P} = \{P_1, ..., P_m\}$ be a partition of K with diameter smaller than r.

Proof. From Lemma 3.3 it is clear that

$$\frac{1}{A} J_\mu T^m(x) \leq \frac{\mu(P)}{\mu(g(P))} \leq A\, J_\mu T^m(x)$$

for all $x \in g(P)$. It follows that

$$\frac{1}{A} \inf_{P \in \mathcal{P}} \mu(P) \leq \mu(g(P)) \cdot J_\mu T^m(x) \leq A \sup_{P \in \mathcal{P}} \mu(P) \quad (+).$$

Recalling that $J_\mu f = \lambda e^{-\psi} \frac{h \circ T}{h}$, it follows that

$$\frac{[J_\mu T^m(x)]^{-1}}{\exp\{-m \log \lambda + S_m \psi(x)\}} =$$

$$\begin{aligned}
&= \exp\left\{ -\left[\sum_{j=0}^{m-1} \log\left(\lambda e^{-\psi} \frac{h \circ T}{h} \right)(T^j(x)) - \log \lambda + \psi(T^j(x)) \right] \right\} \\
&= \exp\left\{ -\left[\sum_{j=0}^{m-1} \log h(T^{j+1}(x)) - \log h(T^j(x)) \right] \right\} \\
&= \exp\{-\log h(T^m(x)) + \log h(x)\} \\
&= \frac{h(x)}{h(T^m(x))} \in \left[\frac{\inf h}{\sup h}, \frac{\sup h}{\inf h} \right].
\end{aligned}$$

This togheter with (+) proves Lemma 4.3. □

Let $\mathcal{P}^{(m)} = \{g(P) \mid g : P \to K$ is a contractive branch of $T^{-m}\}$. Then if $\eta \in \mathcal{M}(T)$

$$H_\eta(T, \mathcal{P}^{(m)}) + \int S_m \psi d\eta = \sum_{g,P} \left[-\eta(g(P)) \log \eta(g(P)) + \int_{g(P)} S_m \psi d\eta \right].$$

Let $Z^m_{g(P)} \psi = \sup_{x \in g(P)} S_m \psi(x)$. Then

$$H_\eta(T, \mathcal{P}^{(m)}) + \int S_m \psi d\eta \leq \sum_{g,P} \eta(g(P)) \Big[-\log \eta(g(P)) + Z^m_{g(P)} \psi \Big].$$

Lemma 4.4.

$$\sum_{p_1,...,p_n\geq 0, p_1+...+p_n=1} p_i(-\log p_i + a_i) \leq \log \sum_{i=1}^{n} e^{a_i}$$

.

The proof of the Lemma will be given soon.
It follows from the Lemma that

$$H_\eta(T, \mathcal{P}^{(m)}) + \int S_m \psi d\eta \leq \log \sum_{g,P} e^{Z^m_{g(P)}\psi}.$$

By a former Lemma we get,

$$e^{Z^m_{g(P)}\psi} \leq \frac{\mu(g(P))e^{m\log^\lambda}}{C_1}.$$

Hence

$$\sum_{g,P} e^{Z^m_{g(P)}\psi} \leq \frac{e^{m\log^\lambda}}{C_1}.$$

Then

$$\frac{1}{m}H_\eta(T, \mathcal{P}^m) + \int \psi d\eta \leq -\frac{\log C_1}{m} + \log^\lambda \quad (*).$$

Proof. Proof of Lemma 4.4 Let $v(x_1, ..., x_n) = -\sum_{i=1}^{n} x_i \log x_i + \sum_{i=1}^{n} a_i x_i$ be a function defined in $\{(x_1, ..., x_n) \mid x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1\}$. Suppose that $p = (p_1, ..., p_n)$ is a maximum of v and that $p_j > 0$ for $1 \leq j \leq n$. Then

$$\frac{\partial v}{\partial e_1 e_j} = 0$$

for $2 \leq j \leq n$. Hence

$$\frac{\partial v}{\partial e_1} = \frac{\partial v}{\partial e_j}$$

for $\partial \leq j \leq n$. It results that

$$-\log p_1 - p_1 \cdot \frac{1}{p_1} + a_1 = -\log p_j - p_j \cdot \frac{1}{p_j} + a_j.$$

Reordering the terms

$$P_j = e^{a_j - a_1} p_1.$$

And since $\sum_{j=1}^{n} p_j = 1$, it results that

$$p_1 = \left(\sum_{j=1}^{n} e^{(a_j - a_1)} \right)^{-1}.$$

Calculating the value of v in this point

$$v(p) = \log \sum_{j=1}^{n} e^{a_j}.$$

Observe that if v assume its maximum in a boundary point $\overline{p} = (\overline{p}_1, ..., \overline{p}_n)$ we can suppose without loss of generality that $\overline{p}_n = 0$. But by similar calculus v cannot assume its maximum in $\{(\overline{p}_1, ..., \overline{p}_{n-1}, 0)$, with $\overline{p}_j > 0$, $1 \leq j \leq n-1\}$ because such a maximum would be smaller than the value obtained above. Reducing the problem in this way we would arrive at the conclusion that v assume its maximum in $(1, 0, ..., 0)$. But this is an absurd because

$$\log \sum_{j=1}^{n} e^{a_j} > \log e^{a_1} = a_1 = v(1, 0, ..., 0).$$

Therefore v assume its maximum in the interior point p described above and

$$v(x_1, ...x_n) \leq \log \left(\sum_{j=1}^{n} e^{a_j} \right).$$

□

Lemma 4.5. *Let K be a compact metric space $\eta \in \mathcal{M}(K)$, $\xi > 0$ and C a Borelian partition. There exists $\delta > 0$ such that $H_\eta(C/D) < \xi$ if D is any partition with diam $D < \delta$.*

Proof. Let $C = \{C_1, ..., C_n\}$. By Lemma 4.3, for any $\xi_1 > 0$ there exists $\delta > 0$ such that if diam $D < \delta$ then there exists $E = \{E_1, ..., E_n\} \subset D$ with $\eta(E_i \Delta C_i) < \xi_1$. The expression

$$H_\eta(C/E) = -\sum_{i,j} \eta(E_i \cap C_j) \log \frac{\eta(E_i \cap C_j)}{\eta(E_i)}$$

depends continuously on the numbers $\eta(E_i \cap C_j)$ and $\eta(E_i)$, and it vanishes when $\eta(C_j \cap E_i) = \delta_{ij}\eta(C_j)$. Therefore if ξ_1 is sufficiently small

$$H_\eta(C/E) < \xi.$$

Hence

$$H_\eta(C/D) \leq H_\eta(C/E) < \xi.$$

□

Lemma 4.6. *Suppose diam* $\mathcal{P} < \delta = \frac{C}{3\lambda}$. *Then* $\mathcal{P} \vee ... \vee f^{-m}\mathcal{P}$ *is thiner than* $\mathcal{P}^{(m)}$.

Proof. Let $g_1(P)$ and $g_2(P)$ be atoms of $\mathcal{P}^{(1)}$. Then diam $g_1(P) < \lambda\delta$ and diam $g_2(\mathcal{P}) < \lambda\delta$. Since there exist $x_1 \in g_1(P)$ and $x_2 \in g_2(P)$ such that $d(x_1, x_2) > C$, it does not exist a $Q \in \mathcal{P}$ such that $Q \cap g_1(P) \neq \emptyset$ and $Q \cap g_2(P) \neq \emptyset$. This proves that $\mathcal{P} \vee f^{-1}(\mathcal{P})$ is thiner than $\mathcal{P}^{(1)}$.

Since diam $\mathcal{P} \vee f^{-1}(\mathcal{P}) < \lambda\delta < \delta$ we can repeat the argument and show that $\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee f^{-2}(\mathcal{P})$ is thinner than $\mathcal{P}^{(2)}$.

The proof is completed by induction. □

Definition 4.7. *A collection* $\mathcal{R} = \{R_1, ..., R_n\}$ *of disjoint open sets is a Markov partition of* f *if*

(1) $\cup_{i=1}^n$*bar* $R_i = K$.

(2) diam $R_i < r$ *for all* $i = 1, ..., n$ *and for all contractive branch* $\varphi : R_i \to K$ *of* f^{-1}

$$\varphi(R_i) \cap R_j \neq \emptyset \Rightarrow \varphi(R_i) \subset R_j$$

for all $1 \leq j \leq n$.

Lemma 4.8. *There are Markov partitions of f with diameter arbitrarily small.*

Proof. Let $\mathcal{B} = \{B_1, ..., B_l\}$ be a covering of K by balls with diameters smaller than ξ. Define for $n \geq 0$, $\mathcal{B}^{(n)} = \{\varphi(B_i) \mid \varphi : B_i \to K$ is a contractive branch of $f^{-n}\}$. For $1 \leq i \leq l$ define inductively $\mathcal{B}_i^{(0)} = B_i$ and

$$B_i^{(r)} = B_i^{(r-1)} \cup \left(\cup_{B \in \mathcal{B}^{(r)}, B \cap B^{(r-1)} \neq \emptyset} B \right).$$

Then diam $B_i^{(r)} \leq \xi + 2\xi\lambda + ... + 2\xi\lambda^r$. Let $\hat{B}_i = \cup_{r=0}^{\infty} B_r^{(r)}$. Hence diam $\hat{B}_i < \frac{\xi}{1-\lambda}$. Observe also that if $\varphi : \hat{B}_i \to K$ is a contractive branch of f^{-1} and $\varphi(B_i) \cap B_j \neq \emptyset$ then $\varphi(\hat{B}_i) \subset \hat{B}_j$, by construction.

Let $\mathcal{R}$ be the collection of the open sets $R \subset K$ such that if $R \cap \hat{B}_j \neq \emptyset$ then $R \subset \hat{B}^j$ and such that R is maximal with that property. $\mathcal{R}$ is a finite collection of disjoint open sets such that $\cup_{R \in \mathcal{R}}$bar $R = K$. For completing the proof of the Lemma, it remains to show that condition (2) of the above definition is valid.

Claim:

Proof. Proof of the Claim:

The $\varphi(\hat{B}_j)$ such that $\varphi(B_j) \cap B_1 \neq \emptyset$ cover $\hat{B}_1$.

Because if $x \in \hat{B}_1$ then $x \in \varphi^{(n)}(D_n)$, where $\varphi^{(n)}$ is a contractive branch of f^{-n} and $D_n \in \{B_1, ..., B_l\}$. Also there exists a sequence $\varphi^{(j)}(D_j)$, $1 \leq j \leq n$ such that $\varphi^{(j)}(D_j) \wedge \varphi^{(j+1)}(D_{j+1}) \neq \emptyset$, $1 \leq j \leq n-1$ and $\varphi^{(1)}(D_1) \cap B_1 \neq \emptyset$. Suppose that $D_1 = B_j$. By the construction of the $\hat{B}_j$ the sequence $\varphi^{(j-1)}(D_j)$, $1 \leq j \leq n$ is entirely contained in $\hat{B}_j$, where $\varphi^{(j-1)} = f \circ \varphi^{(j)}$. Therefore x is in $\varphi(\varphi^{(n-1)} D_n) \subset \varphi(\hat{B}_j)$ and $\varphi(B_j) \cap B_1 \neq \emptyset$. This proves the claim. □

Let $R \in \mathcal{R}$ be such that $\varphi(R) \cap \hat{B}_1 \neq \emptyset$. From the claim above it follows that we can find B_j satisfing $\varphi(B_j) \cap B_1 \neq \emptyset$ and $\varphi(\hat{B}_j) \cap \varphi(R) \neq \emptyset$. Therefore $\hat{B}_j \cap R \neq$ and so $R \subset \hat{B}_j$. Hence $\varphi(R) \subset \varphi(\hat{B}_j) \subset \hat{B}_1$. This proves that $\varphi(R) \subset R'$ for some $R' \in \mathcal{R}$. □

Previous Lemmas permit to consider partitions $\mathcal{P}$ satisfying

$$\vee_{i=0}^{m} T^{-i}\mathcal{P} = \{g(P_i) \mid g : P_i \to K \text{ is a contractive branch of } T^{-m}\} = \mathcal{P}^{(m)}.$$

It is sufficient to take a Markov partition (with the boundaries arbitrarily **distributed**) with diameter less than $c/3\lambda$, as in Lemma 4.8.

Because in this case the atoms of the two partitions are exactly of the form $A = \{x \mid T^i(x) \in P_i, 0 \le i \le m\}$.

Lemma 4.6 shows that $h_\eta(T) = h_\eta(T, \mathcal{P})$, for all $\eta \in \mathcal{M}(T)$. Because if C is any partition of K, take $m > 0$ such that $\mathcal{P} \vee ... \vee T^{-m}\mathcal{P}$ has diameter smaller than δ.

Then

$$h_\eta(T, C) \le h_\eta(T, \mathcal{P} \vee ... \vee T^{-m}\mathcal{P}) + H_\eta(C \mid \mathcal{P} \vee ... \vee T^{-m}\mathcal{P}).$$

Since $h_\eta(T, \mathcal{P} \vee ... \vee T^{-m}\mathcal{P}) = h_\eta(T, \mathcal{P})$ and $H_\eta(C \mid \mathcal{P} \vee ... \vee T^{-m}\mathcal{P}) < \xi$ it results that

$$\sup_C h_\eta(T, C) \le h_\eta(T, \mathcal{P}).$$

Hence

$$h_\eta(T) = h_\eta(T, \mathcal{P}).$$

From these facts and from (*),

$$h_\eta(T) + \int \psi d\eta \le \log \lambda.$$

In the following we shall get this fact and we shall show also that the equality holds only if $\eta = \mu$. The proof above gives an intuitive idea of why μ is the probability that **maximizes** the expression $h_\eta(T) + \int \psi d\eta$. It fact, a previous Lemma say what must be the values of $\eta(P)$ so that the equality above holds and such values are exactly the values of $\mu(P)$, according to a previous Lemma.

We must show that if $\eta \neq \mu$, $\eta \in \mathcal{M}(T)$ then

$$h_\eta(f) + \int \psi d\eta < \log \lambda.$$

1^{st} case - η singular with respect to μ.

In this case, there exists C_1 such that $\mu(C_1) = 0$ and $\eta(C_1) = 1$. If $C = \cap_{N\geq 1} \cup_{j\geq N} T^{-j}(C_1)$, then $T^{-1}(C) = C$, $\mu(C) = 0$ and $\eta(C) = 1$. Using again Lemma 4.8, consider F^m union of atoms of $\mathcal{P}^{(m)}$ such that

$$(\eta + \mu)(F^m \Delta C) \to 0$$

when $n \to \infty$.

Suppose by contradiction that

$$\log \lambda \leq \frac{1}{m}(H_\eta(\mathcal{P}^{(m)}) + \int S_m \psi d\eta).$$

Then

$$m \log \lambda \leq \sum_{B\in\mathcal{P}^{(m)}} (-\eta(B) \log \eta(B) + \int_B S_m \psi d\eta).$$

Let $Z^m_{g(P)}\psi = \sup_{x\in g(P)} S_m\psi(x)$. Then

$$m \log \lambda \leq \sum_{B\in\mathcal{P}^{(m)}} \eta(B)(Z^m_B\psi - \log \eta(B))$$

$$\leq \sum_{B\subset F^m} \eta(B)(Z^m_B\psi - \log \eta(B)) + \sum_{B\subset (F^m)^c} \eta(B)(Z^m_B\psi - \log \eta(B)).$$

Lemma 4.9. *Let $p_j \geq 0$, $j = 1, ..., n$, $s = \sum_{j=1}^n p_j \leq 1$, and $a_1, ..., a_n \in \mathbf{R}$. Then*

$$\sum_{j=1}^n p_j(a_j - \log p_j) \leq s\Big(\log \sum_{i=1}^n e^{a_i} - \log s\Big).$$

Proof. Analogous to Lemma 4.5. □

So

$$m \log \lambda \leq \eta(F^m) \log \sum_{B\subset F^m} e^{Z^m_B\psi} + \eta((F^m)^c) \log \sum_{B\subset (F^m)^c} e^{Z^m_B\psi} + 2C$$

where

$$C = \sup_{o\leq s\leq 1} -s \log s.$$

Then

$$-2C \leq \eta(F^m) \log \sum_{B \subset F^m} e^{Z_B^m \psi - m \log \lambda} + \eta((F^m)^c) \log \sum_{B \subset (F^m)^c} e^{Z_B^m \psi - m \log \lambda}.$$

By Lemma 4.4

$$-2C \leq \eta(F^m) \log \sum_{B \subset F^m} \frac{\mu(B)}{C_1} + \eta((F^m)^c) \log \sum_{B \subset (F^m)^c} \frac{\mu(B)}{C_1} =$$

$$= \log \frac{1}{C_1} + \eta(F^m) \log \mu(F^m) + \eta((F^m)^c) \log \mu((F^m)^c).$$

Making $m \to \infty$, $\eta(F^m) \to \eta(C) = 1$ and $\mu(F^m) \to \mu(C) = 0$. Therefore the expression in the right converges to $-\infty$ and this is a contradiction.

It results that

$$\log \lambda > \frac{1}{m}(H_\eta(\mathcal{P}^{(m)}) + \int S_m \psi d\eta)$$

and therefore using the adequate partition

$$\log \lambda > \frac{1}{m} H_\eta(\mathcal{P} \vee ... \vee T^{-m}\mathcal{P}) + \int \psi d\eta.$$

Since the expression in the right is decreasing with m it follows that

$$h_\eta(T) + \int \psi d\eta < \log \lambda.$$

<u>2^{nd} case</u> - η not singular with respect to μ. As η is not also absolutely continuous with respect to μ, we can decompose it as $\eta = \alpha\eta' + (1-\alpha)\mu'$, where $0 < \alpha < 1$, η' singular and μ' absolutely continuous (with respect to μ). Let A be such that $T^{-1}(A) = A$, $\eta'(A) = 1$ and $\mu(A) = 0$. Then for all B Borel set

$$\begin{aligned} \eta'(T^{-1}(B)) = &\quad \eta'(T^{-1}(B \cap A)) + \eta'(T^{-1}(B \cap A^c)) \\ = &\quad \eta'(T^{-1}(B \cap A)) = \frac{1}{\alpha}\eta(T^{-1}(B \cap A)) \\ = &\quad \frac{1}{\alpha}\eta(B \cap A) = \eta'(B \cap A) = \eta'(B). \end{aligned}$$

Hence $\eta' \in \mathcal{M}(T)$. It results that $\mu' \in \mathcal{M}(T)$ and therefore $\mu' = \mu$.

Lemma 4.10. *Let ν, $\nu' \in \mathcal{M}(T)$ mutually singulars and $0 < \alpha < 1$. Then*

$$h_{\alpha\nu+(1-\alpha)\nu'}(T) = \alpha h_{\nu}(T) + (1-\alpha)h_{\nu'}(T).$$

Proof. Let A be T-invariant such that $\nu(A) = 1$ and $\nu'(A) = 0$.

Let $\mathcal{A} = \{A, A^c\}$ and $\eta = \alpha\nu + (1-\alpha)\nu'$. Then if $\mathcal{R}$ is a partition of K,

$$H_{\eta}(\mathcal{R} \vee \mathcal{A}) = \alpha \sum_{R \in \mathcal{R} \vee \mathcal{A}_{R \subset A}} \nu(R) \log \alpha\nu(R)$$

$$+(1-\alpha) \sum_{R \in \mathcal{R} \vee \mathcal{A}_{R \subset A^c}} \nu'(R) \log(1-\alpha)\nu'(R)$$

$$= \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + \alpha H_{\nu}(\mathcal{R} \vee \mathcal{A}) + (1-\alpha)H_{\nu'}(\mathcal{R} \vee \mathcal{A}).$$

Therefore

$$\begin{aligned}
h_{\eta}(T) = & \lim_{n\to\infty} \frac{1}{n} H_{\eta}((\mathcal{P} \vee \mathcal{A}) \vee ... \vee T^{-n+1}(\mathcal{P} \vee \mathcal{A})) = \\
= & \lim_{n\to\infty} \frac{1}{n} H_{\eta}(\mathcal{P} \vee ... \vee T^{-n+1}\mathcal{P} \vee \mathcal{A}) = \\
= & \lim_{n\to\infty} \frac{1}{n} [\alpha \log \alpha + (1-\alpha)\log(1-\alpha) + \\
+ & \alpha H_{\eta}((\mathcal{P} \vee \mathcal{A}) \vee ... \vee T^{-n+1}(\mathcal{P} \vee \mathcal{A})) + \\
+ & (1-\alpha)H_{\eta'}((\mathcal{P} \vee \mathcal{A}) \vee ... \vee T^{-n+1}(\mathcal{P} \vee \mathcal{A}))] \\
= & \alpha h_{\eta}(T) + (1-\alpha)h_{\nu'}(T).
\end{aligned}$$

□

From the Lemma it follows that

$$\begin{aligned}
h_{\eta}(T) = & \alpha h_{\eta'}(T) + (1-\alpha)h_{\mu}(T) \\
< & \alpha \log \lambda + (1-\alpha)\log \lambda \\
= & \log \lambda.
\end{aligned}$$

This proves item (7) and therefore all main Theorem 4.1.

Theorem 4.11. *Let ψ and φ be Hölder-functions. Then $\mu_{\psi} = \mu_{\varphi}$ iff $\psi - \log \lambda_{\psi} \sim \varphi - \log \lambda_{\varphi}$.*

Proof. If $\mu_\psi = \mu_\varphi$ then $J_{\mu_\psi} T = J_{\mu_\varphi} f$. This means that

$$\lambda_\psi e^{-\psi} \frac{h_\psi \circ T}{h_\psi} = \lambda_\varphi e^{-\varphi} \frac{h_\varphi \circ T}{h_\varphi}.$$

Define $h = \frac{h_\psi}{h_\varphi}$. Then

$$\frac{\lambda_\psi}{\lambda_\varphi} \cdot \frac{h \circ T}{h} = e^{-(\varphi - \psi)}.$$

Therefore

$$(\log h) \circ T - \log h = (\psi - \log \lambda_\psi) - (\varphi - \log \lambda_\varphi).$$

Reciprocally suppose that there exists $H \in C^\circ(K)$ such that

$$H \circ T - H = (\psi - \log \lambda_\psi) - (\varphi - \log \lambda_\varphi).$$

To show that $\mu_\psi = \mu_\varphi$ it is sufficient to show that $\mu_\varphi << \mu_\psi$ because they are ergodic probabilities. For this, it is sufficient to show that $\nu_\varphi << \nu_\psi$. Let $h = \exp H$. Then, if $u \in C^\circ(K)$.

$$\begin{aligned}
(h^{-1}\lambda_\psi^{-1}\mathcal{L}_\psi hu)(x) = & \quad h^{-1}(x) \sum_{y \in T^{-1}(x)} e^{(\psi - \log \lambda_\psi)(y)} h(y) u(y) \\
= & \quad h^{-1}(x) \sum_{y \in T^{-1}(x)} e^{(\psi - \log \lambda_\psi + H)(y)} u(y) \\
= & \quad \sum_{y \in T^{-1}(x)} e^{(\psi - \log \lambda_\psi + H - H \circ T)(y)} u(y) \\
= & \quad \lambda_\varphi^{-1} \mathcal{L}_\varphi u(x)
\end{aligned}$$

Hence

$$h^{-1}\lambda_\psi^{-n}\mathcal{L}_\psi^n h = \lambda_\varphi^{-n}\mathcal{L}_\varphi^n$$

for all $n \geq 0$. On the other hand

$$\begin{aligned}
h_\varphi \int u d\nu_\varphi = & \quad \lim_{n \to \infty} \lambda_\varphi^{-n} \mathcal{L}_\varphi^n u \\
= & \quad h^{-1} \lim_{n \to \infty} \lambda_\psi^{-n} \mathcal{L}_\psi^n uh \\
= & \quad h^{-1} h_\psi \int hud\nu_\psi.
\end{aligned}$$

Therefore if $u \in C^{\circ}(K)$, $u \geq 0$,

$$\int u d\nu_{\varphi} \leq \frac{1}{\inf h_{\varphi}} \parallel h_{\psi} \parallel \cdot \parallel h^{-1} \parallel \cdot \parallel h \parallel \int u d\nu_{\psi}.$$

This shows that $\nu_{\varphi} << \nu_{\psi}$. $\square$

5 Differentiability for Hölder potentials

Remember that in last Chapter we have defined the space $C^\gamma(K, \mathbf{R})$ of γ-Hölder-continuous real valued functions on K.

For $\psi \in C^\gamma(K, \mathbf{R})$ consider $\lambda(\psi) > 0$, $h_\psi \in C^\gamma(K, \mathbf{R})$, $\nu_\psi \in \mathcal{M}(K)$, $\mu_\psi \in \mathcal{M}(K)$ given by Theorem 4.1. The purpose of this Chapter is to prove the following Theorem.

Theorem 5.1. *The functions*

$$C^\gamma(K, \mathbf{R}) \ni \psi \longrightarrow \lambda(\psi) \in \mathbf{R}$$

$$C^\gamma(K, \mathbf{R}) \ni \psi \longrightarrow h_\psi \in C^\gamma(K, \mathbf{R})$$

$$C^\gamma(K, \mathbf{R}) \times C^\gamma(K, \mathbf{R}) \ni (\varphi, \psi) \longrightarrow \int \varphi d\nu_\psi \in \mathbf{R}$$

$$C^\gamma(K, \mathbf{R}) \times C^\gamma(K, \mathbf{R}) \ni (\varphi, \psi) \longrightarrow \int \varphi d\mu_\psi \in \mathbf{R}$$

are real analytic.

Consider a function $\psi_0 \in C^\gamma(K, \mathbf{R})$ and define E_1 as the subspace of $C^\gamma(K, \mathbf{C})$ generated by h_{ψ_0} and E_2 as the subspace of $C^\gamma(K, \mathbf{C})$ defined by

$$\int \varphi d\nu_{\psi_0} = 0$$

for all $\varphi \in E_2$. Define projections $\pi_1 : C^\gamma(K, \mathbf{C}) \to E_1$, $\pi_2 : C^\gamma(K, \mathbf{C}) \to E_2$ by

$$\begin{aligned} \pi_1 \varphi = & \quad h_{\psi_0} \int \varphi d\nu_{\psi_0} \\ \pi_2 = & \quad I - \pi_1 \end{aligned}$$

For the proof of Theorem 5.1 we shall need the following Lemma.

Lemma 5.2. *There exist $B > 0$, $0 < \beta < 1$ and $a > 0$ such that*

$$\| \lambda(\psi_0)^{-n} \mathcal{L}_{\psi_0}^n \varphi \|_{a,\gamma} \leq B \beta^n \| \varphi \|_{a,\gamma}$$

for every $\varphi \in E_2$.

Proof. From Theorem 4.1 we know that $\| \lambda(\psi_0)^{-n}\mathcal{L}^n_{\psi_0}\varphi \|_0$ converges to zero, for any $\varphi \in E_2$. Let B be the closed unit ball in E_2. B is compact in the topology $\| \cdot \|_0$. Therefore, given $\delta > 0$, there exists $N > 0$ such that

$$\| \lambda(\psi_0)^{-n}\mathcal{L}^n_{\psi_0} \|_0 \leq \delta$$

if $n \geq N$ and $\varphi \in B$.

Taking a given by part (4) in the proof of Theorem 4.1, and making $\lambda^\gamma = c$

$$| \lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0}\varphi |_{a,\gamma} \leq (C^N \mid \lambda(\psi_0)^{-N}\mathcal{L}^N_{\psi_0}\varphi |_{a,\gamma} +$$
$$C \| \lambda(\psi_0)^{-N}\mathcal{L}^N_{\psi_0}\varphi \|_0) \| \lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0} 1 \|_0$$
$$\leq \left[c^N(c^N \mid \varphi |_{a,\gamma} + C \| \varphi \|_0) \| \lambda(\psi_0)^{-N}\mathcal{L}^N_{\psi_0} 1 \|_0 + C\delta\right] \| \lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0} 1 \|_0 .$$

But for N large enough $\| \lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0} 1 \|_0 \leq 1 + \| h_{\psi_0} \|_0$.

Then

$$| \lambda^{-2N}\mathcal{L}^{2N}_{\psi_0}\varphi |_{a,\gamma} \leq [c^N(c^N + C)(1 + \| h_{\psi_0} \|_0) + C\delta](\| h_{\psi_0} \|_0 + 1).$$

It follows that if N is large enough and δ small enough

$$| \lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0}\varphi |_{a,\gamma} \leq \frac{1}{2}.$$

Hence, if $\varphi \in B$

$$\| \lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0}\varphi \|_{a,\gamma} \leq \frac{1}{2} + \delta < \frac{2}{3}$$

if we had chosen $\delta < 1/6$. It follows from this that $\lambda(\psi_0)^{-2N}\mathcal{L}^{2N}_{\psi_0}(B) \subset B$ and that

$$\| \lambda(\psi_0)^{-2N \cdot n}\mathcal{L}^{2N \cdot n}_{\psi_0} \|_{a,\gamma} \leq \left(\frac{2}{3}\right)^n .$$

We conclude that there exist $B > 0$ and $0 < \beta < 1$ such that

$$\| \lambda(\psi_0)^{-n}\mathcal{L}^n_{\psi_0}\varphi \|_{a,\gamma} \leq B\beta^n \| \varphi \|_{a,\gamma} .$$

$\square$

Proof of Theorem 5.1. Lemma 5.2 implies that the spectrum $sp(\mathcal{L}_{\psi_0})$ of $\mathcal{L}_{\psi_0} : C^\gamma(K, \mathbf{C}) \hookleftarrow$ is the eigen-value $\lambda(\psi_0)$ plus a set contained in the disc $\{z \mid \mid z \mid < \beta\lambda(\psi_0)\}$. Take circles γ_1 and γ_2 centered at $\lambda(\psi_0)$ and 0 such that its interiors are disjoint and contain $sp(\mathcal{L}_{\psi_0})$. Denote by S the space of continuous linear applications $L : C^\gamma(K, \mathbf{C}) \hookleftarrow$ endowed with the norm topology. Let V be a neighborhood of $\mathcal{L}_{\psi_0}$ in S such that $sp(L)C\, int(\gamma,) \cup int(\gamma)$ for all $L \in V$. Given $L \in V$ define the spectral projections $\pi_i(1) : C^\gamma(K, \mathbf{C})$, $i = 1, 2$, by

$$\pi_i(L) = \frac{1}{2\pi_i} \int_{\gamma_i} (L - zI)^{-1} dz$$

and let $E_1(L) = \pi(L)z(C^\gamma(K, \mathbf{C}))$. It is well know that

$$C^\gamma(K, \mathbf{C}) = E_1(L) \oplus E_2(L)$$

$$I = \pi_1(L) + \pi_2(L)$$

$$\dim E_2(L) = 1$$

$\pi_i(L)$ is a complex analytic function of L.

Take $v^* \in (C^\gamma(K, \mathbf{C}))^*$ such that $\langle v^*, h_{\psi_0} \rangle \neq 0$ and define, for $L \in V$

$$\lambda(L) = \frac{\langle v^*, L\pi_1(L)h_{\psi_0} \rangle}{\langle v^*, \pi_1(L)h_{\psi_0} \rangle}.$$

The denominator is different from zero if V is taken small enough. Then the function $\lambda : V \to \mathbf{C}$ is analytic. Define also $h(L) \in C^\gamma(K, \mathbf{C})$ by

$$h(L) = \pi_1(L) \cdot 1.$$

Observe that $h : V \to C^\gamma(K, \mathbf{C})$ is analytic and that $\pi_1(\mathcal{L}_{\psi_0})1 = h_{\psi_0}$. We can therefore define, restricting the neighborhood V if necessary, $\nu : V \to (C^\gamma(K, \mathbf{C}))^*$, by

$$\langle \nu(L), \varphi \rangle = \frac{\pi_1(L)\varphi}{\pi_1(L)1}.$$

The function that associates to each pair $(L, \varphi) \in V \times C^\gamma(K, \mathbf{C})$ the complex number $\langle \nu(L), \varphi \rangle$ is analytic. Since $E_2(L)$ is unidimensional and invariant and $h(L) \in E_1(L)$, it follows that $Lh(L) = \lambda h(L)$ for some $\lambda \in \mathbf{C}$. But

$$\lambda(L) = \frac{\langle v^*, L\pi_1(L)h_{\psi_0} \rangle}{\langle v^*, \pi_1(L)h_{\psi_0} \rangle} = \lambda.$$

Therefore for any $L \in V$

$$Lh(L) = \lambda(L)h(L).$$

For any $L \in V$ consider $L^* : (C^\gamma(K, \mathbf{C}))^* \hookleftarrow$ the adjoint map. Then

$$\begin{aligned}
\langle L^*\nu(L), \varphi\rangle &= \langle \nu(L), L\varphi\rangle \\
&= \frac{\pi_1(L)L\varphi}{\pi_1(L)1} = \frac{L\pi_1(L)L\varphi}{\pi_1(L)1} \\
&= \lambda(L)\frac{\pi_1(L)\varphi}{\pi_1(L)1} = \lambda(L)\langle \nu(L), \varphi\rangle
\end{aligned}$$

for any $\varphi \in C^\gamma(K, \mathbf{C})$. Hence

$$L^*\nu(L) = \lambda(L)\nu(L).$$

Since the application $C^\gamma(K, \mathbf{C}) \ni \psi \to \mathcal{L}_\psi \in S$ is analytic, it follows that there exists a neighborhood W of ψ_0 in $C^\gamma(K, \mathbf{C})$ such that $\mathcal{L}_\psi \in V$ if $\psi \in W$ and the functions

(1) $W \ni \psi \to \lambda(\mathcal{L}_\psi) \in \mathbf{C}$.

(2) $W \ni \psi \to h(\mathcal{L}_\psi) \in C^\gamma(K, \mathbf{C})$.

(3) $W \times C^\gamma(K, \mathbf{C}) \ni (\psi, \varphi) \to \langle \nu(\mathcal{L}_\psi), \varphi\rangle \in \mathbf{C}$ are analytic. More over

$$\mathcal{L}_\psi h(\psi) = \lambda(\mathcal{L}_\psi)h(\psi)$$

and

$$\mathcal{L}_\psi^*\nu(\mathcal{L}_\psi) = \lambda(\mathcal{L}_\psi)\nu(\mathcal{L}_\psi).$$

We shall show now that if ψ is real then

(4) $\lambda(\mathcal{L}_\psi) = \lambda_\psi$

(5) $h(\mathcal{L}_\psi) = h_\psi$

(6) $\nu(\mathcal{L}_\psi) = \nu_\psi$

Write $h_\psi = ah(\mathcal{L}_\psi) + \varphi$ with $\varphi \in E_2(\mathcal{L}_\psi)$. Then

$$\lambda(\mathcal{L}_\psi)^{-n}\mathcal{L}_\psi^n h_\psi = ah(\mathcal{L}_\psi) + \lambda(\mathcal{L}_\psi)^{-n}\mathcal{L}_\psi^n \varphi.$$

Since $\varphi \in E_2(\mathcal{L}_\psi)$, it follows that $\lambda(\mathcal{L}_\psi)^{-n}\mathcal{L}_\psi)^n\varphi \to 0$ and therefore

$$\lim_{n\to\infty} \lambda(\mathcal{L}_\psi)^{-n}\lambda_\psi^n h_\psi = ah(\mathcal{L}_\psi).$$

And since $h(\mathcal{L}_\psi) \neq 0$, $h_\psi \neq 0$, we obtain $\lambda(\mathcal{L}_\psi) = \lambda_\psi$ and $h_\psi = ah(\mathcal{L}_\psi)$. More over

$$\mathcal{L}_\psi^* \nu_\psi = \lambda_\psi \nu_\psi = \lambda(\mathcal{L}_\psi)\nu_\psi.$$

But $\lambda(\mathcal{L}_\psi)$ is a simple eigen-value of $\mathcal{L}_\psi^*$ (because $\lambda(\mathcal{L}_\psi)$ is a simple eigen-value of $\mathcal{L}_\psi$). It follows that $\nu(\mathcal{L}_\psi) = b\nu_\psi$, for some b. Since $\nu_\psi(1) = 1 = \nu(\mathcal{L}_\psi)1$. We conclude that $b = 1$.

Hence $\nu(\mathcal{L}_\psi) = \nu_\psi$.

Then

$$1 = \langle \nu(\mathcal{L}_\psi), h(\mathcal{L}_\psi)\rangle = a\langle \nu(\mathcal{L}_\psi), h_\psi\rangle = a\langle \nu_\psi, h_\psi\rangle = a.$$

It results that $h_\psi = h(\mathcal{L}_\psi)$, completing the proof of (4), (5), (6) that, together with (1), (2), (3), proves Theorem 5.1

$\square$

6 Hausdorff Dimension and Capacity

Definition 6.1. *Let K be a compact metric space. For $t > 0$, define the t-measure of K as*

$$m_t(K) = \sup_{\epsilon \searrow 0} \inf_{\Gamma_\epsilon} \sum_{B \in \Gamma_\epsilon} (r(B))^t$$

where Γ_ϵ is any collection of balls B with radius $r(B)$ smaller than ϵ.

It is easy to verify that there exists a unique $t_0 \in [0, \infty]$ with the following property:

$m_t(K) = \infty$ if $t < t_0$

$m_t(K) = 0$ if $t > t_0$

This t_0 is called the Hausdorff dimension of K and denoted by $HD(K)$.

We shall collect some well-known facts about the Hausdorff dimension in the next proposition. Since they will not be used in what follows, we shall not prove them.

Proposition 6.2. *(a) If M is a compact manifold, than $HD(M) = \dim M$.*

(b) If $K_1 \subset K_2$ then $HD(K_1) \leq HD(K_2)$.

(c) If $K = K_1 \times K_2$ then $HD(K) \geq HD(K_1) + HD(K_2)$.

Definition 6.3. *Let $N(\epsilon, K)$ be the number of balls of radius ϵ necessary for covering K. Define the upper and lower limit capacities of K by*

$$C^+(K) = \limsup_{\epsilon \searrow 0} \frac{\log N(\epsilon, K)}{-\log \epsilon}$$

$$C^-(K) = \liminf_{\epsilon \searrow 0} \frac{\log N(\epsilon, K)}{-\log \epsilon}$$

Lemma 6.4. *$HD(K) \leq C^-(K) \leq C^+(K)$*

Proof. Given $\delta > 0$ consider a sequence $\epsilon_n \searrow 0$ such that $N(\epsilon_n, K) \leq \epsilon_n^{-(C^-(K)+\delta)}$.

Then, if $t > C^-(K) + \delta$

$$m_t(k) \leq \lim_{n\to\infty} \epsilon_n^t \epsilon_n^{-(C^-(K)+\delta)} = 0$$

Hence $HD(K) \leq C^-(K) + \delta$

Since $\delta > 0$ is arbitrary, the result follows. □

Lemma 6.5. *Suppose that there exist a probability μ over K and numbers $\delta^- > 0$ and $\delta^+ > 0$ with the following property:*

$$\delta^- \leq \liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \leq \limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \leq \delta^+$$

for all $x \in K$. Then $\delta^- \leq HD(K) \leq C^-(K) \leq C^+(K) \leq \delta^+$.

Proof. Given $\delta > 0$ it follows from the first inequality that $\mu(B_r(x)) \leq r^{\delta^- - \delta}$ if r is small enough. Therefore if Γ_r is a collection of balls of radius smaller than r covering K

$$1 \leq \sum_{B \in \Gamma_r} \mu(B) \leq \sum_{B \in \Gamma_r} r(B)^{\delta^- - \delta}.$$

It follows that $HD(K) \geq \delta^- - \delta$. Since $\delta > 0$ is arbitrary it results that $HD(K) \geq \delta^-$.

A set $S \subset K$ is called ϵ-separated if for any x, $y \in S$, $d(x, y) > \epsilon$. Denote by $S(\epsilon, K)$ the maximum number of elements in a ϵ-separated set. It is easy to verify that $N(\epsilon, K) \leq S(\epsilon, K) \leq N(\epsilon/2, K)$.

Hence

$$C^+(K) = \limsup_{\epsilon \searrow 0} \frac{\log S(\epsilon, K)}{-\log \epsilon}$$

$$C^-(K) = \liminf_{\epsilon \searrow 0} \frac{\log S(\epsilon, K)}{-\log \epsilon}$$

Given $\delta > 0$, it follows from the third inequality in the hypothesis of the lemma that $\mu(B_r(x)) \geq r^{\delta^+ + \delta}$ if r is small enough. Take a set $\{x_1, ..., x_{S(r,k)}\}$ r-separated. Then the balls $B(x_i, r/2)$ are disjoint and so

$$1 \geq \sum_i \mu(B(x_i, r/2)) \geq \left(\frac{r}{2}\right)^{\delta^+ + \delta} S(r, k).$$

It follows that $C^+(K) \leq \delta^+ + \delta$. Since δ is arbitrary it results that $C^+(K) \leq \delta^+$. This concludes the proof of the lemma. □

Definition 6.6. *A probability $\mu \in m(k)$ is called a δ-probability if there exists $C > 0$ such that*

$$C^{-1} \leq \frac{\mu(B_r(x))}{r^\delta} \leq C$$

for any $x \in k$ and $r > 0$.

Lemma 6.7. *If K admits a δ-probability then $HD(K) = C^-(K) = C^+(K) = \delta$. Any two δ-probabilities are equivalent.*

Proof. The first part of the lemma is an easy corollary of Lemma 6.5. To prove the second part observe that if μ is a δ-probability then for every $k > 1$ there exists $\alpha(K) > 0$ such that

$$\frac{\mu(B_{kr}(x))}{\mu(B_r(x))} \leq \alpha(K)$$

for any $x \in K$ and $r > 0$. This implies that Vitali's recovering theorem holds for μ (just repeating the usual proof in the case of the Lebesgue measure). $A \subset K$ is a Borel set there exists a covering of A by disjoint balls $B_{r_i}(x_i)$, $\{x_i\} \subset A$, with $\sup_i r_i$ arbitrarily small and such that $\mu(A \setminus \cup_i B_{r_i}(x_i)) = 0$. Let ν be another δ-probability, where the associated constant in Definition 6.6 is given by $\overline{C}$ (and not C). Given $\epsilon > 0$ take $U \supset \overline{A}$ on open set such that $\nu(U) \leq \nu(A) + \epsilon$. Suppose that $\sup_i r_i$ is so small that $U \supset \cup_i B_{r_i}(x_i)$.

Then

$$\mu(A) = \mu(A \cap (\cup_i B_{r_i}(x_i))) \leq \mu(\cup_i B_{r_i}(x_i))$$

$$= \sum_i \mu(B_{r_i}(x_i))$$

$$\leq C \sum_i r_i^\delta$$

$$\leq C\overline{C} \sum_i \nu(B_{r_i}(x_i))$$

$$= C\overline{C}\nu(U_i B_{r_i}(x_i))$$

$$\leq C\overline{C}\nu(U)$$

$$\leq C\overline{C}(\nu(A) + \epsilon)$$

Since $\epsilon > 0$ is arbitrary, this proves that $\mu << \nu$. Inverting the roles of μ and ν we can prove that $\nu << \mu$. It follows then that μ and ν are equivalent. □

7 Differentiability of the Hausdorff Dimension of Julia sets

It this chapter we shall prove differentiability results for Julia sets of rational maps on the Riemann sphere. Let us recall the notation. R_d is the space of rational maps $f : \overline{\mathbb{C}} \hookleftarrow$ of the Riemann sphere of degree $d > 1$, endowed with the C^0 topology. $A_d \subset R_d$ is the set of Axiom A rational maps (see definition in [9] or [2]). $J(f)$ is the Julia set of f and $\delta(f)$ its Hausdorff dimension. For more details on the dynamics of rational maps see [5], [9], [7] or [2].

When $f \in A_d$ there exists a unique f-invariant probability M_f on the Borel σ-algebra of $J(f)$ such that there exists $C > 0$ satisfying

$$C^{-1} \leq \frac{u_f(B_r(x))}{r^{\delta(f)}} \leq C$$

for all $x \in J(f)$ and $r > 0$. The main result in this section is the following

Theorem 7.1. *The functions*
$A_d \ni f \to \delta(f)$
$A_d \ni f \to h_{\mu_f}(f)$ *and*
$A_d \ni f \to \int \varphi d\mu_f \in \mathbf{R}$ *where* $\varphi : \overline{\mathbf{C}} \to \mathbf{R}$ *is any Hölder continuous function, are real analytic.*

In the proof of this theorem we shall use the following proposition:

Proposition 7.2. *Suppose $f \in A_d$ and $\infty \notin J(f)$. Then there exist a neighborhood U of f in A_d and an analytic map $h : U \to C^\gamma(J(f), \mathbf{C})$ such that $g \circ h_g(z) = h_g \circ f(z)$ for every $g \in U$ and $z \in J(f)$. Moreover $J(g) = h_g(J(f))$.*

Proof. Given $f \in A_d$ with $\infty \notin J(f)$ take $\delta > 0$ and a neighborhood U_0 of f such that $d(z, J(f)) < \delta$ implies $g(z) \neq \infty$, for any $g \in U_0$. Given $0 < \gamma < 1$ let W be the ball of radius δ centered at the identity I in $C^\delta(J(f), \mathbf{C})$. Define $\Phi : U_0 \times W \to C^\gamma(J(f), \mathbf{C})$ by $\Phi(g, h) = g \circ h - h \circ f$.

It is straightforward to verify that Φ is anlytic. More over,

$$\frac{\partial \Phi}{\partial h}(f, I)\varphi(z) = f'(z)\varphi(z) - \varphi \circ f(z).$$

We claim that

$$\frac{\partial \Phi}{\partial h}(f, I)$$

is an isomorphism. We shall prove this by showing that

$$L : C^\gamma(J(f), \mathbf{C}) \hookleftarrow$$

given by

$$L\varphi(z) = \sum_{j=0}^{\infty} \varphi(f^j z)[(f^{j+1})'(z)]^{-1} \tag{6}$$

for $\varphi \in C^\gamma(J(f), \mathbf{C})$ is an inverse to

$$\frac{\partial \Phi}{\partial h}(f, I).$$

Observe first that since $f \in A_d$, there exists a metric $\|.\|$ in $J(f)$ such that $f|_{J(f)} : J(f) \hookleftarrow$ is an expanding map. Therefore we can prove a lemma similar to Lemma 3.3.

Lemma 7.3. *There exist $r > 0$ and a constant $A > 0$ such that if $x, y \in J(f)$ with $d(f^j x, f^j y) \leq r$, $0 \leq j \leq n$, then*

$$A^{-1} \leq \frac{d(f^j x, f^j y)}{d(x, y)|(f^j)'(x)|} \leq A$$

$$\left| \frac{(f^n)'(y)}{(f^n)'(x)} - 1 \right| \leq A d(f^n y, f^n x).$$

Proof. Take $r > 0$ such that $B(n, r, x) = g(B_r(f^n x))$ with g a contractive branch of f^{-n} with $g(f^n x) = x$ as in Lemma 3.1. Then if $y \in B(n, r, x)$ and $0 \leq j \leq n$

$$\frac{(f^j)'(y)}{(f^j)'(x)} = \prod_{i=0}^{j-1} \frac{f'(f^i y)}{f'(f^i x)} = \prod_{i=0}^{j-1} \frac{f'(f^i y) - (f'(f^i x)}{f'(f^i x)} + 1.$$

Hence

$$\left|\frac{(f^j)'(y)}{(f^j)'(x)} - 1\right| \leq \left[\prod_{i=0}^{j-1} \frac{|f'(f^i y) - f'(f^i x)|}{|f'(f^i x)|} + 1\right] - 1 \leq$$

$$\leq \left[\prod_{i=0}^{j-1} \frac{|f'(f^i y) - f'(f^i x)|}{c} + 1\right] - 1$$

where

$$c = \inf_{x \in J(f)} |f'(x)| > 0.$$

Then

$$\left|\frac{(f^j)'(y)}{(f^j)'(x)} - 1\right| \leq \left[\prod_{i=0}^{j-1} 1 + \frac{C}{c} d(f^i y, f^i x)\right] - 1$$

where

$$C = \sup_{x \in \text{bar } \mathbf{C}} |f''(x)|.$$

Let $\epsilon_i = 0$ or 1, for $0 \leq i \leq j-1$. Then

$$\left|\frac{(f^j)'(y)}{(f^j)'(x)} - 1\right| \leq \left[\prod_{i=0}^{j-1} 1 + \frac{C}{c} \lambda^{j-i} d(f^j y, f^j x)\right] - 1$$

$$= \sum_{(\epsilon_0, \ldots, \epsilon_{j-1}) \neq (0, \ldots, 0)} \prod_{i=0}^{j-1} \left(\frac{C}{c} \lambda^{j-i} d(f^j y, f^j x)\right)^{\epsilon_i} =$$

$$= d(f^j y, f^j x) \sum_{(\epsilon_0, \ldots, \epsilon_{j-1}) \neq (0, \ldots, 0)} \left(\prod_{i=0}^{j-1} \frac{C}{c} \lambda^{j-i}\right)^{\epsilon_i} d(f^j y, f^j x)^{\sum \epsilon_i - 1}.$$

Since $d(f^j x, f^j y) < r$, $d(f^j y, f^j x)^{\sum \epsilon_i - 1} < 1$. Hence

$$\left|\frac{(f^j)'(y)}{(f^j)'(x)} - 1\right| \leq d(f^j y, f^j x) \left[\left(\prod_{i=0}^{j-1} 1 + \frac{C}{c} \lambda^{j-i}\right) - 1\right].$$

Since the last product is bounded by

$$\prod_{i=0}^{\infty} \left(1 + \frac{C}{c} \lambda^i\right)$$

the second part of the lemma follows. The first part follows easily from the second. $\square$

Let us prove now that L of expression (6) is well defined and continuous. Take $\varphi \in C^\gamma(J(f), \mathbf{C})$ and two points x, $y \in J(f)$ with $d(x,y) < r$. Let N be the greatest integer such that $d(f^j x, f^j y) \le r$, for $0 \le j \le N$.

Observe that

$$C\lambda^{-N} \le (f^N)'(x) \le \frac{rA}{d(x,y)}.$$

Therefore

$$N \le \frac{\log rA - \log C + \log d(x,y)}{\log \lambda^{-1}}.$$

Hence there exists a constant $E > 0$ such that $Nd(x,y) \le Ed(x,y)^\gamma$. Then

$$|L\varphi(x) - L\varphi(y)| \le \sum_0^\infty |[(f^{n+1})'(x)]^{-1}\varphi(f^n x) - [(f^{n+1})'(y)]^{-1}\varphi(f^n y)|$$

$$\le \sum_0^N |[(f^{n+1})'(x)]^{-1}\varphi(f^n x) - \varphi(f^n y)| +$$

$$+ |(f^{n+1})'(x)^{-1} - (f^{n+1})'(y)^{-1}||\varphi(f^n y)| +$$

$$+ \sum_{n>N} |(f^{n+1})'(x)|^{-1}|\varphi(f^n x)| + |(f^{n+1})'(x)||\varphi(f^n y)|$$

Therefore

$$|L\varphi(x) - L\varphi(y)| \le \|\varphi\|_\gamma \sum_0^N |(f^{n+1})'(x)|^{-1} d(f^n x, f^n y)^\gamma +$$

$$+ \|\varphi\|_0 \sum_0^N |(f^{n+1})'(x)|^{-1} \left| 1 - \frac{(f^{n+1})'(x)}{(f^{n+1})'(y)} \right| +$$

$$+ \|\varphi\|_0 \Bigg[|(f^{N+2})'(x)^{-1} \sum_0^\infty |(f^n)'(f^{N+1}x)|^{-1} +$$

$$+ |(f^{N+2})'(y)|^{-1} \sum_0^\infty |(f^n)'(f^{N+1}y)|^{-1} \Bigg]$$

$$\le A^\gamma \|\varphi\|_\gamma d(x,y)^\gamma \sum_0^N |(f^{n+1})'(x)|^{-1+\gamma} +$$

$$
\begin{aligned}
& A\|\varphi\|_0 \sum_0^N |(f^{n+1})'(x)|^{-1} d(f^n y, f^n x) + \\
& + \|\varphi\|_0 C \frac{1}{1-\lambda} [|(f^{N+2})'(x)|^{-1} + |(f^{N+2})'(y)|^{-1}] \\
& \leq A^\gamma \|\varphi\|_\gamma d(x,y)^\gamma C^{1-\gamma} \frac{\lambda^{1-\gamma}}{1-\lambda^{1-\gamma}} + \\
& + A^2 \|\varphi\|_0 N d(x,y) + C \frac{1}{1-\lambda} \|\varphi\|_0 2 \frac{A}{r} d(x,y) \\
& \leq F \|\varphi\|_\gamma d(x,y)^\gamma
\end{aligned}
$$

for same constant $F > 0$.

This estimatives proves that $L\varphi \in C^\gamma(J(f), \mathbf{C})$ and that $|L\varphi|_{r,\gamma} \leq F\|\varphi\|_\gamma$. It is very easy to prove that $\|L\varphi\|_0 \leq B\|\varphi\|_0$ for some constant $B > 0$. We conclude that $\|L\varphi\|_\gamma \leq \max(B, F)\|\varphi\|_\gamma$ and therefore L is continuous.

It is a straightforward verification that L is indeed the bilateral inverse of

$$\frac{\partial \Phi}{\partial h}(f, I).$$

We can now apply the implicit function theorem to obtain a neighborhood U of f and an analytic function $h : U \to C^\gamma(J(f), \mathbf{C})$ such that $h_f = I$ and $\Phi(g, hg) = 0$.

If $hg(x) = hg(y)$ then $hg(f^n x) = hg(f^n y)$ for all $n \leq 0$. This implies that $d(f^n x, f^n y) \leq 2d_0(hg, I)$. Therefore, restricting if necessary the neighborhood U, hg is injective. It remains to prove that $hg(J(f)) = J(g)$.

Every point of $hg(J(f))$ is accumulated by a sequence $\{p_n\}$ of periodic points $p_n \neq p$. This follows immediately from the fact that the same property holds for $J(f)$. Hence $p \in J(g)$. Therefore we have proved that $J(g) \supset hg(J(f))$. But every point $hg(p) \in hg(J(f))$ has d pre-images (namely, the points $hg(q)$, with $q \in f^{-1}(\{p\})$). Then $g^{-1}(hg(J(f))) \subset hg(J(f))$. Since completely invariant set of bar $\mathbf{C}$ contains $J(g)$, $J(g) \subset hg(J(f))$.

This completes the proof of our main result. □

Given $f \in A_d$ and $\psi \in C^\gamma(J(f), \mathbf{R})$ let $\mathcal{L}_\psi : C^0(J(f)) \hookleftarrow$ the Perron-Fröbenius operator associated to $f|_{J(f)} : J(f) \hookleftarrow$ and ψ. Let $\lambda(\psi, f)$ be the eigenvalue of $\mathcal{L}_\psi$ given by Ruelle's theorem. Observe that if U is the neighborhood of f given by Proposition 7.2 then

$$\mathcal{L}_{\psi_0 hg}(\varphi_0 hg) = (\mathcal{L}_\psi \varphi) \circ hg$$

(0) for every $g \in U$, $\varphi \in C^0(J(g))$ and $\psi \in C^\gamma(J(g), \mathbf{R})$. This implies that the application $C^0(J(g)) \ni \varphi \to \varphi \circ hg \in C^0(J(f))$ is an equivalence between $\mathcal{L}_\psi : C^0(J(g)) \hookleftarrow$ and $\mathcal{L}_{\psi \circ hg} : C^0(J(f)) \hookleftarrow$ for every $\psi \in C^\gamma(J(g))$ and $g \in U$. Then it is easy to verify that $\lambda(\psi \circ hg, f) = \lambda(\psi, g)$ for every $g \in U$, $\psi \in C^\gamma(J(g), \mathbf{R})$. Besides, it follows from Ruelle's theorem that

$$\log \lambda(\psi, g) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\psi^n(1)(x)$$

for any $x \in J(g)$.

Lemma 7.4. *If $f \in A_d$ and $\alpha \in \mathbf{R}$ then*

(a) The limit below

$$P(d, f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}x} |(f^n)'(y)|^{-\alpha}$$

exists for any $x \in J(f)$ and independs on $x \in J(f)$.

(b) The function $\mathbf{R} \times A_d \ni (\alpha, f) \to P(\alpha, f)$ is real analytic.

(c) $P(\alpha, f) = \log \lambda(-\alpha \log |f'|, f)$.

Proof. (a) and (c) are immediate from the previous observation. Let us then prove (b). Let U be a neighborhood of f in A_d as in Proposition 7.2. Then if $g \in U$

$$\begin{aligned} P(\alpha, g) &= \log \lambda(-\alpha \log |g'|, g) \\ &= \log \lambda(-\alpha \log |g' \circ hg|, f). \end{aligned}$$

It is easy to see that $U \ni g \to \log |g'\, h\, g|$ is real analytic. By Theorem 5.1 $\psi \ni C^\gamma(J(f), \mathbf{R}) \to \lambda(\psi) \in \mathbb{R}$ is real analytic. So $P(.,.)$ is the composition of analytic functions and hence is analytic. $\square$

Lemma 7.5. *For each $f \in A_d$ there exists $C(f) > 0$ such that*

$$\frac{\partial P}{\partial d}(\alpha, f) \leq -C(f)$$

for any $\alpha \in \mathbf{R}$.

Proof. Given $f \in A_d$, take $x \in J(f)$ and define $P_n : \mathbf{R} \hookleftarrow$ by

$$P_n(\alpha) = \frac{1}{n} \log \sum_{y \in f^{-n}x} |(f^n)'(y)|^{-\alpha}.$$

Then

$$P_n'(\alpha) = \frac{1}{n} \frac{-\sum_{y \in f^{-n}x} |(f^n)'(y)|^{-\alpha} \log |(f^n)'(y)|}{\sum_{y \in f^{-n}x} |(f^n)'(y)|^{-\alpha}}.$$

Since $f \in A_d$, there exists $C(f) > 0$ such that

$$\frac{1}{n} \log |(f^n)'(z)| \geq C(f)$$

for any $z \in J(f)$. Hence $P_n'(\alpha) \leq -C(f)$ for any $\alpha \in \mathbf{R}$. This implies that $P_n(\alpha_1) - P_n(\alpha_2) \leq -C(f)(\alpha_1 - \alpha_2)$ for any $\alpha_1 > \alpha_2$. Taking the limits when n tends to infinity $P(\alpha_1, f) - P(\alpha_2, f) \leq -C(f)(\alpha_1 - \alpha_2)$ for any $\alpha_1 > \alpha_2$.

We conclude that

$$\frac{\partial P}{\partial \alpha}(\alpha, f) \leq -C(f)$$

for any $\alpha \in \mathbb{R}$. □

Observe that $P(0, f) = d$. It follows from the lemma above that there exists a unique $\delta(f) > 0$ such that $P(\delta(f), f) = 0$. We can also apply the implicit functions theorem to the equation $P(\alpha, g) = 0$ and conclude that the function $A_d \ni f \to \delta(f) \in \mathbf{R}$ is real analytic.

From the above we get that for every $\psi \in C^\gamma(J(g), \mathbb{R})$ and $g \in U$

$$(hg) * \nu_{\psi \circ hg} = \nu_\psi$$

$$h_{\psi \circ hg} = h_\psi \circ hg$$

and hence

$$(hg) * \mu_{\psi \circ hg} = \mu_\psi.$$

Define $\mu_g = \mu_\psi$ with $\psi = -\delta(g)\log|g^{,}|$.

Then $\mu_g = (hg) * \mu_{-\delta(g)\log(|g^{,}|\circ hg)}$.

Hence

$$\int \varphi d\mu_g = \int (\varphi \circ hg) d\mu_{-\delta(g)\log(|g^{,}|\circ hg)}$$

for any $\varphi \in C^0(\overline{\mathbf{C}}, \mathbf{R})$.

From this relation and the last part of Theorem 5.1 we conclude that $A_d \ni f \to \int \varphi d\mu_f \in \mathbf{R}$ is real analytic if $\varphi \in C^\gamma(\overline{\mathbb{C}}, \mathbb{R})$ (2)

Note also that

$$\log\lambda(-\delta(g)\log|g'|g) = h_{\mu_g}(g) + \int(-\delta(g)\log|g'|)d\mu_g.$$

But

$$\log\lambda(-\delta(g)\log|g^{,}|g) = P(\delta(g), g) = 0.$$

Hence

$$h_{\mu_g}(g) = \delta(g)\int \log|g^{,}|d\mu_g$$

$$= \delta(g)\int \log(|g^{,}|\circ hg)d\mu_{-\delta(g)\log(|g^{,}|\circ hg)}$$

since $g \to \delta(g)$ is analytic it follows again from the last part of Theorem 5.1 that $g \to h_{\mu_g}(g)$ is analytic. (3)

Proposition 7.6. *If there exists a probability $\nu \in \mathcal{M}(J(f))$ whose Jacobian with respect to f is equal to $|f'|^\delta$, then ν is a δ-probability.*

Proof. Let $r_0 > 0$ be such that there are contractive branches of f^{-n} defined in $B_{r_0}(x)$, for any $x \in J(f)$ and $n > 0$. Take $z \in J(f)$ and contractive branches $\varphi_n : B_{r_0}(f^n z) \to \mathbf{C}$ of f^{-n} such that $\varphi_n(f^n z) = z$. Define $0 < \rho_n \leq r_n$ by

$$r_n = \min\{r|B_{r_0}(z) \supset \varphi_n(B_{r_0}(f^n z))\}$$

$$\rho_n = \max\{r|B_{r_0}(z) \subset \varphi_n(B_{r_0}(f^n z))\}$$

By using Lemma 7.3 we obtain

$$A^{-1}r_0|(f^n)'(z)|^{-1} \leq \rho_n \leq r_n \leq Ar_0|(f^n)'(z)|^{-1}(*)$$

and

$$A^{-2} \leq \frac{|(f^n)'(x)|}{(f^n)'(y)|} \leq A^2 (**)$$

for all $x,\, y \in \varphi_n(B_{r_0}(f^n z))$.

So

$$\nu(B_{r_0}(f^n z)) = \nu(f^n(\varphi_n(B_{r_0}(f^n z)))) = \int_{\varphi_n(B_{r_0}(f^n z))} |(f^n)'|^\delta d\nu.$$

Since ν is positive on open sets, $\nu(B_{r_0}(x)) \geq c > 0$ for some constant $c > 0$ and all $x \in J(f)$. This together with (**) proves that there exist a constant $B > 0$ such that

$$B^{-1} \leq |(f^n)'(z)|^\delta \nu(\varphi_n(B_{r_0}(f^n z))) \leq B.$$

Therefore

$$\nu(B_{\rho_n}(z)) \leq B|(f^n)'(z)|^{-\delta} \leq BA^\delta r_0^{-\delta} \rho_n^\delta$$
$$\nu(B_{r_n}(z)) \geq B^{-1}|(f^n)'(z)|^{-\delta} \geq B^{-1}A^{-\delta} r_0^{-\delta} r_n^\delta.$$

Given $0 < r < \rho_0$ take n such that $r_{r+1} < r < r_n$. Then

$$\nu(B_r(z)) \geq \nu(B_{r_n+1}(z)) \geq B^{-1}A^{-\delta} r_0^{-\delta} r_{n+1}^\delta.$$

It follows easily from (*) that there exists a constant $D > 0$ such that

$$D^{-1} \leq \frac{r_{n+1}}{r_n} \leq D$$

and

$$D^{-1} \leq \frac{\rho_{n+1}}{\rho_n} \leq D.$$

It follows that

$$\nu(B_r(z)) \geq B^{-1}A^{-\delta} r_0^{-\delta} D^\delta r^\delta \overset{def}{=} C^{-1} r^\delta.$$

Using now the sequence $\{\rho_n\}$ in the role of $\{r_n\}$ we can prove that $\nu(B_r(z)) \leq Cr^\delta$.

This proves the proposition. $\square$

Observe now that if $\psi = -\delta(g) \log |g^{\cdot}|$ then ν_ψ has Jacobian $J_{\nu_\psi} g = |g^{\cdot}|^{\delta(g)}$. This implies that ν_ψ is a $\delta(g)$-probability. Therefore $\mu_g = \mu_\psi$ is also a $\delta(g)$-probability. Since any two $\delta(g)$-probability are equivalent and μ_g is ergodic it follows that μ_g is the unique invariant $\delta(g)$-probability in $J(g)$. Therefore μ_g is the probability mentioned in Theorem 7.1. Hence (1), (2) and (3) proves the theorem.

References

[1] V. Baladi, Positive Transfer Operators and decay of correlations, World Scientific (2000)

[2] A. Beardon, Iteration of Rational Functions: Complex Analytic Dynamical Systems, Springer (2000)

[3] R. Bowen, Gibbs States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture notes in Math., volume 470, Springer (1975)

[4] M. Brin and A. Katok, On local entropy. Geometric dynamics, 30-38, Lecture Notes in Math., 1007, Springer (1983)

[5] L. Carleson and T. W. Gamelin, Complex Dynamics, Springer (1993)

[6] R. Mañé, Ergodic Theory and Differentiable dynamics, Springer (1987)

[7] R. Mañé, P. Sad and D. Sullivan, On the dynamics of rational maps. Annales scientifiques de l'École Normale Supérieure 16 (2), 193-217 (1983)

[8] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* Vol 187-188 (1990)

[9] P. Sad, Introdução à dinâmica das funções racionais na esfera de Riemann, XIV Coloq. Bras. Mat - IMPA (1983) on line

[10] M. Viana and K. Oliveira, Foundations of Ergodic Theory, Cambrige Press (2016)

[11] P. Walters, Introduction to Ergodic Theory, Springer (1981)