Large Deviations for Quantum Spin probabilities at temperature zero

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Abstract

We consider certain self-adjoint observables for the KMS state associated to the Hamiltonian $H = \sigma_x \otimes \sigma_x$ over the quantum spin lattice $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots$. For a fixed observable of the form $L \otimes L \otimes L \otimes \ldots$, where $L : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, and for zero temperature one can get a naturally defined stationary probability $\mu$ on the Bernoulli space $\{1, 2\}^\mathbb{N}$. This probability is not Markov. Anyway, for such probability $\mu$ we can show that a Large Deviation Principle is true for a certain class of functions. The result is derived by showing the explicit form of the free energy which is differentiable.

We show that the natural probability $\mu$ at temperature zero is invariant for the shift. We also prove that $\mu$ is not a Gibbs state for a continuous normalized potential but, up to a measurable change of coordinates, is an independent Bernoulli probability on two symbols.

1 Introduction

We analyze from the point of view of Ergodic Theory a probability $\mu$ which appears in a natural way on a physical problem of quantum nature (see [17] and [2]). We get explicit expressions and results (like the one on Large Deviations) for some of these problems. We point out that the systems considered in this work are of quantum nature but the resulting sequences of measurement outcomes are described by classical Ergodic Theory.
On the last section we will show that $\mu$ is independent Bernoulli up to a measurable change of coordinates.

Given a selfadjoint operator $H$ acting on a finite dimensional Hilbert space $\mathcal{H}$ and a temperature $T > 0$, the density operator

$$\rho_{H,T} = \frac{e^{-\frac{1}{T}H}}{Z(T)}$$

where $Z(T) = \text{Tr} \ e^{-\frac{1}{T}H}$, is called the KMS state associated to $H$.

Above $\text{Tr}$ means the trace of the operator.

Among other properties $\rho_{H,T}$ maximizes

$$- \text{Tr} (H \rho) - \text{Tr} (\rho \log \rho),$$

among density operators $\rho$ acting on $\mathcal{H}$.

In this way the KMS state plays in Quantum Statistical Physics (see [3] for general results) the role of the Gibbs state in Thermodynamic Formalism.

We consider the KMS state associated to the Hamiltonian $H = \sigma_x \otimes \sigma_x$ over the quantum spin lattice $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots$ and we will estimate the probabilities of the eigenvalues of a fixed general observable which is in the form $L \otimes L \otimes L \ldots$, where $L : \mathbb{C}^2 \to \mathbb{C}^2$. The (infinite) quantum spin lattice is the set

$$(\mathbb{C}^d)^\mathbb{N} = (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d \otimes \ldots),$$

with a natural metric structure (see [6]).

In order to understand this problem (for each fixed $L$) it is natural to define a probability $\mu_L$ on the Bernoulli space $\{1, 2\}^\mathbb{N}$. Even if the Hamiltonian we consider here is very particular we get a large class of probabilities $\mu_L$. This is so because we consider a general $L$ (the mild restriction is given by Assumption A in section 3). We can ask about the ergodic properties of each of these $\mu_L$. We will address here the problem at temperature zero.

The paper [17] is the main motivation for the present work. We recall the general setting of that paper. For a fixed value $d$, we will consider a complex self-adjoint operator $H$ depending on two variables $H : (\mathbb{C}^d \otimes \mathbb{C}^d) \to (\mathbb{C}^d \otimes \mathbb{C}^d)$. As $H$ depends just on two coordinates for practical purposes all definitions will be on sets of the form

$$(\mathbb{C}^d)^n = (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d).$$

The set of linear operators acting on $\mathbb{C}^d$ will be denoted by $\mathcal{M}_d$. We will call $\omega = \omega_n : \underbrace{\mathcal{M}_d \otimes \mathcal{M}_d \otimes \ldots \otimes \mathcal{M}_d}_{n} \to \mathbb{C}$ a $C^*$-dynamical state if $\omega_n(I^\otimes n) = 1$ and $\omega_n(a) \geq 0$, if $a$ is a non-negative element in the tensor product. It
follows that $\omega_n(L_1 \otimes L_2 \otimes ... \otimes L_n)$ is a positive number if all $L_j$ are positive, $j = 1, 2, ..., n$.

The state $w$ that we will consider here is of the form: take $\beta > 0$ and $H : (\mathbb{C}^d \otimes \mathbb{C}^d) \to (\mathbb{C}^d \otimes \mathbb{C}^d)$ self-adjoint. For fixed $n$ we will consider $H_n = (\mathbb{C}^d)^{\otimes n} \to (\mathbb{C}^d)^{\otimes n}$, where

$$H_n = \sum_{j=0}^{n-2} I^\otimes j \otimes H \otimes I^\otimes (n-j-2).$$

Take

$$\rho_\omega = \rho_{\omega_{\beta, n}} = \frac{1}{Tr(e^{-\beta H_n})} e^{-\beta H_n}.$$ 

Finally, we define

$$\omega_{\beta, n}(L_1 \otimes L_2 \otimes ... \otimes L_n) = \frac{1}{Tr(e^{-\beta H_n})} Tr[e^{-\beta H_n}(L_1 \otimes L_2 \otimes ... \otimes L_n)] =$$

$$= Tr(\rho_\omega L_1 \otimes L_2 \otimes ... \otimes L_n) \quad (1)$$

The Pauli matrices are

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

A Quantum Ising Chain is defined by the Hamiltonian of the form

$$H = -J \,(\sigma_1^x \otimes \sigma_2^x) - h \,(\sigma_1^z \otimes I),$$

where $\sigma_i^x$, for example, means that the Pauli matrix $\sigma^x$ act in the position $i$ of the tensor product. The associated $n$-Hamiltonian is (we use below a not so rigorous notation)

$$H_n = \sum_{i=1}^{n} [ -J \,(\sigma_i^x \otimes \sigma_{i+1}^x) - h \,(\sigma_i^z \otimes I) ].$$ 

In the case $h = 0$ we will say that the Quantum Ising Chain has no magnetic term. We will consider the Quantum Ising Chain with no magnetic term.

Consider $L : \mathbb{C}^d \to \mathbb{C}^d$ a self-adjoint operator and $\lambda_1, \lambda_2, ..., \lambda_d$ denote its real eigenvalues. We suppose that $\psi_j$, $j = 1, 2, .., d$ denotes an orthonormal basis of eigenvectors of $L$. 

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We consider the following observable:

$$L^\otimes n = (L \otimes L \otimes \ldots \otimes L) : (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d) \to (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d).$$

Then,

$$(\psi_{j_1} \otimes \psi_{j_2} \otimes \ldots \otimes \psi_{j_k} \otimes \ldots \otimes \psi_{j_n})$$

is an eigenvector of $L^\otimes n$ associated to the eigenvalue $\lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_k} \ldots \lambda_{j_n}$. Any eigenvalue of $L^\otimes n$ is of this form.

In the generic case all the possible eigenvalues $\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_k}, \ldots, \lambda_{j_n}$ are different and all possible products are also different. In this case there is a bijection of the strings $\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_k}, \ldots, \lambda_{j_n}$ an eigenvalues of the operator acting on the tensor product. But this is not an essential issue.

The values obtained by physical measuring (associated to the observable $L^\otimes n$) in the finite dimensional Quantum Mechanics setting are (of course) the eigenvalues of $L^\otimes n$. The relevant information is the probability of each possible outcome event of measuring $L^\otimes n$.

A general reference on Quantum Mechanics appears in [9].

Given, $\beta > 0$, $H : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ and fixed $L : \mathbb{C}^d \to \mathbb{C}^d$ we can define a probability $\mu = \mu_\beta$ on $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}^N$. This will be done by defining $\mu$ on cylinders of size $n$. That is, for each $n$, we will define $\mu_{\beta,n}$ over $\Omega_n = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}^n$ and then we use Kolmogorov extension theorem to get $\mu_\beta$ on $\Omega$. The zero temperature case is concerned with the limit of $\mu_\beta$, when $\beta \to \infty$.

More precisely, for fixed $n$, we consider the $C^*$-state $w = w_n$ given in (1), and we denote by $P_{\psi_j} : \mathbb{C}^d \to \mathbb{C}^d$ the orthogonal projection on the subspace generated by $\psi_j$. Then, we define the probability $\mu$ on $\Omega_n = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}^n$ by

$$\mu(\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_n}) = w(P_{\psi_{j_1}} \otimes P_{\psi_{j_2}} \otimes \ldots \otimes P_{\psi_{j_n}}).$$

With this information one can get the probability of each possible outcome event of measuring via $L^\otimes n$. Indeed, for fixed $n$ as the eigenvalues of $L^\otimes n$ are of the form $\lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_k} \ldots \lambda_{j_n}$ we just collect the $n$-strings $\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_n}$ which produce each specific value and add their individual probabilities.

If the eigenvalues of $L$ are the numbers 1 and $-1$, then the possible eigenvalues of $L^\otimes n$ are also just the values 1 and $-1$. This is the case we consider here. More precisely, we will consider here the Quantum spin Chain with no magnetic term: $H = \sigma_1^x \otimes \sigma_2^x$ and a general $L : \mathbb{C}^2 \to \mathbb{C}^2$ (with eigenvalues reaching only values 1 and $-1$). We are mainly interested in the zero temperature limit case.
In the case $L$ is a two by two complex selfadjoint matrix, this $\mu$ can be considered defined on the Bernoulli space $\{1,2\}^\mathbb{N}$ if we identify $\lambda_1$ with 1 and $\lambda_2$ with 2.

A natural question is: what ergodic properties are true for such probability $\mu$ on $\{1,2\}^\mathbb{N}$? Is it stationary for the action of the shift $\sigma$? Is a Gibbs state for some continuous normalized potential? Is it true a Large Deviation Principle (L.D.P for short) for a certain class of functions? We refer the reader to [8], [10], [7] and [13] for several results on the topic of Large Deviations for Quantum Spin Systems.

Given $A : \{1,2\}^\mathbb{N} \to \mathbb{R}$, is it true that there exist $M > 0$ such that for fixed small $\epsilon$

$$\mu \left\{ z \text{ such that } \left| \frac{1}{n} \sum_{j=0}^{n-1} A(\sigma^j(z)) - \int A(z) \, d\mu(z) \right| \geq \epsilon \right\} \leq e^{-Mn}?$$

This requires to analyze

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \mu \left\{ z \text{ such that } \left| \frac{1}{n} \sum_{j=0}^{n-1} A(\sigma^j(z)) - \int A(z) \, d\mu(z) \right| \geq \epsilon \right\} \right),$$

or more generally, given a subset $B$ of $\mathbb{R}$, estimate

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \mu \left\{ z \text{ such that } \frac{1}{n} \sum_{j=0}^{n-1} A(\sigma^j(z)) \in B \right\} \right).$$

We say that there exist a large deviation principle for the probability $\mu$ and the function $A$ if there exist a lower-semicontinuous function $I : \mathbb{R} \to \mathbb{R}$ such that for all intervals $B$ we get

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \mu \left\{ z \text{ such that } \frac{1}{n} \sum_{j=0}^{n-1} A(\sigma^j(z)) \in B \right\} \right) = \inf_{s \in B} I(s).$$

A more precise formulation requires to discriminate if $B$ is open or closed set (see [4]).

Denote for $t \in \mathbb{R}$

$$c(t) = \lim_{n \to \infty} \frac{1}{n} \log \int e^{t \left( A(z) + A(\sigma(z)) + A(\sigma^2(z)) + \ldots + A(\sigma^{n-1}(z)) \right)} \, d\mu(z).$$

It is a classical result that if $c$ is differentiable then a Large Deviation Principle is true for $\mu$ and $A$ (see [4]). In this case $I$ is the Legendre transform of $c$. We will show a L.D.P. for $\mu$ in the case $A$ depends just of the first coordinate.

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2 The associated probability \( \mu \) when \( H \) has no magnetic term

For a question of notation we will use the expression \( \sigma_x \) instead of \( \sigma^x \) (as defined before).

When \( H = \sigma_x \otimes \sigma_x \) we want to compute \( U = e^{i\beta H} \) for any complex \( \beta \).

The relation \( \sigma_x \circ \sigma_x = I \) is very helpful.

This estimation is necessary in order to get the associated KMS state. Note that

\[
U = e^{i\beta \sigma_x \otimes \sigma_x} \sum_{j=0}^{\infty} \frac{(i\beta)^j}{j!} (\sigma_x \otimes \sigma_x)^j = \sum_{j=0}^{\infty} \frac{(i\beta)^j}{j!} (\sigma_x \otimes \sigma_x)^j = \cos(\beta) (I \otimes I) + i \sin(\beta) (\sigma_x \otimes \sigma_x).
\]

For Quantum Statistical Mechanics is natural to take \( \beta = i\beta \); where \( \beta \) is real. We now want to compute \( B := e^{-\beta H}, \beta \) real.

For \( \beta \) real we get

\[
B = e^{-\beta \sigma_x \otimes \sigma_x} = \cos(i \beta) (I \otimes I) + i \sin(i \beta) (\sigma_x \otimes \sigma_x).
\]

Let us define \( B_n : (\mathbb{C}^2)^\otimes n \rightarrow (\mathbb{C}^2)^\otimes n \) by \( B_n = e^{-\beta H_n} \). In order to help the reader we will present some examples of the general calculation. As a particular case, note that \( \sigma_x \otimes \sigma_x \otimes I \otimes I \) commutes with \( I \otimes \sigma_x \otimes \sigma_x \otimes I \), etc. In this case we get

\[
B_4 = e^{-\beta [(\sigma_x \otimes \sigma_x \otimes I \otimes I) + (I \otimes \sigma_x \otimes \sigma_x \otimes I) + (I \otimes I \otimes \sigma_x \otimes \sigma_x)]} =
\]

\[
e^{-\beta (\sigma_x \otimes \sigma_x \otimes I \otimes I) \circ e^{-\beta (I \otimes \sigma_x \otimes \sigma_x \otimes I) \circ e^{-\beta (I \otimes I \otimes \sigma_x \otimes \sigma_x)}} =
\]

\[
[\cos(i\beta) (I \otimes I \otimes I \otimes I) + i \sin(i\beta) (\sigma_x \otimes \sigma_x \otimes I \otimes I) \circ \]

\[
[\cos(i\beta) (I \otimes I \otimes I \otimes I) + i \sin(i\beta) (I \otimes \sigma_x \otimes \sigma_x \otimes I) \circ \]

\[
[\cos(i\beta) (I \otimes I \otimes I \otimes I) + i \sin(i\beta) (I \otimes I \otimes \sigma_x \otimes \sigma_x)]. \quad (2)
\]

In this particular case if \( \lambda \) are the eigenvalues of a certain \( L : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \), then

\[
\mu_{\beta,4}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}, \lambda_{j_4}) = w(P_{\psi_{j_1}} \otimes P_{\psi_{j_2}} \otimes P_{\psi_{j_3}} \otimes P_{\psi_{j_4}}),
\]

where \( w(\cdot) = c_4 Tr(e^{(-\beta H_4) \cdot}) \), and \( c_4 = \frac{1}{Tr(e^{-\beta H_4})} \).

For \( n \) in the general case one can show that

\[
\mu_{\beta,n}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}, \ldots, \lambda_{j_n}) =
\]
\[ c_n \text{Tr} \left[ e^{-\beta \left( (\sigma_x \otimes \sigma_x \otimes I \otimes \cdots I) + ... + (I \otimes \sigma_x \otimes \cdots I) \right)} \right] = c_n \text{Tr} \left\{ \left( \cos(i\beta) \left( I \otimes \cdots \otimes I \right) + i \sin(i\beta) \left( \sigma_x \otimes \cdots \otimes I \right) \right) \circ \cdots \circ \left( \cos(i\beta) \left( I \otimes \cdots \otimes I \right) + i \sin(i\beta) \left( \sigma_x \otimes \cdots \otimes I \right) \right) \right\}, \]

where \( c_n = \frac{1}{\text{Tr}(e^{-\beta n \sigma_z})} \).

3 The associated probability \( \mu \) on the zero temperature limit case

As a question of notation we use the expression \( \sigma_z \) instead of \( \sigma^z \).

We will present first a simple case for the purpose of getting a clear picture of the setting for the general \( L \). Suppose \( L = \sigma_z \). In this case \( \mu \) will be the maximal entropy measure.

Let us take for instance the case \( n = 4 \). In this way we have to consider the observable \( C = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \).

The eigenvalues of \( \sigma_z \) are \( 1 \) and \( -1 \) which are associated respectively to the eigenvectors \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). We associate \( 1 \) to \( 1 \) and \( 2 \) to \( -1 \).

The eigenvectors of \( C \) are of the form

\[ e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4}, \]

where, \((j_1, j_2, j_3, j_4) \in \{1, 2\}^4 \). The possible eigenvalues are \( 1 \) and \( -1 \).

We denote by \( P_1 : \mathbb{C}^2 \to \mathbb{C}^2 \) the projection on \( e_1 \) and \( P_2 : \mathbb{C}^2 \to \mathbb{C}^2 \) the projection on \( e_2 \). In this way

\[ P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

The probability \( \mu \) of the element \((j_1, j_2, j_3, j_4)\) is

\[ \mu_{j_1 j_2 j_3 j_4} = w(P_{j_1} \otimes P_{j_2} \otimes P_{j_3} \otimes P_{j_4}) = c_4 \text{Tr} \left( B_4 \right), \]

where \( c_4 = \frac{1}{\text{Tr}(e^{-\beta n \sigma_z})} \).

Remember that \( \text{Tr} \left( A_1 \otimes A_2 \otimes A_3 \otimes A_4 \right) = \text{Tr}(A_1) \text{Tr}(A_2) \text{Tr}(A_3) \text{Tr}(A_4) \) and note that

\[ \sigma_x(P_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]
has trace 0. Moreover,
\[ \sigma_x(P_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]
has trace 0.

In this way
\[
\text{Tr} \left[ (I \otimes I \otimes \sigma_x \otimes \sigma_x)(P_{j_1} \otimes P_{j_2} \otimes P_{j_3} \otimes P_{j_4}) \right] = \text{Tr}(P_{j_1} \otimes P_{j_2} \otimes \sigma_x(P_{j_3}) \otimes \sigma_x(P_{j_4})) = 0.
\]
Using a similar reasoning for \((I \otimes \sigma_x \otimes I \otimes I)\), etc... we finally get that
\[
w(P_{j_1} \otimes P_{j_2} \otimes P_{j_3} \otimes P_{j_4}) = c_4 \text{Tr} \left[ e^{-\beta H_4}(P_{j_1} \otimes P_{j_2} \otimes P_{j_3} \otimes P_{j_4}) \right] = c_4 [\cos(i \beta)]^3.
\]
Therefore \(\mu_{\beta,4}(j_1, j_2, j_3, j_4)\) has the same value for any \((j_1, j_2, j_3, j_4) \in \{1, 2\}^4\).

It is quite easy to extend the above for estimating \(\mu_{\beta,n}(j_1, j_2, j_3, ..., j_n)\), \(n \in \mathbb{N}\). In this way we get that \(\mu_\beta\) is the independent probability on \(\{1, 2\}^n\), with \(P(1) = p_1 = 1/2\) and \(P(2) = p_2 = 1/2\).

Now we will consider a more rich and interesting example. We will define \(\mu\) on \(\{1, 2\}^N\) via another general observable.

**Assumption A:** We consider the self-adjoint operators (observable) of the form
\[
L = \begin{pmatrix}
\cos^2(\theta) - \sin^2(\theta) & 2 \cos(\theta) \sin(\theta) \\
2 \cos(\theta) \sin(\theta) & \sin^2(\theta) - \cos^2(\theta)
\end{pmatrix},
\]
where \(\theta \in (0, \pi/2)\), and the corresponding associated observable
\[
C_n = C = \underbrace{L \otimes L \otimes L \otimes \cdots \otimes L}_{n}.
\]

The eigenvalues of \(L\) are \(\lambda_1 = 1\) and \(\lambda_2 = -1\) which are associated respectively to the unitary eigenvectors \(v_1 = (\cos(\theta), \sin(\theta)) \in \mathbb{C}^2\) and \(v_2 = (-\sin(\theta), \cos(\theta)) \in \mathbb{C}^2\) which are orthogonal.

The case \(\theta = 0\) corresponds to \(L = \sigma_z\) and the case \(\theta = \pi/2\) corresponds to \(L = -\sigma_z\). We will exclude this cases from now.

We associate 1 to \(\lambda_1\) and 2 to \(\lambda_2\). The eigenvectors of \(C\) are of the form
\[
v_{j_1} \otimes v_{j_2} \otimes v_{j_3} \otimes v_{j_4} \otimes \cdots \otimes v_{j_n},
\]
where, \((j_1, j_2, j_3, j_4, ..., j_n) \in \{1, 2\}^n\).

We denote by \(P_1 : \mathbb{C}^2 \rightarrow \mathbb{C}^2\) the projection on \(v_1\) and \(P_2 : \mathbb{C}^2 \rightarrow \mathbb{C}^2\) the projection on \(v_2\). In this way
\[
P_1 = \begin{pmatrix}
\cos(\theta)^2 & \cos(\theta) \sin(\theta) \\
\cos(\theta) \sin(\theta) & \sin(\theta)^2
\end{pmatrix}
\]
\[ P_2 = \begin{pmatrix} 
\sin(\theta)^2 & -\cos(\theta) \sin(\theta) \\
-\cos(\theta) \sin(\theta) & \cos(\theta)^2 
\end{pmatrix}. \]

Note that \( \text{Tr} P_1 = \text{Tr} P_2 = 1 \). Moreover

\[ \sigma_x(P_1) = \begin{pmatrix} 
\cos(\theta) \sin(\theta) & \sin^2(\theta) \\
\cos^2(\theta) & \cos(\theta) \sin(\theta) 
\end{pmatrix} \] has trace \( \beta_1 := \sin(2\theta) \in \mathbb{R} \) and

\[ \sigma_x(P_2) = \begin{pmatrix} 
-\cos(\theta) \sin(\theta) & \cos^2(\theta) \\
\sin^2(\theta) & -\cos(\theta) \sin(\theta) 
\end{pmatrix} \] has trace \( \beta_2 := -\sin(2\theta) \in \mathbb{R} \). Therefore, \( \text{Tr} (\sigma_x(P_2)) = \beta_2 = -\beta_1 \). Note that, if \( \theta \neq \frac{\pi}{4} \) then \( \beta_1, \beta_2 \) both have modulus smaller than 1. The case \( \theta = \frac{\pi}{4} \) produces some results that differ essentially of other parameters.

The probability of the element \([j_1, j_2, j_3, \ldots, j_n]\) (cylinder) is given by \( \mu_{\beta,n}(j_1, j_2, j_3, \ldots, j_n) \), see expression (3).

We point out that the factor of normalization is \( c_n = \frac{1}{\text{Tr}(B_n)} = \frac{1}{\cos^{n-1}(i\beta)2^n} \).

Indeed,

\[ \text{Tr}(B_n) = \text{Tr} \left[ e^{-\beta \left( (\sigma_x \otimes \sigma_x \otimes \ldots \otimes I) + (I \otimes \sigma_x \otimes \ldots \otimes I) + \ldots + (I \otimes I \otimes \ldots \otimes I) \right)} \left( I \otimes \ldots \otimes I \right) \right] = \]

\[ \text{Tr} \left\{ \begin{array}{l}
(\cos(i\beta) (I \otimes \ldots \otimes I) + i \sin(i\beta) (\sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
(\cos(i\beta) (I \otimes \ldots \otimes I) + i \sin(i\beta) (I \otimes \sigma_x \otimes \sigma_x \otimes \ldots \otimes I)) \circ \\
\ldots \circ \\
(\cos(i\beta) (I \otimes \ldots \otimes I) + i \sin(i\beta) (I \otimes \ldots \otimes I \otimes I \otimes \sigma_x \otimes \sigma_x)) \circ \\
(I \otimes I \otimes \ldots \otimes I) 
\end{array} \right\} \]

which is a sum of \( 2^{n-1} \) terms. Only the term \( \cos^{n-1}(i\beta) (I \otimes \ldots \otimes I) \) do not contain a product of \( \sigma_x \). As \( \text{Tr} (\sigma_x(I)) = 0 \), any term which is not \( \cos^{n-1}(i\beta) (I \otimes I \otimes I \otimes \ldots \otimes I) \) will produce a null trace. Moreover \( \text{Tr} (I) = 2 \), therefore \( \text{Tr}(B_n) = \cos^{n-1}(i\beta)2^n \).

Using equation (3) we have that

\[ \mu_{\beta,n}(\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_n}) = \]
\[
\mu_\beta,n(\lambda_1, \lambda_2, \ldots, \lambda_n) \sim \\
= \frac{1}{\cos^{n-1}(i\beta)^2 n} \text{Tr} \left\{ \begin{align*}
& (\cos(i\beta) (I \otimes \ldots \otimes I) + i \sin(i\beta) (\sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
& (\cos(i\beta) (I \otimes \ldots \otimes I) + i \sin(i\beta) (I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
& \quad \ldots \circ \\
& (\cos(i\beta) (I \otimes \ldots \otimes I) + i \sin(i\beta) (I \otimes \ldots \otimes I \otimes \sigma_x \otimes \sigma_x)) \circ \\
& (P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_{j_n})
\end{align*} \right\}.
\]

Using the relation \( i \sin(\beta i) = -\cos(\beta i) + e^{-\beta} \) we get, when \( \beta \) is large,

\[
\mu_\beta,n(\lambda_1, \lambda_2, \ldots, \lambda_n) \sim \\
= \frac{1}{\cos^{n-1}(i\beta)^2 n} \text{Tr} \left\{ \begin{align*}
& ((I \otimes \ldots \otimes I) - (\sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
& (I \otimes \ldots \otimes I - (I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
& \quad \ldots \circ \\
& ((I \otimes \ldots \otimes I) - (I \otimes \ldots \otimes I \otimes \sigma_x \otimes \sigma_x)) \circ \\
& (P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_{j_n})
\end{align*} \right\}.
\]

Therefore, when \( \beta \to \infty \), the limit measure \( \mu_n \) satisfies

\[
\mu_n(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{2^n} \text{Tr} \left\{ \begin{align*}
& ((I \otimes \ldots \otimes I) - (\sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
& (I \otimes \ldots \otimes I - (I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
& \quad \ldots \circ \\
& ((I \otimes \ldots \otimes I) - (I \otimes \ldots \otimes I \otimes \sigma_x \otimes \sigma_x)) \circ \\
& (P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_{j_n})
\end{align*} \right\}.
\]

For example, as \( \text{Tr}(\sigma_x P_{j_i}) = \beta_{j_i} \),

\[
\mu_2(j_1, j_2) = \frac{1}{2^2} \text{Tr} \left[ (I \otimes I - \sigma_x \otimes \sigma_x) \circ (P_{j_1} \otimes P_{j_2}) \right] = \\
= \frac{1}{2^2} \text{Tr} \left[ P_{j_1} \otimes P_{j_2} - \sigma_x P_{j_1} \otimes \sigma_x P_{j_2} \right] = \frac{1}{2^2} \left[ 1 - \beta_{j_1} \beta_{j_2} \right].
\]

Note that if \( \theta \neq \frac{\pi}{4} \), the number \( 1 - \beta_{j_1} \beta_{j_2} \) is always positive because \( |\beta_1| < 1 \) and \( |\beta_2| < 1 \).
Moreover $\mu(1) = \mu_2(1, 1) + \mu_2(1, 2) = 1/2 = \mu(2)$ for getting $\mu$ an invariant probability for the shift (see Corollary 2).

We also get

$$
\mu_3(j_1, j_2, j_3) = \frac{1}{2^3} \text{Tr} \left\{ \begin{array}{c}
((I \otimes I \otimes I) - (\sigma_x \otimes \sigma_x \otimes I)) \circ \\
((I \otimes I \otimes I) - (I \otimes \sigma_x \otimes \sigma_x) \circ) \\
(P_{j_1} \otimes P_{j_2} \otimes P_{j_3})
\end{array} \right\} = \\
= \frac{1}{2^3} \left[ 1 - \beta_{j_1} \beta_{j_2} - \beta_{j_2} \beta_{j_3} + \beta_{j_1} \beta_{j_3} \right].
$$

**Proposition 1.**

$$
\mu_{n+1}(1, j_2, j_3, \ldots, j_n) + \mu_{n+1}(2, j_2, j_3, \ldots, j_n) = \mu_n(j_2, j_3, \ldots, j_n).
$$

**Proof:** Indeed, using (4) and that $P_1 \otimes P_j \otimes P_{j_2} \otimes \ldots \otimes P_n = I \otimes P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_n$, we get

$$
\mu_{n+1}(1, j_2, j_3, \ldots, j_n) + \mu_{n+1}(2, j_2, j_3, \ldots, j_n) = \\
= \frac{1}{2^{n+1}} \text{Tr} \left\{ \begin{array}{c}
((I \otimes \ldots \otimes I) - (\sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \\
((I \otimes \ldots \otimes I) - (I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \circ \ldots \circ \\
((I \otimes \ldots \otimes I) - (I \otimes \ldots \otimes I \otimes \sigma_x \otimes \sigma_x)) \circ \circ \ldots \circ \\
(I \otimes P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_{j_n})
\end{array} \right\}
$$

As $\text{Tr}(\sigma_x) = 0$

$$
= \frac{1}{2^{n+1}} \text{Tr} \left\{ \begin{array}{c}
-(\sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I) \circ \\
((I \otimes \ldots \otimes I) - (I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I)) \circ \circ \ldots \circ \\
((I \otimes \ldots \otimes I) - (I \otimes \ldots \otimes I \otimes \sigma_x \otimes \sigma_x)) \circ \circ \ldots \circ \\
(I \otimes P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_{j_n})
\end{array} \right\} = 0.
$$

Note also that, for any $B_1, \ldots, B_n$ we have

$$
\text{Tr}(I \otimes B_1 \otimes \ldots \otimes B_n) = 2 \text{Tr}(B_1 \otimes \ldots \otimes B_n).
$$
Then
\[ \mu(1, j_1, j_2, \ldots, j_n) + \mu(2, j_1, j_2, \ldots, j_n) = \]
\[ \frac{1}{2^{n+1}} \text{Tr} \left\{ \left( (I \otimes \ldots \otimes I) \circ \right. \right. \]
\[ \left. \left. \left( I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes \ldots \otimes I \right) \circ \right. \right. \]
\[ \left. \left. \left( I \otimes \ldots \otimes I \right) \right. \right. \]
\[ \left. \left. \left. \left( I \otimes P_{j_1} \otimes P_{j_2} \otimes \ldots \otimes P_{j_n} \right) \right. \right. \]
\[ = \mu(j_1, j_2, \ldots, j_n). \]

It follows from the commutativity of the composition of the several terms in expression (4) that
\[ \mu_n(j_1, j_2, j_3, j_4, \ldots, j_n) = \mu_n(j_n, j_{n-1}, j_{n-2}, \ldots, j_2, j_1). \]  

This sequence \( \mu_n, n \in \mathbb{N} \), will define a probability \( \mu \) on the Bernoulli space. Indeed, the probability \( \mu_n \) and \( \mu_{n+1} \) are compatible in the sense that
\[ \mu_n(j_1, j_2, j_3, \ldots, j_n) = \mu_{n+1}(j_1, j_2, j_3, \ldots, j_{n+1}) + \mu_{n+1}(j_1, j_2, j_3, \ldots, j_n, 2). \]

This follows from (5) and Proposition 1.

In this way by Kolmogorov extension theorem we get a probability \( \mu \) on the Bernoulli space \( \{1, 2\}^\mathbb{Z} \) which is stationary for the shift \( \sigma \) (see Proposition 1). 

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Corollary 2. The probability $\mu$ is stationary.

One can easily show that such $\mu$ is not a Markov probability. Moreover, the Stochastic Process associated to $\mu$ is not a two step Markov Chain (see section ??).

4 A Large Deviation Principle for $\mu$

We want to show how to compute $\mu$ recursively in cylinders.

Theorem 3. For any $n > 0$ we have that

$$
\mu(k, j_1, j_2, \ldots, j_n) = \frac{\mu(j_1, j_2, \ldots, j_n)}{2} - \frac{\beta_k \beta_{j_1}}{2^2} \mu(j_2, \ldots, j_n) + \frac{\beta_k \beta_{j_2}}{2^3} \mu(j_3, j_4, \ldots, j_n) - \frac{\beta_k \beta_{j_3}}{2^4} \mu(j_4, j_5, \ldots, j_n) + \ldots + (-1)^{n-2} \beta_k \beta_{j_{n-2}} \mu(j_{n-1}, j_n) + \frac{(-1)^{n-1} \beta_k \beta_{j_{n-1}}}{2^{n+1}} + \frac{(-1)^n \beta_k \beta_{j_n}}{2^{n+1}}.
$$

(6)

Proof: By equation (4) we have

$$
\mu(k, j_1, j_2, \ldots, j_n) = \frac{1}{2^{n+1}} \text{Tr} \left\{ \left( I \otimes \cdots \otimes I \right) - (\sigma_x \otimes \sigma_x \otimes I \otimes \cdots \otimes I) \right\} \circ \left\{ \left( I \otimes \cdots \otimes I \right) - (I \otimes I \otimes I \otimes I \otimes \cdots) \right\} \circ \ldots \circ \left\{ I \otimes \cdots \otimes I \right\} - (I \otimes \cdots \otimes I \otimes I \otimes \cdots \otimes I \otimes \cdots \otimes I \otimes I \otimes \cdots \otimes I) \right\} \circ \left\{ P_k \otimes P_{j_1} \otimes P_{j_2} \otimes \cdots \otimes P_{j_n} \right\}
$$

$$
= \frac{1}{2^{n+1}} \text{Tr} \left\{ \left( I \otimes \cdots \otimes I \right) \right\} \circ \left\{ \left( I \otimes \cdots \otimes I \right) - (I \otimes I \otimes \cdots \otimes I \otimes \cdots \otimes I \otimes \cdots) \right\} \circ \ldots \circ \left\{ I \otimes \cdots \otimes I \right\} - (I \otimes I \otimes \cdots \otimes I \otimes I \otimes \cdots \otimes I \otimes I \otimes \cdots \otimes I) \right\} \circ \left\{ P_k \otimes P_{j_1} \otimes P_{j_2} \otimes \cdots \otimes P_{j_n} \right\}
$$

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\[\begin{align*}
+ \frac{1}{2^{n+1}} \text{Tr} \left\{ 
&\begin{aligned}
&-(\sigma_x \otimes \sigma_x \otimes I \otimes I \otimes \cdots \otimes I) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes I \otimes \cdots \otimes I \otimes I) \right) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes \cdots \otimes I \otimes I) \otimes \cdots \right) \\
&\left( P_k \otimes P_{j_1} \otimes P_{j_2} \otimes \cdots \otimes P_{j_n} \right)
\end{aligned}
\right\}
\end{align*}\]

\[\begin{align*}
= \frac{1}{2} \mu(j_1, j_2, \ldots, j_n) - \\
- \frac{1}{2^{n+1}} \text{Tr} \left\{ 
&\begin{aligned}
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes I \otimes \cdots \otimes I \otimes I) \right) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes \cdots \otimes I \otimes I) \otimes \cdots \right) \\
&\left( \sigma_x P_k \otimes \sigma_x P_{j_1} \otimes P_{j_2} \otimes \cdots \otimes P_{j_n} \right)
\end{aligned}
\right\}
\end{align*}\]

\[\begin{align*}
= \frac{1}{2} \mu(j_1, j_2, \ldots, j_n) - \\
- \frac{1}{2^{n+1}} \text{Tr} \left\{ 
&\begin{aligned}
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) \right) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes I \otimes \cdots \otimes I \otimes I) \right) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes \cdots \otimes I \otimes I) \otimes \cdots \right) \\
&\left( \sigma_x P_k \otimes \sigma_x P_{j_1} \otimes P_{j_2} \otimes \cdots \otimes P_{j_n} \right)
\end{aligned}
\right\}
\end{align*}\]

\[\begin{align*}
= \frac{1}{2} \mu(j_1, j_2, \ldots, j_n) - \\
- \frac{1}{2^{n+1}} \text{Tr} \left\{ 
&\begin{aligned}
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) \right) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes I \otimes \cdots \otimes I \otimes I) \right) \circ \\
&\left( (+ \underbrace{I \otimes \cdots \otimes I}_{n+1}) - (I \otimes I \otimes \cdots \otimes I \otimes I) \otimes \cdots \right) \\
&\left( \sigma_x P_k \otimes \sigma_x P_{j_1} \otimes P_{j_2} \otimes \cdots \otimes P_{j_n} \right)
\end{aligned}
\right\}
\end{align*}\]
Some examples of direct computations:

\[
\mu(k, j_1, j_2) = \frac{\mu(j_1, j_2)}{2} - \frac{\beta_k \beta_j_1}{8} + \frac{\beta_k \beta_j_2}{8},
\]

\[
\mu(k, j_1, j_2, j_3) = \frac{\mu(j_1, j_2, j_3)}{2} - \frac{\beta_k \beta_j_1}{4} + \frac{\beta_k \beta_j_2}{16} + \frac{\beta_k \beta_j_3}{16},
\]
\[
\mu(k, j_1, j_2, j_3, j_4) = \frac{\mu(j_1, j_2, j_3, j_4)}{2} - \frac{\beta_k \beta_{j_1}}{4} \mu(j_2, j_3, j_4) + \frac{\beta_k \beta_{j_2}}{8} \mu(j_3, j_4) - \frac{\beta_k \beta_{j_3}}{32} + \frac{\beta_k \beta_{j_4}}{32}.
\]

Suppose \( A : \{1, 2\} \rightarrow \mathbb{R} \) is a function and we are interested in the sums of the form \( A(x_0) + A(x_1) + A(x_2) + \ldots + A(x_n) \), \( n \in \mathbb{N} \), where \( x = (x_0, x_1, x_2, \ldots, x_n) \).

For the purpose of future use in Large Deviations we are interested for \( t > 0 \) in

\[
Q_n(t) = \int e^{\sum (A(x_0) + A(x_1) + \ldots + A(x_n))} d\mu(x) = 
\sum_{j_0} \sum_{j_1} \ldots \sum_{j_n} e^{\sum (A(j_0) + A(j_1) + \ldots + A(j_n))} \mu(j_0, j_1, \ldots, j_n).
\]

Denote \( \alpha(t) = \sum \beta_{j_k} e^{t A(j_k)} \) and \( \delta(t) = \sum e^{t A(j_k)} \). Note that \( |\alpha(t)| < |\delta(t)| \).

**Theorem 4.** For any \( n \) we have that

\[
Q_n(t) = \frac{1}{2} \delta(t) Q_{n-1}(t) - \frac{1}{4} \alpha(t)^2 Q_{n-2}(t) + \frac{1}{8} \delta(t) \alpha(t)^2 Q_{n-3}(t) - \frac{1}{16} \delta(t)^2 \alpha(t)^2 Q_{n-4}(t) + \frac{1}{32} \delta(t)^3 \alpha(t)^2 Q_{n-5}(t) - \frac{(-1)^{n-1}}{2^{n-2}} \delta(t)^n \alpha(t)^2 Q_2(t) + \frac{(-1)^{n}}{2^{n-1}} \delta(t)^{n-3} \alpha(t)^2 Q_1(t). \tag{7}
\]

As an example we will compute \( Q_3(t) \). From the equation

\[
\mu(j_0, j_1, j_2, j_3) = \frac{\mu(j_1, j_2, j_3)}{2} - \frac{\beta_{j_0} \beta_{j_2}}{4} \mu(j_2, j_3) + \frac{\beta_{j_0} \beta_{j_2}}{16} - \frac{\beta_{j_0} \beta_{j_3}}{16}
\]

we get

\[
Q_3(t) = \sum_{j_0} \sum_{j_1} \sum_{j_2} \sum_{j_3} e^{\sum (A(j_0) + A(j_1) + A(j_2) + A(j_3))} \mu(j_0, j_1, j_2, j_3) = 
\left[ \frac{1}{2} \sum_{j_0} e^{t A(j_0)} \right] \sum_{j_1} \sum_{j_2} \sum_{j_3} e^{\sum (A(j_1) + A(j_2) + A(j_3))} \mu(j_1, j_2, j_3) + 
\]

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\[-\frac{1}{4}\left[ \sum_{j_0} e^{tA(j_0)} \beta_{j_0} \right] \left[ \sum_{j_1} e^{tA(j_1)} \beta_{j_1} \right] \sum_{j_2,j_3} e^{t(A(j_2)+A(j_3))} \mu(j_2,j_3) \]
\[+ \frac{1}{16} \sum_{j_0} \sum_{j_1} \sum_{j_2} \sum_{j_3} e^{t(A(j_0)+A(j_1)+A(j_2)+A(j_3))} \beta_{j_0} \beta_{j_2} \]
\[= \frac{1}{2} \delta(t) \; Q_2(t) - \frac{1}{4} \alpha(t)^2 Q_1(t) \]

**Proof of Theorem 4:**

By definition

\[Q_n(t) = \sum_{j_0} \sum_{j_1} ... \sum_{j_n} e^{t(A(j_0)+A(j_1)+...+A(j_n))} \mu(j_0, j_1, ..., j_n)\]

From the equation (6)

\[
\mu(j_0, j_1, j_2, ..., j_n) = \frac{\mu(j_1, j_2, ..., j_n)}{2} - \frac{\beta_{j_0} \beta_{j_1}}{2^2} \mu(j_2, ..., j_n) + \frac{\beta_{j_0} \beta_{j_2}}{2^4} \mu(j_3, j_1, ..., j_n) - \frac{\beta_{j_0} \beta_{j_3}}{2^4} \mu(j_4, j_5, ..., j_n) + ... + \frac{(-1)^{n-2} \beta_{j_0} \beta_{j_{n-2}}}{2^{n-1}} \mu(j_{n-1}, j_j) + \frac{(-1)^{n-1} \beta_{j_0} \beta_{j_{n-1}}}{2^{n+1}} + \frac{(-1)^n \beta_{j_0} \beta_{j_n}}{2^{n+1}}.
\]

Then

\[Q_n(t) = \frac{1}{2} \sum_{j_0} e^{tA(j_0)} \sum_{j_1,...,j_n} e^{t(A(j_1)+...+A(j_n))} \mu(j_1,...,j_n) - \frac{1}{2^2} \sum_{j_0} e^{tA(j_0)} \beta_{j_0} \sum_{j_1} e^{tA(j_1)} \beta_{j_1} \sum_{j_2,...,j_n} e^{t(A(j_2)+...+A(j_n))} \mu(j_2,...,j_n) \]
\[+ \frac{1}{2^3} \sum_{j_0} e^{tA(j_0)} \beta_{j_0} \sum_{j_1} e^{tA(j_1)} \beta_{j_1} \sum_{j_2,...,j_n} e^{t(A(j_2)+...+A(j_n))} \mu(j_2,...,j_n) \]
\[+ \frac{(-1)^n}{2^{n+1}} \sum_{j_0} e^{tA(j_0)} \beta_{j_0} \sum_{j_1} e^{tA(j_1)} \beta_{j_1} \sum_{j_2,...,j_n} e^{t(A(j_2)+...+A(j_n))} \mu(j_3,...,j_n) \]
\[+ ... + \frac{(-1)^n}{2^{n+1}} \sum_{j_0} e^{tA(j_0)} \beta_{j_0} \sum_{j_1} e^{tA(j_1)} \beta_{j_1} \sum_{j_2,...,j_n} e^{t(A(j_2)+...+A(j_n))} \mu(j_{n-1},j_n) \]

(following the definitions of \(\alpha(t)\) and \(\delta(t)\))

\[= \frac{1}{2} \delta(t) Q_{n-1}(t) - \frac{1}{4} \alpha(t)^2 Q_{n-2}(t) + \frac{1}{8} \delta(t) \alpha(t)^2 Q_{n-3}(t) - \frac{1}{16} \delta(t)^2 \alpha(t)^2 Q_{n-4}(t) + ...\]

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\[
\frac{1}{32} \delta(t)^3 \alpha(t)^2 Q_{n-5}(t) - \cdots + \frac{(-1)^{n-1}}{2^{n-2}} \delta(t)^{n-4} \alpha(t)^2 Q_2(t) + \frac{(-1)^n}{2^{n-1}} \delta(t)^{n-3} \alpha(t)^2 Q_1(t) + \frac{(-1)^n}{2^{n+1}} \alpha(t)^2 \delta(t)^{n-2} = \frac{1}{2} \delta(t) Q_{n-1}(t) \frac{1}{4} \alpha(t)^2 Q_{n-2}(t) + \frac{1}{8} \delta(t) \alpha(t)^2 Q_{n-3}(t) - \frac{1}{16} \delta(t)^2 \alpha(t)^2 Q_{n-4}(t) + \frac{1}{32} \delta(t)^3 \alpha(t)^2 Q_{n-5}(t) - \cdots + \frac{(-1)^{n-1}}{2^{n-2}} \delta(t)^{n-4} \alpha(t)^2 Q_2(t) + \frac{(-1)^n}{2^{n-1}} \delta(t)^{n-3} \alpha(t)^2 Q_1(t).
\]

\[\square\]

**Proposition 5.**

\[Q_{n+2}(t) = \frac{\delta(t)^2 - \alpha(t)^2}{4} Q_n(t).\]

We get as a corollary that \(Q_n(t)\) grows like \(\sqrt{\frac{\delta(t)^2 - \alpha(t)^2}{2}}\). In this way for each fixed \(t\):

\[c(t) = \lim_{n \to \infty} \frac{1}{n} \log Q_n(t) = \log \frac{\sqrt{\delta(t)^2 - \alpha(t)^2}}{2}\]

\[= \frac{1}{2} \log \left( \left[ \sum_{j_0} e^{t A(j_0)} \right]^2 - \left[ \sum_{j_0} \beta_{j_0} e^{t A(j_0)} \right]^2 \right) - \log(2) \quad (8)\]

is clearly differentiable on \(t\).

**Proof.** (of the proposition)

We have that

\[Q_n(t) = \frac{1}{2} \delta(t) Q_{n-1}(t) - \frac{1}{4} \alpha(t)^2 Q_{n-2}(t) + \frac{1}{8} \delta(t) \alpha(t)^2 Q_{n-3}(t) - \frac{1}{16} \delta(t)^2 \alpha(t)^2 Q_{n-4}(t) + \frac{1}{32} \delta(t)^3 \alpha(t)^2 Q_{n-5}(t) - \cdots + \frac{(-1)^{n-1}}{2^{n-2}} \delta(t)^{n-4} \alpha(t)^2 Q_2(t) + \frac{(-1)^n}{2^{n-1}} \delta(t)^{n-3} \alpha(t)^2 Q_1(t).
\]

and using the same formula applied in \(Q_{n-1}(t)\):

\[Q_{n-1}(t) = \frac{1}{2} \delta(t) Q_{n-2}(t) - \frac{1}{4} \alpha(t)^2 Q_{n-3}(t) + \frac{1}{8} \delta(t) \alpha(t)^2 Q_{n-4}(t) - \frac{1}{16} \delta(t)^2 \alpha(t)^2 Q_{n-5}(t) + \frac{1}{32} \delta(t)^3 \alpha(t)^2 Q_{n-6}(t) - \cdots + \frac{(-1)^{n-2}}{2^{n-3}} \delta(t)^{n-5} \alpha(t)^2 Q_2(t) + \frac{(-1)^{n-1}}{2^{n-2}} \delta(t)^{n-4} \alpha(t)^2 Q_1(t).
\]

If we change \(Q_{n-1}\) in the first equation for the right hand side of the second equation we get:

\[Q_n(t) = \frac{\delta(t)^2 - \alpha(t)^2}{4} Q_{n-2}(t).\]

\[\square\]
Consider
\[ c(t) = \lim_{n \to \infty} \frac{1}{n} \log \left( \int e^{t(A(x_0)+A(x_1)+...+A(x_n))} d\mu(x) \right). \]

\( c(t) \) is sometimes called the free energy on the point \( t \) for the probability \( \mu \) and the classical observable \( A \) (see [4]).

As we said before if \( c(t) \) is differentiable for all \( t \in \mathbb{R} \), then a Large Deviation Principle is true (see [4]) for the sums
\[ \frac{1}{n+1} \left( A(x_0)+A(x_1)+...+A(x_n) \right) = \frac{1}{n+1} \left( A(x)+A(\sigma(x))+...+A(\sigma^n(x)) \right). \]

We just show that a L.D.P. is true for such class of \( A \) because we show the explicit form of \( c \) and it is differentiable (see (8)).

5 \( \mu \) is not mixing

We will show that the probability \( \mu \) is not mixing.

Theorem 6. For any \( n \geq 2 \) we have that
\[
\sum_{j_2, \ldots, j_n=1}^{2} \mu(a, b, j_2, \ldots, j_n, c, d) = \mu(a, b)\mu(c, d) + \frac{(-1)^n(\beta_b - \beta_a)(\beta_c - \beta_d)}{2^n}
\]

Proof: Using theorem 3 we get
\[
\mu(a, b, j_2, \ldots, j_n, c, d) = \frac{\mu(b, j_2, \ldots, j_n, c, d)}{2} - \frac{\beta_a \beta_b}{2^n} \mu(j_2, \ldots, j_n, c, d) + \frac{\beta_a \beta_j}{2^n} \mu(j_3, j_4, \ldots, j_n, c, d) - \frac{\beta_a \beta_j}{2^n} \mu(j_2, j_3, j_4, \ldots, j_n, c, d) + \ldots + \frac{(-1)^n \beta_a \beta_j}{2^n+1} \mu(c, d)
\]
Note that
\[
\sum_{j_2=1}^{2} \frac{\beta_a \beta_j}{2^n} \mu(j_3, j_4, \ldots, j_n, c, d) = 0,
\]
as \( \beta_2 = -\beta_1 \) and \( j_2 \) only appears in \( \beta_{j_2} \). And in general, for \( k = 2, \ldots, n \)
\[
\sum_{j_k=1}^{2} \beta_{j_k}(...) = 0.
\]
Therefore we will separate all terms that contains \( \beta_{j_k} \),

\[
\mu(a, b, j_2, \ldots, j_n, c, d) = \frac{\mu(b, j_2, \ldots, j_n, c, d)}{2} - \frac{\beta_a \beta_b}{2^2} \mu(j_2, \ldots, j_n, c, d) + \\
\frac{(-1)^{n+1} \beta_a (\beta_c - \beta_d)}{2^{n+3}} + \sum_{k=2}^{n} \beta_{j_k}(...) = \\
\frac{1}{2} \left( \frac{\mu(j_2, \ldots, j_n, c, d)}{2} - \frac{\beta_b \beta_{j_2}}{2^2} \mu(j_3, \ldots, j_n, c, d) \right) + \\
\frac{1}{2} \left( \frac{\beta_b \beta_{j_3}}{2^3} \mu(j_4, \ldots, j_n, c, d) - \frac{\beta_b \beta_{j_4}}{2^4} \mu(j_5, \ldots, j_n, c, d) \right) + \ldots \\
- \frac{\beta_a \beta_b}{2^2} \mu(j_2, \ldots, j_n, c, d) + \frac{(-1)^{n+1} \beta_a (\beta_c - \beta_d)}{2^{n+3}} + \sum_{k=2}^{n} \beta_{j_k}(...) = \\
\frac{\mu(j_2, \ldots, j_n, c, d)}{2^2} - \frac{\beta_a \beta_b}{2^2} \mu(j_2, \ldots, j_n, c, d) + \\
\frac{(-1)^n \beta_b (\beta_c - \beta_d)}{2^{n+3}} + \frac{(-1)^{n+1} \beta_a (\beta_c - \beta_d)}{2^{n+3}} + \sum_{k=2}^{n} \beta_{j_k}(...) = \\
\mu(a, b) \mu(j_2, \ldots, j_n, c, d) + \frac{(-1)^n (\beta_b - \beta_a)(\beta_c - \beta_d)}{2^{n+3}} + \sum_{k=2}^{n} \beta_{j_k}(...).
\]

Then

\[
\mu(a, b, j_2, \ldots, j_n, c, d) = \\
\mu(a, b) \mu(j_2, \ldots, j_n, c, d) + \frac{(-1)^n (\beta_b - \beta_a)(\beta_c - \beta_d)}{2^{n+3}} + \sum_{k=2}^{n} \beta_{j_k}(...).
\]

We know that \( \mu \) is stationary, so

\[
\sum_{j_2, \ldots, j_n} \mu(j_2, \ldots, j_n, c, d) = \mu(c, d).
\]

It follows that

\[
\sum_{j_2, \ldots, j_n} \mu(a, b, j_2, \ldots, j_n, c, d) = \mu(a, b) \mu(c, d) + \frac{(-1)^n (\beta_b - \beta_a)(\beta_c - \beta_d)}{2^4}.
\]
From the above for \((a, b) = (1, 2)\) and \((c, d) = (2, 1)\) we get that
\[
\lim_{n \to \infty} \mu(\sigma^{-2n}(c, d) \cap (a, b)) \neq \mu(a, b) \mu(c, d),
\]
and this shows that \(\mu\) is not mixing (see [11] or [15]). This also implies that for some continuous functions there is no decay of correlation to 0.

6 \(\mu\) is not Gibbs (for a continuous normalized potential)

Now we investigate other kind of questions.

Consider a generic element \(x = (k_0, k_1, k_2, \ldots, k_n, \ldots) \in \{1, 2\}^\mathbb{N}\).

Expression (6) can be written as
\[
\mu(k_n, k_{n-1}, k_{n-2}, \ldots, k_1, k_0) = \frac{1}{2} \mu(k_{n-1}, k_{n-2}, \ldots, k_1, k_0)
- \frac{\beta_k \beta_{k_n-1}}{4} \mu(k_{n-2}, k_{n-3}, \ldots, k_1, k_0) +
\frac{\beta_k \beta_{k_{n-2}}}{8} \mu(k_{n-3}, k_{n-4}, \ldots, k_1, k_0) - \frac{\beta_k \beta_{k_{n-3}}}{16} \mu(k_{n-4}, k_{n-5}, \ldots, k_1, k_0) + \ldots
+ (-1)^{n-2} \frac{\beta_k \beta_{k_2}}{2^{n-1}} \mu(k_1, k_0) + (-1)^{n-1} \frac{\beta_k \beta_{k_1}}{2^n + 1} + (-1)^n \frac{\beta_k \beta_{k_0}}{2^n + 1}.
\]

Therefore, from (5) we get that
\[
\mu(k_0, k_1, k_2, \ldots, k_{n-1}) = \frac{1}{2} \mu(k_0, k_1, k_2, \ldots, k_{n-2}, k_{n-1})
- \frac{\beta_k \beta_{k_{n-1}}}{4} \mu(k_0, k_1, k_2, \ldots, k_{n-2}, k_{n-1}) +
\frac{\beta_k \beta_{k_{n-2}}}{8} \mu(k_0, k_1, k_2, \ldots, k_{n-3}, k_{n-2}) - \frac{\beta_k \beta_{k_{n-3}}}{16} \mu(k_0, k_1, k_2, \ldots, k_{n-5}, k_{n-4}) + \ldots
+ (-1)^{n-2} \frac{\beta_k \beta_{k_2}}{2^{n-1}} \mu(k_0, k_1) + (-1)^{n-1} \frac{\beta_k \beta_{k_1}}{2^n + 1} + (-1)^n \frac{\beta_k \beta_{k_0}}{2^n + 1}. \tag{9}
\]

Note that
\[
\frac{\mu(k_0, k_1, k_2, \ldots, k_{n-1}, k_n)}{\beta_k} = \frac{\mu(k_0, k_1, k_2, \ldots, k_{n-2}, k_{n-1})}{2 \beta_k}.
\]
\[
\frac{\beta_{k_{n-2}}}{8} \mu(k_0, k_1, k_2, \ldots, k_{n-4}, k_{n-3}) - \frac{\beta_{k_{n-3}}}{16} \mu(k_0, k_1, k_2, \ldots, k_{n-5}, k_{n-4}) + \ldots
\]
\[
+ (-1)^{n-2} \frac{\beta_{k_2}}{2^{n-1}} \mu(k_0, k_1) + (-1)^{n-1} \frac{\beta_{k_1}}{2^{n+1}} + (-1)^n \frac{\beta_{k_0}}{2^{n+1}}. \tag{10}
\]

For example
\[
\frac{\mu(k_0, k_1, k_2, k_3)}{\beta_{k_3}} = \frac{\mu(k_0, k_1, k_2)}{2\beta_{k_3}} - \frac{\beta_{k_2}}{4} \mu(k_0, k_1) + \frac{\beta_{k_1}}{16} - \frac{\beta_{k_0}}{16},
\]

and
\[
\frac{2\mu(k_0, k_1, k_2, k_3, k_4)}{\beta_{k_4}} = \frac{\mu(k_0, k_1, k_2, k_3)}{\beta_{k_4}} - \frac{\beta_{k_3}}{2} \mu(k_0, k_1, k_2) + \frac{\beta_{k_2}}{4} \mu(k_0, k_1) - \frac{\beta_{k_1}}{16} + \frac{\beta_{k_0}}{16}
\]

If we add this two last equations we get
\[
2\mu(k_0, \ldots, k_4) = \left(1 - \frac{\beta_{k_4}}{\beta_{k_3}}\right) \mu(k_0, \ldots, k_3) + \beta_{k_4} \left(\frac{1}{2\beta_{k_3}} - \frac{\beta_{k_3}}{2}\right) \mu(k_0, k_1, k_2)
\]

This can be generalized for each \( n \) as below

**Proposition 7.**
\[
2\mu(k_0, k_1, \ldots, k_n) = \left(1 - \frac{\beta_{k_n}}{\beta_{k_{n-1}}}\right) \mu(k_0, \ldots, k_{n-1}) + \beta_{k_n} \left(\frac{1}{2\beta_{k_{n-1}}} - \frac{\beta_{k_{n-1}}}{2}\right) \mu(k_0, \ldots, k_{n-2})
\]

We can also use the symmetric expression (5) in order to write

**Proposition 8.**
\[
\mu(k_0, k_1, \ldots, k_n) = \frac{1}{2} \left(1 - \frac{\beta_{k_0}}{\beta_{k_1}}\right) \mu(k_1, \ldots, k_n) + \frac{\beta_{k_0}}{2} \left(\frac{1}{2\beta_{k_1}} - \frac{\beta_{k_1}}{2}\right) \mu(k_2, \ldots, k_n)
\]

**Example:** We remember that \( \beta_1 = \sin(2\theta) \) and \( \beta_2 = -\beta_1 \). In the particular case that \( \theta = \pi/4 \) we have \( \beta_1 = 1, \beta_2 = -1 \) and, using proposition 7.
\[ \mu(k_0, k_1, \ldots, k_n) = \frac{1}{2} \left( 1 - \frac{\beta_k}{\beta_{k-1}} \right) \mu(k_0, \ldots, k_{n-1}) \]

If \( k_n = k_{n-1} \) then \( \mu(k_0, k_1, \ldots, k_n) = 0 \). We conclude that \( \mu \) is supported in the periodic orbit of the point \((0, 1, 0, 1, 0, 1, \ldots)\).

We denote
\[ a(k_0, k_1) = \frac{1}{2} \left( 1 - \frac{\beta_{k_0}}{\beta_{k_1}} \right) , \]
\[ b(k_0, k_1) = \frac{1}{4} \beta_{k_0} \left( \frac{1}{\beta_{k_1}} - \beta_{k_1} \right) , \]
where \( k_0, k_1 \in \{1, 2\} \)

\[ \gamma = \frac{1}{4} \left( 1 - \beta_1^2 \right) = \mu(1, 1). \]

From the above proposition
\[ \mu(k_0, k_1, \ldots, k_n) = a(k_0, k_1)\mu(k_1, \ldots, k_n) + b(k_0, k_1)\mu(k_2, \ldots, k_n). \quad (11) \]

The possible values of \( a(k_0, k_1) \) and \( b(k_0, k_1) \) are
a) if \( k_0 = k_1 \), then \( a(k_0, k_1) = 0 \) and \( 0 < b(k_0, k_1) = \frac{1}{4}(1 - \beta_1^2) = \gamma < \frac{1}{4} \)

b) if \( k_0 \neq k_1 \), then \( a(k_0, k_1) = 1 \) and \( -\frac{1}{4} < b(k_0, k_1) = \frac{1}{4}(\beta_1^2 - 1) = -\gamma < 0 \)

**Proposition 9.** Suppose \( \theta \neq \pi/4 \). Then \( \mu \) is positive on cylinders sets.

**Proof.** The probability of cylinders of size 1, 2 and 3 are not zero. Suppose that \( \mu(x_1, \ldots, x_n) > 0 \) for any cylinder set of size \( n \). As \( \mu \) is stationary
\[ \mu(x_0, \ldots, x_n) = \mu(1, x_0, x_1, \ldots, x_n) + \mu(2, x_0, x_1, \ldots, x_n) \geq \mu(x_0, x_0, x_1, \ldots, x_n) \]
\[ = a(x_0, x_0)\mu(x_0, \ldots, x_n) + b(x_0, x_0)\mu(x_1, \ldots, x_n) = \gamma\mu(x_1, \ldots, x_n) > 0. \]

Then the result follows by induction. \( \square \)

We get also a corollary about the Jacobian \( J \) of \( \mu \). Define for \( x = (x_0, x_1, x_2, \ldots) \)
\[ J(x) = \lim_{n \to \infty} \frac{\mu(x_0, \ldots, x_n)}{\mu(x_1, \ldots, x_n)} \]
if the limit exists. It is known that \( J \) is well defined for \( \mu \) a.e. \( x \) and can be seen as the Radon-Nikodym derivative of \( \mu \) over the inverse branches of \( \sigma \) (see [14], [11] or [15]).
From (11)
\[
\frac{\mu(k_0, k_1, ..., k_n)}{\mu(k_1, ..., k_n)} = a(k_0, k_1) + b(k_0, k_1) \frac{\mu(k_2, ..., k_n)}{\mu(k_1, ..., k_n)},
\]
then, we get the following

**Corollary 10.**

\[
J(k_0, k_1, k_2, ...) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{J(k_1, k_2, k_3, ...)}. 
\]  

(13)

According to (13) \( J : \{1, 2\}^\mathbb{N} \to \mathbb{R} \) is an unknown function which satisfies the property:

\[
J(k_0, k_1, k_2, ...) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{J(k_1, k_2, k_3, ...)} = \\
a(k_0, k_1) + b(k_0, k_1) \frac{1}{a(k_1, k_2) + b(k_1, k_2) \frac{1}{J(k_2, k_3, k_4, ...)}}.
\]

**Remark:** We have that \( \gamma \leq J \leq 1 - \gamma \).

Indeed, if we denote by \( J^n(x) := \frac{\mu(x_0, ..., x_n)}{\mu(x_1, ..., x_n)} \), \( x = (x_0, x_1, ...) \). From (12) we get

\[
J^n(x_0, x_1, x_2, ...) = a(x_0, x_1) + b(x_0, x_1) \frac{1}{J^{n-1}(x_1, x_2, x_3, ...)}.
\]

It follows that

\[
1 \geq J^{n+1}(x_0, x_0, x_1, x_2, x_3, ...) = \gamma \frac{1}{J^n(x_0, x_1, x_2, x_3, ...)}
\]

(14)

and then

\[
J^n(x_0, x_1, ...) \geq \gamma
\]

for any \( n \) and \( x = x_0, x_1, ... \).

As the sum \( J^n(1, x_2, x_3, ..) + J^n(2, x_2, x_3, ..) = 1 \) we get that also \( J^n \leq 1 - \gamma \).

**Proposition 11.** \( J \) is not defined in \( 1^\infty = (1, 1, 1, 1, ...) \).
Proof. from (12) we get
\[
\frac{\mu(k_0, k_1, ..., k_n)}{\mu(k_1, ..., k_n)} = a(k_0, k_1) + b(k_0, k_1) \frac{\mu(k_2, ..., k_n)}{\mu(k_1, ..., k_n)},
\]
where \(a(1, 1) = 0\) and \(0 < b(1, 1) = \gamma < 1\).
and then
\[
\frac{\mu(1, ..., 1)_{n+1}}{\mu(1, ..., 1)_n} = \gamma \frac{\mu(1, ..., 1)_{n-1}}{\mu(1, ..., 1)_n},
\]
Note that
\[
\frac{\mu(11)}{\mu(1)} = 2 \gamma.
\]
By induction
\[
\frac{\mu(1, ..., 1)_{n+1}}{\mu(1, ..., 1)_n} = 1/2,
\]
for \(n\) even, and
\[
\frac{\mu(1, ..., 1)_{n+1}}{\mu(1, ..., 1)_n} = 2 \gamma,
\]
for \(n\) odd. Then, \(J(1^\infty)\) does not exist.
\[\square\]

One can consider on \(\Omega = \{1, 2\}^N\) the metric \(d(x, y) = \alpha^{-n}\), where \(n\) is the first symbol of \(x = (x_0, x_1, ..)\) and \(y = (y_0, y_1, ..)\) such that they disagree, and \(\alpha > 1\). Let \(h(\mu)\) be the Kolmogorov-Sinai entropy of \(\mu\).

We will use the following definition of Gibbs state:

**Definition 12.** Suppose \(f\) is a continuous potential \(f : \Omega = \{1, 2\}^N \rightarrow \mathbb{R}\), such that, for all \(y \in \Omega\) we have \(\sum_{\sigma(y) = x} e^{f(y)} = 1\). Then, if \(\nu\) is a \(\sigma\)-invariant probability such that:
\[
\sup_{\rho} \left\{ h(\rho) + \int f \, d\rho \right\} = h(\nu) + \int f \, d\nu,
\]
we say that \(\nu\) is a Gibbs state for \(f\).
Such probabilities are also called $g$-measures.

There exists a continuous function $f$ as above such that has more than one Gibbs state (see [16] and references there).

**Theorem 13.** $\mu$ is not a Gibbs state for a continuous potential.

**Proof:**

Suppose by contradiction that $\mu$ is Gibbs for a continuous function $f$.

First note that

$$\sup_{\rho} \left\{ h(\rho) + \int f \, d\rho \right\} = 0.$$ 

It is known that $h(\rho) = \int \log J_\rho \, d\rho$, and then, it follows from Lemma 3.3 and Lemma 3.4 in [14] that

$$\sup_{\rho} \left\{ h(\rho) + \int f \, d\rho \right\} \leq 0.$$ 

By hypothesis $h(\mu) + \int f \, d\mu = 0$.

This shows the claim.

Denote by $L_f$ the Ruelle operator for $f$ (see [14]). Note that $L_f(1) = 1$.

It follows from Proposition 3.4 in [14] that $\mu$ is a fixed point of the dual $L^*_f$ (it is used just the fact that $f$ is continuous). This means that for any function $\varphi$ we have

$$\int L_f(\varphi) \, d\mu = \int \varphi \, d\mu.$$ 

Taking $\varphi = I_{(i_0,i_1,\ldots,i_n)}$ we get that $L_f(\varphi)(x) = I_{(i_1,i_2,\ldots,i_n)}(x) e^{f(i_0,x)}$.

In this way we get

$$\mu(i_0,i_1,\ldots,i_n) = \int_{(i_1,i_2,\ldots,i_n)} e^{f(i_0,x)} \, d\mu(x),$$

and hence

$$\frac{\mu(i_0,i_1,\ldots,i_n)}{\mu(i_1,i_2,\ldots,i_n)} = \frac{\int_{(i_1,i_2,\ldots,i_n)} e^{f(i_0,x)} \, d\mu(x)}{\mu(i_1,i_2,\ldots,i_n)}.$$ 

The right hand side is an average of values of $e^f$ over decreasing cylinder sets.

Since $e^f$ is assumed to be a continuous function, this converges uniformly in the sequence $(i_0,i_1,\ldots,i_n)$ to $e^{f(i_0,i_1,\ldots,i_n)}$.

Then, for any $i \in \{1,2\}$, we have that for any $x = (i,x_1,\ldots,x_n,\ldots)$ the limit exists

$$\lim_{n \to \infty} \frac{\mu(i,x_1,\ldots,x_n)}{\mu(x_1,\ldots,x_n)} = e^{f(i,x_1,\ldots,x_n,\ldots)}.$$
But this property is not true for \( x = (1^\infty) \) as we just showed above (see Proposition 11).

We will make some final remarks about \( J \)

**Definition 14.** Given \((k_0, k_1, k_3, ..., k_n, ...)\) we denote \( \tilde{J} \) the continuous fraction expansion expression

\[
\tilde{J}(k_0, k_1, k_3, ..., k_n, ...) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{a(k_1, k_2) + b(k_1, k_2) \frac{1}{a(k_2, k_3) + b(k_2, k_3) \frac{1}{...}}} \tag{16}
\]

if this converges, that is, if there exist the limit of the finite truncation.

In this sense \( \tilde{J} \) has an expression in continued fraction (see expression (b) in page 2 in [5] for the general setting).

It is easy to see that the function \( \tilde{J} \) where defined is a solution of (13).

Denote \( p = \frac{1+\beta_1}{2} \) where \( \beta_1 = \sin(2 \theta) \). We will show in the next section that the only possible (convergent) values of the continuous fraction expansion (16) of \( \tilde{J} \) are \( p \) and \( 1-p \).

7 Up to a measurable change of coordinates \( \mu \) is Bernoulli

We will show that the invariant probability \( \mu \) that we get is measurably equivalent to the Bernoulli independent probability on \( \{1, 2\}^\mathbb{N} \) associated to the probabilities \( p \) and \( 1-p \), where \( p = \frac{1+\beta_1}{2} \) and \( \beta_1 = \sin(2 \theta) \). Therefore, the entropy of \( \mu \) is \(-p \log p - (1-p) \log(1-p)\).

Remember that we denote

a) if \( k_0 = k_1 \), then \( a(k_0, k_1) = 0 \) and \( 0 < b(k_0, k_1) = \frac{1}{4}(1 - \beta_1^2) = \gamma < \frac{1}{4} \)

b) if \( k_0 \neq k_1 \), then \( a(k_0, k_1) = 1 \) and \(-\frac{1}{4} < b(k_0, k_1) = \frac{1}{4}(\beta_1^2 - 1) = -\gamma < 0 \)

The measure \( \mu \) can be computed recursively in finite cylinders by \( \mu(1) = \mu(\overline{1}) = \mu(2) = \mu(\overline{2}) = 1/2 \) and, for \( n \geq 1 \), we get for the other cylinder sets

\[
\mu(k_0, k_1, k_2, ..., k_n) = a(k_0, k_1)\mu(k_1, k_2, k_3, ..., k_n) + b(k_1, k_2)\mu(k_2, k_3, ..., k_n).
\]

The Jacobian \( J \) of the invariant probability \( \mu \) given by

\[
J(k_0, k_1, k_2, ..., k_n, ...) = \lim_{n \to \infty} \frac{\mu(k_0, k_1, k_2, ..., k_n)}{\mu(k_1, k_2, ..., k_n)}
\]
exists $\mu$ almost everywhere.

It satisfies the property: given $k = (k_0, k_1, k_2, ..., k_n, ...) \in \{1, 2\}^\mathbb{N}$, then

$$J(k_0, k_1, k_2, ..., k_n, ...) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{a(k_1, k_2) + b(k_1, k_2) \frac{1}{a(k_2, k_3) + b(k_2, k_3) \frac{1}{...}}},$$

(17)

in the case it converges, that is, if there exist the limit of the finite truncation

$$\mu(k_0, k_1, k_2, ..., k_n) = \frac{a(k_0, k_1) + b(k_0, k_1)}{a(k_1, k_2) + b(k_1, k_2) \frac{1}{a(k_2, k_3) + b(k_2, k_3) \frac{1}{...}}}.$$

The following relations follow from the previous equalities:

$$J(k_0, k_1, k_2, ...) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{J(k_1, k_2, k_3, ...)}. \quad (18)$$

and

$$J(k_0, k_1, k_2, ...) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{a(k_1, k_2) + b(k_1, k_2) \frac{1}{J(k_2, k_3, k_4, ...)}}. \quad (19)$$

Now we will look at changes of the sequence of successive pairs in the sequence $k$. We will introduce another code.

If $k_0 = k_1$ then we associated the symbol $a$ and if $k_0 \neq k_1$ we associate the symbol $b$.

From this procedure we get from each given sequence $k \in \{1, 2\}^\mathbb{N}$ a new sequence $m = (m_1, m_2, m_3, ...) \in \{a, b\}^\mathbb{N}$ by looking if there is a change or not in the string $k$ by using the rules

$$\begin{align*}
\begin{array}{c}
\underbrace{11}_a, \\
\underbrace{22}_a, \\
\underbrace{12}_b, \\
\underbrace{21}_b
\end{array}
\end{align*}$$

For example, given a sequence $k$ of the form

$$k = (121122...),$$

then, we associate $m = (b, b, a, b, ..)$

Given $m$ we can find the $k = (k_0, k_1, k_2, ...) \to m$ it is associated if we know the first element $k_0 = 1$ or $k_0 = 2$.

We get the rules:
a) If \( m_1 = a, m_2 = a \), then \( J(m_1, m_2, m_3, \ldots) = J(m_3, m_4, \ldots) \).

b) If \( m_1 = a, m_2 = b \), then \( J(m_1, m_2, m_3, \ldots) = \frac{1}{\gamma - 1 - J(m_3, m_4, \ldots)} \).

c) If \( m_1 = b, m_2 = a \), then \( J(m_1, m_2, m_3, \ldots) = 1 - J(m_3, m_4, \ldots) \).

d) If \( m_1 = b, m_2 = b \), then \( J(m_1, m_2, m_3, \ldots) = 1 - \frac{1}{\gamma - 1 - J(m_3, m_4, \ldots)} \).

For example, in terms of sequences \( k \) the item

I) \( J(1,1,1,k_3,\ldots) = J(1,k_3,\ldots) \) and

II) \( J(2,2,2,k_3,\ldots) = J(2,k_3,\ldots) \)

In this way if \( k = (a_0, a_2, a_3, \ldots) \in \{1,2\}^N \) is an element where the fraction expansion \( J(k) \) limit exists, then this value does not change if we delete 00 (respectively, 11) of the sequence \( k \) where appears 000 (respectively, 111).

Indeed, this corresponds to \( m_1 = a = m_2 \). In other words, this corresponds to a sequence \( a.a \). In this way when looking to sequences \( m \) the parts with \( (aa)^{2n} \), \( n \in \mathbb{N} \) can be deleted (when \( J \) converges.)

This shows that is more appropriate to consider the \( m \) expansion than the \( k \) expansion in this moment.

Moreover, if \( m = (m_1, m_2, m_3, m_4, \ldots) \in \{a,b\}^N \) is an element where the the fraction expansion \( J(k) = J(m) \) limit exists, then one can show that this value does not change if we delete parts of the sequence where appears \( b.a.b \) and \( a.b.a \).

This means when considering \( a.b.a.b \) we get for example:

\[ J(1,1,2,1,k_5,\ldots) = J(1,k_5,\ldots) \]

that is we can delete 1,1,2,2.

For example

\[ J(a,b,a,b,m_5,m_6,\ldots) = J(m_5,m_6,\ldots) \]

\[ J(a,b,a,b,a,b,m_9,m_{10},\ldots) = J(m_9,m_{10},\ldots) \]

\[ J(b,a,b,a,m_5,m_6,\ldots) = J(m_5,m_6,\ldots) \]

and

\[ J(m_1,m_2,a,b,a,b,m_7,m_8,\ldots) = J(m_1,m_2,m_7,m_8,\ldots) \]

Then, it is natural to consider one level up of symbolic representation.

We get a new code introducing a new dictionary where we associate \( \alpha = ab \) and \( \beta = ba \).
As we have seen, we can substitute $aa$ by $\emptyset$.
One can also substitute

$$bb \text{ by } \beta \alpha = baab.$$  

So we can divide an $m$ string in blocks of two letters and make the above substitutions, obtaining a new string $w = (w_1, w_2, w_3, \ldots) \in \{\alpha, \beta\}^N$.

For example we associate

$$m = (a, b, b, a, b, a, a, b, \ldots)$$

to

$$w = (\alpha, \beta, \beta, \alpha, \ldots),$$

and

$$m = (a, b, a, b, a, b, a, b, \ldots)$$

to

$$w = (\alpha, \alpha, \alpha, \beta, \ldots).$$

By considering the compositions of the corresponding functions, one can substitute (delete, indeed) parts of $w$ strings

$$\beta \beta = baba \text{ by } \emptyset$$

$$\alpha \alpha = abab \text{ by } \emptyset.$$  

We present some examples in order to illustrate the code: from the rules about $baba$ and $abab$ we get that in the case $J$ converges

$$J(\alpha, \alpha, \alpha, \beta, w_5, \ldots) = J(\alpha, \beta, w_5, \ldots),$$

$$J(\alpha, \alpha, \alpha, \alpha, \beta, w_6, \ldots) = J(\beta, w_6, \ldots),$$

$$J(\beta, \beta, \beta, \beta, \alpha, w_6, \ldots) = J(\alpha, w_6, \ldots)$$

and

$$J(\beta, \alpha, \alpha, \alpha, \alpha, \beta, w_8, \ldots) = J(\beta, \alpha, \beta, w_8, \ldots).$$
In the finite fraction expansion of odd order of $k$ (which is associated to a certain string $m$ of even order and so to a string $w$) we can delete parts (in the $\alpha, \beta$ dictionary expansion) in such a way that we end up with the estimation of $J$ in a string $w$ of one of the kinds:

a) $(\alpha \beta)^n$ or $(\alpha \beta)^n \alpha$,

or

b) $(\beta \alpha)^n$ or $(\beta \alpha)^n \beta$.

Indeed, note that

$$(\beta \alpha)^{n_1} (\beta \alpha)^{n_2} \alpha (\beta \alpha)^{n_3} \beta (\beta \alpha)^{n_4} \beta (\beta \alpha)^{n_5} \alpha (\beta \alpha)^{n_6} \cdots (\beta \alpha)^{n_{s-1}} \beta (\beta \alpha)^{n_s} =

(\beta \alpha)^{n_1 + n_2 + \cdots + n_s + s/2 - 1} \beta$$

Consider the transformation

$$x \rightarrow f_1(x) = 1 - \frac{1}{\gamma - 1 - \frac{1}{x}}$$

The string $\beta \alpha$ means $baab$, which corresponds to $J(m_1, m_2, m_3, \ldots) = 1 - \frac{1}{\gamma - 1 - \frac{1}{J(m_3, m_4, \ldots)}}$.

Note that if the expansion $J(m_3, m_4, \ldots)$ exists for the string $(m_3, m_4, \ldots)$, then it also exists the one for $J(m_1, m_2, m_3, \ldots)$.

In this way $f_1(J(m_3, m_4, \ldots)) = J(m_1, m_2, m_3, \ldots)$.

Consider now $f_2$ defined by

$$x \rightarrow f_2(x) = \frac{1}{\gamma - 1 - \frac{1}{1-x}}.$$

The string $\alpha \beta$ means $abba$, that is, corresponds to $J(m_1, m_2, m_3, \ldots) = \frac{1}{\gamma - 1 - \frac{1}{1-J(m_3, m_4, \ldots)}}$.

In this way $f_2(J(m_3, m_4, \ldots)) = J(m_1, m_2, m_3, \ldots)$.

Note that the fixed points for both functions $f_1(x) = 1 - \frac{1}{\gamma - 1 - \frac{1}{x}}$ and $f_2(x) = \frac{1}{\gamma - 1 - \frac{1}{1-x}}$ are the same and equal to

$$\frac{1 \pm \sqrt{1 - 4 \gamma}}{2} = \frac{1 \pm \beta_1}{2}.$$

Recall that $p = \frac{1 + \beta_1}{2}$. Note that the interval $[1 - p, p]$ is invariant by $f_1$ and also by $f_2$. The point $p$ is a global attractor for $f_1$ in $(1 - p, p]$, and $1 - p$ is a global attractor for $f_2$ in $[1 - p, p)$.

In the points where $J$ exists the expression (17) is true.
As we have seen in the beginning of our discussion, it is natural to truncate $J(k_0, k_1, k_2, \ldots)$ (at level $r$ for instance) by taking in the last position $r$ in the expansion of $J$ the value $1/2$. As $1/2$ belongs $[1 - p, p]$ (in fact is its center), and the interval $[1 - p, p]$ is left invariant by the diffeomorphisms $g_a(x) = \frac{x}{2}$ and $g_b(x) = 1 - \frac{x}{2}$, when the limit exists the successive truncations should converge to $p$ or to $1 - p$.

Then, the only possible (convergent) values of the continuous fraction expansion (17) of $J$ are $p$ or $1 - p$.

Now we will show that $\mu$ is measurably equivalent to the Bernoulli probability $\mu_p$ with parameter $p$. Denote $A = \{\theta \in \{1, 2\}^\mathbb{N} | J(\theta) = p\}$ and $B = \{\theta \in \{1, 2\}^\mathbb{N} | J(\theta) = 1 - p\}$. From the before reasoning we get $\mu(A \cup B) = 1$. As $\gamma/p = 1 - p$, $1 - \gamma/p = p$, $\gamma/(1 - p) = p$ and $1 - \gamma/(1 - p) = 1 - p$, we get from

$$J(k_0, k_1, k_2, \ldots) = a(k_0, k_1) + b(k_0, k_1) \frac{1}{J(k_1, k_2, k_3, \ldots)},$$

the following properties:

- If $\theta = (k_0, k_1, k_2, \ldots) \in A$ begins with 1, then $2\theta \in A$, $1\theta \in B$,
- If $\theta = (k_0, k_1, k_2, \ldots) \in A$ begins with 2, then $1\theta \in A$, $2\theta \in B$,
- If $\tilde{\theta} = (k_0, k_1, k_2, \ldots) \in B$ begins with 1, then $1\tilde{\theta} \in A$, $2\tilde{\theta} \in B$,

and

- If $\tilde{\theta} = (k_0, k_1, k_2, \ldots) \in B$ begins with 2, then $2\tilde{\theta} \in A$, $1\tilde{\theta} \in B$.

Therefore, $\sigma(A) = \sigma(B) = A \cup B = \{1, 2\}^\mathbb{N}$ (mod 0), and $\sigma|A$ and $\sigma|B$ are injective. Then, since the Jacobian of $\mu$ in $A$ is $p$, for a measurable $X \subset A$, we get $\mu(\sigma(X)) = \mu(X)/p$ and, since the Jacobian of $\mu$ in $B$ is $1 - p$, for measurable $Y \subset B$, we get $\mu(\sigma(Y)) = \mu(Y)/(1 - p)$.

So, we can define a measurable automorphism $h$ preserving probabilities such that is defined for almost every point, $h : (\{1, 2\}^\mathbb{N}, \mu) \to (\{0, 1\}^\mathbb{N}, \mu_p)$, where $\mu_p$ is the Bernoulli independent probability associated to $p$ for 0, and $1 - p$ for 1. More precisely, $h(\theta) = (c_0, c_1, c_2, \ldots)$, where $c_j = 0$, if $\sigma^j(\theta) \in A$, and $c_j = 1$, if $\sigma^j(\theta) \in B$.

References


